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AN OPERATOR SPACE CHARACTERIZATION OF FRÉCHET SPACES NOT CONTAINING l^1

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Dedicated to the memory of Susanne Dierolf

Abstract: The classes of Fréchet spaces not containing l^1, of Gelfand-Phillips spaces, and of dual Gelfand-Phillips spaces are characterized by (pre)compactness criteria for sets of bounded linear operators transforming bounded sets into precompact sets.

Keywords: Locally convex spaces, no-containment of l^1, limited sets, precompact operators.

1. Introduction

For Banach spaces X and Y, the following compactness criterion for subsets of the space $K(X, Y)$ of compact linear operators from X into Y has been established in [6, Theorem 1]:

If X does not containing (an isomorphic copy of) l^1, then a subset H of $K(X, Y)$ is relatively compact in operator norm if and only if

(i) $Hx = \{hx \mid h \in H\}$ *is relatively compact in Y for all $x \in X$;*
(ii) $\|hx_n\| \to 0$ *uniformly over $h \in H$ for all weak nullsequences $(x_n)_n \subset X$.*

The purpose of this note is (a) to extend this result to Fréchet spaces, and, (b) to show that the criterion actually characterizes non-containment of l^1. Also, the method of proof will be shortcut by reducing the characterization to a simple combination of (a linearized version of) the Arzela-Ascoli Theorem and Rosenthal's characterization of non-containment of l^1. As a byproduct, the linearized Arzela-Ascoli Theorem (Proposition 2.1 below) also leads to operator space characterizations of the Gelfand-Phillips property for locally convex spaces or strong duals of such.

Notation and Terminology (A). Given a locally convex space X, X_σ will denote X endowed with the weak topology $\sigma(X, X')$, while X'_b, respectively, X'_τ will denote the dual of X endowed with the strong, respectively, the Mackey topology, and X'_c, respectively, X'_λ will denote the dual endowed with the topology

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of uniform convergence on all compact convex, respectively, all precompact subsets of X. Given a subset C of X, $C^\circ := \{x' \in X' \mid |\langle x', x\rangle| \leqslant 1 \text{ for all } x \in C\}$ will denote its (absolute) polar in X'.

A subset C of X is called *limited* if $\langle c, x_n^*\rangle \to 0$ uniformly over all $c \in C$ for any equicontinuous weak*−nullsequence in X'. Obviously, all precompact subsets of X are limited. X will be called a *Gelfand-Phillips space* (abbreviated by X is (GP)) if, conversely, all limited subsets of X are precompact ([9]).

(B). Given locally convex spaces X and Y, the basic operator space to be considered here is the space $K_b^b(X,Y)$ of all weak-to-weak-continuous linear operators from X into Y that transform bounded subsets of X into precompact subsets of Y, endowed with the topology of uniform convergence on the bounded subsets of X.

Moreover, we shall also consider the ϵ−product $X\epsilon Y$ of X and Y, which is the space $L_e(X'_c, Y)$ of all weak*−weakly continuous linear operators from X' to Y that transform equicontinuous subsets of X' into relatively compact subsets of Y. Note that, for X and Y complete, the completed injective tensor product $X\tilde{\otimes}_\epsilon Y$ is a (closed linear) subspace of $X\epsilon Y$, with equality in case either of X or Y has the approximation property (cf. [10]).

2. Results

The criterion for relative compactness of subsets of $K(X,Y)$ for Banach spaces X and Y with X not containing l^1 alluded to in the Introduction ([6, Thm. 1]) can be extended to the following characterization of Fréchet spaces not containing l^1.

Proposition 2.1. *For a Fréchet space X, the following are equivalent:*

(a) *X does not contain an isomorphic copy of l^1;*
(b) *Given any locally convex (equivalently, any Fréchet, equivalently, any Banach) space Y, a subset H of $K_b^b(X,Y)$ is precompact if and only if*
   (i) *$H(x)$ is precompact in Y for all $x \in X$;*
   (ii) *$h(x_n) \to 0$ in Y uniformly over all $h \in H$ for any weak-nullsequence $(x_n)_n$ in X.*

If, in Proposition 2.1, we replace non-containment of l^1 in X by X'_b being (GP), we get the following variant for just any locally convex space X.

Proposition 2.2. *For a locally convex space X, the following are equivalent:*

(a) *X'_b is a Gelfand-Phillips space;*
(b) *Given any quasi-complete locally convex (equivalently, any Fréchet, equivalently, any Banach) space Y, a subset H of $K_b^b(X,Y)$ is precompact if and only if*
   (i) *$H(x)$ is relatively compact in Y for all $x \in X$;*
   (ii) *$h''x_n'' \to 0$ in Y uniformly over all $h \in H$ for any equicontinuous weak*−nullsequence $(x_n'')_n$ in $X''\ (= (X'_b)')$.*

As an aside, we thus deduce indirectly from a combination of Propositions 2.1 and 2.2 that, for a Fréchet space X, non-containment of l^1 implies that the strong dual X'_b is a Gelfand-Phillips space. This extends the corresponding result for Banach spaces of [3, Cor. 5].

In turn, Proposition 2.2 specializes to the following Banach space result.

Corollary 2.3. *For a Banach space X, the following are equivalent:*

(a) *X^* is a Gelfand-Phillips space.*
(b) *Given any Banach space Y, a subset H of $K(X, Y)$ is relatively compact in operator norm if and only if*
   (i) *$H(x)$ is relatively compact in Y for all $x \in X$;*
   (ii) *$\|h^{**}x_n^{**}\| \to 0$ uniformly over all $h \in H$ for any weak*−nullsequence $(x_n^{**})_n$ in X^{**}.*

Finally, with regard to the Gelfand-Phillips property for a locally convex space X (rather than for its strong dual), we note the following operator characterization corresponding to the ones above.

Proposition 2.4. *For a locally convex space X, the following are equivalent:*

(a) *X is a Gelfand-Phillips space;*
(b) *Given any locally convex (equivalently, any Fréchet, equivalently, any Banach) space Y, a subset H of $X \epsilon Y$ is precompact if and only if*
   (i) *$H(x')$ is precompact in Y for all $x' \in X'$;*
   (ii) *$hx'_n \to 0$ in Y uniformly over all $h \in H$ for any equicontinuous weak*−nullsequence $(x'_n)_n$ in X'.*

Obviously, for X and Y complete locally convex spaces, this result also yields a characterization of X being (GP) by the corresponding criterion for relative compactness of subsets of $X \tilde{\otimes}_\epsilon Y$.

As a further special case, we consider the Banach space $C(K, X)$.

Corollary 2.5. *If K is a compact Hausdorff space, and X a Banach space with the Gelfand-Phillips property then a subset $H \subset C(K, X)$ is relatively compact if and only if*

(i) *H is equicontinuous on K with respect to the weak topology of X;*
(ii) *$\|x_n^* \circ h\|_\infty \to 0$ uniformly over $h \in H$ for all weak* nullsequences $(x_n^*)_n \subset X^*$.*

3. Proofs

All of the results of section 2 will follow from the subsequent linearized version of the Arzela-Ascoli theorem, teamed with suitable known results.

Given locally convex spaces X and Y, and a family $\mathcal{S}$ of bounded subsets of X that cover X, we consider the space $K_{\mathcal{S}}(X, Y)$ of all weak-to-weak continuous linear operators from X into Y that transform the sets $S \in \mathcal{S}$ into precompact subsets of Y, endowed with the topology of uniform convergence on the $S \in \mathcal{S}$. The space X' endowed with the topology of uniform convergence on the $S \in \mathcal{S}$ will be denoted by $X'_{\mathcal{S}}$. For subsets $H \subset K_{\mathcal{S}}(X,Y)$ and $A \subset Y'$, the subset $\bigcup\{h'(A) \mid h \in H\}$ of X' will be denoted by $H'(A)$.

Lemma 3.1. *A subset H of $K_{\mathcal{S}}(X,Y)$ is precompact if and only if*

(i) *$H(x)$ is precompact in Y for all $x \in X$;*
(ii) *$H'(V^\circ)$ is precompact in $X'_{\mathcal{S}}$ for all zero-neighbourhoods V of Y.*

This result is well-known, cf. [2, Corollary, section 3] (compare [7] for special cases). We include a short independent *proof*: *Necessity*: As the $S \in \mathcal{S}$ cover X, (i) follows from continuity of the point evaluations $\delta_x : \{h \mapsto h(x)\}$, $x \in X$. Next, given a zero-neighbourhood V in Y, and $S \in \mathcal{S}$, by precompactness, there exist $h_1, ..., h_n \in H$ such that $H \subset \bigcup_1^n (h_i + W(S,V))$, with $W(S,V) := \{u \in K_{\mathcal{S}}(X,Y) \mid u(S) \subset V\}$. By polarity, this translates into $H'(V^\circ) \subset \bigcup_1^n (h'_i(V^\circ) + S^\circ)$. Noting that $u(S)$, $S \in \mathcal{S}$, being precompact in Y translates into u' being continuous from Y'_λ into $X'_{\mathcal{S}}$, and noting that V° is compact in Y'_λ, (ii) is now immediate.

Sufficiency: Given $S \in \mathcal{S}$, the set S_1 = closed absolutely convex hull of S is uniformly equicontinuous (as a set of linear functionals) on $X'_{\mathcal{S}}$, so that the weak topology on X coincides on S_1 with the topology of uniform convergence on precompact subsets of $X'_{\mathcal{S}}$. Thus, given any closed absolutely convex zero-neighbourhood V in Y, by (ii), there exists a weak zero-neighbourhood U_w in X such that $(H'(V^\circ))^\circ \supset (H'(V^\circ))^\circ \cap S_1 \supset U_w \cap S_1$. By polarity, this implies that $H(U_w \cap S_1) \subset V^{\circ\circ} = V$. Hence, $H_{|S_1}$ is equicontinuous at $0 \in S_1$, and thus, as S_1 is absolutely convex, uniformly equicontinuous from (S_1, weak) to Y, and so is $H_{|S}$ from (S, weak) to Y. Teamed with (i), and by noting that the $S \in \mathcal{S}$ are precompact in X_σ (being bounded in X), the Arzela-Ascoli theorem [1, Théorème 2, § 2, 5.] reveals that H is precompact in $K_{\mathcal{S}}(X,Y)$, thus completing the proof.

The proof of Proposition 2.1 will be a simple combination of Lemma 3.1 with the subsequent characterization of Fréchet spaces not containing l^1. We shall call a subset P of the dual of a locally convex space X *weak***−limited* if $\langle x', x_n\rangle \to 0$ uniformly over $x' \in P$ for any weak nullsequence $(x_n)_n \subset X$. Notice that, according to [4, Ch. V, § 3.3, Exercise 3], weak*−limited subsets of X' are (a) exactly those that are precompact for the topology of uniform convergence on all subsets B of X with the property that any sequence in B has a weak Cauchy subsequence, and (b) exactly those that are precompact in X'_τ in case X is a metrizable locally convex space.

Lemma 3.2. *A Fréchet space X does not contain l^1 if and only if every weak*- -limited subset P of X' is relatively compact in X'_b.*

Proof. It is known that Rosenthal's characterization of non-containment of l^1 ([8]) carries over from Banach to Fréchet spaces, i.e., a Fréchet space X does not contain l^1 if and only if every bounded sequence in X has a weak Cauchy subsequence ([5]). Thus, necessity of the assertion of Lemma 3.2 follows from part

(a) of the above result of [4]. Sufficiency, in turn, follows from part (b) of that same result, in conjunction with the fact (cf. [11, p. 398]) that a Fréchet space X does not contain l^1 if and only if Mackey and strong nullsequences in X' coincide. ■

Proof of Proposition 2.1. Condition (b) (ii) amounts to $\langle x', x_n \rangle \to 0$ uniformly over all $x' \in H'(V^\circ)$. Thus, letting $\mathcal{S} = \mathcal{B_X}$ = all bounded subsets of X in Lemma 3.1, necessity of conditions (i) and (ii) in (b) holds for general X, as $H'(V^\circ)$ is precompact in X'_b by Lemma 3.1. In turn, in case X does not contain l^1, sufficiency of (i) and (ii) follows from combining Lemma 3.1 (for $\mathcal{S} = \mathcal{B_X}$) and Lemma 3.2, as (ii) amounts to $H'(V^\circ)$ being weak$^*-$ limited in X'. Finally, the special case of Y = scalars in (b), teamed with Lemma 3.2, shows that (b) implies (a). ■

Proof of Proposition 2.2. With regard to (ii) of part (b) of Proposition 2.2, notice that, as Y is supposed to be (at least) quasi-complete, the second adjoint of any $u \in K_b^b(X, Y)$ maps back into Y. Other than that, since that same condition simply means that $H'(V^\circ)$ is a limited subset of X'_b for all zero-neighbourhoods V of Y, Lemma 3.1 (for $\mathcal{S} = \mathcal{B_X}$) shows that (a) implies (b), while the reverse implication is simply a specialization of Y to the scalars in (b). ■

Proof of Proposition 2.4. This result is an immediate consequence of Lemma 3.1 for the special case of X being replaced by X'_c, and $\mathcal{S}$ being the family of all equicontinuous subsets of X', teamed with the observation that (ii) of part (b) amounts to $H'(V^\circ)$ being a limited subset of X for all zero-neighbourhoods V of Y. ■

Proof of Corollary 2.5. This is an immediate consequence of Proposition 2.4, and the isometry $C(K, X) = X \tilde{\otimes}_\epsilon C(K)$, given by $\{F \mapsto \{x^* \mapsto x^* \circ F\}\}$, teamed with the scalar Arzela-Ascoli theorem for $C(K)$. ■

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