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# ON THE FOURIER TRANSFORM OF LORENTZ INVARIANT DISTRIBUTIONS

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**Abstract:** We present a new formula for the Fourier transform of a Lorentz invariant temperate distribution. The formula is applied so as to yield the temperate fundamental solution of the Klein-Gordon operator.

**Keywords:** Lorentz invariance, Fourier transforms, temperate distributions, Klein–Gordon operator

## 1. Introduction and notation

Our personal motivation for this paper was a futile attempt to derive the Fourier transform of the function $f([x,x]) = ([x,x]+c^2)^{-1}$, $c^2 \in \mathbf{C} \setminus \mathbf{R}$, i.e., the temperate fundamental solution of the Klein–Gordon operator, by employing Strichartz' formula, see [23, Thm. 1, p 509]. Although Strichartz' formula refers to the more general case of distributions invariant with respect to the pseudo-orthogonal group $O(p,q)$, Lorentz invariance constituting the special case of $p=1$, $q=n-1$, simple insertion of $f(s) = (s+c^2)^{-1}$ does not yield the final result. On the one hand, Strichartz' formula applies formally only to rapidly decreasing test functions $\phi(s) \in \mathcal{S}(\mathbf{R}^1)$, on the other hand, more importantly, the integrals arising from this formula can be evaluated immediately only if $[x,x] > 0$ and the dimension $n$ is odd. We then observed that, for $[x,x] < 0$ and for $n$ even, respectively, the resulting integrals can be simplified by means of the residue theorem.

Due to the importance of the fundamental solutions of the Klein–Gordon operator $[\partial,\partial] - c^2$, it seems justified to reconsider the subject of Fourier transforms of Lorentz invariant distributions. Let us describe now the content and the set-up of this article.

In Section 2, we first review some facts on Lorentz invariant distributions making use of the more general treatments in [14], [4], [24]. In Proposition 1, we

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determine the Fourier transforms of $\delta_s([x,x])$, $s \in \mathbf{R}$. This yields a formula equivalent to Strichartz' formula cited above ([23, Thm. 1, p 509]) if we take into account the representation of a Lorentz invariant test function in the form

$$\phi([x,x]) = \int_{\mathbf{R}} \phi(s)\delta_s([x,x])\,\mathrm{d}s, \qquad \phi \in \mathcal{S}(\mathbf{R}).$$

We compare our formulas in Proposition 1 also with those in [5, Ch. III] and in [25]. In Corollary 1, the particular cases of the dimensions $n = 2, 3, 4$ are listed in more explicit form. In Propositions 2,3,4, representations of the Fourier transforms of Lorentz invariant locally integrable functions $f([x,x])$ are given.

In Section 3, we transform the formulas of Proposition 1 so as to yield simple results also in the cases $[x,x] < 0$ or ($[x,x] > 0$ and even dimension). For the evaluation of Fourier transforms of Lorentz invariant distributions, we have the following table:

| | $[x,x] > 0$ | $[x,x] < 0$ |
|---|---|---|
| $n$ even | Proposition 5 | Proposition 5 |
| $n$ odd | Proposition 1 | Proposition 5 |

In Section 4, we derive the unique temperate fundamental solutions of the iterated Klein–Gordon operator $([\partial,\partial] - c^2)^m$, $m \in \mathbf{N}$, $c^2 \in \mathbf{C} \setminus \mathbf{R}$, see Proposition 6. We therefrom then rederive the temperate fundamental solutions of the Klein–Gordon operators in "low" dimensions, i.e., for $n = 2, 3, 4$.

Let us introduce some notation. We shall always suppose that the space dimension $n$ is at least 2; we write $x_0, \ldots, x_{n-1}$ for the coordinates in the space $\mathbf{R}^n$, and we equip it with the Lorentz metric $[x,y] = x_0y_0 - x_1y_1 - \cdots - x_{n-1}y_{n-1}$.

We employ the standard notation for the distribution spaces $\mathcal{D}'$, $\mathcal{S}'$, the dual spaces of the spaces $\mathcal{D}$, $\mathcal{S}$ of "test functions" and of "rapidly decreasing functions", respectively, see [22], [7]. The Heaviside function is denoted by $Y$, and we write $\delta_s \in \mathcal{D}'(\mathbf{R}^1)$, $s \in \mathbf{R}$, for the delta distribution with support in $s$, which is the derivative of $Y(x-s)$, i.e., $\langle \phi, \delta_s \rangle = \phi(s)$ for $\phi \in \mathcal{D}(\mathbf{R}^1)$. In contrast, $\delta$ without any subscript stands for the delta distribution at the origin. For a distribution $T \in \mathcal{D}' = \mathcal{D}'(\mathbf{R}^n)$, we denote by $\check{T}$ its reflection at the origin.

The pullback $h^*T = T \circ h \in \mathcal{D}'(\Omega)$ of a a distribution $T$ in one variable $t$ with respect to a submersive $\mathcal{C}^\infty$ function $h : \Omega \to \mathbf{R}$, $\Omega \subset \mathbf{R}^n$ open, is defined as in [3, Section 7.2, p. 81], i.e.,

$$\langle \phi, h^*T \rangle = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\Big( \int_{\{x \in \Omega;\, h(x) < t\}} \phi(x)\,\mathrm{d}x \Big), T \Big\rangle, \qquad \phi \in \mathcal{D}(\Omega). \tag{1.1}$$

We use the Fourier transform $\mathcal{F}$ in the form

$$(\mathcal{F}\phi)(\xi) := \int \mathrm{e}^{-\mathrm{i}\xi x}\phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{S}(\mathbf{R}^n),$$

this being extended to $\mathcal{S}'$ by continuity. (Herein and also elsewhere, the Euclidean inner product $(\xi, x) \mapsto \xi x$ is simply expressed by juxtaposition.)

## 2. The Fourier transform of Lorentz invariant distributions

Let us first review some facts concerning the structure of Lorentz invariant temperate distributions, cf. [14], [4]. We denote the proper Lorentz group by $L(\mathbf{R}^n)$, i.e.,

$$L(\mathbf{R}^n) = \{A = (a_{ij})_{0\leqslant i,j\leqslant n-1} \in \mathrm{Gl}(\mathbf{R}^n);\ \det A > 0,\ a_{00} > 0,\\ \text{and } \forall x \in \mathbf{R}^n : [Ax, Ax] = [x, x]\}.$$

The space $\mathcal{S}'_L$ of temperate Lorentz invariant distributions is given by

$$\mathcal{S}'_L = \mathcal{S}'_L(\mathbf{R}^n) = \{T \in \mathcal{S}'(\mathbf{R}^n);\ \forall A \in L(\mathbf{R}^n) : T \circ A = T\}.$$

Obviously, $\mathcal{S}'_L$ is the direct sum of the spaces of even and of odd invariant distributions, i.e., $\mathcal{S}'_L = \mathcal{S}'_{L,+} \oplus \mathcal{S}'_{L,-}$, where

$$\mathcal{S}'_{L,\pm} = \{T \in \mathcal{S}'_L;\ \check{T} = \pm T\},$$

cf. also [14, p. 228], [4, p. 45].

If $T \in \mathcal{S}'_{L,-}$ and $n \geqslant 3$, then $\operatorname{supp} T \subset \{x \in \mathbf{R}^n;\ [x, x] \geqslant 0\}$. This implies that $T$ is determined as a pullback of a one-dimensional distribution supported in $[0, \infty)$, i.e., symbolically, we have

$$T = \operatorname{sign}(x_0) \cdot S([x, x]), \qquad S \in \mathcal{S}'(\mathbf{R}_+).$$

Here

$$\mathcal{S}'(\mathbf{R}_+) = \{S \in \mathcal{S}'(\mathbf{R}^1);\ \operatorname{supp} S \subset [0, \infty)\},$$

and the isomorphism relating $S$ and $T$ is given in a precise way by

$$\mathcal{S}'(\mathbf{R}_+) \xrightarrow{\sim} \mathcal{S}'_{L,-}(\mathbf{R}^n) : S \longmapsto (T : \phi \mapsto \langle N(\phi), S\rangle), \qquad n \geqslant 3,$$

where

$$N(\phi)(t) = \int_{\mathbf{R}^{n-1}} \frac{\phi(\sqrt{t + |x'|^2}, x') - \phi(-\sqrt{t + |x'|^2}, x')}{2\sqrt{t + |x'|^2}}\, \mathrm{d}x', \qquad t > 0.$$

Note that $N(\phi)$ arises by applying formula (1.1) to define $\langle \phi, \operatorname{sign}(x_0) \cdot S([x, x])\rangle$; the application of $S$ to $N(\phi)$ is justified by the fact that $N(\phi)$ can be continued $\mathcal{C}^\infty$ to the whole real line, cf. [4, Thm. 8.2, p. 52].

For $\mathcal{S}'_{L,+}$, the situation is more complicated. Outside the origin, $T \in \mathcal{S}'_{L,+}$ is generated by $S \in \mathcal{S}'(\mathbf{R})$, i.e.,

$$\mathcal{S}'(\mathbf{R}) \xrightarrow{\sim} \{T|_{\mathbf{R}^n\setminus\{0\}};\ T \in \mathcal{S}'_{L,+}\} : S \longmapsto S([x, x]),$$

cf. [4, Lemma 8.1, p. 46]. However, the space $\mathcal{S}'_{L,+}$ itself is isomorphic to the space $H'$ defined in [4, pp. 48, 49]. Note that if $S \in \mathcal{S}'(\mathbf{R})$ with $0 \notin \operatorname{supp} S$, then $T = S([x, x]) \in \mathcal{S}'_{L,+}$ is defined unambiguously by the requirements $0 \notin \operatorname{supp} T$

and $T = S([x,x])$ in $\mathbf{R}^n \setminus \{0\}$. In particular, this is the case for $T = \delta_s([x,x]) = \delta(s-[x,x])$ if $s \in \mathbf{R}\setminus\{0\}$; for $n \geqslant 3$, we can also define $\delta([x,x])$ by continuity, i.e., $\delta([x,x]) = \lim_{s\to 0}\delta_s([x,x])$. Explicitly, we have

$$\langle \phi, \delta([x,x])\rangle = \int_{\mathbf{R}^{n-1}} \frac{\phi(|x'|,x') + \phi(-|x'|,x')}{2|x'|}\,\mathrm{d}x', \qquad \phi \in \mathcal{D}(\mathbf{R}^n),\ n \geqslant 3.$$

It is also clear that the distributions $Y(\pm x_0)\delta_s([x,x]) \in \mathcal{S}'_L$ are well-defined for $s > 0$.

In the following proposition, we determine the Fourier transforms of the distributions $Y(x_0)\delta_s([x,x])$, $s > 0$, and $\delta_s([x,x])$, $s < 0$, which correspond to uniform mass distributions on the upper sheet of the two-sheeted hyperboloid $[x,x] = s$, $s > 0$ and on the one-sheeted hyperboloid $[x,x] = s$, $s < 0$, respectively.

**Proposition 1.**

(1) *For $s > 0$, let $S = Y(x_0)\delta_s([x,x]) \in \mathcal{S}'_L$ be defined as above. Then its Fourier transform $\mathcal{F}S$ is the value at $\lambda = \frac{n-2}{2}$ of the entire distribution-valued function $\lambda \mapsto T_\lambda$, which, for* $\operatorname{Re}\lambda < 1$, *is given by the locally integrable function*

$$\begin{aligned} T_\lambda(x) = {}& (2\pi)^{(n-2)/2} Y(-[x,x]) \Big(\frac{s}{-[x,x]}\Big)^{\lambda/2} K_\lambda\big(\sqrt{-s[x,x]}\big) \\ & - 2^{n/2-2}\pi^{n/2} Y([x,x]) \Big(\frac{s}{[x,x]}\Big)^{\lambda/2} \big[N_{-\lambda}\big(\sqrt{s[x,x]}\big) \\ & + \mathrm{i}\,\mathrm{sign}(x_0) J_{-\lambda}\big(\sqrt{s[x,x]}\big)\big]. \end{aligned} \tag{2.1}$$

*In other words,*

$$\mathcal{F}\big(Y(x_0)\delta_s([x,x])\big) = T_{(n-2)/2}, \qquad s > 0.$$

(2) *For $s < 0$, the Fourier transform of $\delta_s([x,x]) \in \mathcal{S}'_{L,+}$ is the value at $\lambda = \frac{n-2}{2}$ of the entire distribution-valued function $\lambda \mapsto U_\lambda$, which, for* $\operatorname{Re}\lambda < 1$, *is given by the locally integrable function*

$$\begin{aligned} U_\lambda(x) = {}& -2^{(n-2)/2}\pi^{n/2} Y(-[x,x]) \Big(\frac{s}{[x,x]}\Big)^{\lambda/2} N_\lambda\big(\sqrt{s[x,x]}\big) \\ & + 2^{n/2}\pi^{(n-2)/2} Y([x,x]) \cos(\lambda\pi) \Big(\frac{-s}{[x,x]}\Big)^{\lambda/2} K_\lambda\big(\sqrt{-s[x,x]}\big). \end{aligned} \tag{2.2}$$

*In other words,*

$$\mathcal{F}\big(\delta_s([x,x])\big) = U_{(n-2)/2}, \qquad s < 0.$$

**Proof.** (1) If $s > 0$ and $\mathcal{F}_{x_0}$ and $\mathcal{F}_{x'}$ denote the partial Fourier transforms with respect to the variables $x_0$ and $x' = (x_1, \dots, x_{n-1})$, respectively, see [24, §20.5, p. 198], then

$$\mathcal{F}\big(Y(x_0)\delta_s([x,x])\big) = \mathcal{F}_{x_0}\Big(\mathcal{F}_{x'}\big(Y(x_0)\delta_s([x,x])\big)\Big).$$

Since the distributions

$$Y(x_0)\delta(x_0^2 - s - |x'|^2) = \frac{Y(x_0 - \sqrt{s})}{2\sqrt{x_0^2 - s}}\,\delta\big(|x'| - \sqrt{x_0^2 - s}\big)$$

continuously depend on $x_0$ for $n \geqslant 4$, i.e.,

$$Y(x_0)\delta(x_0^2 - s - |x'|^2) \in \mathcal{C}\big(\mathbf{R}^1_{x_0}, \mathcal{S}'(\mathbf{R}^{n-1}_{x'})\big), \qquad n \geqslant 4,$$

are still piecewise continuous in $x_0$ with a jump at $x_0 = \sqrt{s}$ for $n = 3$, and are still locally integrable with respect to $x_0$ for $n = 2$, we can fix the variable $x_0$ in order to calculate the partial Fourier transform with respect to $x'$.

From the Poisson–Bochner formula, see [22, (VII,7;22), p. 259], [19, (7), p. 127], [5, Ch. II, 3.4, p. 198], i.e.,

$$\mathcal{F}\big(\delta(|x'| - R)\big) = (2\pi R)^{(n-1)/2}|x'|^{-(n-3)/2}J_{(n-3)/2}(R|x'|) \in \mathcal{S}'(\mathbf{R}^{n-1}), \quad R > 0, \tag{2.3}$$

we infer that

$$\begin{aligned}\mathcal{F}\big(Y(x_0)\delta_s([x,x])\big) &= 2^{(n-3)/2}\pi^{(n-1)/2} \\ &\quad \times \mathcal{F}_{x_0}\Big(Y\big(x_0 - \sqrt{s}\big)(x_0^2 - s)^{(n-3)/4}|x'|^{(3-n)/2}J_{(n-3)/2}\big(|x'|\sqrt{x_0^2 - s}\big)\Big).\end{aligned}$$

The distribution-valued function

$$\begin{aligned}&\tilde{T}_\lambda : \{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda > -\tfrac{1}{2}\} \longrightarrow \mathcal{S}'(\mathbf{R}^n) : \\ &\qquad \lambda \longmapsto Y\big(x_0 - \sqrt{s}\big)(x_0^2 - s)^{(2\lambda-1)/4}|x'|^{1/2-\lambda}J_{-1/2+\lambda}\big(|x'|\sqrt{x_0^2 - s}\big)\end{aligned}$$

is holomorphic and can analytically be continued to an entire function due to the recursion formula $\frac{\partial \tilde{T}_\lambda}{\partial x_0} = x_0\tilde{T}_{\lambda-1}$. Therefore, $\mathcal{F}\big(Y(x_0)\delta_s([x,x])\big)$ is the value at $\lambda = \frac{n-2}{2}$ of the entire function $\lambda \mapsto 2^{(n-3)/2}\pi^{(n-1)/2}\mathcal{F}_{x_0}(\tilde{T}_\lambda)$, cf. [9, Proposition (2.1.5) (i)], [17, Proposition 1.6.2, p. 28].

For $-\frac{1}{2} < \operatorname{Re}\lambda < 0$ and fixed $x'$, the function $x_0 \mapsto \tilde{T}_\lambda(x_0, x')$ is absolutely integrable. Hence the Fourier transform with respect to $x_0$ can be calculated classically and yields, by [16, 14.57, p. 82; 14.32, p. 176],

$$\begin{aligned}\mathcal{F}_{x_0}(\tilde{T}_\lambda) &= |x'|^{1/2-\lambda}\int_{\sqrt{s}}^{\infty} \mathrm{e}^{-\mathrm{i}x_0 t}(t^2 - s)^{(2\lambda-1)/4}J_{-1/2+\lambda}\big(|x'|\sqrt{t^2 - s}\big)\,\mathrm{d}t \\ &= \sqrt{\frac{2}{\pi}}\,Y(-[x,x])\Big(\frac{s}{-[x,x]}\Big)^{\lambda/2}K_\lambda(\sqrt{-s[x,x]}) \\ &\quad - \sqrt{\frac{\pi}{2}}\,Y([x,x])\Big(\frac{s}{[x,x]}\Big)^{\lambda/2}\big[N_{-\lambda}(\sqrt{s[x,x]}) + \mathrm{i}\operatorname{sign}(x_0)J_{-\lambda}(\sqrt{s[x,x]})\big].\end{aligned}$$

This yields formula (2.1). (2) Similarly, for $s < 0$,

$$\begin{aligned}\mathcal{F}\big(\delta_s([x,x])\big) &= \mathcal{F}_{x_0}\Big(\mathcal{F}_{x'}\big(\delta(x_0^2 - s - |x'|^2)\big)\Big)\\ &= 2^{(n-3)/2}\pi^{(n-1)/2}\\ &\quad\times \mathcal{F}_{x_0}\Big((x_0^2-s)^{(n-3)/4}|x'|^{(3-n)/2}J_{(n-3)/2}\big(|x'|\sqrt{x_0^2-s}\big)\Big).\end{aligned}$$

The distribution-valued function

$$\tilde{U}_\lambda : \mathbf{C} \longrightarrow \mathcal{S}'(\mathbf{R}^n) : \lambda \longmapsto (x_0^2-s)^{(2\lambda-1)/4}|x'|^{1/2-\lambda}J_{-1/2+\lambda}\big(|x'|\sqrt{x_0^2-s}\big)$$

is plainly entire and $\mathcal{F}\big(\delta_s([x,x])\big)$ coincides with $2^{(n-3)/2}\pi^{(n-1)/2}\mathcal{F}_{x_0}(\tilde{U}_{(n-2)/2})$. For Re $\lambda < 1$, this partial Fourier transform with respect to $x_0$ can be calculated classically by fixing $x'$. For $x' \neq 0,\ s < 0$, [16, 14.22, p. 78] furnishes

$$\begin{aligned}\mathcal{F}_{x_0}(\tilde{U}_\lambda) &= 2|x'|^{1/2-\lambda}\int_0^\infty \cos(x_0 t)(t^2-s)^{(2\lambda-1)/4}J_{\lambda-1/2}(|x'|\sqrt{t^2-s})\,\mathrm{d}t\\ &= -\sqrt{2\pi}\,Y(-[x,x])\Big(\frac{s}{[x,x]}\Big)^{\lambda/2}N_\lambda(\sqrt{s[x,x]})\\ &\quad+\sqrt{\frac{8}{\pi}}\,Y([x,x])\cos(\lambda\pi)\Big(\frac{-s}{[x,x]}\Big)^{\lambda/2}K_\lambda(\sqrt{-s[x,x]}).\end{aligned}$$

This implies formula (2.2) and completes the proof. ■

**Remarks.**

(1) Comparing our formulas (2.1) and (2.2) in Proposition 1 with formula (7) in [5, Ch. III, 2.10, p. 294] we note that they are both representations of $\mathcal{F}\big(\delta_s([x,x])\big)$, $s \in \mathbf{R}$, as analytic continuations, but with respect to different parameters: Our formulas are continuations with respect to the index $\lambda$ of the Bessel functions, whereas in [5], the quadratic form $[x,x]$ is interpreted as boundary value of the non-degenerate complex quadratic form $[x,x] + \mathrm{i}\epsilon|x|^2$, $\epsilon > 0$. We also observe that (2.1), (2.2) above yield immediately an explicit result outside the light cone $[x,x] = 0$.

(2) For $s > 0$, the Fourier transforms of $Y(x_0)\delta_s([x,x])$ are special cases of the formulas (II,3;3/4), p. 84, in [25]. There, more generally, $T = \mathcal{F}\big(Y(x_0)\delta_s^{(k)}([x,x])\big)$ is considered. However, the results given in [25] are only partially correct. This can be seen, e.g., by comparing Corollary 1, (b) below with [25, (II,3;4)] in the case $k = 0,\ n = 3$. The method used in [25] consists in decomposing $\mathbf{R}^n$ into three open sets $C_1, C_f, C_b$ and the light cone $\bar{C} = \{x \in \mathbf{R}^n;\ [x,x] = 0\}$, see [25, (I,3;1–4), p. 76]. The restriction of the investigated Fourier transforms to the *closed* set $\bar{C}$ is not defined since, generally, distributions cannot be restricted to closed sets, and this leads to the erroneous term in [25, (II,3;4)]. We also point out that the restriction $T|_{C_1}$, say, which is a $\mathcal{C}^\infty$ function and hence also a distribution in $C_1$

cannot be conceived in a canonical way as a distribution in $\mathbf{R}^n$. Hence a formula as $T = T|_{C_1} + T|_{C_f} + T|_{C_b} + T|_{\bar{C}}$ coming from "adding the results" (see [25, p. 79]) does not make sense. Similarly, formula (II,1;1) in [25] for $\mathcal{F}\big(Y(x_0)\delta_s([x,x])\big)$ in the case $n = 4$ is correct only if interpreted in the sense of our Corollary 1 d) below, i.e., by conceiving $T|_{C_1} + T|_{C_f} + T|_{C_b}$ as a principal value distribution.

Let us yet formulate the results in (2.1) and (2.2) more explicitly in the case of small dimensions $n$.

**Corollary 1.**

(a) *If $s > 0$ and $n = 2$, then*

$$\begin{aligned}\mathcal{F}\big(Y(x_0)\delta_s([x,x])\big) &= Y(-[x,x])K_0(\sqrt{-s[x,x]}) \\ &\quad - \frac{\pi}{2}Y([x,x])\big[N_0(\sqrt{s[x,x]}) \\ &\quad + \mathrm{i}\,\mathrm{sign}(x_0)J_0(\sqrt{s[x,x]})\big] \in L^1_{\mathrm{loc}}(\mathbf{R}^2).\end{aligned}$$

(*In $\mathbf{R}^2$, this formula also encompasses the case of $\mathcal{F}\big(\delta_s([x,x])\big)$, $s < 0$, by reflection.*)

(b) *If $s \geqslant 0$ and $n = 3$, then*

$$\begin{aligned}\mathcal{F}\big(Y(x_0)\delta_s([x,x])\big) &= \frac{\pi Y(-[x,x])}{\sqrt{-[x,x]}}\,\mathrm{e}^{-\sqrt{-s[x,x]}} \\ &\quad - \frac{\pi Y([x,x])}{\sqrt{[x,x]}}\,\big[\sin(\sqrt{s[x,x]}) \\ &\quad + \mathrm{i}\,\mathrm{sign}(x_0)\cos(\sqrt{s[x,x]})\big] \in L^1_{\mathrm{loc}}(\mathbf{R}^3).\end{aligned}$$

(c) *If $s \leqslant 0$ and $n = 3$, then*

$$\mathcal{F}\big(\delta_s([x,x])\big) = \frac{2\pi Y(-[x,x])}{\sqrt{-[x,x]}}\,\cos(\sqrt{s[x,x]}) \in L^1_{\mathrm{loc}}(\mathbf{R}^3).$$

(d) *If $s > 0$ and $n = 4$, then*

$$\begin{aligned}\mathcal{F}\big(Y(x_0)\delta_s([x,x])\big) &= \mathrm{i}\pi^2\,\mathrm{sign}(x_0)\Big[Y([x,x])\sqrt{\frac{s}{[x,x]}}\,J_1\big(\sqrt{s[x,x]}\big) - 2\delta([x,x])\Big] \\ &\quad + \pi\,\mathrm{vp}\bigg(\sqrt{\frac{s}{|[x,x]|}}\Big[2Y(-[x,x])K_1\big(\sqrt{-s[x,x]}\big) \\ &\quad + \pi Y([x,x])N_1\big(\sqrt{s[x,x]}\big)\Big]\bigg) \in \mathcal{D}'(\mathbf{R}^4).\end{aligned}$$

(*Herein the principal value has the following meaning:*

$$\mathrm{vp}\big(f(x)\big) = \lim_{\epsilon \searrow 0}\big(Y(|[x,x]| - \epsilon)f(x)\big),$$

*the limit converging in $\mathcal{D}'(\mathbf{R}^4)$.*)

(e) *If* $s = 0$ *and* $n = 4$, *then*

$$\mathcal{F}\big(Y(x_0)\delta([x,x])\big) = -2\pi \operatorname{vp}\Big(\frac{1}{[x,x]}\Big) - 2\mathrm{i}\pi^2 \operatorname{sign}(x_0)\delta([x,x]).$$

(f) *If* $s < 0$ *and* $n = 4$, *then*

$$\mathcal{F}\big(\delta_s([x,x])\big) = -2\pi \operatorname{vp}\bigg(\sqrt{\Big|\frac{s}{[x,x]}\Big|}\Big[2Y([x,x])K_1\big(\sqrt{-s[x,x]}\big) + \pi Y(-[x,x])N_1\big(\sqrt{s[x,x]}\big)\Big]\bigg).$$

**Proof.** The formulas in (a), (b) and (c) follow immediately from Proposition 1 since $T_\lambda$ and $U_\lambda$ are locally integrable functions for $\operatorname{Re}\lambda < 1$, and this is the case for $\lambda = \frac{n-2}{2}$, $n = 2, 3$.

If $n = 4$, then $\lambda = 1$, and the values of $T_1$ and $U_1$ can be obtained as limits, i.e., $T_1 = \lim_{\lambda \nearrow 1} T_\lambda$, $U_1 = \lim_{\lambda\nearrow 1} U_\lambda$. From the elementary formula

$$\lim_{\lambda\searrow -1} |t|^\lambda \operatorname{sign} t = \operatorname{vp}(t^{-1}) \qquad \text{in } \mathcal{S}'(\mathbf{R}^1_t),$$

we infer that

$$\begin{aligned}
&\lim_{\lambda\nearrow 1}\bigg[2\pi Y(-[x,x])\Big(\frac{s}{-[x,x]}\Big)^{\lambda/2} K_\lambda\big(\sqrt{-s[x,x]}\big) \\
&\qquad - \pi^2 Y([x,x])\Big(\frac{s}{[x,x]}\Big)^{\lambda/2} N_{-\lambda}\big(\sqrt{s[x,x]}\big)\bigg] \\
&= \pi \operatorname{vp}\bigg(\sqrt{\frac{s}{|[x,x]|}}\Big[2Y(-[x,x])K_1\big(\sqrt{-s[x,x]}\big) \\
&\qquad + \pi Y([x,x])N_1\big(\sqrt{s[x,x]}\big)\Big]\bigg).
\end{aligned}$$

This yields the second part in (d), and an analogous reasoning furnishes the formula in (f).

On the other hand, for $s > 0$ and $\operatorname{Re}\lambda < 1$, the function

$$S_\lambda(t) = Y(t)\Big(\frac{s}{t}\Big)^{\lambda/2} J_{-\lambda}(\sqrt{st})$$

is locally integrable in $\mathbf{R}^1_t$ and depends holomorphically on $\lambda$. Since $S_{\lambda+1} = 2\frac{\mathrm{d}}{\mathrm{d}t} S_\lambda$ holds for $\operatorname{Re}\lambda < 0$, the distribution-valued function $\lambda \mapsto S_\lambda$ can analytically be continued to the whole complex $\lambda$-plane. In particular,

$$S_1 = 2\frac{\mathrm{d}}{\mathrm{d}t} S_0 = 2\frac{\mathrm{d}}{\mathrm{d}t}\big[Y(t)J_0(\sqrt{st})\big] = 2\delta - Y(t)\sqrt{\frac{s}{t}}\, J_1(\sqrt{st}),$$

and the composition with $t = [x,x]$ yields the formula in (d). Finally, (e) follows from (d) by performing the limit $s \searrow 0$. The proof is complete. ■

**Remarks.**

(1) For the formula in (d), cf. [20, 29.4, p. 186, and 31.5, p. 200]; [21, pp. 83, 84]; [2, App. E, (E4), p. 334]; [10, Ch. IV, (5.6/7), pp. 136, 137]; [12, (IV,1/2), p. 67]. A part of the formulas in Corollary 1 can also be obtained by specializing formula (5) in [5, Ch. III, 2.9, p. 291].

(2) If, by abuse of notation, we write generally $Y(t)(s/t)^{\lambda/2}J_{-\lambda}(\sqrt{st})$ for the distribution-valued function $S_\lambda,\ \lambda \in \mathbf{C}$, considered in the proof above, then the recursion formula $S_{\lambda+1} = 2\frac{\mathrm{d}}{\mathrm{d}t}S_\lambda$ implies the following equation, which holds in $\mathcal{D}'(\mathbf{R}^1_t)$ for fixed $s > 0$ and $k \in \mathbf{N}$ :

$$Y(t)\Big(\frac{s}{t}\Big)^{k/2} J_{-k}(\sqrt{st}) \\ = 2^k \sum_{j=0}^{k-1} \frac{(-s)^j}{2^{2j} j!}\, \delta^{(k-j-1)}(t) + (-1)^k Y(t)\Big(\frac{s}{t}\Big)^{k/2} J_k(\sqrt{st}).$$

A similar formula appears in [11, (1.12), p. 188], where a proof by development in a power series is given.

The Fourier transforms in Corollary 1 are the analogues in the Lorentz case of the Poisson–Bochner formula (2.3). They yield formulas for the Fourier transforms of Lorentz invariant functions $f([x,x])$. Since the necessary assumptions on $f$ depend on the dimension $n$, we consider the cases $n = 2, 3, 4$ separately.

**Proposition 2.** *Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^1)$ such that $\dfrac{f(s)\log^2|s|}{(1+|s|)^{1/4}}$ is integrable. If $x \in \mathbf{R}^2$ with $[x,x] = x_0^2 - x_1^2$, then $f([x,x]) \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$, and the Fourier transform of $f([x,x])$ is locally integrable, continuous outside the light cone, and given by*

$$\mathcal{F}\big(f([x,x])\big) = \int_{-\infty}^{\infty} f(s)\cdot \big[2Y(-s[x,x])K_0(\sqrt{-s[x,x]}) \\ - \pi Y(s[x,x])N_0(\sqrt{s[x,x]})\big]\,\mathrm{d}s.$$

**Proof.** The mapping

$$\mathbf{R}\setminus\{0\} \longrightarrow \mathcal{S}'(\mathbf{R}^2) : s \longmapsto \delta_s([x,x])$$

is continuous and hence can be integrated against a test function in $\mathcal{D}(\mathbf{R}\setminus\{0\})$. Therefore, the formula in (a) of Corollary 1 yields the result by Fubini's theorem if $f \in \mathcal{D}(\mathbf{R}\setminus\{0\})$. This can then be extended by density to the class of functions in Proposition 2 using Lebesgue's theorem and the asymptotic properties of the Bessel functions at 0 and $\infty$. ∎

**Proposition 3.** *Let $f \in L^1(\mathbf{R}^1)$ and $x \in \mathbf{R}^3$ with $[x,x] = x_0^2 - x_1^2 - x_2^2$. Then $f([x,x]) \in L^1_{\mathrm{loc}}(\mathbf{R}^3)$, and the Fourier transform of $f([x,x])$ is locally integrable,* *continuous outside the light cone, and given by*

$$\mathcal{F}\big(f([x,x])\big) = -\frac{2\pi Y([x,x])}{\sqrt{[x,x]}} \int_0^\infty f(s)\sin\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s + \frac{2\pi Y(-[x,x])}{\sqrt{-[x,x]}} \int_0^\infty \big[f(s)\,\mathrm{e}^{-\sqrt{-s[x,x]}} + f(-s)\cos\big(\sqrt{-s[x,x]}\big)\big]\,\mathrm{d}s.$$

**Proof.** This follows from Corollary 1 (b), (c) in an analogous way as Proposition 2. ■

**Proposition 4.** *Let* $f \in L^1_{\mathrm{loc}}(\mathbf{R}^1)$ *such that* $f(s)|s|^{1/4} \in L^1(\mathbf{R})$. *If* $x \in \mathbf{R}^4$ *with* $[x,x] = x_0^2 - x_1^2 - x_2^2 - x_3^2$, *then* $f([x,x]) \in L^1_{\mathrm{loc}}(\mathbf{R}^4)$, *and the Fourier transform of* $f([x,x])$ *is continuous outside the light cone* $[x,x] = 0$, *and generally a principal value given by*

$$\mathcal{F}\big(f([x,x])\big) = 2\pi\,\mathrm{vp}\bigg(\frac{1}{[x,x]} \int_{-\infty}^\infty f(s)\big[\pi Y(s[x,x])\sqrt{s[x,x]}\,N_1\big(\sqrt{s[x,x]}\big) - 2Y(-s[x,x])\sqrt{-s[x,x]}\,K_1\big(\sqrt{-s[x,x]}\big)\big]\,\mathrm{d}s\bigg). \tag{2.4}$$

(*The meaning of the principal value is as explained in Corollary* 1 (d).)

**Proof.** The parts (d), (e) and (f) of Corollary 1 yield the following representation of $\mathcal{F}\big(\delta_s([x,x])\big)$ :

$$\mathcal{F}\big(\delta_s([x,x])\big) = \mathrm{vp}\bigg(\frac{1}{[x,x]}\cdot g(s[x,x])\bigg), \qquad s \in \mathbf{R}, \tag{2.5}$$

where

$$g(t) = 2\pi^2 Y(t)\sqrt{t}\,N_1(\sqrt{t}) - 4\pi Y(-t)\sqrt{-t}\,K_1(\sqrt{-t}).$$

We observe that $g(t)$ is $\mathcal{C}^\infty$ outside the origin and continuous at $t = 0$. More precisely, the behavior of $g$ at 0 and at $\infty$, respectively, is determined by

$$g(t) = -4\pi + \pi t\log|t| + \mathcal{O}(t) \text{ for } t \to 0, \text{ and } g(t) = \mathcal{O}(|t|^{1/4}) \text{ for } |t| \to \infty,$$

with $\mathcal{O}$ denoting, as usual, Landau's symbol. In particular,

$$\exists C > 0 : \forall t \in \mathbf{R} : |g(t) - g(0)| \leqslant C|t|^{1/4}. \tag{2.6}$$

Let us consider the Banach space

$$\mathcal{M} = \{\mu \text{ Radon measure on } \mathbf{R};\ |s|^{1/4}\mu \text{ is an integrable measure}\}$$

with the norm $\|\mu\| = \int_{\mathbf{R}} |s|^{1/4}\,\mathrm{d}|\mu|(s)$. For $\mu \in \mathcal{M}$, the function

$$h_\mu(u) = \int_{\mathbf{R}} g(su)\,\mathrm{d}\mu(s)$$

is well-defined and continuous. Furthermore, taking $C$ as in (2.6) we obtain

$$|h_\mu(u) - h_\mu(0)| \leqslant C \int_{\mathbf{R}} |su|^{1/4} \, \mathrm{d}|\mu|(s) = C\|\mu\| \cdot |u|^{1/4},$$

which implies that

$$\mathrm{vp}\left( \frac{h_\mu([x,x])}{[x,x]} \right) \in \mathcal{S}'(\mathbf{R}^4)$$

is well-defined for $\mu \in \mathcal{M}$.

By (2.5),

$$\mathcal{F}\big(\mu([x,x])\big) = \mathrm{vp}\left( \frac{h_\mu([x,x])}{[x,x]} \right) \tag{2.7}$$

holds for $\mu = \delta_s$, $s \in \mathbf{R}$. Note, however, that the vector space $V$ of linear combinations of $\delta_s$, $s \in \mathbf{R}$, is not dense in $\mathcal{M}$ with respect to the norm topology. In contrast, $V$ is dense in $\mathcal{M}$ if we equip it with the weak topology $\sigma$ with respect to the space

$$\{f \in \mathcal{C}(\mathbf{R});\ (1+|s|)^{-1/4} f(s) \text{ is bounded}\}.$$

Since (2.7) holds for each $\mu$ in $V$ and both sides depend, as elements of $\mathcal{S}'(\mathbf{R}^4)$ with the weak topology, continuously on $\mu$ in $\mathcal{M}$ with respect to $\sigma$, we conclude that (2.7) holds for each $\mu \in \mathcal{M}$. In particular, for $f \in L^1_{\mathrm{loc}}(\mathbf{R}^1)$ such that $f(s)|s|^{1/4} \in L^1(\mathbf{R})$, we infer that

$$\mathcal{F}\big(f([x,x])\big) = \mathrm{vp}\left( \frac{1}{[x,x]} \int_{-\infty}^{\infty} f(s) g(s[x,x]) \, \mathrm{d}s \right),$$

and this completes the proof. ■

## 3. Representations of $\mathcal{F}\big(f([x,x])\big)$ derived by contour deformation

In Section 2, we deduced formulas for $\mathcal{F}\big(f([x,x])\big)$ which apply to "arbitrary" functions $f$ satisfying suitable, dimension-dependent growth conditions. In this section, we shall, in contrast, assume that $f$ is the boundary value of a meromorphic function $f(z)$ defined in the complex upper half-plane $\operatorname{Im} z > 0$, and we shall express $\mathcal{F}\big(f([x,x])\big)$ in part by residues.

**Proposition 5.** *Let $f(z)$ be meromorphic in the complex upper half-plane $\operatorname{Im} z > 0$ with the poles $z_1, \ldots, z_m$, $m \in \mathbf{N}_0$, and assume that $f$ is of polynomial growth, i.e.,*

$$\exists N \geqslant 0 : \forall z \in \mathbf{C} \text{ with } \operatorname{Im} z > 0 : |f(z)| \leqslant N(1+|z|)^N.$$

*We suppose, furthermore, that $f(s + \mathrm{i}\epsilon)$ converges locally uniformly for $\epsilon \searrow 0$ and that the limit fulfills $f(s)(1+|s|)^{(n-3)/4} \in L^1(\mathbf{R})$.*

*Then* $S := f([x,x]) \in \mathcal{S}'(\mathbf{R}^n)$ *and* $\mathcal{F}S$ *is given in* $[x,x] \neq 0$ *by*

$$\begin{aligned}
\mathcal{F}S &= 2\mathrm{i}(2\pi)^{n/2}\Big(Y(-[x,x]) + \mathrm{e}^{-(n-2)\pi\mathrm{i}/2}Y([x,x])\Big) \\
&\quad \times \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j}\Big[f(z)\Big(\frac{z}{-[x,x]}\Big)^{(n-2)/4} K_{(n-2)/2}\big(\sqrt{-z[x,x]}\big)\Big] \\
&\quad + \frac{\mathrm{i}}{2}\,(2\pi)^{n/2}Y(-[x,x]) \int_{-\infty}^{0} f(s)\Big(\frac{s}{[x,x]}\Big)^{(n-2)/4} J_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s \\
&\quad + \frac{\mathrm{i}}{2}\,(2\pi)^{n/2}Y([x,x])\Big\{\frac{2}{\pi}\sin\big(\tfrac{n-2}{2}\pi\big) \\
&\quad \times \int_{-\infty}^{0} f(s)\Big(-\frac{s}{[x,x]}\Big)^{(n-2)/4} K_{(n-2)/2}\big(\sqrt{-s[x,x]}\big)\,\mathrm{d}s \\
&\quad + \int_{0}^{\infty} f(s)\Big(\frac{s}{[x,x]}\Big)^{(n-2)/4}\Big[J_{-(n-2)/2}\big(\sqrt{s[x,x]}\big) \\
&\quad - 2\mathrm{e}^{-(n-2)\pi\mathrm{i}/2} J_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s\Big]\Big\}.
\end{aligned}$$

(*With respect to the residues, we note that* $z^{(n-2)/4}K_{(n-2)/2}(\sqrt{z})$ *can be considered as a holomorphic function of* $z \in \mathbf{C} \setminus (-\infty, 0]$.)

**Proof.** First observe that $\mathcal{F}\big(\delta_s([x,x)\big)$ is, according to formulas (2.1) and (2.2) in Proposition 1, infinitely differentiable in the region $G$ of $\mathbf{R}^n$ where $[x,x] \neq 0$; furthermore, for fixed $x$ in $G$, $\mathcal{F}\big(\delta_s([x,x)\big)$ is bounded by a multiple of $(1+|s|)^{(n-3)/4}$ if $|s| \to \infty$. Hence we can employ the formula

$$\mathcal{F}S = \mathcal{F}\big(f([x,x)\big) = \int_{-\infty}^{\infty} f(s)\mathcal{F}\big(\delta_s([x,x)\big)\,\mathrm{d}s$$

in $G$.

Setting $G = G_+ \cup G_-$ with $G_\pm = \{x \in \mathbf{R}^n;\ \pm[x,x] > 0\}$, and $W_\pm := \mathcal{F}\big(f([x,x)\big)|_{G_\pm}$, respectively, we obtain

$$\begin{aligned}
W_- &= 2^{n/2}\pi^{(n-2)/2}|[x,x]|^{(2-n)/2} \\
&\quad \times \Big[\int_{0}^{\infty} f(s)(-s[x,x])^{(n-2)/4} K_{(n-2)/2}\big(\sqrt{-s[x,x]}\big)\,\mathrm{d}s \\
&\quad - \frac{\pi}{2}\int_{-\infty}^{0} f(s)(s[x,x])^{(n-2)/4} N_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s\Big] \\
&= 2^{n/2}\pi^{(n-2)/2}|[x,x]|^{(2-n)/2} \\
&\quad \times \Big[\int_{-\infty}^{\infty} f(s)(-s[x,x])^{(n-2)/4} K_{(n-2)/2}\big(\sqrt{-s[x,x]}\big)\,\mathrm{d}s \\
&\quad + \frac{\mathrm{i}\pi}{2}\int_{-\infty}^{0} f(s)(s[x,x])^{(n-2)/4} J_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s\Big]. \qquad (3.1)
\end{aligned}$$

In the integral over $(-\infty, 0]$ we have used the formula

$$(t+\mathrm{i}0)^{\lambda/2} K_\lambda(\sqrt{t+\mathrm{i}0}) = -\frac{\pi}{2}|t|^{\lambda/2}\big[N_\lambda(\sqrt{|t|}) + \mathrm{i}J_\lambda(\sqrt{|t|})\big], \qquad \lambda = \frac{n-2}{2},$$

valid for $t = -s[x,x] < 0$, see [6, 8.407.2, 8.476.8].

Finally, we apply the residue theorem to the integral in (3.1) which contains the function $K_{(n-2)/2}\big(\sqrt{-s[x,x]}\big)$, and we infer that the following equation holds for $[x,x] < 0$ :

$$\begin{aligned}\mathcal{F}S = {} & 2\mathrm{i}(2\pi)^{n/2}\sum_{j=1}^{m} \operatorname*{Res}_{z=z_j}\Big[f(z)\Big(\frac{z}{-[x,x]}\Big)^{(n-2)/4} K_{(n-2)/2}\big(\sqrt{-z[x,x]}\big)\Big] \\ & + \frac{\mathrm{i}}{2}\,(2\pi)^{n/2}\int_{-\infty}^{0} f(s)\Big(\frac{s}{[x,x]}\Big)^{(n-2)/4} J_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s.\end{aligned}$$

Similarly, Proposition 1 furnishes

$$\begin{aligned} W_+ = {} & 2^{n/2}\pi^{(n-2)/2}[x,x]^{(2-n)/2} \\ & \times\Big[\cos\big(\tfrac{n-2}{2}\pi\big)\int_{-\infty}^{0} f(s)(-s[x,x])^{(n-2)/4} K_{(n-2)/2}\big(\sqrt{-s[x,x]}\big)\,\mathrm{d}s \\ & -\frac{\pi}{2}\int_0^\infty f(s)(s[x,x])^{(n-2)/4} N_{-(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s\Big] \\ = {} & 2^{n/2}\pi^{(n-2)/2}[x,x]^{(2-n)/2} \\ & \times\Big[\mathrm{e}^{-(n-2)\pi\mathrm{i}/2}\int_{-\infty}^{\infty} f(s)(-s[x,x])^{(n-2)/4} K_{(n-2)/2}\big(\sqrt{-s[x,x]}\big)\,\mathrm{d}s \\ & +\mathrm{i}\sin\big(\tfrac{n-2}{2}\pi\big)\int_{-\infty}^{0} f(s)(-s[x,x])^{(n-2)/4} K_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s \\ & +\frac{\mathrm{i}\pi}{2}\int_0^\infty f(s)(s[x,x])^{(n-2)/4} \\ & \times\big[J_{-(n-2)/2}\big(\sqrt{s[x,x]}\big) - 2\mathrm{e}^{-(n-2)\pi\mathrm{i}/2} J_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\big]\mathrm{d}s\Big].\end{aligned}$$

Here we have used the identity

$$\begin{aligned}(t-\mathrm{i}0)^{\lambda/2} K_\lambda(\sqrt{t-\mathrm{i}0}) = {} & -\frac{\pi}{2}|t|^{\lambda/2}\mathrm{e}^{\mathrm{i}\lambda\pi}\big[N_{-\lambda}(\sqrt{|t|}) \\ & + \mathrm{i}J_{-\lambda}(\sqrt{|t|}) - 2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\lambda\pi}J_\lambda(\sqrt{|t|})\big], \qquad \lambda = \frac{n-2}{2},\end{aligned}$$

valid for $t = -s[x,x] < 0$ (see [6, 8.407.2, 8.476.1, 8.476.3]), in order to replace the function $N_{-(n-2)/2}\big(\sqrt{s[x,x]}\big)$. If the integral involving $K_{(n-2)/2}\big(\sqrt{-s[x,x]}\big)$ is expressed by residues, we obtain the terms of the formula in Proposition 5 referring to the region $G_+$. This completes the proof. ∎

## 4. Temperate fundamental solutions of the iterated Klein–Gordon operator

The iterated Klein–Gordon operator $(\partial_0^2 - \Delta_{n-1} - c^2)^m = ([\partial, \partial] - c^2)^m$, $c \in \mathbf{C}$, $m \in \mathbf{N}$, is hyperbolic in the direction $x_0 = t$, and hence this operator possesses one and only one fundamental solution with support in the half-space $x_0 \geqslant 0$, see [18, pp. 89, 90], [22, (VI,5;30), p. 179], [12], [10], [8, Thm. 12.5.1, p. 120]. This fundamental solution is calculated best by means of the many-dimensional Laplace transformation, see [26, § 9], [13]. Note that the Laplace transformation can be applied since this fundamental solution has its support inside a convex cone.

In contrast, we are aiming here at deriving a *temperate* fundamental solution $E$ of $(\partial_0^2 - \Delta_{n-1} - c^2)^m$ by *Fourier transformation.* Under the assumption of $c^2 \in \mathbf{C} \setminus \mathbf{R}$, we have $([x, x] + c^2)^{-m} \in \mathcal{O}_M(\mathbf{R}^n)$ and hence $E = (-1)^m \mathcal{F}^{-1}\big(([x, x] + c^2)^{-m}\big) \in \mathcal{O}'_C(\mathbf{R}^n)$, and this is the only temperate fundamental solution of $(\partial_0^2 - \Delta_{n-1} - c^2)^m$. Except for the case of odd $n$ and $[x, x] > 0$, the application of Proposition 5 proves to be advantageous to that of Proposition 1.

**Proposition 6.** *Let* $c \in \mathbf{C}$ *with* $\operatorname{Re} c > 0$, $\operatorname{Im} c \neq 0$, *and define* $z^\lambda = \mathrm{e}^{\lambda \log z}$ *by* $\operatorname{Im}(\log z) \in (-\pi, \pi)$ *for* $z \in \mathbf{C} \setminus (-\infty, 0]$.

(1) *The holomorphic distribution-valued function*

$$E_\lambda : \mathbf{C} \longrightarrow \mathcal{O}'_C(\mathbf{R}^n) : \lambda \longmapsto \mathcal{F}^{-1}\big(([x, x] + c^2)^{-\lambda}\big)$$

*has, for* $\operatorname{Re} \lambda > \frac{n}{2} - 1$, *the representation*

$$E_\lambda(x) = \frac{\mathrm{e}^{-\mathrm{i}\, \mathrm{sign}(\operatorname{Im} c)(n-1)\pi/2}}{(2\pi)^{n/2} 2^{\lambda - 1} \Gamma(\lambda)} \Big(\frac{c}{\sqrt{[x, x]}}\Big)^{n/2 - \lambda} K_{n/2 - \lambda}\big(c\sqrt{[x, x]}\big) \in L^1_{\mathrm{loc}}(\mathbf{R}^n). \tag{4.1}$$

*Here we set* $\sqrt{[x, x]} = -\mathrm{i}\ \mathrm{sign}(\operatorname{Im} c)\sqrt{-[x, x]}$ *if* $[x, x] < 0$.

(2) $E_\lambda : \mathbf{C} \to \mathcal{O}'_C(\mathbf{R}^n)$ *is a group homomorphism from the additive group* $\mathbf{C}$ *into the convolution group* $\mathcal{O}'_C$, *i.e.,* $E_\lambda * E_\mu = E_{\lambda + \mu}$ *for each* $\lambda, \mu \in \mathbf{C}$.

(3) *In particular,* $E_{-m} = (-[\partial, \partial] + c^2)^m \delta$, *and the only temperate fundamental solution of the iterated Klein–Gordon operator* $(\partial_0^2 - \Delta_{n-1} - c^2)^m$ *is* $E = (-1)^m E_m$.

**Proof.** Due to the assumptions on $c$, $\operatorname{Im}([x, x] + c^2) \neq 0$ for $x \in \mathbf{R}^n$, and hence the powers $([x, x] + c^2)^{-\lambda}$, $\lambda \in \mathbf{C}$, are defined by means of the determination of the logarithm given in the proposition. Therefore $\lambda \mapsto E_\lambda$ is an entire function with values in $\mathcal{O}'_C$. Since, obviously,

$$([x, x] + c^2)^{-\lambda} \cdot ([x, x] + c^2)^{-\mu} = ([x, x] + c^2)^{-\lambda - \mu}, \qquad \lambda, \mu \in \mathbf{C},$$

the mapping $\lambda \mapsto E_\lambda$ is a group homomorphism from$(\mathbf{C}, +)$ into $(\mathcal{O}'_C, *)$.

In order to calculate $E_\lambda$, let us first assume that both $\operatorname{Re} c$ and $\operatorname{Im} c$ are positive, and let us distinguish four cases according to whether $n$ is even or odd, and whether $[x, x]$ is positive or negative, respectively.

If $n$ is odd and $[x,x] > 0$, then Proposition 1 yields, by [6, 8.465.2] and [15, 4.23, p. 36], the following for $\operatorname{Re}\lambda > \frac{n-1}{4}$ :

$$\begin{aligned}
E_\lambda &= \mathcal{F}^{-1}\big(([x,x]+c^2)^{-\lambda}\big)\\
&= -\frac{1}{2(2\pi)^{n/2}[x,x]^{(n-2)/4}}\int_0^\infty \frac{s^{(n-2)/4}}{(s+c^2)^\lambda}\, N_{-(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s\\
&= \frac{(-1)^{(n-1)/2}}{(2\pi)^{n/2}[x,x]^{(n-2)/4}}\int_0^\infty \frac{u^{n/2}}{(u^2+c^2)^\lambda}\, J_{(n-2)/2}\big(u\sqrt{[x,x]}\big)\,\mathrm{d}u\\
&= \frac{(-1)^{(n-1)/2}}{(2\pi)^{n/2}2^{\lambda-1}\Gamma(\lambda)}\Big(\frac{c}{\sqrt{[x,x]}}\Big)^{n/2-\lambda} K_{n/2-\lambda}\big(c\sqrt{[x,x]}\big).
\end{aligned}$$

On the other hand, if $n$ is arbitrary, $[x,x] < 0$ and $\operatorname{Re}\lambda > \frac{n+1}{4}$, then Proposition 5 furnishes with $f(z) = (z+c^2)^{-\lambda}$

$$\begin{aligned}
E_\lambda &= \frac{\mathrm{i}}{2}(2\pi)^{-n/2}\int_{-\infty}^0 (s+c^2)^{-\lambda}\Big(\frac{s}{[x,x]}\Big)^{(n-2)/4} J_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\,\mathrm{d}s\\
&= \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\lambda\pi}}{(2\pi)^{n/2}(-[x,x])^{(n-2)/4}}\int_0^\infty \frac{u^{n/2}}{(u^2-c^2)^\lambda}\, J_{(n-2)/2}\big(u\sqrt{-[x,x]}\big)\,\mathrm{d}u\\
&= \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\lambda\pi}}{(2\pi)^{n/2}2^{\lambda-1}\Gamma(\lambda)}\Big(\frac{-\mathrm{i}c}{\sqrt{-[x,x]}}\Big)^{n/2-\lambda} K_{n/2-\lambda}\big(-\mathrm{i}c\sqrt{-[x,x]}\big).
\end{aligned}$$

Due to

$$\Big(\frac{-\mathrm{i}c}{\sqrt{-[x,x]}}\Big)^{n/2-\lambda} = \Big(\mathrm{e}^{-\mathrm{i}\pi}\frac{c}{\sqrt{[x,x]}}\Big)^{n/2-\lambda} = \Big(\frac{c}{\sqrt{[x,x]}}\Big)^{n/2-\lambda}\mathrm{e}^{\mathrm{i}\lambda\pi}\mathrm{e}^{-\mathrm{i}n\pi/2},$$

the last expression coincides with the result in (4.1) for $[x,x] < 0$.

Finally, let us consider the case of $n$ even and $[x,x]$ positive. Then Proposition 5 and [6, 8.404.2] yield

$$\begin{aligned}
E_\lambda &= \frac{\mathrm{i}\,[x,x]^{(2-n)/4}}{2(2\pi)^{n/2}}\int_0^\infty \frac{s^{(n-2)/4}}{(s+c^2)^\lambda}\\
&\quad\times\Big[J_{-(n-2)/2}\big(\sqrt{s[x,x]}\big) - 2\mathrm{e}^{(n-2)\pi\mathrm{i}/2}J_{(n-2)/2}\big(\sqrt{s[x,x]}\big)\Big]\,\mathrm{d}s\\
&= \frac{\mathrm{i}\,(-1)^{n/2}}{(2\pi)^{n/2}([x,x])^{(n-2)/4}}\int_0^\infty \frac{u^{n/2}}{(u^2+c^2)^\lambda}\, J_{(n-2)/2}\big(u\sqrt{[x,x]}\big)\,\mathrm{d}u\\
&= \frac{\mathrm{e}^{-\mathrm{i}\pi(n-1)/2}}{(2\pi)^{n/2}2^{\lambda-1}\Gamma(\lambda)}\Big(\frac{c}{\sqrt{[x,x]}}\Big)^{n/2-\lambda} K_{n/2-\lambda}\big(c\sqrt{[x,x]}\big).
\end{aligned}$$

So in each case, we have obtained the result announced in Proposition 6, at least for $[x,x] \neq 0$. Let us observe that, for $\operatorname{Re}\lambda > \frac{n}{2} - 1$,

$$\Big(\frac{c}{\sqrt{[x,x]}}\Big)^{n/2-\lambda} K_{n/2-\lambda}\big(c\sqrt{[x,x]}\big) \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$$

due to

$$K_\nu(z) \sim \tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}, \qquad z \to 0,\ \operatorname{Re}\nu > 0,$$

see [1, 9.6.9, p. 375]. For sufficiently large $\operatorname{Re}\lambda$, the distribution $E_\lambda$ must be locally integrable, as one can see by approximation from $\mathcal{F}^{-1}\big((\alpha^2 x_0^2 - x_1^2 - \cdots - x_{n-1}^2 - c^2)^\lambda\big)$, $\alpha = 1 + \mathrm{i}\epsilon$, $\epsilon \to 0$. Therefore, formula (4.1) in the proposition holds for all complex $\lambda$ satisfying $\operatorname{Re}\lambda > \frac{n}{2} - 1$ by analytic continuation. To conclude the proof, we employ the relation $E_{\lambda,c} = \overline{E_{\bar\lambda,\bar c}}$ in order to reduce the case of $\operatorname{Im} c < 0$ to that of $\operatorname{Im} c > 0$. ■

**Remarks.**

(1) [5] arrives at a formula comparable to (4.1) using analytic continuation with respect to the coefficients of the quadratic form $[x, x]$, see [5, Ch. III, 2.8, (8), p. 289].

(2) Let us emphasize again (cfälso the introduction) that the formulas (2.1), (2.2) in Proposition 1, respectively the one in [23, Thm. 1, p. 509], which refers to $\mathcal{F}\big(\phi([x, x])\big)$, $\phi \in \mathcal{S}(\mathbf{R})$, do not directly yield the simple result in Proposition 6, except for the case of $n$ odd and $[x, x] > 0$.

**Corollary 2.** *Let $c \in \mathbf{C}$ with $\operatorname{Re} c > 0$, $\operatorname{Im} c \neq 0$, and set, as in Proposition 6, $\sqrt{[x, x]} = -\mathrm{i}\operatorname{sign}(\operatorname{Im} c)\sqrt{-[x, x]}$ for $[x, x] < 0$. Let $E$ denote the uniquely determined temperate fundamental solution of $\partial_0^2 - \Delta_{n-1} - c^2$.*

(a) *If $n = 2$, then*

$$E = \frac{\mathrm{i}\operatorname{sign}(\operatorname{Im} c)}{2\pi} K_0\big(c\sqrt{[x, x]}\big) \in L^1_{\mathrm{loc}}(\mathbf{R}^2).$$

(b) *If $n = 3$, then*

$$E = \frac{\mathrm{e}^{-c\sqrt{[x,x]}}}{4\pi\sqrt{[x, x]}} \in L^1_{\mathrm{loc}}(\mathbf{R}^3).$$

(c) *If $n = 4$, then*

$$E = -\frac{\mathrm{i}\operatorname{sign}(\operatorname{Im} c)}{4\pi^2} \operatorname{vp}\bigg(\frac{c}{\sqrt{[x, x]}} K_1\big(c\sqrt{[x, x]}\big)\bigg) + \frac{1}{4\pi}\delta([x, x]).$$

**Proof.** For $n = 2$ or $n = 3$, Proposition 6 immediately yields the results, since then $E = -E_1 \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. For $n = 4$, in contrast, we have to determine $E = -E_1 = -\lim_{\lambda \searrow 1} E_\lambda$, since $E_\lambda \in L^1_{\mathrm{loc}}(\mathbf{R}^4)$ holds only for $\operatorname{Re}\lambda > \frac{n}{2} - 1 = 1$.

If $n = 4$, $\operatorname{Re}\lambda > 1$ and $\operatorname{Im} c > 0$, then

$$\begin{aligned} E_\lambda(x) &= \frac{\mathrm{i}}{4\pi^2 \cdot 2^{\lambda-1}\Gamma(\lambda)} \Big(\frac{c}{\sqrt{[x, x]}}\Big)^{2-\lambda} K_{2-\lambda}\big(c\sqrt{[x, x]}\big) \\ &= \frac{\mathrm{i}}{2^{\lambda+1}\pi^2\Gamma(\lambda)} \lim_{\epsilon \searrow 0}\Big[([x, x] - \mathrm{i}\epsilon)^{\lambda-2}\big(c\sqrt{[x, x]}\big)^{2-\lambda} K_{2-\lambda}\big(c\sqrt{[x, x]}\big)\Big] \end{aligned}$$

because of $\sqrt{[x,x]} = -\mathrm{i}\sqrt{[x,x]}$ for $[x,x] < 0$, which implies

$$\lim_{\epsilon\searrow 0}\Big[\big([x,x]-\mathrm{i}\epsilon\big)^{\lambda-2}\big(c\sqrt{[x,x]}\big)^{2-\lambda}\Big] = \Big(\frac{c}{\sqrt{[x,x]}}\Big)^{2-\lambda}.$$

Sokhotski's formula furnishes for the boundary values

$$\lim_{\epsilon\searrow 0}\big([x,x]-\mathrm{i}\epsilon\big)^{\lambda-2} =: \big([x,x]-\mathrm{i}0\big)^{\lambda-2}$$

the following limit relation in $\mathcal{S}'(\mathbf{R}^4)$ :

$$\lim_{\lambda\searrow 1}\big([x,x]-\mathrm{i}0\big)^{\lambda-2} = \mathrm{vp}\Big(\frac{1}{[x,x]}\Big) + \mathrm{i}\pi\delta([x,x]).$$

Since the function $f(t) = ctK_1(ct)$, $t \in \mathbf{R}$, is $\mathcal{C}^1$, it can be multiplied with the principal value and with the delta function, and therefore

$$E = -\frac{\mathrm{i}}{4\pi^2}\,\mathrm{vp}\bigg(\frac{c}{\sqrt{[x,x]}}\,K_1\big(c\sqrt{[x,x]}\big)\bigg) + \frac{1}{4\pi}\,\delta([x,x]).$$

As before, for $\operatorname{Im} c < 0$, we use $E_{1,c} = \overline{E_{1,\bar{c}}}$. ■

**Remark.** For $n = 4$, the limits with respect to $c = \mathrm{i}\epsilon$, $\pm\epsilon \searrow 0$, yield the following fundamental solutions $E_\pm$ of the wave operator $\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 = \Box_4$ :

$$E_\pm = \mp\frac{\mathrm{i}}{4\pi^2}\,\mathrm{vp}\Big(\frac{1}{[x,x]}\Big) + \frac{1}{4\pi}\,\delta([x,x]) = \mp\frac{\mathrm{i}}{4\pi^2}\,\mathrm{vp}\big(([x,x]\mp\mathrm{i}0)^{-1}\big).$$

Note that $\mathcal{F}\big(\delta([x,x])\big) = -4\pi\,\mathrm{vp}\big([x,x]^{-1}\big)$ by Corollary 1 (e), and hence $\Box_4\,\mathrm{vp}\big([x,x]^{-1}\big) = 0$, i.e., $\mathrm{vp}\big([x,x]^{-1}\big)$ is a solution of the homogeneous wave equation in $\mathbf{R}^4$. On the other hand,

$$\operatorname{Re} E_\pm = \frac{1}{4\pi}\,\delta([x,x]) = \frac{\delta(|x_0|-|x'|)}{8\pi|x_0|}, \qquad x' = (x_1,x_2,x_3)^T,$$

originates as convex combination of the retarded and the advanced fundamental solution $\delta(x_0 \mp |x'|)/(4\pi|x'|)$.

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