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ON THE LAPLACE TRANSFORM FOR VECTOR VALUED HYPERFUNCTIONS

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Dedicated to Professor Dr. H.G. Tillmann
on the occasion of his 85 birthday

Abstract: We introduce a Laplace transform for Laplace hyperfunctions valued in a complete locally convex space X. In this general case the Laplace transform is a compatible family of holomorphic functions with values in local Banach spaces. Especially interesting is the case where $X = L_b(E, F)$ is the space of operators between locally convex spaces. In the forthcoming paper [6] this will be applied to solve the abstract Cauchy problem for operators in complete ultrabornological locally convex spaces (like spaces of smooth functions and distributions) extending results of Komatsu for operators in Banach spaces.

Keywords: Abstract Cauchy problem, Laplace hyperfunctions, Laplace distributions, Laplace transform, Laplace inversion formula, exponential growth.

1. Introduction

The solution of the abstract Cauchy problem is a classical part of the theory of differential equations. A standard approach to this problem for operators in Banach spaces is the use of Laplace transform for operator valued (generalized) functions. This has been developed in several settings of generalized functions, the most general being the approach of Komatsu (see [9, 10]) in the framework of Laplace hyperfunctions.

Though many standard operators of analysis are naturally defined on locally convex spaces like holomorphic functions, $C^\infty-$ functions or spaces of distributions, a solution of the abstract Cauchy problem for operators in locally convex spaces by means of conditions on the resolvent (and using the Laplace transform as the relevant tool) is missing, perhaps since a corresponding reasoning was considered to be impossible because of simple examples like the following:

Let $C : H(\mathbb{C}) \to H(\mathbb{C})$ be the multiplication operator defined by $C(f)(z) := zf(z)$ for $f \in H(\mathbb{C})$. The $L(H(\mathbb{C}))-$valued continuous function $T : [0, \infty[\to$

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$L_b(H(\mathbb{C})), T(t)(f)(z) := e^{tz}f(z)$, clearly gives an (at most exponentially increasing) solution $F(t) := T(t)f$ of the following abstract Cauchy problem:

$$\begin{cases} F'(t) = CF, \\ F(0) = f. \end{cases}$$

Nevertheless the operator $\lambda - C : H(\mathbb{C}) \to H(\mathbb{C})$ is for no $\lambda \in \mathbb{C}$ surjective, hence the resolvent set is void, and moreover the Laplace transform

$$L(T)(\lambda)(f)(z) := \int_0^\infty e^{-\lambda t}T(t)f(z)\, dt = \frac{f(z)}{\lambda - z}$$

is not defined as an operator in $H(\mathbb{C})$. Similar examples are also given in [1, p. 164] ($C := x\frac{d}{dx}$ in the Schwartz space $\mathcal{S}(\mathbb{R})$) and in [3, p. 125] ($C := \frac{d}{dz}$ in $H(\mathbb{C})$).

In spite of these examples we will develop in the present paper a suitable notion of Laplace transform for Laplace hyperfunctions valued in a complete (ultra)bornological locally convex spaces (which becomes especially transparent for Fréchet spaces and which can also be applied to the simple example above).

In this way we will modify and extend the approach of Komatsu to cover operators in spaces typical for analytic applications: various spaces of smooth functions or distributions.

The results of the present paper are used in the forthcoming paper [6] to provide a solution of the abstract Cauchy problem for operators in complete ultrabornological locally convex spaces using a suitably general notion of a resolvent for operators in locally convex spaces.

Unlike Komatsu [10] (who used boundary values of exponentially increasing holomorphic functions) we will define Laplace hyperfunctions by duality, i.e., as the dual space of a natural space of holomorphic test functions. We then introduce a Laplace transform for these Laplace hyperfunctions with values in a complete locally convex space X which is the projective limit of its local Banach spaces $(X_\gamma)_{\gamma\in\Gamma}$. The appropriate notion of Laplace transform then is a family of holomorphic functions $(L_\gamma(T))_{\gamma\in\Gamma}$ with values in X_γ satisfying a suitable compatibility condition and some exponential growth condition on sectorial domains in the complex plane.

Since we precisely describe the Laplace image of our space of test functions, we can also prove a Laplace inversion formula for Laplace hyperfunctions in our general setting. In this way we completely characterize the Laplace transforms of vector valued Laplace hyperfunctions (see Theorem 2.4 and Corollary 3.5).

This provides a full extension and improvement of Komatsu's results [9, 10] on Laplace transform of operator valued Laplace hyperfunctions in Banach spaces to our general setting which is required in applications to spaces of smooth or analytic functions as well as to various spaces of distributions and will be explored in the forthcoming paper [6].

Our Laplace transform also provides a general frame for the study of the Laplace transform for vector valued weighted generalized functions. The case of weighted vector valued distributions is treated in Section 4, where also many more examples are given which are connected to semigroups of operators providing solutions of the Cauchy problem for certain differential operators with variable coefficients. It is worth noting that a theory of vector valued hyperfunctions of Sato type is developed in [5].

Specifically, let E and F be locally convex spaces , where E is bornological with system $\mathcal{B}^E$ of bounded absolutely convex subsets (and corresponding normed spaces $E_B, B \in \mathcal{B}^E$) and where F is complete with the topology defined by a seminorm system $\{\|\cdot\|_\alpha, \alpha \in A\}$ (with corresponding local Banach spaces $F_\alpha, \alpha \in\ A$). Let $X = L_b(E,F)$ be the space of continuous linear operators from E to F. Then $L_b(E,F) = \operatorname{proj}_{(B,\alpha)\in\mathcal{B}^E\times A} L(E_B, F_\alpha)$, that is, the local Banach spaces for $X = L_b(E,F)$ are provided by the projective spectrum $(L(E_B,F_\alpha))_{(B,\alpha)\in\mathcal{B}^E\times A}$ of spaces of operators on Banach spaces. Thus the Laplace transform of a Laplace hyperfunction with values in $X = L_b(E,F)$ is a family of holomorphic functions with values in $L(E_B,F_\alpha), (B,\alpha) \in \mathcal{B}^E \times A$. The Laplace transform of $L_b(E,F)-$valued Laplace hyperfunctions essentially simplifies in the important case where E and F are Fréchet spaces (or, dually, E and F are $(DFS)-$spaces, respectively), e.g. if E and F are the space of holomorphic functions or $C^\infty-$ functions or tempered distributions (see Corollaries 2.9, 2.10 and 3.6).

We provide several examples where our Laplace transform exists and is calculable while the Laplace transform in the classical sense (i.e. as one holomorphic function defined on a fixed open set in $\mathbb{C}$ with values in $L_b(E)$) does not exist. Nevertheless, in many cases, our generalized Laplace transform is a very natural "true" Laplace transform given by an integral over $[0,\infty)$.

For the theory of hyperfuntions see [14], [15], [8] as well as [12] and [13]. For non-explained notions from functional analysis see [11].

2. A general Laplace transform and Laplace hyperfunctions

In [10] Komatsu introduced a general definition for the Laplace transform of generalized functions valued in Banach spaces. In this section we will extend and modify this definition (which was based on boundary values) in two ways: we will consider generalized functions defined on a natural space of test functions (instead of vector valued boundary values) which seems to be more natural in the present context, and moreover the values will be taken in a complete locally convex space. As it turns out, the Laplace transform is a compatible family of holomorphic functions defined on a directed family of domains in $\mathbb{C}$ containing large angular domains rather than on a single domain (see Theorem 2.4). This crucial notion is introduced as follows:

Let X be a complete locally convex space defined by a projective spectrum $\mathscr{X} := (X_\gamma)_{\gamma\in\Gamma}$ of locally convex spaces $(X_\gamma, \|\cdot\|_\gamma)$ with linking maps $\kappa_\gamma : X \to X_\gamma$ and $\kappa_\nu^\gamma : X_\gamma \to X_\nu$ for $\gamma \geqslant \nu$. Correspondingly, let $\mathscr{G} := (G_\gamma)_{\gamma\in\Gamma}$ be a *directed* family of domains in $\mathbb{C}$, that is,

$$\emptyset \neq G_\gamma \subseteq G_\nu \qquad \text{if } \gamma \geqslant \nu.$$

Definition 2.1. *Let $\mathscr{X} = (X_\gamma)_{\gamma\in\Gamma}$ and $\mathscr{G} = (G_\gamma)_{\gamma\in\Gamma}$ be a projective spectrum of locally convex spaces and a corresponding directed family of domains in $\mathbb{C}$, respectively. A family $\mathscr{S} = (S_\gamma)_{\gamma\in\Gamma}$ is called a spectral-valued (or $\mathscr{X}$-valued) holomorphic function (denoted by $\mathscr{S} : \mathscr{G} \to \mathscr{X}$) if*

(i) *$S_\gamma : G_\gamma \to X_\gamma$ is holomorphic;*
(ii) *(compatibility) $\forall\ \gamma \geqslant \nu : \quad \kappa_\nu^\gamma \circ S_\gamma = S_\nu\big|_{G_\gamma}$.*

We will show that the natural definition of a Laplace transform for a large space of $X-$valued generalized functions will lead to an $\mathscr{X}-$ valued holomorphic function $\mathscr{L} : \mathscr{G} \to \mathscr{X}$.

To motivate the definition of Laplace hyperfunctions given below we remark that the local Banach spaces of the corresponding space of test functions should at least contain the functions

$$f_\lambda(z) := \exp(-\lambda z)$$

for λ in some angular domain in $\mathbb{C}$. Moreover, to obtain the largest class of Laplace hyperfunctions on $[0,\infty[$, the test functions should be defined on small angular neighborhoods of $[0,\infty[$. These observations lead to the following definition of the test function space:

$$H := \operatorname*{ind}_K (\operatorname*{proj}_k H_{K,k}) = \operatorname*{ind}_K H_K$$

where

$$H_{K,k} := \{f \in H(\Omega_K) : \|f\|_{K,k} := \sup_{z\in\Omega_K} |f(z)| \exp(k \operatorname{Re} z) < \infty\}$$

and

$$\Omega_K := \left\{ z \in \mathbb{C} : |\operatorname{Im} z| < \frac{\operatorname{Re} z}{K} + \frac{1}{K^2} \right\}.$$

We also define

$$H^0_{K,k} := \overline{H_K}^{H_{K,k}} \subseteq H_{K,k},$$

as well as

$$V_{K,k} := \left\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda > k + \frac{|\operatorname{Im} \lambda|}{K} \right\}.$$

The following conventions will be used throughout the paper:

$$z = x + iy \text{ and } \lambda = \eta + i\zeta, \qquad \text{where } x,\ y,\ \eta,\ \zeta \text{ are reals.}$$

Also, the constants C and C_j may change from line to line without further notice.

The following Lemma shows that H satisfies the conditions required above:

Lemma 2.2. *Let $K > 1$, then $f_\lambda \in H^0_{K,k}$ for any $\lambda \in V_{K,k}$.*

Proof. For $\lambda \in V_{K,k}$ let us define

$$f_{j,\lambda}(z) := \exp\left(-\lambda z - \frac{z^2}{2j}\right).$$

With the convention for z and λ from above we get for any $k \in \mathbb{N}$ and $z \in \Omega_K$:

$$\begin{aligned} |f_{j,\lambda}(z)| \exp(kx) &= \exp\left(-\eta x + \zeta y - \frac{(x^2 - y^2)}{2j} + kx\right) \\ &\leqslant \exp\left(-\eta x + \zeta y - \frac{\left(x^2 - \frac{x^2}{K^2} - \frac{2x}{K^3} - \frac{1}{K^4}\right)}{2j} + kx\right). \end{aligned}$$

The right hand side is bounded on Ω_K for fixed $\lambda \in V_{K,k}$ and $K > 1$. Thus $f_{j,\lambda} \in H_K$. We now get the following estimate for $x \geqslant 0$:

$$\begin{aligned} |f_{j,\lambda}(z) - f_\lambda(z)| \exp(kx) &= \exp(-\eta x + \zeta y + kx) \left|1 - \exp\left(-\frac{z^2}{2j}\right)\right| \\ &\leqslant \exp\left(-\eta x + \frac{|\zeta| x}{K} + \frac{|\zeta|}{K^2} + kx\right) \left|1 - \exp\left(-\frac{z^2}{2j}\right)\right|. \end{aligned}$$

If $\operatorname{Re} \lambda = \eta > k + \frac{|\zeta|}{K}$, then $\exp\left(-\eta x + \frac{|\zeta| x}{K} + \frac{|\zeta|}{K^2} + kx\right)$ tends to zero as x tends to $+\infty$. On the other hand, the term $\left|1 - \exp\left(-\frac{z^2}{2j}\right)\right|$ is bounded on Ω_K uniformly with respect to j and tends to zero uniformly on

$$\Omega_K \cap \{z = x + iy : x \leqslant x_0\}$$

for any x_0 as $j \to \infty$. Thus $\|f_{j,\lambda} - f_\lambda\|_{K,k} \to 0$ as $j \to \infty$. ∎

Definition 2.3. *A continuous linear operator $T : H \to X$ will be called an X-valued Laplace hyperfunction.*

We now define the Laplace transform $\mathscr{L}(T)$ for an X-valued hyperfunction

$$T : H \to X,$$

where X is a complete locally convex space given by the projective spectrum $\mathscr{X} = (X_\gamma)_{\gamma \in \Gamma}$ of Banach spaces X_γ as above: for any $\gamma \in \Gamma$ and any $K \in \mathbb{N}$ we have continuous linear mappings

$$\kappa_\gamma \circ T \circ i_K : H_K \to X_\gamma$$

where $i_K : H_K \to H$ is the canonical embedding. Hence there is $k = k(\gamma, K)$ such that $\kappa_\gamma \circ T \circ i_K$ has a unique continuous extension

$$T_{\gamma,K,k} : H^0_{K,k} \to X_\gamma$$

since X_γ is complete. By Lemma 2.2,

$$L_\gamma(T)(\lambda) := T_{\gamma,K,k}(f_\lambda) \quad \text{ is defined for } \lambda \in V_{K,k}.$$

By the properties of inductive and projective limits we may assume that

$$k(\nu, K) \leqslant k(\gamma, J) \qquad \text{if } \gamma \geqslant \nu \text{ and } J \geqslant K. \tag{2.1}$$

Set

$$G_\gamma := \cup_{K\in\mathbb{N}} V_{K,k(\gamma,K)}.$$

Theorem 2.4. *Let X be a complete locally convex space defined by the projective spectrum $\mathscr{X} = (X_\gamma)_{\gamma\in\Gamma}$ of Banach spaces and let*

$$T : H \to X$$

be continuous. Let $\mathscr{L}(T) := (L_\gamma(T))_{\gamma\in\Gamma}$ and $\mathscr{G} := (G_\gamma)_{\gamma\in\Gamma}$ be chosen for T as above. Then $\mathscr{G}$ is a directed family of domains in $\mathbb{C}$ and $\mathscr{L}(T) : \mathscr{G} \to \mathscr{X}$ is a well defined holomorphic $\mathscr{X}-$valued function such that

$$\begin{aligned} &\forall\ \gamma\in\Gamma\ \forall\ K\in\mathbb{N}\ \exists\ k: \quad G_\gamma \supseteq V_{K,k} \quad \textit{and} \\ &\qquad\qquad \sup_{\lambda\in V_{K,k}} \|L_\gamma(T)(\lambda)\|_\gamma \exp\left(-\frac{\operatorname{Re}\lambda}{K}\right) < \infty. \end{aligned} \tag{2.2}$$

Definition 2.5. *The spectral valued holomorphic function $\mathscr{L}(T)$ defined above is called Laplace transform of T.*

Definition 2.6. *The set of all holomorphic $\mathscr{X}-$valued maps $\mathscr{S} : \mathscr{G} \to \mathscr{X}$ that satisfy the condition (2.2) is denoted by $H_{exp}(\mathscr{X})$.*

Notice that $H_{exp}(\mathscr{X})$ is a vector space canonically.

Since the sets $G \subseteq \mathbb{C}$ of the above form will play an essential role later on we define:

Definition 2.7. *An open set $G \subseteq \mathbb{C}$ is called conoidal if for every $K \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $G \supseteq V_{K,k}$.*

Proof of Theorem 2.4. (a) By (2.1) we have for $\gamma \geqslant \nu$

$$V_{K,k(\nu,K)} \supseteq V_{K,k(\gamma,K)}$$

and hence

$$G_\gamma \subseteq G_\nu.$$

Thus, $\mathscr{G}$ is a directed family of domains.

(b) We clearly have

$$\|L_\gamma(T)(\lambda)\|_\gamma \leqslant \|f_\lambda\|_{K,k} \cdot \|T_{\gamma,K,k}\|_{L(H^0_{K,k},X_\gamma)} \qquad \text{for } \lambda \in V_{K,k}.$$

Observe that

$$\|f_\lambda\|_{K,k} = \sup_{z\in\Omega_K} |\exp(-\lambda z)| \exp(k \operatorname{Re} z) = \sup_{z\in\Omega_K} \exp(-\eta x + \zeta y + kx).$$

Let $|y| < \frac{x}{K} + \frac{1}{K^2}$ and $\lambda \in V_{K,k}$. Thus $\eta > k + \frac{|\zeta|}{K}$ and there is $\delta > 0$ such that $\eta - \delta = k + \frac{|\zeta|}{K}$.

For $x \geqslant 0$ we have

$$\begin{aligned} \exp(-\eta x + \zeta y + kx) &\leqslant \exp\left(-\eta x + |\zeta|\left(\frac{x}{K} + \frac{1}{K^2}\right) + kx\right) \\ &\leqslant \exp\left(-\delta x + \frac{|\zeta|}{K^2}\right) \leqslant \exp\left(-\delta x + \frac{\eta - k}{K}\right) \\ &\leqslant \exp\left(\frac{\eta}{K}\right) \exp(-\delta x). \end{aligned} \tag{2.3}$$

For $-\frac{1}{K} \leqslant x \leqslant 0$ we have

$$\begin{aligned} \exp(-\eta x + \zeta y + kx) &\leqslant \exp\left(-\eta x + kx + |\zeta|\left(\frac{x}{K} + \frac{1}{K^2}\right)\right) \\ &\leqslant \exp\left(-\frac{1}{K}\left(-\eta + k + \frac{|\zeta|}{K}\right) + \frac{|\zeta|}{K^2}\right) \\ &= \exp\left(\frac{\eta - k}{K}\right) \leqslant \exp\left(\frac{\eta}{K}\right). \end{aligned}$$

We have proved that

$$\sup_{\lambda\in V_{K,k}} \|L_\gamma(T)(\lambda)\|_\gamma \exp\left(-\frac{\operatorname{Re}\lambda}{K}\right) < \infty.$$

(c) We will prove that $L_\gamma(T)$ is holomorphic on $V_{K,k}$. It suffices to show that in $H^0_{K,k}$

$$\lim_{\mu\to\lambda} \frac{f_\lambda - f_\mu}{\lambda - \mu} = g_\lambda \in H^0_{K,k},$$

where

$$g_\lambda(z) := -z \exp(-\lambda z).$$

Let us define

$$\begin{aligned} I &:= \left|\frac{f_\lambda(z) - f_\mu(z)}{\lambda - \mu} - g_\lambda(z)\right| \exp(k \operatorname{Re} z) \\ &= |\exp(-\lambda z)| \exp(k \operatorname{Re} z) \left|\frac{1 - \exp(-(\mu - \lambda) z)}{\lambda - \mu} + z\right|. \end{aligned}$$

Clearly, for $z \in \Omega_K$, $x \leqslant x_0$, this tends uniformly to zero as $\mu \to \lambda$. For $x \geqslant x_0$, $z \in \Omega_K$ we have $|z| \leqslant Mx$. Moreover, by (2.3), we also have if $|\mu - \lambda| < \frac{\delta}{2M}$:

$$\begin{aligned}
I &\leqslant \exp\left(\frac{\eta}{K}\right) \exp(-\delta x) \left| \sum_{n=2}^{\infty} \frac{(-(\mu-\lambda)z)^n}{(\lambda-\mu)n!} \right| \\
&\leqslant \exp\left(\frac{\eta}{K}\right) \exp(-\delta x) \sum_{n=2}^{\infty} \frac{|\mu-\lambda|^n |z|^n}{|\mu-\lambda| n!} \\
&\leqslant \exp\left(\frac{\eta}{K}\right) \exp(-\delta x) |\mu-\lambda| \sum_{n=2}^{\infty} \frac{|\mu-\lambda|^{n-2} M^n x^n}{n!} \\
&\leqslant \exp\left(\frac{\eta}{K}\right) \exp(-\delta x) \left| \frac{(\mu-\lambda)(2M)^2}{\delta^2} \right| \sum_{n=0}^{\infty} \frac{\left(\frac{\delta}{2}x\right)^n}{n!} \\
&\leqslant \exp\left(\frac{\eta}{K}\right) \exp\left(-\frac{\delta}{2}x\right) \left(\frac{2M}{\delta}\right)^2 |\mu-\lambda| \to 0 \quad \text{uniformly for } x \geqslant x_0 \text{ as } \mu \to \lambda.
\end{aligned}$$

(d) By (c) and the identity theorem, $L_\gamma(T)$ is well defined on G_γ if we show that

$$T_{\gamma,K,k(\gamma,K)}(f_x) = T_{\gamma,J,k(\gamma,J)}(f_x)$$

if $J \geqslant K$ and $x > k(\gamma, J)$. But this is clear since $H_{K,k(\gamma,J)}$ is embedded in $H_{J,k(\gamma,J)}$ and in $H_{K,k(\gamma,K)}$ and the respective extensions of $\kappa_\gamma \circ T \circ i_J$ and $\kappa_\gamma \circ T \circ i_K$ are unique. ■

Of course our definition of Laplace transform depends on the choice of G_γ (that is the choice of $k(\gamma, K)$). So the Laplace transform should be considered rather as a family of germs of holomorphic functions defined on domains of the above type near $]k, \infty[$ for large k ($\mathscr{L}(T)$ is clearly determined by the values on $]k, \infty[$ for large k).

In applications of this paper in [6] we are mainly interested in the case where X is the space $L_b(E, F)$ of continuous linear operators between locally convex spaces endowed with the topology of uniform convergence on bounded sets: let E and F be locally convex spaces, where F is complete and hence

$$F := \operatorname*{proj}_{\alpha \in A} F_\alpha$$

with local Banach spaces F_α. Let E be bornological, i.e.

$$E = \operatorname*{ind}_{B \in \mathcal{B}^E} E_B$$

where $\mathcal{B}^E$ is the system of bounded closed absolutely convex subsets of E and $E_B := \operatorname{span}(B)$ endowed with the gauge norm corresponding to B. Then $L_b(E, F)$ is complete and we have the topological identity

$$L_b(E, F) = \operatorname*{proj}_{(B,\alpha) \in (\mathcal{B}^E, A)} L(E_B, F_\alpha), \tag{2.4}$$

i.e. the local Banach spaces for $L_b(E,F)$ are the spaces $L(E_B, F_\alpha)$ of continuous operators endowed with the operator norm and the index system in Theorem 2.4 is $\Gamma := \mathscr{B}^E \times A$. To prove (2.4) we notice that the natural inclusion of $L_b(E,F)$ into $\operatorname{proj}_{(B,\alpha)\in(\mathscr{B}^E,A)} L(E_B, F_\alpha)$ is surjective since E is bornological.

If the spectrum $\mathscr{X} = (X_\gamma)_{\gamma\in\Gamma}$ is very big it looks as if the Laplace transform is a hopelessly complicated object with huge families $(G_\gamma)_{\gamma\in\Gamma}$ and $(L_\gamma(T))_{\gamma\in\Gamma}$. Fortunately, for typical cases, even though Γ is uncountable it might happen that the families $(G_\gamma)_{\gamma\in\Gamma}$ and $(L_\gamma(T))_{\gamma\in\Gamma}$ consist of countable many objects if J below is countable.

Proposition 2.8. *Let $X = \operatorname{proj}_{j\in J} X_j$ where $X_j = \operatorname{ind}_{n\in\mathbb{N}} X_{j,n}$ are LB-spaces with Banach spaces $(X_{j,n}, \|\cdot\|_{j,n})$. Assume that $X_j = \operatorname{proj}_{\gamma\in\Gamma_j} X_\gamma$ for Banach spaces X_γ and define projective spectra representing X by*

$$\mathscr{X} := (X_\gamma)_{\gamma\in\bigcup_{j\in J}\Gamma_j} \qquad \text{and} \qquad \mathscr{Y} := (X_j)_{j\in J}.$$

For every Laplace hyperfunction $T : H \to X$ there are a directed family of conoidal sets $\mathscr{U} := (U_j)_{j\in J}$ and a $\mathscr{Y}-$valued holomorphic function

$$\mathscr{L} = (L_j(T))_{j\in J} : \mathscr{U} \to \mathscr{Y},$$

such that

$$\begin{aligned} &\forall\ j\in J\ \forall\ K\in\mathbb{N}\ \exists\ k,l\in\mathbb{N}: \quad U_j \supseteq V_{K,k} \\ &\text{and} \qquad \forall\ \gamma\in\Gamma_j\ \exists\ C: \quad \sup_{\lambda\in V_{K,k}} \|L_j(T)(\lambda)\|_\gamma \exp\left(-\frac{\operatorname{Re}\lambda}{K}\right) \leqslant \\ &\qquad\qquad\qquad\qquad C\cdot \sup_{\lambda\in V_{K,k}} \|L_j(T)(\lambda)\|_{j,l} \exp\left(-\frac{\operatorname{Re}\lambda}{K}\right) < \infty \end{aligned}$$

and such that the Laplace transform $\mathscr{L}(T) : \mathscr{G} \to \mathscr{X}$ satisfies:

$$\forall\ j\in J\quad \forall\ \gamma\in\Gamma_j : \quad G_\gamma = U_j \quad \text{and} \quad L_\gamma(T) = i_\gamma^j \circ L_j(T),$$

where $i_\gamma^j : X_j \to X_\gamma$ is the standard linking map for $\gamma\in\Gamma_j$.

Proof. For every $K\in\mathbb{N}$

$$i_j \circ T \circ i_K : H_K \to X_j = \operatorname*{ind}_{n\in\mathbb{N}} X_{j,n}$$

is continuous since H_K is bornological. Since H_K is a Fréchet space, by Grothendieck factorization theorem, there is l such that

$$i_j \circ T \circ i_K : H_K \to X_{j,l}$$

is continuous. Hence for some $k := k(j,K)$ we get a continuous extension:

$$T_{j,K,k} : H^0_{K,k} \to X_{j,l} \hookrightarrow X_j.$$

As in the proof of Theorem 2.4 this provides a holomorphic function

$$L_j(T) : G_j := \bigcup_{K\in\mathbb{N}} V_{K,k(j,K)} \to X_j$$

satisfying all the requirements of the Proposition. ■

The above result is especially useful for the so-called PLS-spaces X (i.e., projective limits of sequences of duals of Fréchet-Schwartz spaces = DFS-spaces [4]) since then J is countable and thus the Laplace transform is a sequence of holomorphic functions defined on a decreasing sequence of domains. This is so, for instance, for X being the space of distributions or the space of real analytic functions [4] or $X = L_b(E,F)$, E, F DFS-spaces [5, Prop. 4.3].

For $X = L_b(E,F)$ with arbitrary Fréchet spaces E and F the assumptions of Proposition 2.8 are not satisfied since $L_b(E,F)$ need not be a projective limit of LB-spaces, nevertheless we get with the notation from Theorem 2.4:

Corollary 2.9. *Let E and F be Fréchet spaces with increasing system $(\| \ \|_n)_{n\in\mathbb{N}}$ of seminorms. Let $\mathscr{Y} := (L(E,F_n))_{n\in\mathbb{N}}$. Then for every Laplace hyperfunction $T : H \to L_b(E,F)$ there are a directed family of domains $\mathscr{U} := (U_n)_{n\in\mathbb{N}}$ and a $\mathscr{Y}-$valued holomorphic function $\mathscr{L} := (L_n(T))_{n\in\mathbb{N}} : \mathscr{U} \to \mathscr{Y}$ such that*

$$\forall\ n,K\in\mathbb{N}\quad \exists\ k,m\in\mathbb{N} : \quad U_n \supseteq V_{K,k} \qquad \textit{and} \qquad \sup_{\lambda\in V_{K,k}} \|L_n(T)(\lambda)\|_{L(E_m,F_n)} e^{-\operatorname{Re}\lambda/K} < \infty. \tag{2.5}$$

and such that the Laplace transform $\mathscr{L}(T)$ satisfies

$$\forall\ B\in\mathcal{B}^E \quad \forall\ n\in\mathbb{N} : \quad G_{(B,n)} = U_n \quad \textit{and} \quad L_{(B,n)}(T)(\lambda) = L_n(T)(\lambda)\circ i_B \tag{2.6}$$

where $i_B : E_B \to E$ is the canonical inclusion.

Proof. Since E is barrelled, bounded sets in $L_b(E,F)$ are equicontinuous, thus $L_b(E,F) = \operatorname{proj}_{n\in\mathbb{N}} L_b(E,F_n)$ has the same bounded sets as $\operatorname{proj}_{n\in\mathbb{N}}\big(\operatorname{ind}_{l\in\mathbb{N}} L(E_l,F_n)\big)$. Since H is bornological any linear operator from H is continuous if and only if it maps bounded sets into bounded sets. Thus $T : H \to L_b(E,F)$ is continuous if and only if

$$T : H \to \operatorname*{proj}_{n\in\mathbb{N}}\big(\operatorname*{ind}_{l\in\mathbb{N}} L(E_l,F_n)\big)$$

is continuous. Apply Proposition 2.8. ■

Corollary 2.10. *Let $E := \operatorname{ind}_n E^n$ and $F := \operatorname{ind}_n F^n$ be (DFS)-spaces and let $\mathscr{Y} := (L(E^n,F))_{n\in\mathbb{N}}$. Then for every Laplace hyperfunction $T : H \to L_b(E,F)$ there are a directed family of domains $\mathscr{U} := (U_n)_{n\in\mathbb{N}}$ and a $\mathscr{Y}-$ valued holomorphic function $\mathscr{L} := (L_n(T))_{n\in\mathbb{N}} : \mathscr{U} \to \mathscr{Y}$ such that*

$$\forall\ n,K\in\mathbb{N}\ \exists\ k,m\in\mathbb{N} : \quad U_n \supseteq V_{K,k} \qquad \textit{and} \qquad \sup_{\lambda\in V_{K,k}} \|L_n(T)(\lambda)\|_{L(E^n,F^m)} e^{-\operatorname{Re}\lambda/K} < \infty \tag{2.7}$$

and such that the Laplace transform satisfies

$$\forall\ n \in \mathbb{N}\quad \forall\ \alpha \in A: \quad G_{(n,\alpha)} = U_n \quad \text{and} \quad L_{(n,\alpha)}(T)(\lambda) = j_\alpha \circ L_n(T)(\lambda) \quad (2.8)$$

where $j_\alpha : F \to F_\alpha$ is the canonical spectral mapping.

Proof. Observe that $L_b(E, F)$ can be identified topologically with $L_b(F', E')$ by taking adjoints. Apply Corollary 2.9 or directly Prop. 2.8 since $L_b(F', E')$ is a PLS-space [5, Prop. 4.3]. ■

3. A general Laplace inversion formula

We will show in this section that a natural Laplace transform can be defined on our test function space H and that $H_{exp}(\mathscr{X})$ naturally operates on an $(LF)-$space of holomorphic functions which is the Laplace image of H. This will lead to a Laplace inversion formula for $T \in L_b(H, X)$ in our setting (see Theorem 3.4 and Corollary 3.6).

To start with we study a suitable (anti) Laplace transform $\check{L}$ on H. The (anti) Laplace image of H is in fact a quotient space of holomorphic functions in the right half plane $\mathbb{C}_+$ defined as follows: let

$$\widehat{H} := \operatorname*{ind}_K (\widehat{H}_K / N_K)$$

where

$$\widehat{H}_K := \left\{ f \in H(\mathbb{C}_+) \mid \forall k \in \mathbb{N} : |f|_{K,k} := \sup_{z \in \omega_{K,k}} |f(z)| \exp\left(\frac{\text{Re } z}{K}\right) < \infty \right\}$$

for

$$\omega_{K,k} := \mathbb{C}_+ \setminus V_{K,k}$$

and

$$N_K := \{f \in H(\mathbb{C}_+) : |f|_K := \sup_{z \in \mathbb{C}_+} |f(z)| \exp\left(\frac{\text{Re } z}{K}\right) < \infty\}.$$

Notice that N_K is a closed subspace of $\widehat{H}_K$ since for any $f \in N_K$

$$|f|_K = |f|_{K,1} + \sup_{z \in V_{K,1}} |f(z)| e^{\text{Re } z/K} \leqslant |f|_{K,1} + \sup_{z \in \partial V_{K,1}} |f(z)| e^{\text{Re } z/K} \leqslant 2|f|_{K,1}$$

(to see the second estimate we apply the maximum principle on $\overline{V}_{K,1}$ to $f(z) \exp((1/K - \varepsilon)z)$ for $\varepsilon > 0$).

As it turns out the obvious definition $\int_0^\infty e^{\lambda t} f(t) dt$ of the (anti) Laplace transform is not suitable for our test function space H since it does not take into account that the functions in H are defined on cones starting left of 0. Instead we define for $f \in H_K$

$$\check{L}_K(f)(\lambda) := \int_{-1/(2K)}^{\infty} e^{\lambda t} f(t) dt = \int_{\gamma_{K,\text{sign}(\text{Im } \lambda)}} e^{\lambda t} f(t) dt \qquad \text{for } \lambda \in \mathbb{C}$$

by Cauchy's integral theorem where $\gamma_{K,\pm}$ is the ray parametrized by

$$\gamma_{K,\pm}(t) := t \pm i(t/K + 1/(2K^2)), \qquad t \geqslant -1/(2K).$$

The ambiguity of the definition above leads to the quotient structure of the (anti) Laplace image of H. The basic result is the following

Theorem 3.1. *The mappings $\check{L}_K, K \geqslant 1$, define a topological isomorphism $\check{L} : H \to \widehat{H}$:*

$$H \supset H_K \ni f \mapsto \check{L}_{2K}(f) + N_{2K} \in \widehat{H_{2K}}/N_{2K} \subset \widehat{H}.$$

Proof. (a) For $f \in H_K$ and $\lambda = \eta + i\xi \in \mathbb{C}_+$ we get by the definition of $\|\cdot\|_{K,k}$ on $H_{K,k}$

$$\begin{aligned}|\check{L}_K(f)(\lambda)| &\leqslant C_1 \|f\|_{K,k+1} \sup_{t \geqslant -1/(2K)} e^{t(\eta - |\xi|/K - k) - |\xi|/(2K^2)} \\ &\leqslant C_2 \|f\|_{K,k+1} e^{-\operatorname{Re}\lambda/(2K)}\end{aligned}$$

if $0 \leqslant \eta \leqslant |\xi|/K + k$, i.e. if $\lambda \in \omega_{K,k} := \mathbb{C}_+ \setminus V_{K,k}$. Hence

$$|\check{L}_K(f)(\lambda)|_{2K,k} \leqslant C \|f\|_{K,k+1} \tag{3.1}$$

and $\check{L}_K : H_K \to \widehat{H}_{2K}$ is defined and continuous.

(b) For $J > K$ we have

$$\begin{aligned}|\check{L}_K(f)(\lambda) - \check{L}_J(f)(\lambda)| &= \left| \int_{-1/(2K)}^{-1/(2J)} e^{\lambda t} f(t) dt \right| \\ &\leqslant C_1 \|f\|_{K,1} e^{-\operatorname{Re}\lambda/(2J)} \qquad \text{if } \lambda \in \mathbb{C}_+.\end{aligned}$$

Therefore,

$$\check{L}_K(f)(\lambda) - \check{L}_J(f)(\lambda) : H_K \to N_{2J}$$

is continuous, hence $\check{L} : H \to \widehat{H}$ is defined and continuous.

(c) The inverse M of $\check{L}$ is also of Laplace type and is defined as follows: for $g \in \widehat{H}_K$ let

$$M_K(g)(z) := \frac{1}{2\pi i} \int_{\partial V_{K,1}} e^{-z\lambda} g(\lambda) d\lambda = \frac{1}{2\pi i} \int_{\partial V_{K,k}} e^{-z\lambda} g(\lambda) d\lambda \tag{3.2}$$

if $z = x + iy \in \Omega_{2K}$ and $k \in \mathbb{N}$, where $\partial V_{K,k}$ has clockwise orientation. The second equation holds by the Cauchy integral theorem.

Using the parametrization

$$t \mapsto k + |t| + iKt, \qquad t \in \mathbb{R}, \tag{3.3}$$

for $\partial V_{K,k}$ we get

$$\begin{aligned}\left|\int_{\partial V_{K,k}} e^{-z\lambda} g(\lambda) d\lambda\right| &\leqslant C_1 e^{-kx} |g|_{K,k} \int_{-\infty}^{\infty} e^{|t|(-x+K|y|-1/K)} dt \\ &\leqslant C_2 e^{-kx} |g|_{K,k} \int_{-\infty}^{\infty} e^{|t|(-x/2-3/(4K))} dt \\ &\leqslant C_2 e^{-kx} |g|_{K,k} \int_{-\infty}^{\infty} e^{-|t|/(4K)} dt \\ &\leqslant C_3 e^{-kx} |g|_{K,k} \qquad \text{if } x+iy \in \Omega_{2K}.\end{aligned} \tag{3.4}$$

Since $M_K(g) = 0$ for $g \in N_K$ by the Cauchy integral theorem,

$$M_K : \widehat{H}_K / N_K \to H_{2K} \quad \text{ is defined and continuous.}$$

Again by Cauchy's integral theorem we see that $M_K(g)(x) = M_J(g)(x)$ for $x > 0$ if $g \in M_K$ and $J > K$. Thus,

$$M : \widehat{H} = \operatorname*{ind}_K (\widehat{H}_K / N_K) \to H = \operatorname*{ind}_K H_K \quad \text{ is defined and continuous.}$$

(d) We will prove now that $M \circ \check{L} = \mathrm{id}$ on H. To see this we consider the parts of $\partial V_{K,k}$ in the upper and in the lower half plane separately, and correspondingly the different definitions of $\gamma_{K,\mathrm{sign}(\mathrm{Im}\,\lambda)}$. Let $f \in H_K$ and $x \geqslant 1$. Since $f \in H_{K,k}$ for any k, we may change the order of integration and get

$$\begin{aligned} M(\check{L}(f))(x) &= \frac{1}{2\pi i} \int_{\partial V_{2K,1,+}} e^{-x\lambda} \int_{\gamma_{K,+}} e^{\lambda\tau} f(\tau) d\tau d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{\partial V_{2K,1,-}} e^{-x\lambda} \int_{\gamma_{K,-}} e^{\lambda\tau} f(\tau) d\tau d\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma_{K,+}} f(\tau) \int_{\partial V_{2K,1,+}} e^{(\tau-x)\lambda} d\lambda d\tau \\ &\quad + \frac{1}{2\pi i} \int_{\gamma_{K,-}} f(\tau) \int_{\partial V_{2K,1,-}} e^{(\tau-x)\lambda} d\lambda d\tau \\ &= \frac{1}{2\pi i} \left(-\int_{\gamma_{K,+}} \frac{f(\tau) e^{\tau-x}}{\tau-x} d\tau + \int_{\gamma_{K,-}} \frac{f(\tau) e^{\tau-x}}{\tau-x} d\tau \right) \\ &= \frac{1}{2\pi i} \int_{\partial\Omega_K + 1/(2K)} \frac{f(\tau) e^{\tau-x}}{\tau - x} d\tau = f(x) \end{aligned}$$

by the Cauchy integral formula.

(e) The equality $\check{L} \circ M = \mathrm{id}$ on $\widehat{H}$ follows by similar ideas complemented by some new arguments: for $g \in \widehat{H}_K$ and $0 < a < 1/2$ fixed we thus get for $2a < x < 1$ using (3.3)

$$\begin{aligned}\check{L}(M(g))(x) &= \frac{1}{2\pi i}\int_{-1/(4K)}^{\infty} e^{xt}\int_{\partial V_{K,1}} e^{-t\lambda} g(\lambda)d\lambda dt \\ &= \frac{1}{2\pi i}\int_{\partial V_{K,1}} g(\lambda)\int_{-1/(4K)}^{\infty} e^{(x-\lambda)t} dt d\lambda \\ &= \underbrace{\frac{1}{2\pi i}\int_{\partial V_{K,1}} \frac{g(\lambda)}{\lambda - x} e^{(\lambda-x)/(4K)} d\lambda}_{h(x):=} \\ &= g(x) + \underbrace{\frac{1}{2\pi i}\int_{\partial V_{K,a}} \frac{g(\lambda)}{\lambda - x} e^{(\lambda-x)/(4K)} d\lambda}_{G(x):=} = g(x) + G(x),\end{aligned}$$

by the Cauchy integral formula.

The function G can be extended to a holomorphic function on $\mathbb{C}_+$ by

$$G(z) := \frac{1}{2\pi i}\int_{\partial V_{J,a}} \frac{g(\lambda)}{\lambda - z} e^{(\lambda-z)/(4K)} d\lambda \qquad \text{if } z \in V_{J,2a},\ a > 0,\ J \geqslant K.$$

The definition is independent of a and $J \geqslant K$ by Cauchy's theorem.

We will show that $G \in N_{4K}$ (and therefore $\check{L} \circ M(g) = g$ in $\widehat{H}$): for $z \in V_{K,1}$ the integration contour $\partial V_{K,1/2}$ is contained in $\omega_{K,1}$, hence

$$\begin{aligned}|G(z)| &\leqslant C_1 |g|_{K,1} \int_{-\infty}^{\infty} \left| e^{(1/2+|t|+iKt-z)/(4K)} \right| e^{-(1/2+|t|)/K} dt \\ &\leqslant C_2 |g|_{K,1} e^{-\operatorname{Re} z/(4K)} \text{ if } z \in V_{K,1}.\end{aligned} \tag{3.5}$$

Also h can be extended to a holomorphic function on $\mathbb{C}_+$ as follows

$$h(z) := \frac{1}{2\pi i}\int_{\partial V_{K,k+1}} \frac{g(\lambda)}{\lambda - z} e^{(\lambda-z)/(4K)} d\lambda \qquad \text{if } z \in \omega_{K,k},\ k \in \mathbb{N}.$$

The definition is independent of k by Cauchy's theorem. Hence

$$\begin{aligned}|h(z)| &\leqslant C_1 |g|_{K,2} \int_{-\infty}^{\infty} |e^{(2+|t|+iKt-z)/(4K)}| e^{-(2+|t|)/K} dt \\ &\leqslant C_2 |g|_{K,2} e^{-\operatorname{Re} z/(4K)} \int_{-\infty}^{\infty} e^{-|t|/(2K)} dt \\ &\leqslant C_2 |g|_{K,2} e^{-\operatorname{Re} z/(4K)} \qquad \text{if } z \in \omega_{K,1}.\end{aligned}$$

Since $g \in \widehat{H}_K$ we thus get by the identity theorem

$$|G(z)| \leqslant |h(z)| + |g(z)| \leqslant C_3 |g|_{K,2} e^{-\operatorname{Re} z/(4K)} \qquad \text{if } z \in \omega_{K,1} = \mathbb{C}_+ \setminus V_{K,1}.$$

By (3.5) we thus conclude that $G \in N_{4K}$ as desired. ■

As a first application of Theorem 3.1 and its proof we will show now that H is somehow the minimal space satisfying the conditions required for the Laplace test functions in the beginning of section 2 (see the remarks after Definition 2.1).

Lemma 3.2. *Let $K > 1$, then*

$$H_{K,k+2} \subseteq \overline{\operatorname{span}\{f_\lambda : \lambda \in V_{2K,k}\}}^{H_{4K,k}} \subseteq H^0_{4K,k}.$$

Proof. The second inclusion was shown in Lemma 2.2. To prove the first inclusion we use the Laplace inversion formula in H from Theorem 3.1: For $f \in H_{K,k+2}$ we get

$$f(z) = \frac{1}{2\pi i} \int_{\partial V_{2K,k+1}} e^{-z\lambda} \check{L}_K(f)(\lambda) d\lambda \qquad \text{if } z \in \Omega_{4K}$$

by (3.1), (3.4) and (d) in the proof of Theorem 3.1. Moreover, for any $\varepsilon > 0$ there is $j \in \mathbb{N}$ such that by (3.4)

$$\begin{aligned} &\left| f(z) - \frac{1}{2\pi i} \int_{\partial V_{2K,k+1} \cap \{|\operatorname{Re}\lambda| \leqslant j\}} e^{-z\lambda} \check{L}_K(f)(\lambda) d\lambda \right| e^{k \operatorname{Re} z} \\ &\qquad = \left| \frac{1}{2\pi i} \int_{\partial V_{2K,k+1} \cap \{|\operatorname{Re}\lambda| \geqslant j\}} e^{-z\lambda} \check{L}_K(f)(\lambda) d\lambda \right| e^{k \operatorname{Re} z} < \varepsilon \qquad \text{if } z \in \Omega_{4K}. \end{aligned}$$

Clearly, the Riemann sums of $\frac{1}{2\pi i} \int_{\partial V_{2K,k+1} \cap \{|\operatorname{Re}\lambda| \leqslant j\}} e^{-z\lambda} \check{L}_K(f)(\lambda) d\lambda$ are in $\operatorname{span}\{f_\lambda \mid \lambda \in V_{2K,k}\}$ and they converge with respect to $\|\cdot\|_{4K,k}$. ■

Notice that $H_{exp}(\mathscr{X})$ naturally operates on $\widehat{H}$ as follows: for $\mathscr{S} \in H_{exp}(\mathscr{X})$ and $g \in \widehat{H}_K$ let

$$(\langle Y(\mathscr{S}), g\rangle)_\gamma := \int_{\partial V_{K,k+1}} g(\lambda) S_\gamma(\lambda) d\lambda \in X_\gamma \qquad \text{for } \gamma \in \Gamma$$

if $V_{2K,k} \subset G_\gamma$ i.e. if $k := k(2K, \gamma)$, where $\partial V_{K,k+1}$ has clockwise orientation and is parametrized by (3.3).

Lemma 3.3. *Let $\mathscr{S} \in H_{exp}$.*

(a) *The map $(\langle Y(\mathscr{S}), \,\cdot\, \rangle)_\gamma : \widehat{H} \to X_\gamma$ is well defined and continuous.*
(b) *The mappings $(\langle Y(\mathscr{S}), \,\cdot\, \rangle)_\gamma, \gamma \in \Gamma$, define $Y(\mathscr{S}) \in L(\widehat{H}, X)$.*

Proof. (a): Let $g \in \widehat{H}_K$. Since $\partial V_{K,k+1} \subset V_{2K,k} \cap \omega_{K,k+2}$ we get by the assumption on $S_\gamma(\lambda)$ using (3.3)

$$\left\| \int_{\partial V_{K,k+1}} g(\lambda) S_\gamma(\lambda) d\lambda \right\|_\gamma \leqslant C_1 |g|_{K,k+2} \int_{-\infty}^{\infty} e^{-(|t|+k+1)/(2K)} dt \leqslant C_2 |g|_{K,k+2}.$$

If $g \in N_K$ then the estimate $|g(\lambda)| \leqslant C_3 e^{-\operatorname{Re}\lambda/K}$ also holds on $V_{2K,k}$. This implies by Cauchy's integral theorem that $\int_{\partial V_{K,k+1}} g(\lambda) S_\gamma(\lambda) d\lambda = 0$. Thus $(\langle Y(\mathscr{S}), \cdot \rangle)_\gamma : \widehat{H}_K / N_K \to X_\gamma$ is well defined and continuous.

Using Cauchy's integral theorem again we show that for $g \in \widehat{H}_K$ and $J > K$ we may shift the path of integration in the definition of $(\langle Y(\mathscr{S}), \cdot \rangle)_\gamma$ first to $\partial V_{K,k(2J,\gamma)+1}$ and then to $\partial V_{J,k(2J,\gamma)+1}$. This shows that

$$(\langle Y(\mathscr{S}), \cdot \rangle)_\gamma : \widehat{H} = \operatorname*{ind}_K \widehat{H}_K / N_K \to X_\gamma$$

is welldefined and continuous.

(b): This follows from the compatibility assumption in Definition 2.1 and Cauchy's theorem since we may shift the path of integration from $\partial V_{K,k(2K,\gamma)+1}$ to $\partial V_{K,k(2K,\nu)+1}$ if $\gamma \geqslant \nu$. ■

We can now state and prove the Laplace inversion formula for $T \in L(H, X)$:

Theorem 3.4. *The mapping*

$$Z : L_b(H, X) \to L_b(\widehat{H}, X), \qquad Z(T) := Y(\mathscr{L}(T))$$

is a linear topological isomorphism. More precisely we have the following Laplace inversion formula:

$$\kappa_\gamma \circ T(f) = \frac{1}{2\pi i}\big(\langle Z(T), \check{L}(f)\rangle\big)_\gamma = \frac{1}{2\pi i}\int_{\partial V_{2K,k+1}} \check{L}(f)(\lambda) L_\gamma(T)(\lambda) d\lambda \tag{3.6}$$

if $f \in H_K$, $T \in L(H, X)$ and $V_{4K,k} \subset G_\gamma$ i.e. if $k := k(4K, \gamma)$.

Proof. (a) Z is defined and linear by Lemma 3.3 and Theorem 2.4.

We first prove (3.6): for $f \in H_K$ we get by (3.1), (3.2) and part d) of the proof of Theorem 3.1

$$\begin{aligned}
\kappa_\gamma \circ T(f) &= T_{\gamma,4K,k} \circ i_{4K}^K f = T_{\gamma,4K,k} \circ M(\check{L}(f)) \\
&= T_{\gamma,4K,k}\left(\frac{1}{2\pi i}\int_{\partial V_{2K,k+1}} f_\lambda \check{L}(f)(\lambda) d\lambda\right) \\
&= \frac{1}{2\pi i}\int_{\partial V_{2K,k+1}} T_{\gamma,4K,k}(f_\lambda)\, \check{L}(f)(\lambda) d\lambda \\
&= \frac{1}{2\pi i}\int_{\partial V_{2K,k+1}} L_\gamma(T)(\lambda) \check{L}(f)(\lambda) d\lambda = \frac{1}{2\pi i}(\langle Y(\mathscr{L}(T)), \check{L}(f)\rangle)_\gamma \\
&= \kappa_\gamma\left(\frac{1}{2\pi i}\langle Z(T), \check{L}(f)\rangle\right)
\end{aligned}$$

by the definition of $L_\gamma(T)(\lambda)$ since the Riemann sums of the integral converge in $H_{4K,k}$.

(b) The map Z is injective by (3.6). To show that Z is surjective we fix $\widetilde{T} \in L(\widehat{H}, X)$ and set $T := \frac{1}{2\pi i}\widetilde{T} \circ \check{L} \in L(H, X)$ by Theorem 3.1. Then

$$\widetilde{T}(\check{L}(f)) = 2\pi i T(f) = Z(T)(\check{L}(f)) \qquad \text{for any } f \in H,$$

by (3.6). Hence $\widetilde{T} = Z(T)$ by Theorem 3.1.

The topological statement follows from (3.6) and Theorem 3.1. ■

We finally get the following converse of Theorem 2.4.

Corollary 3.5. *Let $\mathscr{S} \in H_{exp}(\mathscr{X})$ and set*

$$\kappa_\gamma \circ T(f) := \frac{1}{2\pi i} \int_{\partial V_{4K,k+1}} \check{L}(f)(\lambda) S_\gamma(\lambda) d\lambda \tag{3.7}$$

if $f \in H_K$ and $V_{4K,k} \subset G_\gamma$ i.e. if $k := k(4K, \gamma)$. Then this defines the unique $T \in L(H, X)$ such that $\mathscr{S} = \mathscr{L}(T)$.

Proof. *Uniqueness.* This is evident from Theorem 3.4 since we conclude from $\mathscr{L}(T_1) = \mathscr{L}(T_2)$ that

$$T_1(f) = \frac{1}{2\pi i} Z(T_1)(\check{L}f) = \frac{1}{2\pi i} Y(\mathscr{L}(T_1))(\check{L}f) = \frac{1}{2\pi i} Y(\mathscr{L}(T_2))(\check{L}f) = T_2(f)$$

if $f \in H$.

Existence. If $\mathscr{S} \in H_{exp}(X)$ then $Y(\mathscr{S}) \in L(\widehat{H}, X)$ and $T := \frac{1}{2\pi i} Y(\mathscr{S}) \circ \check{L} \in L(H, X)$ by Lemma 3.3 and Theorem 3.1. This gives the formula for T as above. To show that $\mathscr{L}(T) = \mathscr{S}$ it suffices to show that $L_\gamma(T)(x) = S_\gamma(x)$ for any γ and large real x. For $f_{j,x}(z) := \exp(-xz - z^2/(2j))$ as in the proof of Lemma 2.2 we get

$$\begin{aligned} L_\gamma(T)(x) &= \lim_{j\to\infty} \kappa_\gamma \circ T(f_{j,x}) = \frac{1}{2\pi i} \lim_{j\to\infty} \underbrace{Y(\mathscr{S})(\check{L}(f_{j,x}))}_{\in X_\gamma} \\ &= \lim_{j\to\infty} \frac{1}{2\pi i} \int_{\partial V_{2K,k+1}} \check{L}(f_{j,x})(\lambda) S_\gamma(\lambda) d\lambda. \end{aligned}$$

To calculate the limit we use Lebesgue's theorem of dominated convergence twice: since by definition

$$\check{L}(f_{j,x})(\lambda) = \int_{\gamma_{K,\mathrm{sign}(\mathrm{Im}\,\lambda)}}^{\infty} e^{(\lambda - x)\tau - \tau^2/(2j)} d\tau,$$

we have to estimate for $t \geqslant 0$ and $\tau \geqslant -1/(2K)$

$$|e^{(k+1-x+t\pm 2iKt)(\tau \pm i(\tau/K + 1/(2K^2))) - (\tau \pm i(\tau/K+1/(2K^2)))^2/(2j)}| \leqslant C_1 e^{(k+1-x)\tau - t/(2K)}. \tag{3.8}$$

Since $\lim_{j\to\infty} e^{(\lambda-x)\tau-\tau^2/(2j)} = e^{(\lambda-x)\tau}$ pointwise on $\gamma_{K,\mathrm{sign}(\mathrm{Im}\,\lambda)}$ we thus get for $x \geqslant k+2$

$$\lim_{j\to\infty} \check{L}(f_{j,x})(\lambda) = \int_{\gamma_{K,\mathrm{sign}(\mathrm{Im}\,\lambda)}}^{\infty} e^{(\lambda-x)\tau} d\tau = \frac{e^{(x-\lambda)/(2K)}}{x-\lambda}$$

pointwise for $\lambda \in \partial V_{2K,k+1}$. The formula (3.8) and the estimate for $S_\gamma(\lambda)$ on $V_{2K,k}$ also imply that

$$\|S_\gamma(\lambda)\|_\gamma |\check{L}(f_{j,x})(\lambda)| \leqslant C_2 e^{-\mathrm{Re}\,\lambda/(4K)}$$

on $\partial V_{2K,k+1} \subset V_{4K,k}$. Using Lebesgue's theorem again we get

$$L_\gamma(T)(x) = \frac{1}{2\pi i} \int_{\partial V_{2K,k+1}} \frac{e^{(x-\lambda)/(2K)}}{x-\lambda} S_\gamma(\lambda) d\lambda = S_\gamma(x)$$

by Cauchy's theorem and the orientation of $\partial V_{2K,k+1}$. ■

For Fréchet spaces and (DFS)−spaces we get the converse of Corollaries 2.9 and 2.10:

Corollary 3.6.

a) *Let E and F be Fréchet spaces with increasing system $(\| \ \|_n)_{n\in\mathbb{N}}$ of seminorms. Let $\mathscr{Y} := (L(E,F_n))_{n\in\mathbb{N}}$ and let $\mathscr{U} := (U_n)_{n\in\mathbb{N}}$ be a directed family of domains. Then for any $\mathscr{Y}$−valued holomorphic function $\mathscr{S} : \mathscr{U} \to \mathscr{Y}$ satisfying* (2.5) *there is a unique Laplace hyperfunction $T : H \to L_b(E,F)$ such that $\mathscr{L}(T) = \mathscr{S}$.*

b) *Let $E := \mathrm{ind}_n E^n$ and $F := \mathrm{ind}_n F^n$ be (DFS)-spaces. Let $\mathscr{Y} := (L(E^n,F))_{n\in\mathbb{N}}$ and let $\mathscr{U} := (U_n)_{n\in\mathbb{N}}$ be a directed family of domains. Then for any $\mathscr{Y}$−valued holomorphic function $\mathscr{S} : \mathscr{U} \to \mathscr{Y}$ satisfying* (2.7) *there is a unique Laplace hyperfunction $T : H \to L_b(E,F)$ such that $\mathscr{L}(T) = \mathscr{S}$.*

Proof. a) Define $G_{(B,n)}$ and $S_{(B,n)} : G_{(B,n)} \to L(E,F_n)$ for any $B \in \mathcal{B}^E$ by (2.6). Then $\widetilde{S} := (S_{(B,n)})_{(B,n)\in\mathcal{B}^E\times\mathbb{N}} \in H_{exp}(\mathscr{X})$ for $\mathscr{X} := (L(E_B,F_n))_{(B,n)\in\mathcal{B}^E\times\mathbb{N}}$ and the conclusion follows from Corollary 3.5.

b) This is proved similarly. ■

4. Examples of Laplace hyperfunctions and Laplace transforms

Since our Laplace test function space H is continuously embedded in the space $A(K)$ of analytic germs near a compact $K \subset [0,\infty[$, the Laplace transform developed sofar applies to any vector valued hyperfunction with compact support, i.e. to any $T \in L(A(K),X)$, and therefore also to any of the standard vector valued generalized functions with compact support.

Also, corresponding results for the Laplace transform of vector valued generalized functions of exponential growth can be easily obtained from the preceding results. We only discuss the distribution case in some detail, the modifications needed for vector valued exponentially increasing ultradistributions are left to the reader.

The space $\mathcal{K}'$ of Laplace distributions is by definition the dual space of

$$\mathcal{K} := \{f \in C^\infty([0,\infty[) \mid \forall k \in \mathbb{N} : \|f\|_k := \sup_{x\in[0,\infty[,|\alpha|\leqslant k} |f^{(\alpha)}(x)|e^{kx} < \infty\}.$$

We also need the global version

$$\mathcal{K}_\mathbb{R} := \{f \in C^\infty(\mathbb{R}) \mid \forall k \in \mathbb{N} : \|f\|_k := \sup_{x\in\mathbb{R},|\alpha|\leqslant k} |f^{(\alpha)}(x)|e^{k|x|} < \infty\}.$$

The transition from the preceding results to the case of weighted distributions is provided by the global version of H defined by

$$H_\mathbb{R} := \operatorname*{ind}_K(\operatorname*{proj}_k H_{\mathbb{R},K,k}) = \operatorname*{ind}_K H_{\mathbb{R},K}$$

where

$$H_{\mathbb{R},K,k} := \{f \in H(\widetilde{\Omega}_K) : \|f\|_{K,k} := \sup_{z\in\widetilde{\Omega}_K} |f(z)| \exp(k|\operatorname{Re} z|) < \infty\}$$

and the conic neighborhoods of $\mathbb{R}$ are defined by

$$\widetilde{\Omega}_K := \left\{z \in \mathbb{C} : |\operatorname{Im} z| < \frac{|\operatorname{Re} z|}{K} + \frac{1}{K^2}\right\}.$$

We gather the needed facts in the following Lemma. Let the Fourier transform be defined by

$$\widehat{f}(z) := \int_\mathbb{R} f(x)e^{-ixz}dx \qquad \text{for } z \in \mathbb{C} \text{ and } f \in H_\mathbb{R}.$$

Lemma 4.1.

(a) *Let $f \in H_{\mathbb{R},K}$. Then $\widehat{f}$ is an entire function such that for all $k \in \mathbb{N}$ there is $C > 0$ such that*

$$|\widehat{f}(z)|e^{|z|/(4K^2)} \leqslant C\|f\|_{K,k} \qquad \textit{if} \ \ |\operatorname{Im} z| \leqslant k - 1 + |\operatorname{Re} z|/K$$

(b) *$H_\mathbb{R}$ is densely embedded in $\mathcal{K}_\mathbb{R}$.*

(c) *Let $f \in \mathcal{K}_\mathbb{R}$. Then $\widehat{f}$ is an entire function such that for all $k \in \mathbb{N}$ there is $C > 0$ such that*

$$|\widehat{f}(z)||z^k| \leqslant C\|f\|_k \qquad \textit{if} \ \ |\operatorname{Im} z| \leqslant k - 1$$

Proof. (a) By the Cauchy integral theorem we have for $f \in H_{\mathbb{R},K}$

$$\widehat{f}(z) := \int_{\gamma_z} f(\xi)e^{-i\xi z}d\xi$$

where $\gamma_z(t) := t - i\operatorname{sign}(\operatorname{Re} z)(|t|/K + 1/(2K^2)), t \in \mathbb{R}$. This easily gives the desired estimate.

(b) This follows by convolution with the normalized Gaussians $g_n(z) := c_n \exp(-nz^2)$.

(c) This is obvious. ∎

Our result for the Laplace transform of vector valued exponentially bounded distributions is as follows. Notice that the sufficient condition (4.2) is seemingly weaker than the necessary condition (4.1). This effect has already been observed for weighted distributions with values in Banach spaces, see e.g. Komatsu [10, Theorem 9].

Theorem 4.2. *Let X be a complete locally convex space defined by the projective spectrum $\mathscr{X} = (X_\gamma)_{\gamma\in\Gamma}$ of Banach spaces.*

(a) *Let $T \in L(\mathcal{K}, X)$. Then $\mathscr{L}(T) : \mathscr{G} \to \mathscr{X}$ is a holomorphic $\mathscr{X}-$valued function such that*

$$\forall\ \gamma \in \Gamma\ \exists\ k: \quad G_\gamma = V_k := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > k\} \text{ and}$$
$$\sup_{\lambda\in V_k} \|L_\gamma(T)(\lambda)\|_\gamma (1+|\lambda|)^{-k} < \infty. \tag{4.1}$$

(b) *Conversely, let $\mathscr{S} : \mathscr{G} \to \mathscr{X}$ be an $\mathscr{X}-$valued holomorphic map such that*

$$\forall\ \gamma \in \Gamma\ \forall\ K \in \mathbb{N}\ \exists\ k: \quad G_\gamma = V_k \text{ and} \tag{4.2}$$
$$\sup_{\lambda\in V_k} \|S_\gamma(\lambda)\|_\gamma (1+|\lambda|)^{-k} e^{-\operatorname{Re}\lambda/K} < \infty.$$

Then there is a unique $T \in L(\mathcal{K}, X)$ such that $\mathscr{L}(T) = \mathscr{S}$.

(c) *For $T \in L(\mathcal{K}, X)$ the following Laplace inversion formula holds for $f \in \mathcal{K}$:*

$$\kappa_\gamma \circ T(f) = \frac{1}{2\pi i}\int_{\partial V_{k+1}} \widehat{E(f)}(i\lambda) L_\gamma(T)(\lambda) d\lambda \tag{4.3}$$

where $E(f) \in \mathcal{K}_\mathbb{R}$ is an extension of f and $k = k(\gamma)$ is as in (a).

Proof. (a) Since $T \in L(\mathcal{K}, X)$, for any γ there is k such that

$$T : (\mathcal{K}, \|\cdot\|_k) \to X_\gamma$$

is continuous. We may assume that $k(\gamma) \leqslant k(\nu)$ if $\gamma \leqslant \nu$. Since $H_{K,k}$ is continuously embedded in $\mathcal{K}_k := \{f \in C^k([0,\infty[) \mid \|f\|_k < \infty\}$ for any K, $L_\gamma(T)(\lambda)$ is defined on $V_k = \cup_{K\in\mathbb{N}} V_{K,k}$ by the construction before Theorem 2.4 and it defines an $\mathscr{X}-$valued holomorphic map on $\mathscr{G}$ by Theorem 2.4. Moreover, for $\lambda \in V_{K,k}$ we have with $f_{j,\lambda}$ as in the proof of Lemma 2.2

$$|L_\gamma(T)(\lambda)| = \lim_{j\to\infty} |\kappa_\gamma \circ T(f_{j,\lambda})| \leqslant C \limsup_{j\to\infty} \|f_{j,\lambda}\|_k = C\|f_\lambda\|_k \leqslant C(1+|\lambda|)^k$$

since $H_{K,k}$ is continuously embedded in $\mathcal{K}_k$ and thus, by Lemma 2.2, $\lim_{j\to\infty} f_{j,\lambda} = f_\lambda =: \exp(-\lambda\cdot\,)$ also with respect to $\|\cdot\|_k$.

(b) *Uniqueness.* If $T_j \in L(\mathcal{K}, X)$ satisfy $\mathscr{L}(T_1) = \mathscr{S} = \mathscr{L}(T_2)$ then $T_j \in L(H, X)$ and hence $T_1 \mid_H = T_2 \mid_H$ by Corollary 3.5 and therefore $T_1 \mid_{\mathcal{K}_\mathbb{R}} = T_2 \mid_{\mathcal{K}_\mathbb{R}}$ by Lemma 4.1b). Thus $T_1 = T_2$ since $\mathcal{K} = \mathcal{K}_\mathbb{R}/\mathcal{K}_0$ where $\mathcal{K}_0 := \{f \in \mathcal{K} \mid f(x) = 0 \text{ if } x \geqslant 0\}$.

Existence. The assumption implies that $\mathscr{S} \in H_{exp}(\mathscr{X})$. Hence by Corollary 3.5 there is $T \in L(H, X)$ such that $\mathscr{L}(T) = \mathscr{S}$ and T is defined by

$$\kappa_\gamma \circ T(f) = \frac{1}{2\pi i} \int_{\partial V_{4K,k+1}} \check{L}(f)(\lambda) S_\gamma(\lambda) d\lambda \tag{4.4}$$

if $f \in H_K$ and $V_{4K,k} \subset G_\gamma$ i.e. if $k := k(4K, \gamma)$.

Notice that for $f \in H_{\mathbb{R},K}$ (and $\check{L}_K(f)$ defined before Theorem 3.1) we have

$$\check{L}_K(f)(z) - \widehat{f}(iz) = \int_{-\infty}^{-1/(2K)} f(x) e^{xz} dx =: g(z),$$

where $g \in N_{2K}$ and therefore

$$\int_{\partial V_{4K,k+1}} g(\lambda) S_\gamma(\lambda) d\lambda = 0$$

by the Cauchy integral theorem, hence we get for $f \in H_{\mathbb{R},K}$

$$\kappa_\gamma \circ T(f) = \frac{1}{2\pi i} \int_{\partial V_{4K,k+1}} \widehat{f}(i\lambda) S_\gamma(\lambda) d\lambda.$$

For $f \in H_{\mathbb{R},J}$ we may choose K large enough such that by Lemma 4.1 (a) and the bounds on $S_\gamma(\lambda)$ the path of integration may be shifted by the Cauchy integral theorem to get for some j

$$\kappa_\gamma \circ T(f) = \frac{1}{2\pi i} \int_{\partial V_{j+1}} \widehat{f}(i\lambda) S_\gamma(\lambda) d\lambda. \tag{4.5}$$

The latter formula extends T to a continuous linear mapping $T : \mathcal{K}_{\mathbb{R}} \to X$ by Lemma 4.1 (c). If $f \in \mathcal{K}_{\mathbb{R}}$ satisfies $f(x) = 0$ for any $x \geqslant \delta$, $\delta < 0$, then $\widehat{f}(iz) \in N_R$ for some R and hence $T(f) = 0$. Since the set of these f is dense in $\mathcal{K}_0$ we thus have $T\,|_{\mathcal{K}_0} = 0$, that is, $T : \mathcal{K} = \mathcal{K}_{\mathbb{R}}/\mathcal{K}_0 \to X$ is continuous.

(c) For $T \in L(\mathcal{K}, X)$ and $f \in \mathcal{K}$ we clearly have $T(f) = T(E(f))$ since $f = E(f)\,|_{[0,\infty[}$. The second equality in (4.3) now follows from the proof of (b) above by setting $S_\gamma(\lambda) = L_\gamma(T)(\lambda)$ in (4.5) for $E(f)$ instead of f. ■

Examples of more regular $L(E)-$valued Laplace hyperfunctions are easily obtained from the following proposition treating a situation corresponding to C_0-semigroups.

Proposition 4.3. *Let $\{T(t) \mid t \geqslant 0\}$ be a pointwise continuous family (i.e., continuous with respect to the variable t and the topology of pointwise convergence in $L(E)$) of continuous linear mappings in a complete (ultra)bornological space E such that*

$$\forall\ \alpha \in A\ \forall\ B \in \mathcal{B}^E\ \exists\ C_j = C_j(B, \alpha)\ \forall\ t \geqslant 0 : \quad \sup_{g \in B} \|T(t)g\|_\alpha \leqslant C_1 e^{C_2 t}. \tag{4.6}$$

Then the mapping $T : H \to L_b(E,E)$ *defined by*

$$T(f)(g) := \int_0^\infty f(t)T(t)g\ dt \qquad \text{if } f \in H \text{ and } g \in E$$

is continuous. The Laplace transform $\mathscr{L}(T) = (L_{(B,\alpha)}(T)(\lambda))$ *is defined on*

$$\mathscr{G} = (G_{(B,\alpha)}) \text{ for } G_{(B,\alpha)} = \{\lambda \mid \operatorname{Re}(\lambda) > C_2(B,\alpha)\}.$$

Moreover

$$\|L_{(B,\alpha)}(\lambda)\|_{L(E_B,E_\alpha)} \leqslant \frac{C}{\operatorname{Re}\lambda - C_2(B,\alpha)}.$$

In [1] Babalola considered families of operators satisfying

$$\forall\ \alpha \in A\ \exists\ \beta \in A\ \exists\ C_j = C_j(\beta,\alpha)\ \forall\ t \geqslant 0 : \quad \|T(t)g\|_\alpha \leqslant C_1 e^{C_2 t}\|g\|_\beta \quad (4.7)$$

which is (at least formally) stronger than (4.6) (compare also Proposition 4.13).

Let us observe that the families $T(t)$ satisfying (4.7) have the continuity estimates not depending on t. Especially, in the case of Frechet spaces E,

$$\sigma_t(n) := \inf\{k :\ \exists\ C\ \forall\ g \quad \|T(t)g\|_n \leqslant C\|g\|_k\}$$

does not depend on $t \in [0,\infty)$. On $C^\infty(\mathbb{R})$ the family $T(t)$, $T(t)g(x) := g(e^t x)$ has continuity estimates depending on t since a suitable version of the first condition of (4.9) below is not satisfied.

Let us collect some examples of families like in Proposition 4.3 above.

Example 4.4. Semigroups of composition operators

Let $\psi_t : \mathbb{R}^d \to \mathbb{R}^d$ be a semigroup of diffeomorphisms, i.e.,

$$\psi_{t+s}(x) = \psi_t(\psi_s(x)), \qquad \text{for } t, s > 0. \tag{4.8}$$

We define the family of maps:

$$T(t) := C_{\psi_t}, \qquad C_{\psi_t}(g)(x) := g(\psi_t(x))$$

and we can consider them on various function spaces.

(a) Assume that

$$\begin{aligned} &\forall\ k\ \exists\ C_k\ \forall\ t \in [0,\infty)\ \forall\ |x| \leqslant k \quad |\psi_t(x)| \leqslant C_k, \\ &\forall\ \alpha : \quad \text{the functions } (t,x) \mapsto \frac{\partial^\alpha \psi_t(x)}{\partial x^\alpha} \text{ are continuous.} \end{aligned} \tag{4.9}$$

Then the family of maps

$$T(t) : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d) \qquad T(t)g(x) := g(\psi_t(x))$$

satisfies (4.7). Indeed, (4.9) implies that

$$\forall\ k\ \exists\ D_k\ \forall\ t\in[0,1]:\quad \sup_{|x|\leqslant k}\sup_{|\alpha|\leqslant k}\left|\frac{\partial^\alpha\psi_t(x)}{\partial x^\alpha}\right|\leqslant D_k.$$

Since

$$\begin{aligned}\frac{\partial}{\partial x_i}\psi_t(x)&=\frac{\partial}{\partial x_i}(\underbrace{\psi_{t/n}\circ\psi_{t/n}\circ\cdots\circ\psi_{t/n}(x)}_{n-\text{times}})\\&=\left(\frac{\partial}{\partial x_i}\psi_{t/n}\right)\circ\psi_{(1-1/n)t}(x)\left(\frac{\partial}{\partial x_i}\psi_{t/n}\right)\circ\psi_{(1-2/n)t}(x)\ldots\frac{\partial}{\partial x_i}\psi_{t/n}(x)\end{aligned}$$

then for $t\in[0,n]$, $|x|\leqslant k$ we get

$$\left|\frac{\partial}{\partial x_i}\psi_t(x)\right|\leqslant D^n_{C_k}.$$

Thus for $t\in[n-1,n]$ and $|x|\leqslant k$ we have

$$\left|\frac{\partial}{\partial x_i}\psi_t(x)\right|\leqslant D_{C_k}D^t_{C_k}=D_{C_k}e^{wt}\tag{4.10}$$

for $w=\log D_{C_k}$.

Let us observe that for $C^\infty-$functions f and g we have

$$\frac{\partial^\alpha}{\partial x^\alpha}f\circ g(x)=\sum_{|\beta|\leqslant|\alpha|}\left(\frac{\partial^\alpha}{\partial x^\alpha}f\right)(g(x))\cdot P_\beta(x),\tag{4.11}$$

where for every β the function P_β is a polynomial of derivatives of g of order $\leqslant|\alpha|$.

Therefore, an estimate like (4.10) also holds for $\partial^\alpha\psi_t$ (with ω depending on α). It is not difficult to show that

$$\|T(t)g\|_k\leqslant Ce^{wt}\|g\|_{C_k},$$

for suitable C and w, where

$$\|f\|_k:=\sup_{|x|\leqslant k}\sup_{|\alpha|\leqslant k}\left|\frac{\partial^\alpha f(x)}{\partial x^\alpha}\right|.$$

The continuity condition in (4.9) and (4.11) imply that $t\mapsto T(t)$ is pointwise continuous.

(b) If instead of (4.9) we assume:

$$\begin{aligned}&\exists\ \delta>0\ \forall\ t\in[0,\delta]\ \exists\ C\ \forall\ x\in\mathbb{R}^d:\quad(1+|x|)\leqslant C(1+|\psi_t(x)|)\\&\exists\ \delta>0\ \forall\ t\in[0,\delta]\ \forall\ k\ \exists\ C_k:\quad\sup_{x\in\mathbb{R}^d}\sup_{|\alpha|\leqslant k,\alpha\neq0}\left|\frac{\partial^\alpha\psi_t(x)}{\partial x^\alpha}\right|\leqslant C_k,\\&\forall\ t_0,\alpha\ \exists\ l\ \forall\ \varepsilon>0\ \exists\ \delta>0\ \forall\ x\in\mathbb{R}^d,|t-t_0|<\delta:\\&\qquad\qquad\left|\frac{\partial^\alpha}{\partial x^\alpha}\psi_t(x)-\frac{\partial^\alpha}{\partial x^\alpha}\psi_{t_0}(x)\right|\leqslant\varepsilon(1+|x|)^l,\end{aligned}\tag{4.12}$$

then the family of maps $T(t):\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ defined as in (a) satisfies (4.7).

Using two first conditions as in the proof of part (a) we can show that

$$\begin{aligned} \sup_{x\in\mathbb{R}^d} \frac{(1+|x|)}{(1+|\psi_t(x)|)} &\leqslant Ce^{wt} \\ \sup_{x\in\mathbb{R}^d} \sup_{|\alpha|\leqslant k,\alpha\neq 0} \left|\frac{\partial^\alpha \psi_t(x)}{\partial x^\alpha}\right| &\leqslant Ce^{wt} \end{aligned} \tag{4.13}$$

for suitably chosen C and w. The formula (4.11) and the last condition in (4.12) implies that $t \mapsto T(t)$ is pointwise continuous. The conditions (4.13) imply that for some C and w:

$$\|T(t)g\|_k = \sup_{|\alpha|\leqslant k, l\leqslant k} \sup_{x\in\mathbb{R}^d} \left|\frac{\partial^\alpha}{\partial x^\alpha} g(\psi_t(x))\right| (1+|x|)^l \leqslant \|g\|_k Ce^{wt}.$$

(c) If $\psi_t : K \to K$, K a compact subset in $\mathbb{R}^d$ with smooth boundary then $T(t) : C^\infty(K) \to C^\infty(K)$. If we assume

$$\forall\ \alpha : \quad \text{the functions } (t,x) \mapsto \frac{\partial^\alpha \psi_t(x)}{\partial x^\alpha} \text{ are continuous} \tag{4.14}$$

then the family $T(t)$ satisfies (4.7).

(d) Of course, we could also treat the family of transposed operators ${}^tT(t)$ on the tempered distributions $\mathcal{S}(\mathbb{R}^d)'_b$ in part (a) or distributions with compact support in part (b) as well on $(C^\infty(K))'_b$ in part (c).

(e) If $K \subseteq \mathbb{R}^d$ is a compact set with smooth boundary then there is a continuous linear extension operator $V : C^\infty(K) \to C^\infty(\mathbb{R}^d)$ such that for a fixed compact set L, $K \Subset L$ for every $f \in C^\infty(K)$ holds $\operatorname{supp} V(f) \Subset L$. We can define operator:

$$T(t) : C^\infty(K) \to C^\infty(K), \qquad T(t)g(y) := (Vg)(\psi_t(y)), y \in K.$$

If we assume

$$\forall\ \alpha : \quad \text{the functions } (t,x) \mapsto \frac{\partial^\alpha \psi_t(x)}{\partial x^\alpha} \text{ are continuous} \tag{4.15}$$

then the family $T(t)$ satisfies (4.7).

Example 4.5. Special composition operators

Let $Q : \mathbb{R} \to \mathbb{R}$ be a C^∞ function on $\mathbb{R}$ with bounded derivatives of any order and such that $0 < \frac{1}{C} < Q' < C$ on $\mathbb{R}$, $Q(0) = 0$. Then we define:

$$\psi_t(x) := Q^{-1}(e^{at}Q(x))$$

which is an example of a diffeomorphism like in Example 4.4. The condition (4.12) is easy to check. Since,

$$\frac{\partial}{\partial t}\psi_t(x) = a\frac{Q(x)}{Q'(x)} \cdot \frac{\partial}{\partial x}\psi_t(x), \qquad \psi_0(x) = x,$$

the function f, $f(t) := T(t)g$, solves the abstract Cauchy problem:

$$\begin{cases} \frac{d}{dt}f &= Af, \\ f(0) &= g, \end{cases}$$

where $A : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, $Af(x) := a\frac{Q(x)}{Q'(x)}\frac{d}{dx}f(x)$, $T(t)g(x) = g\left(Q^{-1}\left(e^{at}Q(x)\right)\right)$.

The special case $Q(x) = x$ for $a = 1$ (i.e., $Af(x) = x\frac{d}{dx}f(x)$) is considered in Babalola's paper [1, Section 6] as well as in [7, Ch. IV.3.3].

The general Laplace transform of T exists by Proposition 4.3 and Theorem 2.4. However, T does not admit an operator-valued Laplace transform in the usual sense in $\mathcal{S}(\mathbb{R})$ i.e. the integral

$$L(T)(g)(\lambda) := \int_0^\infty e^{-\lambda t}T(t)g\, dt \qquad \text{if } g \in \mathcal{S}(\mathbb{R}) \tag{4.16}$$

does not exist in $\mathcal{S}(\mathbb{R})$ for any $\lambda \in \mathbb{R}$. Indeed, (4.16) gives for $y > 0$

$$\begin{aligned} L(T)(g)(\lambda)(y) &= \int_0^\infty e^{-\lambda t}(T(t)g)(y)dt \\ &= \int_0^\infty e^{-\lambda t}g(Q^{-1}(e^{at}Q(y)))dt \\ &= \int_y^\infty \left(\frac{Q(y)}{Q(z)}\right)^{\frac{\lambda}{a}} g(z)\frac{Q'(z)}{aQ(z)}dz, \end{aligned}$$

where $z = Q^{-1}(e^{at}Q(y))$. If $\operatorname{supp} g \Subset (0,\infty)$ then for y small enough

$$L(T)(g)(\lambda)(y) = (Q(y))^{\frac{\lambda}{a}} \int_{\mathbb{R}} \frac{g(z)Q'(z)}{(Q(z))^{\frac{\lambda}{a}+1}\,a}dz.$$

It is clear that if $\frac{\lambda}{a}$ is not an integer and $n \in \mathbb{N}$, $n > \frac{\lambda}{a} \in \mathbb{R}$ then $\frac{d^n}{dy^n}\left[L(T)(g)(\lambda)\right](y)$ is not continuous as $y \to 0$. Of course, for Re $\frac{\lambda}{a} > n$ the integral exists.

On the other hand, we have

$$\begin{aligned} \left(L(T)(\lambda)g\right)(y) := T(f_\lambda)g(y) &:= \int_0^\infty e^{-\lambda t}g(Q^{-1}(e^{at}Q(y)))\, dt \\ &= \operatorname{sgn} y|Q(y)|^{\frac{\lambda}{a}} \int_y^{\mathrm{sgn(y)}\infty} g(z)|Q(z)|^{-\frac{\lambda}{a}-1}\frac{Q'(z)}{a}dz \qquad \text{if } g \in \mathcal{S}(\mathbb{R}) \end{aligned}$$

which converges for large $\operatorname{Re}(\lambda)$ as a continuous linear operator between the local Banach spaces

$$\mathcal{S}_k(\mathbb{R}) := \{f \in C^k(\mathbb{R}) \mid \sup_{j\leqslant k, l\leqslant k, x\in\mathbb{R}} |x^j f^{(l)}(x)| < \infty\}.$$

This shows how the Laplace transform in the general sense of Theorem 2.4 is obtained in this concrete case.

Example 4.6. Semigroups of generalized shifts

(a) A more general example is given as follows. Let $P : \mathbb{R} \to \mathbb{R}$ be a smooth function with discrete sequence of zeros:

$$\dots < x_{-n} < x_{-n+1} < \dots < x_{-1} < x_0 < x_1 < \dots < x_n < x_{n+1} < \dots .$$

If the sequence of zeros is bounded from above (i.e., there is the biggest zero x_k), then we assume additionally in the interval (x_k, ∞) that

$$\exists\ C > 0 \quad |P(x)| < C|x|$$

and in this case we denote $x_{k+1} := \infty$. Analogous assumptions should hold if there is the smallest zero of P.

We define F to be a primitive of $\frac{1}{P}$ on (x_{n-1}, x_n). Clearly, $F : (x_{n-1}, x_n) \to (-\infty, \infty)$ is a strictly monotone smooth bijection. We define

$$\psi_{t,P}(x) := F^{-1}(t + F(x)).$$

These maps form a semigroup of diffeomorphisms on (x_{n-1}, x_n). Observe that the definition does not depend on the choice of primitive F! We can repeat the same procedure on all intervals between the zeros (and our assumptions assure that it works also on possibly existing infinite intervals). It is not difficult to see that defining $\psi_{t,P}(x_n) = x_n$ makes it a homeomorphism

$$\psi_{t,P} : \mathbb{R} \to \mathbb{R}.$$

Moreover, the value $\psi_{t,P}$ depends only on values of P for arguments between x and $\psi_{t,P}(x)$. Example 4.5 is a particular case of the present example — just take $P = \frac{Q}{Q'}$.

Lemma 4.7. *Let $P \in C^\infty(\mathbb{R})$, $P(0) = 0$, $P'(0) \neq 0$ and $m \in \mathbb{N}$. There are polynomials Q_1, Q_2 such that*

$$Q_1(0) = Q_2(0) = 0, \quad Q_1'(0) = Q_2'(0) = 1, \quad Q_1^{(k)}(0) = Q_2^{(k)}(0) \quad \textit{for } k = 1, \dots, m$$

and on some neighborhood of zero

$$P'(0)\frac{Q_1}{Q_1'} \leqslant P \leqslant P'(0)\frac{Q_2}{Q_2'}.$$

Proof. Since $P \in C^\infty(\mathbb{R})$ thus

$$P(x) = P'(0)x + a_2x^2 + \dots + a_nx^n + R_n(x),$$

where $\frac{R_n(x)}{x^n} \to 0$ as $x \to 0$. For every polynomial of the form

$$W(x) = a_1x + a_2x^2 + \dots + a_kx^k, \qquad a_1 \neq 0$$

we find a polynomial

$$Q(x) = x + b_2 x^2 + \cdots + b_l x^l, \qquad l < k$$

such that

$$W(x)Q'(x) - a_1 Q(x) \tag{4.17}$$

is a polynomial with root at zero of order at least l. To obtain (b_j) we calculate (4.17) and solve the corresponding linear equations for $b_2, \dots, b_l$. Notice that the first p coefficients of Q depend only on the first p coefficients of W.

Apply this procedure to the Taylor polynomial W of order $2n+2$ for the function $P(x) + \varepsilon x^n$, where $l = n+2$. Then

$$\begin{aligned}(P(x) + \varepsilon x^n)\, Q'(x) - P'(0)Q(x) &= W(x)Q'(x) - P'(0)Q(x) + R_{2n+2}(x)Q'(x)\\ &= \tilde{R}_{n+1}(x) \qquad \text{where } \frac{\tilde{R}_{n+1}(x)}{x^{n+1}} \to 0 \text{ as } x \to 0.\end{aligned}$$

Thus

$$P(x) + \varepsilon x^n - P'(0)\frac{Q(x)}{Q'(x)} \leqslant \varepsilon x^{n+1}$$

on a neighborhood of zero. We take $Q_2 := Q$. The construction of Q_1 is analogous, where we take $P(x) - \varepsilon x^n$ instead of $P(x) + \varepsilon x^n$. Hence Q_1 and Q_2 have equal first $n-1$ coefficients. ∎

Lemma 4.8. *If $P_1 \leqslant P_2$ then $\psi_{t,P_1} \leqslant \psi_{t,P_2}$.*

Proof. If $0 < P_1 \leqslant P_2$ then for $\psi_{t,P_1}(x) =: u$ we get $u \geqslant x$ for $t \geqslant 0$ and

$$\int_x^u \frac{1}{P_2(v)} dv \leqslant \int_x^u \frac{1}{P_1(v)} dv = t.$$

Hence

$$\psi_{t,P_2}(x) \geqslant u = \psi_{t,P_1}(x).$$

If $0 > P_2(x) \geqslant P_1(x)$ then $u \leqslant x$ and

$$\int_u^x \left(\frac{-1}{P_2(v)}\right) dv \geqslant \int_u^x \left(\frac{-1}{P_1(v)}\right) dv = t \geqslant 0.$$

Therefore

$$\psi_{t,P_2}(x) \geqslant u = \psi_{t,P_1}(x).$$

Finally, if $P_1 \leqslant 0 \leqslant P_2$ then $\psi_{t,P_1}(x) \leqslant x \leqslant \psi_{t,P_2}(x)$. ∎

Lemma 4.9. *If $P \in C^\infty(\mathbb{R})$ has only discrete zeros of order 1, then $\psi_{t,P} \in C^\infty(\mathbb{R})$ and it is a diffeomorphism. Moreover, the function Ψ_P, $\Psi_P(t,x) := \psi_{t,P}(x)$, is a C^∞-function of two variables.*

Proof. Assume that $x_k = 0$ and $m \in \mathbb{N}$. By Lemmas 4.8 and 4.7,

$$\psi_{t,P'(0)\frac{Q_1}{Q_1'}} \leqslant \psi_{t,P} \leqslant \psi_{t,P'(0)\frac{Q_2}{Q_2'}} \tag{4.18}$$

on a neighborhood of zero. On the other hand, on a neighborhood of x_k

$$\psi_t(x) := \psi_{t,P'(0)\frac{Q}{Q'}}(x) = Q^{-1}(e^{P'(0)t}Q(x)).$$

Clearly, $\psi_t \in C^\infty$ and its m first derivatives at zero depend on the m first derivatives of Q at zero. Thus $\psi_{t,P}$ is between two C^∞-functions with the same m first derivatives at zero. Hence, $\psi_{t,P}$ is m-times differentiable at zero and $\psi'_{t,P}(0) = e^{P'(0)t}$. We can repeat the same procedure for each x_k.

Since $\psi_{t,P}$ is strictly increasing between the zeros of P and has a non-zero derivative at the zeros of P, thus its inverse exists and is smooth.

Clearly, Ψ_P is smooth at (t,x) for any $t \geqslant 0$, $x \neq x_k$, $k \in \mathbb{N}$. Moreover, at $x = x_k$ the functions $\frac{\partial^j}{\partial t^j}\psi_{t,P'(0)\frac{Q_1}{Q_1'}}$ and $\frac{\partial^j}{\partial t^j}\psi_{t,P'(0)\frac{Q_2}{Q_2'}}$ have the same values and the same first $m-j$ derivatives with respect to x. By (4.18) the function Ψ_P has all partial derivatives of order $\leqslant m$ at any point (t, x_k). Since it holds for any $m \in \mathbb{N}$, the function Ψ_P is a C^∞-function of two variables. ■

To give a more concrete example let us take $P(x) = \sin x$, then

$$\psi_{t,P}(x) = 2\arctan(e^t \tan(x/2)) + 2\pi \cdot E\left(\frac{x+\pi}{2\pi}\right)$$

where $E(x)$ denotes the integer part of x. This is the same as

$$\psi_{t,P}(x) = \arccos\left(\frac{(1-e^{2t}) + (1+e^{2t})\cos x}{(1+e^{2t}) + (1-e^{2t})\cos x)}\right) + \pi E\left(\frac{x}{\pi}\right).$$

In fact we have proved above that if on any neighborhood of a zero x_k of P there is a smooth function Q such that

$$P'(x_k)\frac{Q(x)}{Q'(x)} = P(x), \qquad Q'(x_k) = 1$$

then $\psi_{t,P}(x)$ is infinitely differentiable with respect to the two variables t and x around (t, x_k). This implies the following conclusion:

Proposition 4.10. *Let P have infinitely many discrete zeros $(x_k)_{k\in\mathbb{Z}}$ unbounded from below and from above, all zeros are of order one. Then the family of maps*

$$T(t) : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad T(t)g(x) := g(\psi_{t,P}(x))$$

satisfies

$$\|T(t)g\|_k \leqslant C_{k,1} \cdot e^{C_{k,2}t}\|g\|_k \qquad \textit{for } t \geqslant 0$$

for

$$\|h\|_k := \sup_{x \in [x_{-k}, x_k]} \sup_{l \leqslant k} |h^{(l)}(x)|.$$

In fact $T(t)$ are isomorphisms of $C^\infty(\mathbb{R})$. An analogous statement holds for $l < m$ if we consider

$$T(t) : C^\infty[x_l, x_m] \to C^\infty[x_l, x_m].$$

Proof. The maps $\psi_{t,P}$ satisfy the conditions (4.9) and (4.14) of Example 4.4. ■

Clearly f, $f(t) := T(t)g$, solves the abstract Cauchy problem:

$$\begin{cases} \frac{d}{dt} f &= Af, \\ f(0) = g, \end{cases}$$

where $A : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $Ah(x) := P(x)\frac{d}{dx}h(x)$.

The Laplace transform $L(T)(\lambda)$ for T as above never is a smooth function for most of the values λ. Indeed, for $y \in (x_n, x_{n+1})$:

$$\begin{aligned} L(T)(\lambda)(g)(y) &= \int_0^\infty e^{-\lambda t} g(F^{-1}(t + F(y)))dt \\ &= \int_y^{x_{n+1}} e^{\lambda(F(y)-F(z))} g(z) \frac{dz}{P(z)} \\ &= e^{\lambda F(y)} \int_y^{x_{n+1}} e^{-\lambda F(z)} g(z) \frac{dz}{P(z)}. \end{aligned}$$

If $\operatorname{supp} g \Subset (x_n, x_{n+1})$ then for y close to x_n we have

$$\begin{aligned} L(T)(\lambda)(g)(y) &= e^{\lambda F(y)} \int_{x_n}^{x_{n+1}} e^{-\lambda F(z)} g(z) \frac{dz}{P(z)} \\ &= e^{\lambda F(y)} \cdot \text{const}\,. \end{aligned}$$

If for a fixed $\lambda_0 \in \mathbb{R}$ this is a smooth function at x_n then for $\lambda \in \mathbb{R}$ such that λ/λ_0 is not rational the function $L(T)(\lambda)(g)(y)$ cannot be smooth at x_n. Thus $L(T)(\lambda)$ is not a map from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ or from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$. It only exists in the generalized way.

(b) It is not clear what should be assumed on P that $T(t)$ defined as above satisfies the corresponding condition in $\mathcal{S}(\mathbb{R})$. But there are plenty of examples when this is so, for instance, $P(x) = x$ or $P(x) = \sin x$ or more generally P is periodic satisfying the assumptions of Proposition 4.10.

(c) In case if $K = [a, b] \Subset \mathbb{R}$ is properly contained in the interval $[x_l, x_m]$ we can consider $T(t) : C^\infty(K) \to C^\infty(K)$ defined as in Example 4.4 (e).

Problem 4.11. Clarify for which P the operator $T(t)$ defined above satisfies assumptions of Proposition 4.3 on $\mathcal{S}$? What about P with zeros of higher order for the C^∞-case?

Problem 4.12. For which $P \in C^\infty(\mathbb{R})$ the function $(t,y) \mapsto \psi_{t,P}(y)$ is smooth as a function of two variables?

Proposition 4.10 solves the latter Problem for P having only zeros of order one.

Let us finally clarify the difference between the condition (4.7) and (4.6).

Proposition 4.13. *(a) There is a family of operators $T(t)$ on the space of finite sequences ϕ which satisfies (4.6) but not (4.7).*

(b) Let E be a Fréchet space with system $(\|\cdot\|)_m, m \in \mathbb{N}$, of seminorms defining the topology of E. Then a family $T(t)$ satisfies (4.6) if and only if it satisfies (4.7).

Proof. (a): Let us take

$$T(t)(x) := \big(e^{nt}x_n\big)_{n\in\mathbb{N}} \qquad \text{for } x = (x_n) \in \phi.$$

The operator T satisfies (4.6) but not (4.7).

(b): Let us define the space

$$\hat{E} := \{f = (f_t)_{t\in\mathbb{R}_+} \in E^{\mathbb{R}_+} : \ \forall\ m\ \exists\ n \in \mathbb{N} \quad \|f\|_{m,n} := \sup_t \|f_t\|_m e^{-nt} < \infty\}.$$

Thus topologically

$$\hat{E} = \operatorname*{proj}_m\, k(E_m),$$

where E_m is a step space of E and $k(E_m)$ is the coechelon E_m-valued space $\mathrm{ind}_n\, \ell_\infty(v_n, E_m)$, $v_n(t) := e^{-nt}$. If (4.6) is satisfied then the map

$$\hat{T} : E \to \hat{E}, \qquad \hat{T}(g) := (T(t)g)_{t\in\mathbb{R}_+}$$

is a bounded map. Since E is bornological, $\hat{T}$ is continuous, hence

$$\hat{T}_m : E \to k(E_m)$$

is continuous for any $m \in \mathbb{N}$. Of course, $k(E_m)$ is an LB-space. Since E is a Fréchet space, by the Grothendieck factorization theorem (see [11, 24.33]), there is $n \in \mathbb{N}$ such that $\hat{T}_m(E) \subset \ell_\infty(v_n, E_m)$ and hence $\hat{T}_m : E \to \ell_\infty(v_n, E_m)$ is continuous by the closed graph theorem. This is exactly (4.7). ■

References

[1] V.A. Babalola, Semigroups of operators on locally convex spaces, *Trans. Amer. Math. Soc.* **199** (1974), 163–179.

[2] V.A. Babalola, Integration of evolution equations in a locally convex space, *Studia Math.* **50** (1974), 117–125.

[3] B. Dembart, On the theory of semigroups of operators on locally convex spaces, *J. Funct. Anal.* **16** (1974), 123–160.

[4] P. Domański, Classical PLS-spaces: spaces of distributions, real analytic functions and their relatives, *Orlicz Centenary Volume, Banach Center Publications*, **64**, Institute of Mathematics, Polish Academy of Sciences, Warszawa 2004, pp. 51–70.

[5] P. Domański and M. Langenbruch, Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions, *Publ. RIMS Kyoto Univ.* **44**(4) (2008), 1097–1142.

[6] P. Domański and M. Langenbruch, On the abstract Cauchy problem for operators in locally convex spaces, preprint 2009.

[7] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York 2000.

[8] H. Komatsu, Hyperfunctions and linear partial differential equations, *Lect. Notes Math.* **287** (1972), 180–191.

[9] H. Komatsu, Laplace transform of hyperfunctions - A new foundation of Heavyside calculus -, *J. Fac. Sci. Tokyo Sect. IA Math.* **34** (1987), 805–820.

[10] H. Komatsu, Operational calculus and semigroups of operators, *Springer LN Math.* **1540** (1993), 213–234.

[11] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford 1997.

[12] M. Morimoto, *An Introduction to Sato's Hyperfunctions*, AMS, Providence 1993.

[13] Y. Saburi, Fundamental properties of modified Fourier hyperfunctions, *Tokyo J. Math.* **8** (1985), 231–273.

[14] M. Sato, Theory of hyperfunctions, *J. Fac. Sci. Univ. Tokyo, Sec. I* **8** (1959/60), 139–193, 387–436.

[15] P. Schapira, Theorie des hyperfonctions, *Lect. Notes in Math.* **126**, Springer-Verlag, Berlin-Heidelberg-New York 1970 (French), Russian transl., Mir, Moskva 1972.

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