Functiones et Approximatio
42.1 (2010), 51–58

JACKSON q-MAHLER MEASURES

Miki Hirano, Nobushige Kurokawa

Abstract: In this note, we define a q-analogue of the Mahler measures by using the Jackson integral which we call *the Jackson q-Mahler measures.* Especially we study their classical limit for polynomials of one variable.

Keywords: Mahler measure, q-analogue, Jackson integral.

1. Introduction

For a polynomial $f(x_1, \cdots, x_n) \in \mathbf{C}[x_1, \cdots, x_n]$, the (logarithmic) Mahler measure $m(f)$ of f defined by

$$\begin{aligned} m(f) &= \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n})| d\theta_1 \cdots d\theta_n \\ &= \mathrm{Re} \int_0^1 \cdots \int_0^1 \log f(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}) d\theta_1 \cdots d\theta_n, \end{aligned}$$

was introduced by Mahler in order to study transcendental numbers [10]. Besides the original motivation, this measure has remarkable relations to special values of zeta functions and topological entropies of dynamical systems (*cf.* Boyd [1], Deninger [2], and Lind-Schmidt-Ward [9]).

In general, evaluating the Mahler measures is a challenging problem and requires various ideas. As an approach to obtain explicit formulas, one of the authors constructed in his paper [5] its q-analogue by the q-logarithm functions. See also the references [8] and [3] for the q-Mahler measures of this direction.

In this note, we consider another q-analogue of Mahler measures via the Jackson integral

$$\int_0^1 g(t) d_q t = (1-q) \sum_{j=1}^{\infty} g(q^j) q^{j-1}, \qquad 0 < q < 1.$$

2000 Mathematics Subject Classification: 11M06

(This definition of the Jackson integral is slightly different from that in the textbook [4]): For $f(x_1,\cdots,x_n)\in\mathbf{C}[x_1,\cdots,x_n]$ and $q=(q_1,\cdots,q_n)\in(0,1)^n\backslash S_f$, we define *the Jackson q-Mahler measure* of f by

$$\begin{aligned} m^q(f) &= \int_0^1\cdots\int_0^1 \log|f(e^{2\pi i\theta_1},\cdots,e^{2\pi i\theta_n})|d_{q_1}\theta_1\cdots d_{q_n}\theta_n \\ &= (1-q_1)\cdots(1-q_n)\sum_{j_1,\cdots,j_n=1}^{\infty}\log|f(e^{2\pi i q_1^{j_1}},\cdots,e^{2\pi i q_n^{j_n}})|\cdot q_1^{j_1-1}\cdots q_n^{j_n-1}, \end{aligned}$$

where the exceptional set S_f of the parameters is defined by

$$S_f=\left\{q\in(0,1)^n \;\middle|\; f(e^{2\pi i q_1^{j_1}},\cdots,e^{2\pi i q_n^{j_n}})=0 \text{ for some } (j_1,\cdots,j_n)\in(\mathbf{Z}_{>0})^n\right\}.$$

Remark that the Jackson q-Mahler measure is completely different from the previous q-analogue.

Our first concern for the Jackson q-Mahler measures is their classical limits $q\to(1,\cdots,1)$. By the definition, the classical limit of the Jackson integral gives the Riemann integral

$$\lim_{q\to1}\int_0^1 g(t)d_qt=\int_0^1 g(t)dt,$$

if $g(t)$ is continuous in the closed interval $[0,1]$. However, if not, the above equation is not trivial. Thus, our first problem is whether the equation

$$\lim_{q\to(1,\cdots,1),\,q\notin S_f} m^q(f)=m(f),$$

holds or not.

In this note, we consider the Jackson q-Mahler measures for the polynomials of one variable and prove the above mentioned equation holds:

Theorem 1. *Let $f\in\mathbf{C}[x]$. Then we have*

$$\lim_{q\to1,\,q\notin S_f} m^q(f)=m(f).$$

Here S_f is the exceptional set of the parameters for f.

The non-trivial parts of this theorem are evaluations of the Jackson q-Mahler measures and their classical limits for the polynomial $x-\alpha$ of degree one with $|\alpha|=1$. These are expressed by infinite series including the zeta value at even integers and can be evaluated by using the double sine function $\mathcal{S}_2(x)$.

The authors thank Matilde Lalin and Hiroyuki Ochiai for valuable comments.

2. Evaluation of $m^q(x-\alpha)$

For $\alpha\in\mathbf{C}$ with $|\alpha|=1$, the integrand $\log|e^{2\pi i\theta}-\alpha|$ of the Jackson q-Mahler measure $m^q(x-\alpha)$ is not continuous in the closed interval $[0,1]$. In this section, we compute $m^q(x-\alpha)$ in order to get its classical limit.

First, we consider the case of $\alpha = 1$. Remark that the integrand $\log|e^{2\pi i\theta} - 1|$ is continuous in $(0,1)$ and tends to $-\infty$ when $\theta \to 0$ and 1. By the definition, we have

$$\begin{aligned} m^q(x-1) &= \int_0^1 \log|e^{2\pi i\theta} - 1| d_q\theta = \int_0^1 \log|2\sin(\pi\theta)| d_q\theta \\ &= \log 2 + \int_0^1 \log|\sin(\pi\theta)| d_q\theta. \end{aligned}$$

Using an evaluation of the Jackson integral of the logarithmic sine function in the paper [6], we have the following proposition.

Proposition 1. *For $0 < q < 1$, we have*

$$m^q(x-1) = \log(2\pi) + \frac{\log q}{1-q} - \sum_{k=1}^{\infty} \frac{\zeta(2k) q^{2k}}{k[2k+1]_q}.$$

Here $[n]_q = \dfrac{1-q^n}{1-q}$ and $\zeta(s)$ is the Riemann zeta function.

Here, the difference between this proposition and Theorem 1 in [6] is caused by the definition of the Jackson integral.

Next, we consider the case of $\alpha = e^{2\pi i\theta_0}$ with $0 < \theta_0 < 1$. Then the exceptional set for $f(x) = x - \alpha$ is given by

$$S_{\theta_0} = \left\{ \theta_0^{\frac{1}{m}} \,\middle|\, m \in \mathbf{Z}_{>0} \right\}.$$

In the following discussion, we assume $q \notin S_{\theta_0}$. Let

$$M = M(q) = \max\{m \in \mathbf{Z}_{>0} \,|\, q^m > \theta_0\}.$$

Then we have the inequality $q^{M+1} < \theta_0 < q^M$, and

$$\lim_{q \to 1, q \notin S_{\theta_0}} q^M = \theta_0. \tag{1}$$

Similarly to the case of $m^q(x-1)$, we have

$$\begin{aligned} m^q(x-\alpha) &= \int_0^1 \log|e^{2\pi i\theta} - e^{2\pi i\theta_0}| d_q\theta = \log 2 + \int_0^1 \log|\sin(\pi(\theta - \theta_0))| d_q\theta \\ &= \log 2 + (1-q) \sum_{j=1}^{\infty} \log|\sin(\pi(q^j - \theta_0))| \cdot q^{j-1}. \end{aligned}$$

In the right hand side, we remark that the sign-change inside the absolute value symbol occurs once around $j = M$. The infinite product expression of the sine function and the Taylor expansion of the logarithmic function lead the formula

$$\log(\sin(\pi x)) = \log(\pi x) + \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2}\right) = \log(\pi x) - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k} \cdot \left(\frac{x^2}{n^2}\right)^k \tag{2}$$
$$= \log(\pi x) - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} \cdot x^{2k},$$

which is valid for $0 < x < 1$. By using this formula, $m^q(x-\alpha)$ can be expressed as the sum of $\log(2\pi)$ and

$$(1-q)\left\{\sum_{j=1}^{M} \log(q^j - \theta_0) \cdot q^{j-1} + \sum_{j=M+1}^{\infty} \log(\theta_0 - q^j) \cdot q^{j-1} \right.$$
$$\left. - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} \sum_{j=1}^{\infty} (q^j - \theta_0)^{2k} q^{j-1} \right\}.$$

Since the equation

$$\log(q^j - \theta_0) = j \log q + \log\left(1 - \frac{\theta_0}{q^j}\right) = j \log q - \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\theta_0}{q^j}\right)^k,$$

holds when $1 \leqslant j \leqslant M$, we have

$$\sum_{j=1}^{M} \log(q^j - \theta_0) \cdot q^{j-1} = \log q \sum_{j=1}^{M} j q^{j-1} - \sum_{k=1}^{\infty} \frac{\theta_0^k}{k} \sum_{j=1}^{M} q^{(-k+1)j-1}$$
$$= \log q \cdot \frac{1 - Mq^M(1-q) - q^M}{(1-q)^2}$$
$$- \theta_0 \cdot \frac{\log q^M}{q \log q} - \sum_{k=2}^{\infty} \frac{\theta_0^k}{k} \left(\frac{1}{q^k - q} - \frac{q^M}{q^{Mk}(q^k - q)}\right).$$

Similarly, we have

$$\sum_{j=M+1}^{\infty} \log(\theta_0 - q^j) \cdot q^{j-1} = \log \theta_0 \sum_{j=M+1}^{\infty} q^{j-1} - \sum_{k=1}^{\infty} \frac{1}{k\theta_0^k} \sum_{j=M+1}^{\infty} q^{(k+1)j-1}$$
$$= \log \theta_0 \cdot \frac{q^M}{(1-q)} - \sum_{k=1}^{\infty} \frac{1}{k\theta_0^k} \cdot \frac{q^{(M+1)k+M}}{1 - q^{k+1}}.$$

Together with the identity

$$(1-q)\sum_{j=1}^{\infty} (q^j - \theta_0)^{2k} q^{j-1} = \int_0^1 (x - \theta_0)^{2k} d_q x,$$

we have the following proposition.

Proposition 2. *Let $0<\theta_0<1$. For each $0<q<1$ with $q\notin S_{\theta_0}$, we have*

$$\begin{aligned}
m^q(x-e^{2\pi i\theta_0}) = \log(2\pi) &+ \frac{\log q}{1-q} - q^M\log q^M - \frac{\log q}{1-q}\cdot q^M \\
&- \frac{(1-q)\theta_0\log q^M}{q\log q} + \sum_{k=2}^{\infty}\frac{\theta_0^k}{k}\frac{1}{q[k-1]_q} - \sum_{k=2}^{\infty}\frac{1}{k}\left(\frac{\theta_0}{q^M}\right)^k\frac{q^M}{q[k-1]_q} \\
&+ q^M\log\theta_0 - \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{q^{M+1}}{\theta_0}\right)^k\frac{q^M}{[k+1]_q} \\
&- \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\int_0^1 (x-\theta_0)^{2k}d_qx.
\end{aligned}$$

3. Classical limit of $m^q(x-\alpha)$

In this section, we compute the classical limit $q\to 1$ of the Jackson q-Mahler measure $m^q(x-\alpha)$ with $|\alpha|=1$.

First, we discuss the case of $\alpha=1$. From Proposition 1, we have

$$\begin{aligned}
\lim_{q\to 1} m^q(x-1) &= \lim_{q\to 1}\left\{\log(2\pi)+\frac{\log q}{1-q}-\sum_{k=1}^{\infty}\frac{\zeta(2k)q^{2k}}{k[2k+1]_q}\right\} \\
&= \log(2\pi)-1-\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(2k+1)}.
\end{aligned}$$

The infinite series in the right hand side is equal to $\log(2\pi)-1$, which is evaluated in the paper [6]. Thus we have the following assertion.

Proposition 3. *We have*

$$\lim_{q\to 1} m^q(x-1)=0.$$

For $\alpha=e^{2\pi i\theta_0}$ with $0<\theta_0<1$, the classical limit of the Jackson q-Mahler measure $m^q(x-\alpha)$ is given as

$$\begin{aligned}
\lim_{q\to 1,\, q\notin S_{\theta_0}} m^q(x-\alpha) &= \log(2\pi)-1+(1-\theta_0)\log(1-\theta_0)+\theta_0\log\theta_0 \\
&\quad -\left\{\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(2k+1)}(1-\theta_0)^{2k+1}+\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(2k+1)}\theta_0^{2k+1}\right\},
\end{aligned}$$

from Proposition 2 together with the fact (1) and the identity

$$\sum_{k=2}^{\infty}\frac{\theta_0^k}{k(k-1)} = \sum_{k=1}^{\infty}\frac{\theta_0^{k+1}}{k} - \sum_{k=1}^{\infty}\frac{\theta_0^k}{k} + \theta_0 = (1-\theta_0)\log\theta_0+\theta_0.$$

Now we evaluate the sum of two infinite series in the right hand side by using the double sine function:

$$\mathcal{S}_2(x) = e^x \prod_{n=1}^{\infty} \left\{ \left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}} \right)^n e^{2x} \right\}.$$

For $|x| < 1$, we have

$$\begin{aligned}
\log \mathcal{S}_2(x) &= x + \sum_{n=1}^{\infty} \left\{ n \log\left(1 - \frac{x}{n}\right) - n \log\left(1 + \frac{x}{n}\right) + 2x \right\} \\
&= x + \sum_{n=1}^{\infty} \left\{ -n \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x}{n}\right)^k + n \sum_{k=1}^{\infty} \frac{1}{k} \left(-\frac{x}{n}\right)^k + 2x \right\} \\
&= x - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} x^{2k+1}.
\end{aligned}$$

From this and the equation (2), we obtain the following expression for $|x| < 1$.

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)} x^{2k+1} &= 2 \left(\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k} x^{2k+1} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} x^{2k+1} \right) \qquad (3) \\
&= x \left(\log(2\pi x) - \log \mathcal{S}_1(x) \right) + \log \mathcal{S}_2(x) - x \\
&= x(\log(2\pi) - 1) + x \log x + \log\left(\mathcal{S}_1(x)^{-x} \mathcal{S}_2(x) \right),
\end{aligned}$$

where $\mathcal{S}_1(x) = 2\sin(\pi x)$. Since the double sine function satisfies the equalities (*cf.* [7])

$$\mathcal{S}_2(1+x) = -\mathcal{S}_2(x)\mathcal{S}_1(x), \qquad \mathcal{S}_2(-x) = \mathcal{S}_2(x)^{-1},$$

the identity $\mathcal{S}_2(1-x) = \mathcal{S}_2(x)^{-1}\mathcal{S}_1(x)$ holds and thus we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)} x^{2k+1} + \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)} (1-x)^{2k+1} \\
&\quad = \log(2\pi) - 1 + x \log x + (1-x)\log(1-x) \\
&\qquad + \log\left(\mathcal{S}_1(x)^{-x} \mathcal{S}_2(x) \mathcal{S}_1(1-x)^{-(1-x)} \mathcal{S}_2(1-x) \right) \\
&\quad = \log(2\pi) - 1 + x \log x + (1-x)\log(1-x), \qquad |x| < 1.
\end{aligned}$$

Applying this for $x = \theta_0$, we obtain the following assertion.

Proposition 4. *Let* $0 < \theta_0 < 1$. *Then we have*

$$\lim_{q \to 1,\, q \notin S_{\theta_0}} m^q(x - e^{2\pi i \theta_0}) = 0.$$

4. Proof of theorem

In this section, we consider the classical limit of the Jackson q-Mahler measure $m^q(f)$ for any polynomial $f \in \mathbf{C}[x]$.

It follows from the definition that

$$m^q(fg) = m^q(f) + m^q(g), \qquad f, g \in \mathbf{C}[x], \ \ q \notin S_f \cup S_g.$$

Therefore, if $f(x) = a \prod_{j=1}^{n}(x - \alpha_j)$ with $a \neq 0$, then we have

$$m^q(f) = \log|a| + \sum_{j=1}^{n} m^q(x - \alpha_j), \qquad q \notin S_f.$$

For $\alpha \in \mathbf{C}$ with $|\alpha| \neq 1$, the equation

$$\lim_{q \to 1} m^q(x - \alpha) = m(x - \alpha),$$

holds trivially, since the function $\log|e^{2\pi i\theta} - \alpha|$ is continuous in $[0, 1]$. In the case of $|\alpha| = 1$, the classical limit of $m^q(x - \alpha)$ and $m(x - \alpha)$ with are also coincide; both of them are zero by propositions in the previous section and Jensen's formula. Thus, we have

$$\lim_{q \to 1,\, q \notin S_f} m^q(f) = m(f).$$

for arbitrary $f \in \mathbf{C}[x]$ and the proof of theorem is complete.

References

[1] D.W. Boyd, *Mahler's measure and special values of L-functions*, Experiment. Math. **7** (1998), 37–82.

[2] C. Deninger, *Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbf{Z}^n$-actions*, J. Amer. Math. Soc. **10** (1997), 259–281.

[3] Y. Gon, N. Kurokawa, H. Oyanagi, *Multiple q-Mahler measures and zeta functions*, J. of Number Theory **124** (2007), 328–345.

[4] V. Kac, P. Cheung, *Quantum Calculus*, Springer-Verlag, New York (2002).

[5] N. Kurokawa, *A q-Mahler measure*, Proc. Japan Acad. **80** Ser. A (2004), 70–73.

[6] N. Kurokawa, K. Mimachi, M. Wakayama, *Jackson's integral of the Hurwitz zeta function*, Rend. Circ. Mat. Palermo (2) **56** (2007), 43–56.

[7] N. Kurokawa, S. Koyama, *Multiple sine functions*, Forum Math. **15** (2003), 839–876.

[8] N. Kurokawa, H. Ochiai, *Mahler measures via the crystallization*, Comment. Math. Univ. St. Pauli **54** (2005), 121–137.

[9] D. Lind, K. Schmidt, T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. **101** (1990), 593–629.

[10] K. Mahler, *An application of Jensen's formula to polynomials*, Mathematika **7** (1960), 98–100.

Addresses: Miki Hirano: Department of Mathematics, Faculty of Science, Ehime University, Bunkyo-cho, Matsuyama, Ehime, 790-8577, Japan;
Nobushige Kurokawa: Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo, 152-8551 Japan.

E-mail: hirano@math.sci.ehime-u.ac.jp, kurokawa@math.titech.ac.jp

Received: 8 April 2009