Functiones et Approximatio
41.1 (2009), 15–40

SOME EMBEDDINGS AND EQUIVALENT NORMS OF THE $\mathcal{L}_{p,q}^{\lambda,s}$ SPACES

Douadi Drihem

Abstract: The aim of this paper is to give some properties for the $\mathcal{L}_{p,q}^{\lambda,s}$ spaces, especially concerning embeddings and equivalent norms based of maximal functions and local means.

Keywords: Besov spaces, Campanato spaces, Triebel-Lizorkin spaces, $\mathcal{L}_{p,q}^{\lambda,s}$ spaces, local means, maximal functions.

1. Introduction

G. Bourdaud [1] showed that an function $f \in L^2_{\mathrm{loc}}$ belongs to $BMO(\mathbb{R}^n)$ (Bounded Mean Oscillation) if and only if

$$\int_{\mathbb{R}^n} (1+|x|)^{-n-1} |f(x)|\, dx < \infty$$

and

$$\sup_{B_J} \frac{1}{|B_J|} \sum_{j \geqslant J} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \mid L^2(B_J) \right\|^2 < \infty, \tag{1.1}$$

where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B_J of $\mathbb{R}^n$ with radius 2^{-J} and $\{\Psi, \varphi_j\}_{j \in \mathbb{N}}$ is the smooth dyadic resolution of unity in $\mathbb{R}^n$ (see Definition 2.1 below). This gives that the important space $BMO(\mathbb{R}^n)$ can be described by the Littlewood-Paley decomposition. The idea of G. Bourdaud used by [7] to get the Littlewood-Paley characterization for Campanato spaces $\dot{\mathcal{L}}^{2,\lambda}$ (which contain as special case the space $BMO(\mathbb{R}^n)$) and their local versions L_2^{α}, where [7] showed that if $0 \leqslant \lambda < n+2$, the Campanato space $\dot{\mathcal{L}}^{2,\lambda}$ coincides algebraically and topologically with the space $\dot{\mathcal{L}}_{2,2}^{\lambda,0}(\mathbb{R}^n)$, which can be obtained by replacing in (1.1), $|B_J|$ by $|B_J|^{\lambda/n}$.

In this work, the focus is to give some properties for the spaces $\mathcal{L}_{p,q}^{\lambda,s}$ ($s \in \mathbb{R}$, $\lambda \geqslant 0$, $0 < p, q < \infty$), this class of function spaces is defined as the set of all

2000 Mathematics Subject Classification: 46E35.

tempered distributions f, such that

$$\left\| f \mid \mathcal{L}_{p,q}^{\lambda,s} \right\| = \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \sum_{j \geqslant J^+} 2^{jsq} \left\| \mathcal{F}^{-1} \left(\varphi_j \mathcal{F} f \right) \mid L^p \left(B_J \right) \right\|^q \right)^{1/q} < \infty,$$

where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B_J of $\mathbb{R}^n$ with radius 2^{-J} and $\{\varphi_j\}_{j \in \mathbb{N}_0}$ is the smooth dyadic resolution of unity in $\mathbb{R}^n$.

We want to present here, briefly, the contents of our work. In Section 2 we recall the definitions of the different spaces and some necessary tools. In Section 3 our results on the embedding problems are presented. It is shown that if $0 < t < p < \infty$, $s \in \mathbb{R}$, $r - n/t = s - n/p$ and $0 < q, q_1 < \infty$, then

$$\mathcal{L}_{t,q_1}^{\lambda,r} \hookrightarrow \mathcal{L}_{p,q}^{\lambda,s},$$

if and only if $0 < q_1 \leqslant q < \infty$, when we did not use (directly) Bernstein's inequality to obtain this result. Also we will establish some embeddings between these spaces, under some restrictions, and the Besov spaces and the Triebel-Lizorkin spaces. In particular, for $\lambda \geqslant nq/p$, $0 < p, q < \infty$ and $s, \sigma \in \mathbb{R}$, we have

$$\mathcal{L}_{p,q}^{\lambda,s} \not\hookrightarrow B_{1,\infty}^{\sigma}.$$

Section 4 is the core of this paper when we prove an useful characterization of $\mathcal{L}_{p,q}^{\lambda,s}$ spaces based on so-called local means and maximal functions. Furthermore we will give some equivalent norms of the Local approximation spaces $\mathcal{L}_2^{\alpha}$ for $-n/2 \leqslant \alpha < 1$. The proof has as a starting point the technique used by H.-Q. Bui, M. Paluszyński and M. Taibleson, see [2] and [3], and the simplified version of their papers given by V. S. Rychkov in [9].

2. Definitions and basic properties

As usual, $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + ... + \alpha_n$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

For $v \in \mathbb{Z}, x_0 \in \mathbb{R}^n$, we set $B_v = \{y \in \mathbb{R}^n : |y - x_0| < 2^{-v}\}$, $C_v = B_{v-2} \diagdown B_{v-1}$ and $v^+ = \max\{v, 0\}$, where in the proof of each result we will use the notation B_v for all balls with the same centre x_0 and radius 2^{-v}. The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $x \cdot y = x_1 y_1 + ... + x_n y_n$.

We denote by $|\Omega|$ the n-dimensional Lebesgue measure of $\Omega \subseteq \mathbb{R}^n$. For any measurable subset $\Omega \subseteq \mathbb{R}^n$ the Lebesgue space $L^p(\Omega)$, $0 < p \leqslant \infty$ consists of all measurable functions for which

$$\|f \mid L^p(\Omega)\| = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} < \infty, \quad 0 < p < \infty$$

and

$$\|f \mid L^{\infty}(\Omega)\| = \sup_{x\in\Omega} |f(x)| < \infty.$$

If $\Omega = \mathbb{R}^n$ we put $L^p(\mathbb{R}^n) = L^p$ and $\|f \mid L^p(\mathbb{R}^n)\| = \|f\|_p$.

By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$ or $\overset{\vee}{f}$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

If $s \in \mathbb{R}, 0 < q \leqslant \infty$ and $J \in \mathbb{Z}$, then ℓ^s_{q,J^+} is the set of all sequences $\{f_k\}_{k\geqslant J^+}$ of complex numbers such that

$$\left\|\{f_k\}_{k\geqslant J^+} \mid \ell^s_{q,J^+}\right\| = \left(\sum_{k\geqslant J^+} 2^{ksq} |f_k|^q\right)^{1/q} < \infty,$$

with the obvious modification if $q = \infty$. Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. We write $X \not\hookrightarrow Y$ if there is an f such that $f \in X$ but $f \notin Y$. We shall use c to denote positive constant which may differ at each appearance.

In this section we recall some definitions and some necessary tools.

Definition 2.1. *Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $0 \leqslant \Psi(x) \leqslant 1$ and*

$$\Psi(x) = 1 \quad \textit{for} \ \ |x| \leqslant 1 \qquad \textit{and} \qquad \Psi(x) = 0 \quad \textit{for} \ \ |x| \geqslant \tfrac{3}{2},$$

we put $\varphi_0(x) = \Psi(x)$, $\varphi_1(x) = \Psi(x/2) - \Psi(x)$ and

$$\varphi_j(x) = \varphi_1\left(2^{-j+1}x\right) \quad \textit{for} \ j = 2, 3, ...$$

then we have $\operatorname{supp}\varphi_j \subset \left\{x \in \mathbb{R}^n : 2^{j-1} \leqslant |x| \leqslant 3\cdot 2^{j-1}\right\}$, $\varphi_j(x) = 1$ *for* $3\cdot 2^{j-2} \leqslant |x| \leqslant 2^j$ *and*

$$\Psi(x) + \sum_{j\geqslant 1} \varphi_j(x) = 1 \quad \textit{for all} \ x \in \mathbb{R}^n.$$

The system of functions $\{\varphi_j\}$ is called a smooth dyadic resolution of unity. We define the convolution operators Δ_j by the following:

$$\Delta_j f = \overset{\vee}{\varphi}_j * f, \quad j \in \mathbb{N} \qquad \textit{and} \qquad \Delta_0 f = \overset{\vee}{\Psi} * f, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{j\geqslant 0} \Delta_j f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

Definition 2.2. *Let $s \in \mathbb{R}$, $J \in \mathbb{Z}$, $0 < p, q \leqslant \infty$ and $\Omega \subseteq \mathbb{R}^n$. Then $\ell^s_{q,J^+}(L^p(\Omega))$ is the set of all sequences $\{f_k\}_{k \geqslant J^+}$ of complex-valued Lebesgue measurable functions such that*

$$\left\| \{f_k\}_{k \geqslant J^+} \mid \ell^s_{q,J^+}(L^p(\Omega)) \right\| = \left(\sum_{k \geqslant J^+} 2^{ksq} \| f_k \mid L^p(\Omega) \|^q \right)^{1/q} < \infty,$$

with the obvious modification if $p = \infty$ and/or $q = \infty$.

Note that when $J = 0$ and $\Omega = \mathbb{R}^n$ we have $\ell^s_{q,0}(L^p(\mathbb{R}^n)) = \ell^s_q(L^p)$.

We now define the spaces $\mathcal{L}^{\lambda,s}_{p,q}$ which will be our main object of study.

Definition 2.3. *Let $s \in \mathbb{R}$, $\lambda \geqslant 0$ and $0 < p, q < \infty$. The space $\mathcal{L}^{\lambda,s}_{p,q}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\| = \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \{\Delta_j f\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}(L^p(B_J)) \right\|^q \right)^{1/q} < \infty, \qquad (2.1)$$

where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B_J of $\mathbb{R}^n$ with radius 2^{-J}.

Note that the space $\mathcal{L}^{\lambda,s}_{p,q}$ equipped with the norm (2.1) is a quasi-Banach space.

Now we recall the definition of Local approximation spaces.

Definition 2.4. *Let $1 \leqslant p < +\infty$, $\alpha \geqslant -\frac{n}{p}$ and $N = \max(-1, [\alpha])$. $[\alpha]$ is the integer part of α. We say $f \in \mathcal{L}^\alpha_p$ if and only if $f \in L^p_{loc}(\mathbb{R}^n)$ and for some constant $M = M(f)$, for every cube Q of length side δ, there exists a polynomial P_Q of degree $\leqslant N$ ($P_Q = 0$ if $N = -1$) such that*

$$\left(\sup_Q \frac{1}{|Q|^{1+\alpha p/n}} \int_Q |f - P_Q|^p \, dx \right)^{1/p} < M, \quad \text{if} \quad 0 < \delta < 1$$

and

$$\left(\sup_Q \frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p} < M, \quad \text{if} \quad \delta = 1.$$

We denote by $\left\| f \mid \mathcal{L}^\alpha_p \right\|$ the infimum of the constants M as above.

The following theorem gives a some important properties of the spaces $\mathcal{L}^{\lambda,s}_{2,2}$ which are proved in [7].

Theorem 2.5. *Let $s \in \mathbb{R}$ and $-n/2 \leqslant \alpha < 1$. Then*

$$\mathcal{L}^{2\alpha+n,0}_{2,2} = \mathcal{L}^\alpha_2 \qquad \text{and} \qquad \mathcal{L}^{2\alpha+n,s}_{2,2} = I^s(\mathcal{L}^\alpha_2).$$

Here $I^s(\mathcal{L}^\alpha_2)$ denotes the image of $\mathcal{L}^\alpha_2$ under I^s (the Riesz potential operator).

Remark 2.6. Some import properties of the homogeneous counterpart of the space $\mathcal{L}^{\lambda,s}_{p,q}$ are given in [7].

Now we recall the definitions of the spaces $B^s_{p,q}$ and $F^s_{p,q}$.

Definition 2.7.

(i) *Let $s \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $B^s_{p,q}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\left\| f \mid B^s_{p,q} \right\| = \left\| \{\Delta_j f\}_{j \geqslant 0} \mid \ell^s_q (L^p) \right\| < \infty.$$

(ii) *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leqslant \infty$. The Triebel-Lizorkin space $F^s_{p,q}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\left\| f \mid F^s_{p,q} \right\| = \left\| \{\Delta_j f\}_{j \geqslant 0} \mid L^p \left(\ell^s_q\right) \right\| < \infty.$$

Here if $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leqslant \infty$, then $L^p\left(\ell^s_q\right)$ is the set of all sequences $\{f_k\}_{k \geqslant 0}$ of complex-valued Lebesgue measurable functions such that

$$\left\| \{f_k\}_{k \geqslant 0} \mid L^p \left(\ell^s_q\right) \right\| = \left\| \left\| \{2^{ks} f_k\}_{k \geqslant 0} \mid \ell_q \right\| \right\|_p < \infty,$$

with the obvious modification if $q = \infty$.

Now we recall some lemmas which are useful for us.

Lemma 2.8. *Let $0 < a < 1$, $J \in \mathbb{Z}$ and $0 < q \leqslant \infty$. Let $\{\varepsilon_k\}$ be a sequences of positive real numbers, such that*

$$\left\| \{\varepsilon_k\}_{k \geqslant J^+} \mid \ell^0_{q,J^+} \right\| = I < \infty.$$

The sequences $\left\{\delta_k : \delta_k = \sum_{j=0}^{k} a^{k-j} \varepsilon_j\right\}$ and $\left\{\eta_k : \eta_k = \sum_{j=k}^{\infty} a^{j-k} \varepsilon_j\right\}$, are in ℓ^0_{q,J^+} with

$$\left\| \{\delta_k\}_{k \geqslant J^+} \mid \ell^0_{q,J^+} \right\| + \left\| \{\eta_k\}_{k \geqslant J^+} \mid \ell^0_{q,J^+} \right\| \leqslant c\, I.$$

c depends only on a and q.

This Lemma follows from Young's inequality in ℓ^0_{q,J^+}. The following result is from [12, Chapter V, Theorem 5].

Lemma 2.9. *Let $0 < p < \infty$. Then for all $N, j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$ we have*

$$|\Delta_j f(x)|^p \leqslant c\, 2^{jn} \int_{\mathbb{R}^n} |\Delta_j f(y)|^p \left(1 + 2^j |x - y|\right)^{-Np} dy,$$

where $c > 0$ is independent of j.

The following lemma is from [9, Lemma 1].

Lemma 2.10. *Let ω, $\mu \in \mathcal{S}(\mathbb{R}^n)$ and $M \geqslant -1$, an integer such that $\int_{\mathbb{R}^n} x^\alpha \mu(x)\, dx = 0$ for all $|\alpha| \leqslant M$. Then for any $N > 0$, there is a constant $c_N > 0$ so that*

$$\sup_{z \in \mathbb{R}^n} \left| t^{-n} \mu\left(t^{-1} \cdot\right) * \omega(z) \right| (1 + |z|)^N \leqslant c_N\, t^{M+1}.$$

3. Embeddings

The aim of this section is to generalize some embeddings given in [6].

Theorem 3.1. *Let* $0 < t < p < \infty$, $s \in \mathbb{R}$, $r - n/t = s - n/p$ *and* $0 < q, q_1 < \infty$. *Then*

$$\mathcal{L}^{\lambda,r}_{t,q_1} \hookrightarrow \mathcal{L}^{\lambda,s}_{p,q}$$

if and only if $0 < q_1 \leqslant q < \infty$.

Theorem 3.2. *Let* $s, \sigma \in \mathbb{R}$, $\lambda \geqslant 0$, $0 < p, q, q_1 < \infty$ *and* $0 < t \leqslant \infty$.

(i) *Let* $0 < t \leqslant p < \infty$. *Then*

$$B^{s+\varepsilon}_{t,q} \hookrightarrow \mathcal{L}^{\lambda,s}_{p,q}$$

if and only if $\varepsilon \geqslant \lambda/q + n/t - n/p$.

(ii) *We have*

$$B^{s}_{p,q_1} \hookrightarrow \mathcal{L}^{0,s}_{p,q}$$

if and only if $0 < q_1 \leqslant q < \infty$.

(iii) *Let* $\lambda > 0$. *Then*

$$B^{s+\varepsilon}_{p,\infty} \hookrightarrow \mathcal{L}^{\lambda,s}_{p,q}$$

if and only if $\varepsilon \geqslant \lambda/q$.

(iv) *If* $\lambda \geqslant nq/p$ *and* $\varepsilon > \lambda/q + n/t - n/p$, *we have*

$$\mathcal{L}^{\lambda,s}_{p,q} \not\hookrightarrow B^{s+\varepsilon}_{t,q}, \quad 1 < t \leqslant \infty.$$

(v) *If* $\lambda \geqslant nq/p$ *and* $\sigma \in \mathbb{R}$, *we have*

$$\mathcal{L}^{\lambda,s}_{p,q} \not\hookrightarrow B^{\sigma}_{1,\infty}.$$

Remark 3.3. For $1 \leqslant p, q < \infty$, sufficiency part of Theorem 3.2(i), when $\lambda \geqslant nq/p$, was proved in [6] (by the embedding $B^{s+\lambda/q+n/t-n/p}_{t,q} \hookrightarrow B^{s+\lambda/q-n/p}_{\infty,q} \hookrightarrow \mathcal{L}^{\lambda,s}_{p,q}$). Furthermore, by the embedding $F^{s+\lambda/\max(p,q)}_{p,q} \hookrightarrow B^{s+\lambda/\max(p,q)}_{p,\max(p,q)}$, sufficiency part of Theorem 3.2(ii)–(iii) yields

$$F^{s+\lambda/\max(p,q)}_{p,q} \hookrightarrow \mathcal{L}^{\lambda,s}_{p,\max(p,q)},$$

when was proved in [6].

Proof of Theorem 3.1. *Necessity part.* We suppose that $q_1 > q$. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \hat{\rho} \subset \{\xi \in \mathbb{R}^n : 3/4 < |\xi| < 1\}$. For $x \in \mathbb{R}^n$ we put

$$f(x) = \sum_{k \geqslant 0} \frac{2^{-k(s-n/p+\lambda/q_1)}}{(k+1)^{1/q}} \rho\left(2^k x\right).$$

Due to the support properties of the function ρ we have for any $j \in \mathbb{N}_0$ and any $x \in \mathbb{R}^n$ that $\Delta_j f(x) = \frac{2^{-j(s-n/p+\lambda/q_1)}}{(j+1)^{1/q}} \rho\left(2^j x\right)$ and this implies

$$\begin{aligned} \left\| f \mid \mathcal{L}^{\lambda,r}_{t,q_1} \right\|^{q_1} &\leqslant \sup_{J\in\mathbb{Z}} 2^{\lambda J} \sum_{j\geqslant J^+} \frac{2^{-j(s-n/p-r+\lambda/q_1)q_1}}{(j+1)^{q_1/q}} \left\|\rho\left(2^j\cdot\right)\right\|_t^{q_1} \\ &\leqslant \sup_{J\in\mathbb{Z}} 2^{\lambda J} \sum_{j\geqslant J^+} \frac{2^{-j\lambda}}{(j+1)^{q_1/q}} < \infty. \end{aligned}$$

On the other hand we observe that ($B_J(0)$ is the ball centered at 0 and of radius 2^{-J})

$$\begin{aligned} \left\|\Delta_j f \mid L^p\left(B_J(0)\right)\right\| &= \frac{2^{-j(s+\lambda/q_1)}}{(j+1)^{1/q}} \left\|\rho \mid L^p\left(B_{J-j}(0)\right)\right\| \\ &\geqslant \frac{2^{-j(s+\lambda/q_1)}}{(j+1)^{1/q}} \left\|\rho \mid L^p\left(B_0(0)\right)\right\|, \qquad J\in\mathbb{Z},\ j\geqslant J^+ \\ &= c\ \frac{2^{-j(s+\lambda/q_1)}}{(j+1)^{1/q}} \end{aligned}$$

and this leads to

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q}\right\|^q \geqslant c\ \sup_{J>0} 2^{\lambda J} \sum_{j\geqslant J} \frac{2^{-j\lambda q/q_1}}{j+1} \geqslant c \times \begin{cases} \sup\limits_{J>0} \frac{2^{J\lambda(1-q/q_1)}}{J+1} = \infty & \text{if } \lambda>0 \\ \sup\limits_{J>0} \sum\limits_{j\geqslant J} \frac{1}{j+1} = \infty & \text{if } \lambda = 0. \end{cases}$$

Hence $0<q_1\leqslant q<\infty$ is necessary.

Sufficiency part. If $J\in\mathbb{Z}$ is fixed and if $v\in\mathbb{Z}$, let B_J, B_v are balls of $\mathbb{R}^n$ with the same centre x_0 and of radius 2^{-J} and 2^{-v} respectively. We decompose $\mathbb{R}^n$ as the union of the sets B_{J-1} and $C_{J-i} = B_{J-i-2}\diagdown B_{J-i-1}$, $\mathbb{R}^n = B_{J-1}\cup(\cup_{i\geqslant 0} C_{J-i})$. Hence for all $x\in B_J$, $y\in\mathbb{R}^n$ and $N, j\in\mathbb{N}_0$

$$\begin{aligned} \left|\Delta_j f(y)\right| \left(1+2^j|x-y|\right)^{-N} &= \left|\Delta_j f(y)\right| \left(1+2^j|x-y|\right)^{-N} \chi_{\mathbb{R}^n}(y) \\ &= \left|\Delta_j f(y)\right| \left(1+2^j|x-y|\right)^{-N} \chi_{B_{J-1}}(y) \\ &\quad + \sum_{i\geqslant 0} \left|\Delta_j f(y)\right| \left(1+2^j|x-y|\right)^{-N} \chi_{C_{J-i}}(y). \end{aligned}$$

Here χ_E is the characteristic function of the set E. Let $d=\min(1,t,q)$, Lemma 2.9, the above equality and Minkowski's inequality give

$$\begin{aligned} \left|\Delta_j f(x)\right|^d &\leqslant c\ 2^{jnd/t} \left\| \Delta_j f(\cdot)\left(1+2^j|x-\cdot|\right)^{-N} \right\|_t^d \\ &\leqslant c\ 2^{jnd/t} \left\| \Delta_j f(\cdot)\left(1+2^j|x-\cdot|\right)^{-N} \mid L^t\left(B_{J-1}\right) \right\|^d \\ &\quad + c\ 2^{jnd/t} \sum_{i\geqslant 0} \left\| \Delta_j f(\cdot)\left(1+2^j|x-\cdot|\right)^{-N} \mid L^t\left(C_{J-i}\right) \right\|^d. \end{aligned}$$

Taking the $L^{p/d}(B_J)$-quasi-norm, we obtain that $\|\Delta_j f \mid L^p(B_J)\|^d$ can be estimated by

$$
\begin{aligned}
& c\ 2^{jnd/t} \left(\int_{B_J} \left\| \Delta_j f(\cdot) \left(1 + 2^j |x - \cdot|\right)^{-N} \mid L^t(B_{J-1}) \right\|^p dx \right)^{d/p} \\
& + c\ 2^{jnd/t} \sum_{i \geqslant 0} \left(\int_{B_J} \left\| \Delta_j f(\cdot) \left(1 + 2^j |x - \cdot|\right)^{-N} \mid L^t(C_{J-i}) \right\|^p dx \right)^{d/p} \\
= {} & \left(I^1_{j,J}\right)^d + \sum_{i \geqslant 0} \left(I^2_{j,J-i}\right)^d .
\end{aligned}
\tag{3.1}
$$

We first estimate $I^1_{j,J}$. The Minkowski inequality provides

$$
\begin{aligned}
I^1_{j,J} &\leqslant c\ 2^{jn/t} \left(\int_{B_{J-1}} |\Delta_j f(y)|^t \left\| \left(1 + 2^j |\cdot - y|\right)^{-N} \mid L^p(B_J) \right\|^t dy \right)^{1/t} \\
&\leqslant c\ 2^{jn(1/t-1/p)} \left\| \Delta_j f \mid L^t(B_{J-1}) \right\| = c\ 2^{j(r-s)} \left\| \Delta_j f \mid L^t(B_{J-1}) \right\|,
\end{aligned}
$$

where we used

$$
\left\| \left(1 + 2^j |\cdot - y|\right)^{-N} \mid L^p(B_J) \right\| \leqslant \left\| \left(1 + 2^j |\cdot|\right)^{-N} \right\|_p \leqslant c\ 2^{-jn/p},
$$

for any $Np > n$. Next we are going to estimate the second term in (3.1). Note that if $x \in B_J$ and $y \in C_{J-i}$, we have $|x-y| \geqslant |y - x_0| - |x - x_0| > 2^{i-J}$, which gives (with $N = N_1 + N_2$)

$$
\begin{aligned}
\left(1 + 2^j |x - y|\right)^{-N} &\leqslant 2^{(J-j-i)N_1} \left(1 + 2^j |x - y|\right)^{-N_2} \\
&\leqslant 2^{-iN_1} \left(1 + 2^j |x - y|\right)^{-N_2},
\end{aligned}
$$

where we used $2^{(J-j)N_1} \leqslant 1$, (because of $j \geqslant J^+$). For any $N_2 p > n$ we have

$$
\left\| \left(1 + 2^j |\cdot - y|\right)^{-N_2} \mid L^p(B_J) \right\| \leqslant \left\| \left(1 + 2^j |\cdot|\right)^{-N_2} \right\|_p \leqslant c\ 2^{-jn/p}.
$$

The Minkowski inequality and the last estimate give

$$
\begin{aligned}
I^2_{j,J-i} &\leqslant c\ 2^{jn/t - iN_1} \left(\int_{C_{J-i}} |\Delta_j f(y)|^t \left\| \left(1 + 2^j |\cdot - y|\right)^{-N_2} \mid L^p(B_J) \right\|^t dy \right)^{1/t} \\
&\leqslant c\ 2^{jn(1/t-1/p) - iN_1} \left\| \Delta_j f \mid L^t(B_{J-i-2}) \right\| \\
&= c\ 2^{j(r-s) - iN_1} \left\| \Delta_j f \mid L^t(B_{J-i-2}) \right\|.
\end{aligned}
$$

Consequently for any $J \in \mathbb{Z}$ and any $j \geqslant J^+$ there is a constant $c > 0$ independent of J and j such that

$$
\|\Delta_j f \mid L^p(B_J)\|^d \leqslant c\ 2^{jd(r-s)} \sum_{i \geqslant -1} 2^{-i^+ dN_1} \left\| \Delta_j f \mid L^t(B_{J-i-2}) \right\|^d .
$$

Applying the $\ell^{sd}_{q/d,J^+}$-quasi-norm and using the embedding

$$\ell^s_{q,(J-i-2)^+}\left(L^p\left(B_{J-i-2}\right)\right) \hookrightarrow \ell^s_{q,J^+}\left(L^p\left(B_{J-i-2}\right)\right), \quad i \in \mathbb{N}_0 \cup \{-1\}, \tag{3.2}$$

we get that the $\ell^s_{q,J^+}\left(L^p\left(B_J\right)\right)$-quasi-norm of $\Delta_j f$ is bounded by

$$c\left(\sum_{i\geqslant -1} 2^{-i^+ dN_1}\left\|\{\Delta_j f\}_{j\geqslant (J-i-2)^+} \mid \ell^r_{q,J^+}\left(L^t\left(B_{J-i-2}\right)\right)\right\|^d\right)^{1/d}$$
$$\leqslant c\, 2^{-J\lambda/q}\left\|f \mid \mathcal{L}^{\lambda,r}_{t,q_1}\right\|\left(\sum_{i\geqslant -1} 2^{i^+ d(\lambda/q-N_1)}\right)^{1/d},$$

where we used $\ell^s_{q_1,J^+} \hookrightarrow \ell^s_{q,J^+}$. In view of the equality $|B_J|^{\lambda/n} = 2^{-J\lambda}$, it follows that

$$\left\|f \mid \mathcal{L}^{\lambda,s}_{p,q}\right\| \leqslant c\left\|f \mid \mathcal{L}^{\lambda,r}_{t,q_1}\right\|\left(\sum_{i\geqslant -1} 2^{i^+ d(\lambda/q-N_1)}\right)^{1/d}.$$

We conclude the desired estimate by taking N_1 any integer $> \lambda/q$. ■

Proof of Theorem 3.2. *Proof of* (i). *Necessity part.* Let ρ be the function defined in Theorem 3.1. For $x \in \mathbb{R}^n$ we put

$$f(x) = \sum_{k\geqslant 0} 2^{-k(s+\gamma-n/t)}\rho\left(2^k x\right).$$

It is easy to see that

$$\left\|f \mid B^{s+\varepsilon}_{t,q}\right\| = c\left\|\left\{2^{-j(\gamma-\varepsilon)}\right\}_{j\geqslant 0} \mid \ell_q\right\| < \infty \iff \varepsilon < \gamma$$

and

$$\left\|\Delta_j f \mid L^p\left(B_J(0)\right)\right\| \geqslant c\, 2^{-j(s+\gamma-n/t+n/p)}, \qquad J \in \mathbb{Z},\ j \geqslant J^+$$

and this leads to

$$\left\|f \mid \mathcal{L}^{\lambda,s}_{p,q}\right\|^q \geqslant c\sup_{J>0} 2^{\lambda J}\sum_{j\geqslant J} 2^{-j(\gamma-n/t+n/p)q} \geqslant c\sup_{J>0} 2^{-J(\gamma-n/t+n/p-\lambda/q)q}.$$

We finish the proof of the necessity part by taking $\varepsilon < \gamma < n/t - n/p + \lambda/q$.

Sufficiency part. By the embeddings

$$B^{s+\varepsilon}_{t,q} \hookrightarrow B^{s+\lambda/q+n/t-n/p}_{t,q} \hookrightarrow B^{s+\lambda/q}_{p,q},$$

it is sufficient to prove that

$$B^{s+\lambda/q}_{p,q} \hookrightarrow \mathcal{L}^{\lambda,s}_{p,q}.$$

For $J \leqslant 0$ we have

$$2^{\lambda J} \left\| \{\Delta_j f\}_{j \geqslant J^+} \mid \ell^s_{q,J^+} \left(L^p \left(B_J \right) \right) \right\|^q \leqslant c\, 2^{\lambda J} \sum_{j \geqslant 0} 2^{jsq} \left\| \Delta_j f \right\|_p^q$$
$$\leqslant c \left\| f \mid B^s_{p,q} \right\|^q \leqslant c \left\| f \mid B^{s+\lambda/q}_{p,q} \right\|^q,$$

by the embedding $B^{s+\lambda/q}_{p,q} \hookrightarrow B^s_{p,q}$. For $J > 0$ we have

$$2^{\lambda J} \left\| \{\Delta_j f\}_{j \geqslant J^+} \mid \ell^s_{q,J^+} \left(L^p \left(B_J \right) \right) \right\|^q \leqslant c\, 2^{\lambda J} \sum_{j \geqslant 0} 2^{-\lambda J} 2^{j(s+\lambda/q)q} \left\| \Delta_j f \right\|_p^q$$
$$= c \left\| f \mid B^{s+\lambda/q}_{p,q} \right\|^q,$$

this proves (i).

Proof of (ii). *Necessity part.* We suppose that $q_1 > q$. Let ρ be the function defined in Theorem 3.1. We put

$$g(x) = \sum_{k \geqslant 0} \tfrac{2^{-k(s-n/p)}}{(k+1)^{1/q}} \rho \left(2^k x \right), \qquad x \in \mathbb{R}^n.$$

It is not difficult to see that

$$\left\| g \mid B^s_{p,q_1} \right\| = c \left\| \left\{ \tfrac{1}{(j+1)^{1/q}} \right\}_{j \geqslant 0} \mid \ell_{q_1} \right\| < \infty$$

and

$$\left\| \Delta_j g \mid L^p \left(B_J \left(0 \right) \right) \right\| \geqslant c\, \tfrac{2^{-js}}{(j+1)^{1/q}}, \qquad J \in \mathbb{Z},\ j \geqslant J^+$$

and this leads to

$$\left\| g \mid \mathcal{L}^{0,s}_{p,q} \right\|^q \geqslant c \sum_{j \geqslant 0} \tfrac{1}{j+1} = \infty.$$

Hence $0 < q_1 \leqslant q < \infty$ is necessary.

Sufficiency part. If $J \leqslant 0$, then

$$\left\| \{\Delta_j f\}_{j \geqslant J^+} \mid \ell^s_{q,J^+} \left(L^p \left(B_J \right) \right) \right\|^q \leqslant c \sum_{j \geqslant 0} 2^{jsq} \left\| \Delta_j f \right\|_p^q \leqslant c \left\| f \mid B^s_{p,q} \right\|^q$$

and if $J > 0$ we get

$$\left\| \{\Delta_j f\}_{j \geqslant J^+} \mid \ell^s_{q,J^+} \left(L^p \left(B_J \right) \right) \right\|^q \leqslant c \sum_{j \geqslant J} 2^{jsq} \left\| \Delta_j f \right\|_p^q \leqslant c \left\| f \mid B^s_{p,q} \right\|^q.$$

Proof of (iii). Is analogous to the proof of the sufficiency part of (i).

Proof of (iv). Let f be a function like in proof of (i). Note that for all $j \in \mathbb{N}_0$

$$\left\| \Delta_j f \mid L^p \left(B_J \right) \right\| \leqslant \left\| \Delta_j f \right\|_p = c\, 2^{-j(s+\gamma-n/t+n/p)}.$$

Hence the proof of $\mathcal{L}^{\lambda,s}_{p,q} \nrightarrow B^{s+\varepsilon}_{t,q}$ can be obtained by taking γ satisfying $n/t - n/p + \lambda/q < \gamma < \varepsilon$.

Proof of (v). Define

$$f(x) = \sum_{k \geqslant 0} 2^{-k\gamma+n} \prod_{v=1}^{n} e^{ix_v 2^{k-\epsilon+\beta}} \frac{\sin 2^{k-\epsilon-4} x_v}{x_v},$$

where $\epsilon = \log_2 \sqrt{n}$, $\beta \in \mathbb{R}$ satisfying $\log_2 13 < \beta + 4 < \log_2 15$ and γ will be chosen later. It follows that

$$\hat{f}(\xi) = \sum_{k \geqslant 0} 2^{-k\gamma} \prod_{v=1}^{n} \chi_{[-2^{k-\epsilon-4}, 2^{k-\epsilon-4}]} \left(\xi_v - 2^{k-\epsilon+\beta}\right) = \sum_{k \geqslant 0} 2^{-k\gamma} g_k(\xi).$$

We set $C_J(x_0) = \{x \in \mathbb{R}^n : |x_v - (x_0)_v| < 2^{-J}, v = 1, 2, ..., n\}$. The embedding $B_J \subset C_J(x_0)$, yields

$$\begin{aligned}
&\left\| \prod_{v=1}^{n} e^{i \cdot_v 2^{k-\epsilon+\beta}} \frac{\sin 2^{k-\epsilon-4} \cdot_v}{\cdot_v} \mid L^p(B_J) \right\| \\
&\qquad \leqslant \left(\prod_{v=1}^{n} \int_{|x_v - (x_0)_v| < 2^{-J}} \left| \frac{\sin 2^{k-\epsilon-4} x_v}{x_v} \right|^p dx_v \right)^{1/p} \\
&\qquad \leqslant c\, 2^{kn} \left(\prod_{v=1}^{n} \int_{(x_0)_v - 2^{-J}}^{(x_0)_v + 2^{-J}} dx_v \right)^{1/p} = c\, 2^{n(k-J/p)}.
\end{aligned}$$

The last estimate and the fact that the function g_k is supported in $\{\xi \in \mathbb{R}^n : (2^\beta - 2^{-4}) \cdot 2^k \leqslant |\xi| \leqslant (2^\beta + 2^{-4}) \cdot 2^k\}$ give

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|^q \leqslant c \sup_{J \in \mathbb{Z}} 2^{J(\lambda - nq/p)} \sum_{j \geqslant J^+} 2^{j(s-\gamma+n)q}.$$

Notice that $\lambda - nq/p \geqslant 0$, so by taking γ satisfy $\gamma > \max(n, s + n - n/p + \lambda/q)$, we get $f \in \mathcal{L}^{\lambda,s}_{p,q}$. As above we obtain

$$\begin{aligned}
\left\| f \mid B^{\sigma}_{1,\infty} \right\| &= c \sup_{j \geqslant 0} 2^{j(\sigma-\gamma)} \left\| \prod_{v=1}^{n} e^{i \cdot_v 2^{j-\epsilon+\beta}} \frac{\sin 2^{j-\epsilon-4} \cdot_v}{\cdot_v} \right\|_1 \\
&= c \left\| \frac{\sin \cdot}{\cdot} \right\|_1^n \sup_{j \geqslant 0} 2^{j(\sigma-\gamma+n)}.
\end{aligned}$$

Since $\left\| \frac{\sin \cdot}{\cdot} \right\|_1 = \infty$, then $f \notin B^{\sigma}_{1,\infty}$.

Which completes the proof of theorem. ∎

Remark 3.4. Note that, (iv) is valid for all $0 < t \leqslant 1$, but under this assumption (v) is a more general result than (iv).

4. Equivalent quasi-norms

In order to formulate the main result of this paper, let us consider $k_0, k \in \mathcal{S}(\mathbb{R}^n)$ and $S \geqslant -1$ an integer such that for an $\varepsilon > 0$

$$\left|\overset{\wedge}{k_0}(\xi)\right| > 0 \qquad \text{for } \ |\xi| < 2\varepsilon \tag{4.1}$$

$$\left|\hat{k}(\xi)\right| > 0 \qquad \text{for } \ \frac{\varepsilon}{2} < |\xi| < 2\varepsilon \tag{4.2}$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x)\, dx = 0 \qquad \text{for any } \ |\alpha| \leqslant S. \tag{4.3}$$

Here (4.1) and (4.2) are Tauberian conditions, while (4.3) are moment conditions on k. We recall the notation

$$k_t(x) = t^{-n} k\left(t^{-1}x\right), \quad k_j(x) = k_{2^{-j}}(x), \qquad \text{for } \ t > 0 \text{ and } j \geqslant 1.$$

For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in B_J$ we denote (Peetre's maximal functions)

$$k_j^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|k_j * f(y)|}{(1 + 2^j |x-y|)^a}, \quad k_{j,J}^{*,a} f(x) = \sup_{y \in B_J} \frac{|k_j * f(y)|}{(1 + 2^j |x-y|)^a}, \quad j \in \mathbb{N}_0 \tag{4.4}$$

and

$$\Delta_j^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Delta_j f(y)|}{(1 + 2^j |x-y|)^a}, \quad \Delta_{j,J}^{*,a} f(x) = \sup_{y \in B_J} \frac{|\Delta_j f(y)|}{(1 + 2^j |x-y|)^a}, \quad j \in \mathbb{N}_0.$$

Usually $k_j * f$ is called local mean.

We are able now to state the main result of this paper.

Theorem 4.1. *Let $\lambda \geqslant 0$, $0 < p, q < \infty$, $s < S+1$ and $a > \max(n/p, \lambda/q)$. Then*

$$\left\|f \mid \mathcal{L}_{p,q}^{\lambda,s}\right\|' = \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\|\left\{k_j^{*,a} f\right\}_{j \geqslant J^+} \mid \ell_{q,J^+}^s\left(L^p(B_J)\right)\right\|^q\right)^{1/q} \tag{4.5}$$

and

$$\left\|f \mid \mathcal{L}_{p,q}^{\lambda,s}\right\|'' = \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\|\left\{k_j * f\right\}_{j \geqslant J^+} \mid \ell_{q,J^+}^s\left(L^p(B_J)\right)\right\|^q\right)^{1/q} \tag{4.6}$$

are equivalent quasi-norms in $\mathcal{L}_{p,q}^{\lambda,s}$.

Theorem 4.2. *Let $\lambda \geqslant 0$, $0 < p, q < \infty$, $s < S + 1 - \lambda/q$ and $a > n/p$. Then*

$$\left\|f \mid \mathcal{L}_{p,q}^{\lambda,s}\right\|''' = \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\|\left\{k_{j,J}^{*,a} f\right\}_{j \geqslant J^+} \mid \ell_{q,J^+}^s\left(L^p(B_J)\right)\right\|^q\right)^{1/q} \tag{4.7}$$

is an equivalent quasi-norm in $\mathcal{L}_{p,q}^{\lambda,s}$.

Remark 4.3. In view of the inequalities

$$k^{*,a}_{j,J} f(x) \leqslant k^{*,a}_{j} f(x)$$

and

$$k_j * f(x) \leqslant k^{*,a}_{j,J} f(x),$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any $x \in B_J$, it follows from Theorem 4.1 that, in the case of $a > \max(n/p, \lambda/q)$, (4.7) is a simple consequence of (4.5) and (4.6).

Corollary 4.4. *Let $-n/2 \leqslant \alpha < 1$ and $a > 0$. Then $\left\| f \mid \mathcal{L}^{2\alpha+n,0}_{2,2} \right\|'$ (in case of $a > \max(n/2, n/2+\alpha)$), $\left\| f \mid \mathcal{L}^{2\alpha+n,0}_{2,2} \right\|''$ and $\left\| f \mid \mathcal{L}^{2\alpha+n,0}_{2,2} \right\|'''$ (in case of $a > n/2$ and $S > -1 + n/2 + \alpha$) are equivalent norms in $\mathcal{L}^{\alpha}_{2}$.*

The proof of Corollary 4.4 is immediate because $\mathcal{L}^{2\alpha+n,0}_{2,2} = \mathcal{L}^{\alpha}_{2}$ for any $-n/2 \leqslant \alpha < 1$.

To begin the proof of theorems, let us give some auxiliary results.

Let $\mu_0, \mu \in \mathcal{S}(\mathbb{R}^n)$ be two positive functions on $\mathbb{R}^n$ such that

$$\mu_0(\xi) = 1 \quad \text{if} \quad |\xi| \leqslant 2 \qquad \text{and} \qquad \operatorname{supp} \mu_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 4\} \tag{4.8}$$

and

$$\mu(\xi) = 1 \quad \text{if} \quad 1/2 \leqslant |\xi| \leqslant 2 \qquad \text{and} \qquad \operatorname{supp} \mu \subset \{\xi \in \mathbb{R}^n : 1/4 \leqslant |\xi| \leqslant 4\}. \tag{4.9}$$

For any $j \geqslant 1$ and $\xi \in \mathbb{R}^n$ we define $\mu_j(\xi) = \mu(2^{-j}\xi)$.

The results below will play a key role in the proof of Theorems 4.1 and 4.2.

Theorem 4.5. *Let $s \in \mathbb{R}$, $\lambda \geqslant 0$, $0 < p, q < \infty$ and $a > \max(n/p, \lambda/q)$. Then*

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\| = \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \Delta^{*,a}_{j} f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}(L^p(B_J)) \right\|^q \right)^{1/q}, \tag{4.10}$$

is an equivalent quasi-norm in $\mathcal{L}^{\lambda,s}_{p,q}$.

Theorem 4.6. *Let $s \in \mathbb{R}$, $\lambda \geqslant 0$, $0 < p, q < \infty$ and $a > n/p$. Then*

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\| = \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \Delta^{*,a}_{j,J} f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}(L^p(B_J)) \right\|^q \right)^{1/q}, \tag{4.11}$$

is an equivalent quasi-norm in $\mathcal{L}^{\lambda,s}_{p,q}$.

Theorem 4.7. *Let $s \in \mathbb{R}$, $\lambda \geqslant 0$, $0 < p, q < \infty$ and under the above assumptions on the functions μ_0 and μ then*

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|_{\mu} = \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \overset{\vee}{\mu}_j * f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}(L^p(B_J)) \right\|^q \right)^{1/q},$$

is an equivalent quasi-norm in $\mathcal{L}^{\lambda,s}_{p,q}$.

Proof of Theorem 4.5. *Step 1.* It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any $x \in B_J$ we have

$$|\Delta_j f(x)| \leqslant \Delta_j^{*,a} f(x).$$

This shows that the right-hand side in (2.1) is less than the right-hand side in (4.10).

Step 2. Let $f \in \mathcal{L}_{p,q}^{\lambda,s}$, we want to prove that the quasi-norm in (4.10) can be estimated by the quasi-norm in (2.1). From Lemma 2.9, we have for any $y \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$

$$2^{-jn/p} |\Delta_j f(y)| \leqslant c \left(\int_{\mathbb{R}^n} |\Delta_j f(z)|^p \left(1 + 2^j |y - z|\right)^{-ap} dz \right)^{1/p}. \tag{4.12}$$

Let $d = \min(1, p, q)$. By an argument similar to the proof of Theorem 3.1, we have

$$\begin{aligned} 2^{-jnd/p} |\Delta_j f(y)|^d &\leqslant c \left\| \frac{\Delta_j f(\cdot)}{\left(1 + 2^j |y - \cdot|\right)^a} \mid L^p(B_{J-1}) \right\|^d \\ &\quad + c \sum_{i \geqslant 0} \left\| \frac{\Delta_j f(\cdot)}{\left(1 + 2^j |y - \cdot|\right)^a} \mid L^p(C_{J-i}) \right\|^d \\ &= \left(I_{j,J}^1(y)\right)^d + \sum_{i \geqslant 0} \left(I_{j,J-i}^2(y)\right)^d. \end{aligned} \tag{4.13}$$

We set

$$I_{j,J}^{3,*,a}(x) = \sup_{y \in \mathbb{R}^n} \frac{I_{j,J}^1(y)}{\left(1 + 2^j |x - y|\right)^a}, \quad I_{j,J}^{4,*,a}(x) = \sup_{y \in \mathbb{R}^n} \frac{I_{j,J-i}^2(y)}{\left(1 + 2^j |x - y|\right)^a}.$$

By using the elementary inequality

$$\left(1 + 2^j |y - z|\right)^{-a} \leqslant \left(1 + 2^j |y - x|\right)^a \left(1 + 2^j |x - z|\right)^{-a}, \tag{4.14}$$

for any x, y, z in $\mathbb{R}^n$, we get that $\frac{I_{j,J}^1(y)}{(1+2^j|x-y|)^a}$ is bounded by

$$c \left\| \Delta_j f(\cdot) \left(1 + 2^j |x - \cdot|\right)^{-a} \mid L^p(B_{J-1}) \right\|.$$

Take the sup over $y \in \mathbb{R}^n$, the $L^p(B_J)$-quasi-norm and using that the function $x \mapsto (1 + |x|)^{-ap}$ is in L^1 (since $a > n/p$) we get

$$\begin{aligned} \left\| I_{j,J}^{3,*,a} \mid L^p(B_J) \right\| &\leqslant c \left(\int_{B_{J-1}} |\Delta_j f(z)|^p \left\| \left(1 + 2^j |\cdot - z|\right)^{-ap} \right\|_1 dz \right)^{1/p} \\ &\leqslant c\, 2^{-jn/p} \left\| \Delta_j f \mid L^p(B_{J-1}) \right\|. \end{aligned}$$

Next we are going to estimate the second term on the right-hand side of (4.13). By (4.14) we can estimate $\frac{I^2_{j,J-i}(y)}{(1+2^j|x-y|)^a}$ by

$$c\left\|\Delta_j f(\cdot)\left(1+2^j|x-\cdot|\right)^{-a} \mid L^p(C_{J-i})\right\|.$$

Hence we obtain

$$\begin{aligned}\left\|I^{4,*,a}_{j,J} \mid L^p(B_J)\right\| &\leqslant c\left(\int_{C_{J-i}}|\Delta_j f(z)|^p\left\|\left(1+2^j|\cdot-z|\right)^{-ap} \mid L^p(B_J)\right\|dz\right)^{1/p}\\ &\leqslant c\,2^{(J-j-i)a-Jn/p}\left\|\Delta_j f \mid L^p(B_{J-i-2})\right\|\\ &\leqslant c\,2^{-(jn/p+ia)}\left\|\Delta_j f \mid L^p(B_{J-i-2})\right\|,\end{aligned}$$

where have we used $|x-z|\geqslant|z-x_0|-|x-x_0|\geqslant 2^{i-J}$ (for any $z\in C_{J-i}$ and any $x\in B_J$) and $2^{(j-J)(n/p-a)}\leqslant 1$. Consequently, for any $J\in\mathbb{Z}$ and any $j\geqslant J^+$ there is a constant $c>0$ independent of J and j such that

$$\left\|\Delta^{*,a}_j f \mid L^p(B_J)\right\|^d \leqslant c\sum_{i\geqslant -1}2^{-i^+da}\left\|\Delta_j f \mid L^p(B_{J-i-2})\right\|^d. \tag{4.15}$$

Applying the $\ell^{sd}_{q/d,J^+}$-quasi-norm in (4.15) and using the embedding (3.2), we get that the $\ell^s_{q,J^+}(L^p(B_J))$-quasi-norm of $2^{J\lambda/q}\Delta^{*,a}_j f$ is bounded by

$$\begin{aligned}&\left(\sum_{i\geqslant -1}2^{i^+d(\lambda/q-a)}\left\|\left\{2^{(J-i-2)\lambda/q}\Delta_j f\right\}_{j\geqslant(J-i-2)^+} \mid \ell^s_{q,(J-i-2)^+}\left(L^p(B_{J-i-2})\right)\right\|^d\right)^{1/d}\\ &\qquad\leqslant c\left\|f \mid \mathcal{L}^{\lambda,s}_{p,q}\right\|\left(\sum_{i\geqslant -1}2^{i^+d(\lambda/q-a)}\right)^{1/d}.\end{aligned}$$

We conclude the desired estimate using $\left\{2^{i^+d(\lambda/q-a)}\right\}_{i\geqslant -1}\in\ell_1$ and the equation $|B_J|^{-\lambda/n}=2^{J\lambda}$. ■

Proof of Theorem 4.6. *Step 1.* It is easy to see that for any $f\in\mathcal{S}'(\mathbb{R}^n)$ and any $x\in B_J$ we have

$$|\Delta_j f(x)|\leqslant\Delta^{*,a}_{j,J}f(x).$$

This shows that the right-hand side in (2.1) is less than the right-hand side in (4.11).

Step 2. We follow the argument in Step 2 of the proof of Theorem 4.5. We set

$$\widetilde{I^{3,*,a}_{j,J}}(x)=\sup_{y\in B_J}\frac{I^1_{j,J}(y)}{\left(1+2^j|x-y|\right)^a},\quad \widetilde{I^{4,*,a}_{j,J-i}}(x)=\sup_{y\in B_J}\frac{I^2_{j,J-i}(y)}{\left(1+2^j|x-y|\right)^a}.$$

Letting $a = a + N$ in (4.12) we obtain

$$\left\| \widetilde{I^{3,*,a}_{j,J}} \mid L^p(B_J) \right\| \leqslant \left\| I^{3,*,a}_{j,J} \mid L^p(B_J) \right\| \leqslant c\, 2^{-jn/p} \left\| \Delta_j f \mid L^p(B_{J-1}) \right\|.$$

Since $|y - z| \geqslant 2^{i-J}$ (for any $z \in C_{J-i}$ and any $y \in B_J$) and $2^{(J-j)N} \leqslant 1$, we get

$$\begin{aligned} \left(1 + 2^j |y - z|\right)^{-(a+N)} &\leqslant 2^{(J-j-i)N} \left(1 + 2^j |y - z|\right)^{-a} \\ &\leqslant 2^{-iN} \left(1 + 2^j |y - x|\right)^{a} \left(1 + 2^j |x - z|\right)^{-a}. \end{aligned}$$

This implies

$$\begin{aligned} \left\| \widetilde{I^{4,*,a}_{j,J-i}} \mid L^p(B_J) \right\| &\leqslant c\, 2^{-iN} \left\| \left(1 + 2^j |\cdot|\right)^{-ap} \right\|_p \left\| \Delta_j f \mid L^p(B_{J-i-2}) \right\| \\ &\leqslant c\, 2^{-iN-jn/p} \left\| \Delta_j f \mid L^p(B_{J-i-2}) \right\|. \end{aligned}$$

Hence it follows that

$$2^{J\lambda/q} \left\| \left\{ \Delta^{*,a}_{j,J} f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}\left(L^p(B_J)\right) \right\| \leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\| \left(\sum_{i \geqslant -1} 2^{i^+ d(\lambda/q - N)} \right)^{1/d},$$

where $d = \min(1, p, q)$. Then we conclude the desired estimate by taking N any integer $> \lambda/q$ and the fact that $|B_J|^{-\lambda/n} = 2^{\lambda J}$. ■

Proof of Theorem 4.7. Let $\{\varphi_j\}$ be a smooth partition of unity from Definition 2.1 and let $\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|_\varphi$ be the norm from Definition 2.3. Since for any $j \in \mathbb{N}_0$ clearly $\mu_j(\xi) = 1$ on $\operatorname{supp} \varphi_j$ we get

$$\Delta_j f(x) = \overset{\vee}{\varphi}_j * \overset{\vee}{\mu}_j * f(x) = \int_{\mathbb{R}^n} \overset{\vee}{\varphi}_j(y)\, \overset{\vee}{\mu}_j * f(x - y)\, dy, \qquad x \in B_J.$$

Let $d = \min(1, p, q)$. By the same arguments as in the proof of Theorem 3.1, we obtain

$$\left\| \Delta_j f \mid L^p(B_J) \right\|^d \leqslant c \sum_{i \geqslant -1} 2^{-i^+ dN} \left\| \overset{\vee}{\mu}_j * f \mid L^p(B_{J-i-2}) \right\|^d,$$

where $N \in \mathbb{N}$ and $c > 0$ is independent of J and j. Taking the $\ell^{sd}_{q/d,J^+}$-quasi-norm we get that $\left\| \{\Delta_j f\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}\left(L^p(B_J)\right) \right\|^d$ can be estimated by

$$\begin{aligned} c \sum_{i \geqslant -1} 2^{-i^+ dN} &\left\| \left\{ \overset{\vee}{\mu}_j * f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}\left(L^p(B_{J-i-2})\right) \right\|^d \\ &\leqslant c \sum_{i \geqslant -1} 2^{-i^+ dN} \left\| \left\{ \overset{\vee}{\mu}_j * f \right\}_{j \geqslant (J-i-2)^+} \mid \ell^s_{q,(J-i-2)^+}\left(L^p(B_{J-i-2})\right) \right\|^d \\ &\leqslant c\, 2^{-\lambda J d/q} \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|^d_\mu, \end{aligned}$$

where have we used the embedding (3.2) and N sufficiently large. Then we conclude the desired estimate by the fact that $|B_J|^{-\lambda/n} = 2^{\lambda J}$. To prove the reverse inequality we note that due the support properties of the functions μ_j we have

$$\overset{\vee}{\mu}_j * f = \sum_{k=j-2}^{k=j+3} \overset{\vee}{\mu}_j * \Delta_k f.$$

As above and by Lemma 2.8, we obtain the desired estimate. Consequently, $\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|_{\mu}$ and $\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|_{\varphi}$ are equivalent. ■

Proof of Theorem 4.1. The idea of the proof is from V. S. Rychkov [9].

Step 1. Take any pair of functions ϕ_0 and $\phi \in \mathcal{S}(\mathbb{R}^n)$, so that for an $\acute{\varepsilon} > 0$,

$$\left| \overset{\wedge}{\phi}_0 (\xi) \right| > 0 \quad \text{for} \quad |\xi| < 2\acute{\varepsilon} \tag{4.16}$$

$$\left| \overset{\wedge}{\phi} (\xi) \right| > 0 \quad \text{for} \quad \frac{\acute{\varepsilon}}{2} < |\xi| < 2\acute{\varepsilon}, \tag{4.17}$$

and define for any $a > 0$ the functions $\phi_j^{*,a} f$ as in (4.4). Then there is a constant $c > 0$ such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|' \leqslant c \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \phi_j^{*,a} f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+} \left(L^p (B_J) \right) \right\|^q \right)^{1/q}. \tag{4.18}$$

To prove (4.18) we estimate $\left\{ k_j^{*,a} f \right\}_{j \geqslant J^+}$ in $\ell^s_{q,J^+}\left(L^p(B_J)\right)$-quasi-norm, when we consider the cases $J \geqslant 1$ (or $J \leqslant 0$ and $j \geqslant 1$) and ($J \leqslant 0$ and $j = 0$) separately. This estimate can be obtained by the same arguments as in [9].

Step 2. We will prove in this step that there is a constant $c > 0$ such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|' \leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|''. \tag{4.19}$$

Analogously to (4.1), (4.2) find two functions Λ, $\psi \in \mathcal{S}(\mathbb{R}^n)$, so that for an $\varepsilon > 0$

$$\operatorname{supp} \overset{\wedge}{\Lambda} \subset \left\{ \xi \in \mathbb{R}^n : |\xi| < 2\varepsilon \right\}, \qquad \operatorname{supp} \overset{\wedge}{\psi} \subset \left\{ \xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon \right\}$$

and for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$

$$f = \Lambda_j * (k_0)_j * f + \sum_{m \geqslant j+1} \psi_m * k_m * f.$$

Hence

$$k_j * f = \Lambda_j * (k_0)_j * k_j * f + \sum_{m \geqslant j+1} k_j * \psi_m * k_m * f.$$

By an argument similar to the [9] we deduce that for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $x \in B_J$ and $j \in \mathbb{N}$

$$\left(k_j^{*,a} f(x)\right)^r \leqslant c \sum_{m \geqslant j} 2^{(j-m)Nr+mn} \int_{\mathbb{R}^n} \frac{|k_m * f(z)|^r}{(1+2^m |x-z|)^{ar}} dz,$$

where $0 < r < \infty, N \in \mathbb{N}$ can be still be taken arbitrarily large and $c > 0$ is independent of j. Together with the corresponding estimate for $k_0^{*,a} f$. Writing for any $m \geqslant j$

$$\int_{\mathbb{R}^n} \frac{|k_m * f(z)|^r}{(1+2^m |x-z|)^{ar}} dz = \int_{B_{J-1}} (\cdots)\, dz + \sum_{i \geqslant 0} \int_{C_{J-i}} (\cdots)\, dz. \tag{4.20}$$

It possible to choose r so that $n \max\left(\frac{1}{a}, \frac{1}{a+n/p-\lambda/q}\right) < r < p$, we make such a choice and fix r for the rest of the proof. Now the function $z \mapsto \frac{1}{(1+|z|)^{ar}}$ is in L^1 and we may use the majorant property for the Hardy-Littlewood maximal operator M, see E. M. Stein and G. Weiss [11],

$$\left(|g|^r * \frac{1}{(1+|\cdot|)^{ar}}\right)(x) \leqslant c \left\| \frac{1}{(1+|\cdot|)^{ar}} \right\|_1 M\left(|g|^r\right)(x).$$

It follows that the right-hand side of (4.20) is bounded by

$$2^{-mn} M\left(|k_m * f|^r \chi_{B_{J-1}}\right)(x) + \sum_{i \geqslant 0} 2^{Jn-(iar+mn)} \left\| k_m * f \mid L^r\left(B_{J-i-2}\right)\right\|^r,$$

where we have used $|x-z| \geqslant |z-x_0| - |x-x_0| \geqslant 2^{i-J}$ and $2^{(m-J)(n-ar)} \leqslant 1$. Since the Hardy-Littlewood maximal operator M maps $L^\rho(\mathbb{R}^n)$ into $L^\rho(\mathbb{R}^n)$ for $1 < \rho \leqslant \infty$ and since

$$\left\| k_m * f \mid L^r\left(B_{J-i-2}\right)\right\| \leqslant c\, 2^{(i-J)(n/r-n/p)} \left\| k_m * f \mid L^p\left(B_{J-i-2}\right)\right\|,$$

we get by Lemma 2.8 and the embedding (3.2) (where $d = \min(1, q/r)$)

$$\begin{aligned}
&\left\| \left\{k_j^{*,a} f\right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}\left(L^p\left(B_J\right)\right)\right\|^d \\
&\qquad = \left\| \left\{\left(k_j^{*,a} f\right)^r\right\}_{j \geqslant J^+} \mid \ell^{sr}_{q/r,J^+}\left(L^{p/r}\left(B_J\right)\right)\right\|^{d/r} \\
&\qquad \leqslant c \Bigg(\sum_{i \geqslant -1} 2^{-i^+ d(ar-n(1-r/p))} \\
&\qquad\qquad \times \left\| \left\{k_j * f\right\}_{j \geqslant (J-i-2)^+} \mid \ell^s_{q,J^+}\left(L^p\left(B_{J-i-2}\right)\right)\right\|^{dr} \Bigg)^{1/r} \\
&\qquad \leqslant c\, 2^{-\lambda J d/q} \left\| \left\{2^{-i^+ d(a-n(1/r-1/p)-\lambda/q)}\right\}_{i \geqslant -1} \mid \ell_r \right\| \left(\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q}\right\|''\right)^d.
\end{aligned}$$

Whence, multiplying through by $2^{\lambda J d/q}$, using $|B_J|^{-\lambda/n} = 2^{\lambda J}$, and taking the sup over all $J \in \mathbb{Z}$ and all balls B_J of $\mathbb{R}^n$ with radius 2^{-J}, we get the estimate (4.19) by the fact that $\left\{2^{-i^+ d(a-n(1/r-1/p)-\lambda/q)}\right\}_{i \geqslant -1} \in \ell_r$.

Step 3. We will prove in this step that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ the following estimates are true:

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|' \leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\| \tag{4.21}$$

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\| \leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|''. \tag{4.22}$$

Let μ_0 and $\mu \in \mathcal{S}(\mathbb{R}^n)$ be two positive functions on $\mathbb{R}^n$ satisfying (4.8) and (4.9). Let $\hat{\phi}_0 = \mu_0$ and $\hat{\phi} = \mu$. The inequality in (4.21) is proved by the chain of the estimates

$$\begin{aligned}
\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|' &\leqslant c \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \phi_j^{*,a} f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}\left(L^p\left(B_J\right)\right) \right\|^q \right)^{1/q} \\
&\leqslant c \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \phi_j * f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}\left(L^p\left(B_J\right)\right) \right\|^q \right)^{1/q} \\
&\leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|,
\end{aligned}$$

where the first inequality is (4.18), see Step 1, the second inequality is (4.19) (with ϕ and ϕ_0 instead of k and k_0), see Step 2, and finally the last inequality follows by Theorem 4.7. The proof of (4.22) is by the following chain

$$\begin{aligned}
\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\| &\leqslant c \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \phi_j^{*,a} f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}\left(L^p\left(B_J\right)\right) \right\|^q \right)^{1/q} \\
&\leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|' \leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|'',
\end{aligned}$$

where the first inequality is an obvious consequence of (4.10), the second inequality is (4.18), see Step 1, with the roles of k_0 and k respectively ϕ_0 and ϕ interchanged, and finally the last inequality is (4.19), see Step 2. Hence the theorem is proved. ∎

Proof of Theorem 4.2. By Theorem 4.6 and the inequalities

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\| \leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|'' \leqslant c \left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|''',$$

we need only to prove that there is a constant $c > 0$ such that

$$\left\| f \mid \mathcal{L}^{\lambda,s}_{p,q} \right\|''' \leqslant c \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \phi_{j,J}^{*,a} f \right\}_{j \geqslant J^+} \mid \ell^s_{q,J^+}\left(L^p\left(B_J\right)\right) \right\|^q \right)^{1/q}, \tag{4.23}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any pair of functions ϕ_0 and $\phi \in \mathcal{S}(\mathbb{R}^n)$, satisfy (4.16) and (4.17).

Define for any $a>0$ the functions $\phi_{j,J}^{*,a}f$ as in (4.4). It follows from (4.16) and (4.17) that there exist two functions $\Phi,\psi\in\mathcal{S}(\mathbb{R}^n)$ so that

$$\begin{gathered}\operatorname{supp}\overset{\wedge}{\Phi}\subset\{\xi\in\mathbb{R}^n:|\xi|<2\acute{\varepsilon}\}\\ \operatorname{supp}\overset{\wedge}{\psi}\subset\left\{\xi\in\mathbb{R}^n:\frac{\acute{\varepsilon}}{2}<|\xi|<2\acute{\varepsilon}\right\}\\ \overset{\wedge}{\Phi}(\xi)\overset{\wedge}{\phi}_0(\xi)+\sum_{m\geqslant 1}\overset{\wedge}{\psi}\left(2^{-m}\xi\right)\overset{\wedge}{\phi}\left(2^{-m}\xi\right)=1,\qquad\xi\in\mathbb{R}^n\end{gathered}$$

and so for any $f\in\mathcal{S}'(\mathbb{R}^n)$

$$f=\Phi*\phi_0*f+\sum_{m\geqslant 1}\psi_m*\phi_m*f.$$

Consequently, we have for any $j\in\mathbb{N}_0$

$$k_j*f=k_j*\Phi*\phi_0*f+\sum_{m\geqslant 1}k_j*\psi_m*\phi_m*f.$$

Let

$$k_{j,J,m}^{*,a}f(x)=\sup_{y\in B_J}\frac{|k_j*\psi_m*\phi_m*f(y)|}{(1+2^j|x-y|)^a},\qquad m\geqslant 1$$

and

$$k_{j,J,0}^{*,a}f(x)=\sup_{y\in B_J}\frac{|k_j*\Phi*\phi_0*f(y)|}{(1+2^j|x-y|)^a}.$$

We are going now to estimate $\left\{k_{j,J}^{*,a}f\right\}_{j\geqslant J^+}$ in $\ell_{q,J^+}^s(L^p(B_J))$-quasi-norm.

The case $\mathbf{J\geqslant 1}$ (or $\mathbf{J\leqslant 0}$ and $\mathbf{j\geqslant 1}$).

First, let $m>j$. Writing for any $u\in\mathbb{R}^n$ and any $m>j$

$$k_j*\psi_m(u)=2^{jn}\left(k*\psi_{2^{j-m}}\right)\left(2^ju\right),$$

we get by Lemma 2.10, that for any integer $K\geqslant -1$ and any $M>0$ there is a constant $c>0$ independent of j and m

$$|k_j*\psi_m(z)|\leqslant c\,\frac{2^{(j-m)(K+1)+jn}}{(1+2^j|z|)^M},\quad z\in\mathbb{R}^n.$$

This implies

$$\begin{aligned}|k_j*\psi_m*\phi_m*f(y)|&\leqslant c\,2^{(j-m)(K+1)+jn}\int_{\mathbb{R}^n}|\phi_m*f(z)|\left(1+2^j|y-z|\right)^{-M}dz\\&=c\,2^{(j-m)(K+1)+jn}\left(\int_{B_{J-1}}(\cdots)\,dz+\sum_{i\geqslant 0}\int_{C_{J-i}}(\cdots)\,dz\right)\\&=c\,2^{(j-m)(K+1)+jn}\left(D_{j,J,m}^1(y)+\sum_{i\geqslant 0}D_{j,J-i,m}^2(y)\right).\end{aligned}\tag{4.24}$$

First, we note that

$$|\phi_m * f(h)| \leqslant (1 + 2^m |v - h|)^d \phi^{*,d}_{m,J} f(v), \tag{4.25}$$

for all $v, h \in B_J$ and all $d \geqslant 0$. Using the elementary estimate

$$\begin{aligned}(1 + 2^m |x - z|)^a &\leqslant (1 + 2^m |x - y|)^a (1 + 2^m |y - z|)^a \\ &\leqslant 2^{2(m-j)a} \left(1 + 2^j |x - y|\right)^a \left(1 + 2^j |y - z|\right)^a,\end{aligned} \tag{4.26}$$

we get by (4.25) for $z, x, a, J-1$ (in place of h, v, d, J, respectively) and (4.26)

$$\begin{aligned}D^1_{j,J,m}(y) &\leqslant 2^{2(m-j)a} \left(1 + 2^j |x - y|\right)^a \phi^{*,a}_{m,J-1} f(x) \int_{\mathbb{R}^n} \left(1 + 2^j |h|\right)^{a-M} dh \\ &\leqslant c\, 2^{2(m-j)a - jn} \left(1 + 2^j |x - y|\right)^a \phi^{*,a}_{m,J-1} f(x),\end{aligned} \tag{4.27}$$

where we have used $M > a + n$. By using (4.25) for $z, x, a, J - i - 2$ (in place of h, v, d, J, respectively), $|y - z| \geqslant |z - x_0| - |y - x_0| \geqslant 2^{i-J}$ for any $y \in B_J$, $z \in C_{J-i}$, $\int_{\mathbb{R}^n} \left(1 + 2^j |z|\right)^{-M+a+N} dz \leqslant c\ 2^{-jn}$, for any $N \in \mathbb{N}$ and any $M > a + N + n$ and since $2^{(J-j)N} \leqslant 1$, we obtain

$$\begin{aligned}D^2_{j,J-i,m}(y) &\leqslant c\, 2^{2(m-j)a} \left(1 + 2^j |x - y|\right)^a \phi^{*,a}_{m,J-i-2} f(x) \\ &\quad \times \int_{C_{J-i}} \left(1 + 2^j |y - z|\right)^{a-M} dz \\ &\leqslant c\, 2^{2(m-j)a - iN - jn} \left(1 + 2^j |x - y|\right)^a \phi^{*,a}_{m,J-i-2} f(x).\end{aligned} \tag{4.28}$$

In (4.27) and (4.28) multiplying through by $\left(1 + 2^j |x - y|\right)^{-a}$, and taking the sup over $y \in B_J$ we get for any integer $K > 2a + s$, $m > j \geqslant J^+$ and any $x \in B_J$

$$k^{*,a}_{j,J,m} f(x) \leqslant c\, 2^{(j-m)(s+1)} \sum_{i \geqslant -1} 2^{-i^+ N} \phi^{*,a}_{m,J-i-2} f(x),$$

Let now $m \leqslant j$. Writing for any $u \in \mathbb{R}^n$ and any $m \leqslant j$

$$k_j * \psi_m(u) = 2^{mn} \left(k_{2^{m-j}} * \psi\right)(2^m u),$$

we get by Lemma 2.10, that for any $M > 0$ there is a constant $c > 0$ independent of j and m

$$|k_j * \psi_m(z)| \leqslant c\, \frac{2^{(m-j)(S+1)+mn}}{\left(1 + 2^m |z|\right)^M}, \qquad z \in \mathbb{R}^n.$$

In part for technical reasons, we prove this case in two separate cases:

Case 1: $m \leqslant J^+ \leqslant j$. We have only case $J \geqslant 1$ needs to be verified. As in (4.24), we can get for all $y \in B_J$

$$|k_j * \psi_m * \phi_m * f(y)| \leqslant c\, 2^{(m-j)(S+1)+mn} \left(D^1_{m,m,m}(y) + \sum_{i \geqslant 0} D^2_{m,m-i,m}(y) \right).$$

Letting $j=m, J=m$ in the estimate of $D^1_{j,J,m}$ and $D^2_{j,J-i,m}$ in the case $m>j$, we can get for any $N\in\mathbb{N}$

$$k^{*,a}_{j,J,m}f(x)\leqslant c\,2^{(j-m)(S+1)}\sum_{i\geqslant -1}2^{-i^+N}\phi^{*,a}_{m,m-i-2}f(x).$$

Case 2: $J^+<m\leqslant j$. As in Case 1, but we use the decomposition as in (4.24), we can get for any $N\in\mathbb{N}$

$$k^{*,a}_{j,J,m}f(x)\leqslant c\,2^{(m-j)(S+1)}\sum_{i\geqslant -1}2^{-i^+N}\phi^{*,a}_{m,J-i-2}f(x).$$

Hence for all $f\in\mathcal{S}'(\mathbb{R}^n)$, all $j\geqslant J\geqslant 1$ (or $J\leqslant 0$ and $j\geqslant 1$) and all $N\in\mathbb{N}$

$$\begin{aligned}&\Big\|\sum_{m\geqslant 1}k^{*,a}_{j,J,m}f\mid L^p(B_J)\Big\|^d\\&\quad\leqslant c\sum_{i\geqslant -1}2^{-i^+dN}\Bigg(\sum_{m\leqslant J^+\leqslant j}2^{(m-j)(S+1)d}\ \big\|\phi^{*,a}_{m,m-i-2}f\mid L^p(B_{m-i-2})\big\|^d\\&\qquad+\sum_{m>J^+}\big\|\phi^{*,a}_{m,J-i-2}f\mid L^p(B_{J-i-2})\big\|^d\times\begin{cases}2^{(j-m)(s+1)d} & \text{if}\quad m>j\\2^{(m-j)(S+1)d} & \text{if}\quad J^+<m\leqslant j\end{cases}\Bigg),\end{aligned}$$

where $d=\min(1,p,q)$ and $c>0$ is independent of J and j. By Lemma 2.8, the above expression in $\ell^{sd}_{q/d,J^+}$-quasi-norm is dominated by

$$\begin{aligned}&c\sum_{i\geqslant -1}2^{-i^+dN}\Bigg(\Big\|\big\{\big\|\phi^{*,a}_{j,j-i-2}f\mid L^p(B_{j-i-2})\big\|\big\}_{j\geqslant J^+}\mid\ell^s_{q,J^+}\Big\|^d\\&\quad+\Big\|\Big\{\phi^{*,a}_{j,J-i-2}f\Big\}_{j\geqslant J^+}\mid\ell^s_{q,J^+}(L^p(B_{J-i-2}))\Big\|^d\Bigg)\\&\leqslant c\sum_{i\geqslant -1}2^{-i^+dN}\Big\|\Big\{\phi^{*,a}_{j,J-i-2}f\Big\}_{j\geqslant J^+}\mid\ell^s_{q,J^+}(L^p(B_{J-i-2}))\Big\|^d\\&\leqslant c\,2^{-\lambda Jd/q}\Big\|\Big\{2^{i^+d(\lambda/q-N)}\Big\}_{i\geqslant -1}\mid\ell_1\Big\|\\&\quad\times\left(\sup_{B_J}\frac{1}{|B_J|^{\lambda/n}}\Big\|\Big\{\phi^{*,a}_{j,J}f\Big\}_{j\geqslant J^+}\mid\ell^s_{q,J^+}(L^p(B_J))\Big\|^q\right)^{d/q}\\&\leqslant c\,2^{-\lambda Jd/q}\left(\sup_{B_J}\frac{1}{|B_J|^{\lambda/n}}\Big\|\Big\{\phi^{*,a}_{j,J}f\Big\}_{j\geqslant J^+}\mid\ell^s_{q,J^+}(L^p(B_J))\Big\|^q\right)^{d/q},\end{aligned}$$

where we used

$$\begin{aligned}&\left\|\left\{\phi^{*,a}_{j,J-i-2}f\right\}_{j\geqslant J^{+}} \mid \ell^{s}_{q,J^{+}}\left(L^{p}\left(B_{J-i-2}\right)\right)\right\| \\ &\qquad\qquad\hookrightarrow \left\|\left\{\left\|\phi^{*,a}_{j,j-i-2}f \mid L^{p}\left(B_{j-i-2}\right)\right\|\right\}_{j\geqslant J^{+}} \mid \ell^{s}_{q,J^{+}}\right\|,\end{aligned}$$

the embedding (3.2) and N any integer $> \lambda/q$.

We are going now to estimate $\left\{k^{*,a}_{j,J,0}f\right\}_{j\geqslant J^{+}}$ in $\ell^{s}_{q,J^{+}}\left(L^{p}\left(B_{J}\right)\right)$-quasi-norm. We get by Lemma 2.10, that for any $M>0$ there is a constant $c>0$ independent of j

$$\left|k_{j}*\Phi\left(z\right)\right| \leqslant c\ \frac{2^{-j(S+1)+jn}}{\left(1+2^{j}\left|z\right|\right)^{M}}, \qquad z\in\mathbb{R}^{n}.$$

As in case $m>j$, we can write

$$\left|k_{j}*\Phi*\phi_{0}*f\left(y\right)\right| \leqslant c\ 2^{-j(S+1)+jn}\left(D^{1}_{j,0,0}\left(y\right)+\sum_{i\geqslant 0}D^{2}_{j,-i,0}\left(y\right)\right), \qquad y\in B_{J}.$$

So by a simple modification of arguments used in case $m>j$, we can get for all $j\geqslant J\geqslant 1$ (or $J\leqslant 0$ and $j\geqslant 1$) and all $N\in\mathbb{N}$

$$\left\|k^{*,a}_{j,J,0}f \mid L^{p}\left(B_{J}\right)\right\|^{d} \leqslant c\ 2^{-jd(S+1)}\sum_{i\geqslant -1}2^{-i^{+}dN}\left\|\phi^{*,a}_{0,-(i+2)}f \mid L^{p}\left(B_{J}\right)\right\|^{d},$$

where $d=\min\left(1,p\right)$. In view of the inequalities

$$\left\|\phi^{*,a}_{0,-(i+2)}f \mid L^{p}\left(B_{J}\right)\right\| \leqslant \left\|\phi^{*,a}_{0,-(i+2)}f \mid L^{p}\left(B_{-(i+2)}\right)\right\|,$$

for $J\geqslant 1$ or $J\leqslant 0$ and $i\geqslant -\left(J+2\right)$, and

$$\left\|\phi^{*,a}_{0,-(i+2)}f \mid L^{p}\left(B_{J}\right)\right\| \leqslant \left\|\phi^{*,a}_{0,J}f \mid L^{p}\left(B_{J}\right)\right\|,$$

for $J\leqslant 0$ and $i<-\left(J+2\right)$, it follows that for all $j\geqslant J\geqslant 1$ (or $J\leqslant 0$ and $j\geqslant 1$), $\left\|k^{*,a}_{j,J,0}f \mid L^{p}\left(B_{J}\right)\right\|^{d}$ is bounded by

$$c\ 2^{-jd(S+1)}\sum_{i\geqslant -1}2^{-i^{+}dN}\left\|\phi^{*,a}_{0,\min(J,-(i+2))}f \mid L^{p}\left(B_{\min(J,-(i+2))}\right)\right\|^{d}.$$

Since for all $\nu\leqslant 0, 0<q<\infty$ and $\lambda,l\geqslant 0$

$$\begin{aligned}&\left\|\phi^{*,a}_{l,v}f \mid L^{p}\left(B_{v}\right)\right\| \\ &\quad\leqslant c\ 2^{-\lambda v/q-ls}\left(\sup_{B_{J}}\frac{1}{\left|B_{J}\right|^{\lambda/n}}\left\|\left\{\phi^{*,a}_{j,J}f\right\}_{j\geqslant J^{+}} \mid \ell^{s}_{q,J^{+}}\left(L^{p}\left(B_{J}\right)\right)\right\|^{q}\right)^{1/q},\end{aligned} \tag{4.29}$$

we get

$$
\begin{aligned}
&\left\| \left\{ k_{j,J,0}^{*,a} f \right\}_{j \geqslant J^+} \mid \ell_{q,J^+}^{s} \left(L^p \left(B_J \right) \right) \right\| \\
&\qquad = \left\| \left\{ \left\| k_{j,J,0}^{*,a} f \mid L^p \left(B_J \right) \right\|^d \right\}_{j \geqslant J^+} \mid \ell_{q/d,J^+}^{sd} \right\|^{1/d} \\
&\qquad \leqslant c \, 2^{J^+(s-S-1)} \left(\sum_{i \geqslant -1} 2^{-i^+ dN - d \min(J,-(i+2))\lambda/q} \right)^{1/d} \\
&\qquad\quad \times \left(\sup_{B_J} \frac{1}{\left| B_J \right|^{\lambda/n}} \left\| \left\{ \phi_{j,J}^{*,a} f \right\}_{j \geqslant J^+} \mid \ell_{q,J^+}^{s} \left(L^p \left(B_J \right) \right) \right\|^q \right)^{1/q} \\
&\qquad \leqslant c \, 2^{J^+(s-S-1) - \min(J,0)\lambda/q} \\
&\qquad\quad \times \left(\sup_{B_J} \frac{1}{\left| B_J \right|^{\lambda/n}} \left\| \left\{ \phi_{j,J}^{*,a} f \right\}_{j \geqslant J^+} \mid \ell_{q,J^+}^{s} \left(L^p \left(B_J \right) \right) \right\|^q \right)^{1/q},
\end{aligned}
$$

where we used N any integer $> \lambda/q$. Hence, we get for all $f \in \mathcal{S}' \left(\mathbb{R}^n \right)$ and all balls B_J ($J \geqslant 1$ or ($J \leqslant 0$ and $j \geqslant 1$))

$$
\begin{aligned}
\frac{1}{\left| B_J \right|^{\lambda/nq}} \left\| \left\{ k_{j,J}^{*,a} f \right\}_{j \geqslant J^+} \mid \ell_{q,J^+}^{s} \left(L^p \left(B_J \right) \right) \right\| &\leqslant c \left(2^{J^+(s-S-1) + (J - \min(J,0))\lambda/q} + 1 \right) \\
&\quad \times \left(\sup_{B_J} \frac{1}{\left| B_J \right|^{\lambda/n}} \left\| \left\{ \phi_{j,J}^{*,a} f \right\}_{j \geqslant J^+} \mid \ell_{q,J^+}^{s} \left(L^p \left(B_J \right) \right) \right\|^q \right)^{1/q}.
\end{aligned}
$$

Since $s < S+1-\lambda/q$, then the proof of case $J \geqslant 1$ (or $J \leqslant 0$ and $j \geqslant 1$) is complete.

The case $\mathbf{J \leqslant 0}$ and $\mathbf{j = 0}$.

In this case we did not use (4.3) to obtain the desired estimate, so by the fact that $D^\alpha \overset{\wedge}{\psi} (0) = 0$ for any $\alpha \in \mathbb{N}^n$, we can get for any $m \geqslant 1$, all $x \in B_J$ and any $N, K \in \mathbb{N}$

$$
k_{0,J,m}^{*,a} f (x) \leqslant c \, 2^{-m(K+1)} \sum_{i \geqslant -1} 2^{-i^+ N} \phi_{m,J-i-2}^{*,a} f (x),
$$

where $c > 0$ is independent of J and m. By the estimate

$$
\left| k_0 * \phi (z) \right| \leqslant \frac{c}{\left(1 + |z| \right)^M}, \quad z \in \mathbb{R}^n, M \in \mathbb{N}
$$

and by a simple modification of arguments used above we can get for all $x \in B_J$ and all $N \in \mathbb{N}$

$$
k_{0,J,0}^{*,a} f (x) \leqslant c \sum_{i \geqslant -1} 2^{-i^+ N} \phi_{0,J-i-2}^{*,a} f (x),
$$

where $c > 0$ is independent of J. By the embedding $L^p(B_{J-i-2}) \hookrightarrow L^p(B_J)$, (4.29) and by taking K any integer $> \max(-s-1, 0)$ and N any integer $> \lambda/q$, we obtain

$$\begin{aligned}
\left\| k^{*,a}_{0,J} f \mid L^p(B_J) \right\|^d &\leqslant c \sum_{m\geqslant 0} 2^{-md(K+1)} \sum_{i\geqslant -1} 2^{-i^+ dN} \left\| \phi^{*,a}_{m,J-i-2} f \mid L^p(B_{J-i-2}) \right\|^d \\
&\leqslant c\, 2^{-J\lambda d/q} \left\| \left\{ 2^{i^+ d(\lambda/q-N)} \right\}_{i\geqslant -1} \mid \ell_1 \right\| \\
&\quad \times \left\| \left\{ 2^{-md(K+1+s)} \right\}_{m\geqslant 0} \mid \ell_1 \right\| \\
&\quad \times \left(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \left\| \left\{ \phi^{*,a}_{j,J} f \right\}_{j\geqslant J^+} \mid \ell^s_{q,J^+}(L^p(B_J)) \right\|^q \right)^{d/q} \\
&\leqslant c\, 2^{-J\lambda d/q} \Bigg(\sup_{B_J} \frac{1}{|B_J|^{\lambda/n}} \\
&\quad \times \left\| \left\{ \phi^{*,a}_{j,J} f \right\}_{j\geqslant J^+} \mid \ell^s_{q,J^+}(L^p(B_J)) \right\|^q \Bigg)^{d/q},
\end{aligned}$$

where $d = \min(1, p)$. Multiplying through by $2^{J\lambda d/q}$ and using $|B_J|^{-\lambda/n} = 2^{J\lambda}$ we get the desired estimate.

The estimate (4.23) is now as a simple consequence of cases ($J \geqslant 1$ (or $J \leqslant 0$ and $j \geqslant 1$)) and ($J \leqslant 0$ and $j = 0$).

The proof of Theorem 4.2 is thus complete. ■

Acknowledgment. The author would like to thank the referee(s) for the most helpful remarks and corrections which led to the improvements, the results, and the presentation of this paper.

References

[1] G. Bourdaud, *Analyse fonctionnelle dans l'espace Euclidien*, Publ. Math. Univ. Paris. VII, 23, 1987.

[2] H.Q. Bui, M. Paluszyński, M. Taibleson, *A maximal characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces*, Studia. Math. **119** (1996), 219–246.

[3] H.Q. Bui, M. Paluszyński, M. Taibleson, *Characterization of the Besov-Lipschitz and Triebel-Lizorkin spaces. The case* $q < 1$, J. Fourier. Anal. Appl. **3**, special issue (1997), 837–846.

[4] S. Campanato, *Proprietà di Hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa. **17** (1963), 175–188.

[5] S. Campanato, *Proprietà di una famiglia di spazi funzionali*, Ann. Scuola. Norm. Sup. Pisa. **18** (1964), 137–160.

[6] A. El Baraka, *An embedding theorem for Campanato spaces*, Electron. J. Diff. Eqns. **66** (2002), 1–17.
[7] A. El Baraka, *Littlewood-Paley characterization for Campanato spaces*, J. Funct. Spaces. Appl. **4**(2) (2006), 193–220.
[8] M. Frazier, B. Jawerth, G. Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS Regional Conference Series in Mathematics **79**, Amer. Math. Soc. Providence, RI, 1991.
[9] V.S. Rychkov, *On a theorem of Bui, Paluszyński, and Taibleson*, Proc. Steklov. Inst. Math. **227** (1999), 280–292.
[10] W. Sickel and H. Triebel, *Hölder inequalities and sharp embeddings in function spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type*, Z. Anal. Anwendungen. **14**(1), (1995), 105–140.
[11] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, NJ, 1971.
[12] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Math. 1381, Springer, Berlin, 1989.
[13] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983.
[14] H. Triebel, *Theory of function spaces II*, Birkhäuser, Basel, 1992.
[15] H. Triebel, *Theory of Function Spaces III*, Birkhäuser, Basel, 2006.

Address: Douadi Drihem: Department of Mathematics L.M.P.A., M'Sila University, P.O. Box 166, M'Sila 28000 Algeria.

E-mail: douadidr@yahoo.fr

Received: 22 October 2007; **revised:** 17 November 2008