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G-DENSE CLASSES OF ELLIPTIC EQUATIONS IN THE PLANE

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: We show that, for Ω a bounded convex domain of $\mathbb{R}^2$, any 2×2 symmetric matrix $A(x)$ with $\det A(x) = 1$ for a.e. $x \in \Omega$ satisfying the ellipticity bounds

$$\frac{|\xi|^2}{H} \leqslant \langle A(x)\xi, \xi\rangle \leqslant H|\xi|^2$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^2$ can be approximated, in the sense of G-convergence, by a sequence of matrices of the type

$$\begin{pmatrix} \gamma_j(x) & 0 \\ 0 & \frac{1}{\gamma_j(x)} \end{pmatrix}$$

with

$$H - \sqrt{H^2-1} \leqslant \gamma_j(x) \leqslant H + \sqrt{H^2-1}\,.$$

Keywords: G- convergence, quasiconformal maps.

1. Introduction

In [16] A. Marino and S. Spagnolo proved the following approximation result with respect to G-convergence (see Section 3) of the elliptic operator

$$L = \operatorname{div}(A(x)\nabla\ ) \tag{1.1}$$

by a sequence of *isotropic* operators

$$L_j = \operatorname{div}(\beta_j(x)\,\mathbf{I}\nabla\ )\,. \tag{1.2}$$

where $\mathbf{I} = (\delta_{ij})$ is the $n\times n$ identity matrix.

Theorem 1.1. *Let $A = A(x)$ be a symmetric $n\times n$ matrix satisfying the ellipticity condition* $(K \geqslant 1)$

$$\frac{|\xi|^2}{K} \leqslant \langle A(x)\xi, \xi\rangle \leqslant K|\xi|^2 \tag{1.3}$$

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for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then there exists a sequence of coefficients $\beta_j = \beta_j(x)$ satisfying the bounds

$$\frac{1}{cK} \leqslant \beta_j(x) \leqslant cK \tag{1.4}$$

for $c = c(n) > 1$ such that

$$\beta_j(x)\,\mathbf{I} \xrightarrow{G} A(x)$$

We notice that the loss of ellipticity in the G-approximation, which is expressed by the the presence of constants $c(n)$ in (1.4) cannot be avoided. This follows from the sharp result of Piccinini-Spagnolo [19] which attributes Hölder continuity exponent

$$\alpha = \frac{4}{\pi} \arctan \frac{1}{K}$$

to all local solutions $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to *isotropic* equations

$$\operatorname{div}(\beta(x)\mathbf{I}\nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n$$

with $\frac{1}{K} \leqslant \beta(x) \leqslant K$, while the best Hölder continuity exponent pertaining to solutions $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of general elliptic equations

$$\operatorname{div}(A(x)\nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n$$

with $\frac{\mathbf{I}}{K} \leqslant A(x) \leqslant K\mathbf{I}$ and ${}^tA = A$, is only

$$\bar{\alpha} = \frac{1}{K} < \frac{4}{\pi} \arctan \frac{1}{K}\,.$$

A more precise result of isotropic approximation holds for $n = 2$ ([24], [20]) if we additionally assume

$$\det A(x) = 1 \quad a.e.\ x \in \Omega \tag{1.5}$$

Theorem 1.2. *Let $A(x)$ be a 2×2 symmetric matrix satisfying (1.3) and (1.5) for $x \in \Omega \subset \mathbb{R}^2$. Then there exists $\beta_j(x)$ satisfying*

$$\frac{1}{K} \leqslant \beta_j(x) \leqslant K \quad a.e.\ x \in \Omega$$

such that

$$\beta_j(x)\,\mathbf{I} \xrightarrow{G} A(x)$$

if and only if

$$\frac{|\xi|^2}{\frac{1}{2}\left(K + \frac{1}{K}\right)} \leqslant \langle A(x)\xi, \xi\rangle \leqslant \frac{1}{2}\left(K + \frac{1}{K}\right)|\xi|^2 \tag{1.6}$$

In this paper we also restrict ourselves to the case $n = 2$ and look for a G-dense class in the family of diagonal *anisotropic* matrices which satisfy (1.5).

Let us recall that for $n = 2$ the pointwise condition $\det A(x) = 1$ is preserved under the G- convergence ([10]).

Our main result is the following

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and for $x \in \Omega$ let $A(x)$ satisfy the same assumption as in Theorem 1.2. Then there exists a sequence $\gamma_j(x)$ verifying*

$$\frac{1}{K} \leqslant \gamma_j(x) \leqslant K$$

such that

$$\begin{pmatrix} \gamma_j(x) & 0 \\ 0 & \frac{1}{\gamma_j(x)} \end{pmatrix} \xrightarrow{G} A(x)$$

if and only if $A(x)$ satisfies (1.6).

Corollary 1.1. *Given a symmetric matrix valued function*

$$A : x \in \Omega \mapsto A(x) \in \mathbb{R}^{2\times 2}$$

such that $(H \geqslant 1)$

$$\frac{|\xi|^2}{H} \leqslant \langle A(x)\xi, \xi\rangle \leqslant H|\xi|^2$$

$$\det A(x) = 1$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^2$. Then there exist $\gamma_j, \beta_j \colon\ \Omega \to [0, +\infty)$ such that

$$H - \sqrt{H^2 - 1} \leqslant \gamma_j(x) \leqslant H + \sqrt{H^2 - 1}$$

$$H - \sqrt{H^2 - 1} \leqslant \beta_j(x) \leqslant H + \sqrt{H^2 - 1}$$

and

$$\begin{pmatrix} \gamma_j(x) & 0 \\ 0 & \frac{1}{\gamma_j(x)} \end{pmatrix} \xrightarrow{G} A(x)$$

$$\begin{pmatrix} \beta_j(x) & 0 \\ 0 & \beta_j(x) \end{pmatrix} \xrightarrow{G} A(x)$$

$$\begin{pmatrix} \frac{1}{\beta_j(x)} & 0 \\ 0 & \frac{1}{\beta_j(x)} \end{pmatrix} \xrightarrow{G} A(x)$$

Let us mention other approximation results of the isotropic case in the more general setting of degenerate elliptic equations ([8],[21],[12]).

The influence of B. Bojarski on our paper not only goes back to his seminal work of 1957 ([3]) but also refers to his very recent existence theorem of primary pairs of quasiconformal mappings ([4]).

2. Transition from isotropic to anisotropic matrices

Let $C(y)$ be a real matrix satisfying (1.3) for a.e. $y \in \Omega$, $\Omega \subset \mathbb{R}^2$ a bounded convex domain, and suppose that $s(y) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution of the equation

$$\operatorname{div}(C(y)\nabla s) = 0 \quad \text{in } \Omega. \tag{2.1}$$

Let $t(y) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be the *stream function* of s, i.e.

$$\nabla t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C(y)\nabla s \tag{2.2}$$

It is well known ([1]) that the mapping $G = (s,t)\colon \Omega \to G(\Omega)$ is K-quasiregular, that is

$$|DG(y)|^2 \leqslant K\, J(y,G) \quad \text{a.e. } y \in \Omega\,.$$

Recall from [5] that if G is a homeomorphism, it is named a K-quasiconformal map and that its inverse is also K-quasiconformal.

Then we have

Lemma 2.1. *Let the matrix $C(y)$ be isotropic, i.e. for a.e. $y \in \Omega$*

$$C(y) = \begin{pmatrix} a(y) & 0 \\ 0 & a(y) \end{pmatrix} \tag{2.3}$$

with

$$\frac{1}{K} \leqslant a(y) \leqslant K$$

and let the mapping $G = s + \sqrt{-1}t$, defined by solutions to (2.1) and (2.2), be a $W^{1,2}$-homeomorphism with its inverse. If $F = G^{-1} = u + \sqrt{-1}v$ denotes its inverse, then the functions $u(x)$ and $v(x)$ satisfy the following equations

$$\begin{cases} \operatorname{div}(B(x)\nabla u) = 0 \\ \nabla v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B(x)\nabla u \end{cases} \tag{2.4}$$

where $B(x)$ is the matrix with $\det B = 1$ defined by

$$B(x) = \begin{pmatrix} \frac{1}{a(F(x))} & 0 \\ 0 & a(F(x)) \end{pmatrix} \tag{2.5}$$

Proof. We take the advantage of the well known transition formulas from the complex Beltrami coefficients μ_C and ν_C in the equation

$$h_{\bar z} = \mu_C h_z + \nu_C \overline{h_z}$$

to the coefficient matrix $C = (c_{ij})$ of the elliptic equation in the real coordinates ([2], Chapter 10)

$$\mu_C = \frac{c_{22} - c_{11} - 2ic_{12}}{1 + \operatorname{tr} C + \det C}, \qquad \nu_C = \frac{1 - \det C}{1 + \operatorname{tr} C + \det C},$$

where C is the matrix associated to real part φ of the mapping h, i.e. $\operatorname{div}(C\nabla\varphi) = 0$. If $C = (a_{ij})$ is of the special diagonal form like in (2.3) then

$$\mu_C = \frac{a_{22} - a_{11} - 2ia_{12}}{1 + \operatorname{tr} C + \det C} = \frac{a - a}{1 + 2a + a^2} = 0$$

since $a_{12} = 0$, $\operatorname{tr} C = 2a$ and $\det C = a^2$. Moreover

$$\nu_C = \frac{1 - a^2}{1 + 2a + a^2} = \frac{1 - a}{1 + a}.$$

This means that G satisfies the equation

$$G_{\bar z} = \frac{1 - a}{1 + a}\overline{G_z}$$

and, by a well known result on the composition for Beltrami coefficients ([2], p.280) the inverse $F = G^{-1}$ satisfies

$$F_{\bar w}(w) = \frac{a(G^{-1}(w)) - 1}{1 + a(G^{-1}(w))} F_w(w). \tag{2.6}$$

If we consider this equation having the form of a homogeneous Beltrami equation,

$$F_{\bar w}(w) = \mu_B(w) F_w(w)$$

we deduce $\det B(w) = 1$, since $\nu_B = \frac{1 - \det B}{1 + \operatorname{tr} B + \det B} = 0$ in our case. Moreover $\mu_B(w)$ is real, hence $b_{12} = 0$ and

$$\mu_B = \frac{b_{22} - b_{11}}{2 + \operatorname{tr} B} = \frac{b_{22} - \frac{1}{b_{22}}}{2 + b_{22} + \frac{1}{b_{22}}} = \frac{b_{22} - 1}{b_{22} + 1}. \tag{2.7}$$

Comparing (2.6) and (2.7) we deduce the equality

$$b_{22}(x) = a(F(x))$$

and (2.5) follows immediately together with (2.3). ∎

Next Lemma provides a connection between the second order PDE's satisfied by the real part of the Sobolev homeomorphism $f = (u, v)$ and by the real part of its inverse $g = f^{-1} = (s, t)$, when A is a 2×2 constant symmetric matrix with $\det A = 1$. More precisely, we have

Lemma 2.2. *Let $A = (a_{ij})$ be a constant real matrix and suppose that* $\det A = 1$. *Then $u(x)$ and $v(x)$ are $W^{1,2}_{\mathrm{loc}}$ solutions to*

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 \\ \nabla v = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) A\nabla u \end{cases} \tag{2.8}$$

if and only if $s(y)$ and $t(y)$ are $W^{1,2}_{\mathrm{loc}}$ solutions to

$$\begin{cases} \operatorname{div}(A^{-1}\nabla s) = 0 \\ \nabla t = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) A^{-1}\nabla s \end{cases} \tag{2.9}$$

Proof. As g is the inverse of f, the differential matrices are related by

$$\begin{cases} u_{x_1} = \frac{t_{y_2}}{J_g} \\ u_{x_2} = -\frac{t_{y_1}}{J_g} \end{cases} \qquad \begin{cases} v_{x_1} = -\frac{s_{y_2}}{J_g} \\ v_{x_2} = \frac{s_{y_1}}{J_g} \end{cases} \tag{2.10}$$

where J_g denotes the Jacobian determinant of g. One can easily check that the second equality in (2.8) can be written equivalently as

$$\begin{pmatrix} a_{12} & a_{22} \\ -a_{11} & -a_{12} \end{pmatrix} \nabla v = \nabla u$$

that is

$$\begin{cases} u_{x_1} = a_{12} v_{x_1} + a_{22} v_{x_2} \\ u_{x_2} = -a_{11} v_{x_1} - a_{12} v_{x_2} \end{cases} \tag{2.11}$$

Inserting (2.10) into (2.11) we get

$$\begin{cases} t_{y_1} = a_{12} s_{y_1} - a_{11} s_{y_2} \\ t_{y_2} = a_{22} s_{y_1} - a_{12} s_{y_2} \end{cases} \tag{2.12}$$

which means that

$$\nabla t = \begin{pmatrix} a_{12} & -a_{11} \\ a_{22} & -a_{12} \end{pmatrix} \nabla s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix} \nabla s$$

and the proof is complete. ∎

3. G-convergence of elliptic equations

Let K_j be a sequence of equiintegrable functions $K_j : \Omega \to [1, +\infty)$ and let $A_j = A_j(x)$ be a sequence of symmetric matrices with $\det A_j = 1$ a.e. satisfying the ellipticity bounds

$$\frac{|\xi|^2}{K_j(x)} \leqslant \langle A_j(x)\xi, \xi\rangle \leqslant K_j(x)|\xi|^2 \tag{3.1}$$

Assume $u_j \in W^{1,1}_{\mathrm{loc}}(\Omega)$ are uniformly *finite energy* solutions to the equations

$$\operatorname{div} A_j(x)\nabla u_j = 0 \quad \text{in } \Omega \tag{3.2}$$

i.e. are very weak solutions which satisfy the conditions

$$\int_\Omega \langle A_j(x)\nabla u_j, \nabla u_j\rangle \, dx \leqslant M \qquad \forall j \in \mathbb{N} \tag{3.3}$$

By (3.1) and (3.3), if we choose any Borel subset E of Ω, Hölder's inequality implies

$$\int_E |\nabla u_j|\,dx \leqslant \left(\int_E K_j\,dx\right)^{\frac{1}{2}} \left(\int_\Omega \langle A_j(x)\nabla u_j, \nabla u_j\rangle\,dx\right)^{\frac{1}{2}} \leqslant \sqrt{M}\left(\int_E K_j\,dx\right)^{\frac{1}{2}}$$

Hence $|\nabla u_j|$ are equiintegrable as well and there exists a subsequence u_{j_k} such that

$$u_{j_k} \rightharpoonup u \quad \text{weakly in } W^{1,1}_{\mathrm{loc}}(\Omega)$$

The question to see if there exists an elliptic matrix $A(x)$ satisfying bounds of the type (3.1) such that u is a finite energy solution to

$$\operatorname{div} A(x)\nabla u = 0 \quad \text{in } \Omega$$

is the interesting departing point of generalized theory of G-convergence ([8], [12], [21]). Let us consider the following classical definition concerning the special case $K_j(x) \leqslant K$ which corresponds to equiuniformly elliptic operators ([9], [22], [24], [16]).

Definition 3.1. *The sequence of symmetric matrices $A_j(x)$ satisfying* (3.1) *with $1 \leqslant K_j(x) \leqslant K < \infty$ is said to G-converge to the symmetric matrix $A(x)$, i.e. $A_j \xrightarrow{G} A$, if for any $\xi \in \mathbb{R}^2$ the (unique) solutions $u_j \in W^{1,2}(\Omega)$ to the Dirichlet problems*

$$\begin{cases} \operatorname{div}(A_j(x)\nabla u_j) = 0 & \text{in } \Omega \\ u_j(x) = \langle \xi, x\rangle & \text{on } \partial\Omega \end{cases}$$

converge weakly in $W^{1,2}$ to the (unique) solution $u \in W^{1,2}(\Omega)$ to the Dirichlet problem

$$\begin{cases} \operatorname{div}(A(x)\nabla u) = 0 & \text{in } \Omega \\ u(x) = \langle \xi, x\rangle & \text{on } \partial\Omega \end{cases}$$

We recall that G-convergence of A_j to A implies the weak convergence of local solutions $v_j \in W^{1,2}_{\mathrm{loc}}(\Omega)$

$$\operatorname{div}(A_j\nabla v_j) = 0$$

to local solutions $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$

$$\operatorname{div}(A\nabla v) = 0\,.$$

The following result will be useful in the sequel

Theorem 3.1 ([10]). *Let $A_j(x) = {}^tA_j(x)$ be a sequence of 2×2 matrix valued functions defined for $x \in \Omega \subset \mathbb{R}^2$ satisfying* (3.1). *Then*

$$A_j \xrightarrow{G} A \qquad \textit{iff} \qquad \frac{A_j}{\det A_j} \xrightarrow{G} \frac{A}{\det A}$$

Before passing to the proof of Theorem 1.3 let us state the following result from [7] (Theorem 6.1)

Theorem 3.2. *Let $f_0\colon\ x \in \Omega \mapsto x \in \Omega$ be the identity map on the bounded convex domain $\Omega \subset \mathbb{R}^2$. Then every quasiregular map $f\colon\ \Omega \to \mathbb{R}^2$ satisfying the boundary condition*

$$Re(f - f_0) \in W_0^{1,2}(\Omega) \tag{3.4}$$

is a homeomorphism.

Proof of Theorem 1.3. By well known locality properties of G-convergence [18] it is not restrictive to prove the Theorem in the special case that A is a constant matrix, provide we use the approximation theorem from [15], Section III.5.3 of any measurable complex function $\mu(x)$ by sequence of step functions $\mu_j(x)$ with respect to a.e. convergence, with the property

$$\sup_{x\in\Omega} |\mu_j(x)| \leqslant \sup_{x\in\Omega} |\mu(x)|\,. \tag{3.5}$$

More precisely, given the symmetric matrix $A(x)$ such that $\det A(x) = 1$ let us suppose

$$\frac{|\xi|^2}{\frac{1}{2}\left(K + \frac{1}{K}\right)} \leqslant \langle A(x)\xi, \xi\rangle \leqslant \frac{1}{2}\left(K + \frac{1}{K}\right)|\xi|^2 \tag{3.6}$$

for all $\xi \in \mathbb{R}^2$. Let us introduce the complex coefficient

$$\mu(x) = \frac{a_{11}(x) - a_{22}(x) - 2ia_{12}(x)}{a_{11}(x) + a_{22}(x) + 2}$$

and notice that $|\mu(x)| \leqslant \frac{1}{2}\left(K + \frac{1}{K}\right)$.

Let us denote by $\mu_j(x)$ the approximation step-coefficients with the property (3.5) and define the step-functions

$$a_{11}^{(j)}(x) = \frac{1 - 2\operatorname{Re}\mu_j(x) + |\mu_j(x)|^2}{1 - |\mu_j(x)|^2}$$

$$a_{22}^{(j)}(x) = \frac{1 + 2\operatorname{Re}\mu_j(x) + |\mu_j(x)|^2}{1 - |\mu_j(x)|^2}$$

$$a_{12}^{(j)}(x) = -\frac{2\operatorname{Im}\mu_j(x)}{1 - |\mu_j(x)|^2}$$

The matrices $A_j(x) = (a_{ik}^{(j)}(x))$ satisfy uniformly the bounds (3.6). Since

$$\mu_j(x) \to \mu(x) \quad \text{a.e.}$$

we have also

$$a_{ik}^{(j)}(x) \to a_{ik}(x) \quad \text{a.e.}$$

and then (see [2] p.171 and [22])

$$A_j = (a_{ik}^{(j)}(x)) \xrightarrow{G} A.$$

Using the fact that G- convergence is derived by a metric we conclude by a diagonal process.

So let us assume that A is constant. Since the inverse matrix A^{-1} satisfies the same ellipticity bounds (3.6), we can apply to A^{-1} the isotropic approximation Theorem 1.2 obtaining a sequence

$$\frac{1}{K} \leqslant \beta_j(y) \leqslant K, \quad y \in \Omega \tag{3.7}$$

such that

$$\begin{pmatrix} \beta_j(y) & 0 \\ 0 & \beta_j(y) \end{pmatrix} \xrightarrow{G} A^{-1}. \tag{3.8}$$

Moreover by Theorem 3.1 we have also

$$\begin{pmatrix} \frac{1}{\beta_j(y)} & 0 \\ 0 & \frac{1}{\beta_j(y)} \end{pmatrix} \xrightarrow{G} A^{-1}. \tag{3.9}$$

Let $s_j(y) \in y_1 + W_0^{1,2}(\Omega)$ be the solutions to the Dirichlet problems

$$\begin{cases} \operatorname{div} \left(\begin{smallmatrix} \beta_j(y) & 0 \\ 0 & \beta_j(y) \end{smallmatrix} \right) \nabla s_j = 0 \quad \text{in } \Omega \\ s_j(y) \in y_1 + W_0^{1,2}(\Omega) \end{cases} \tag{3.10}$$

and let us couple them with their stream functions $t_j(y)$ defined by ([1])

$$\nabla t_j(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_j(y) & 0 \\ 0 & \beta_j(y) \end{pmatrix} \nabla s_j(y) \tag{3.11}$$

hence

$$\operatorname{div} \begin{pmatrix} \frac{1}{\beta_j(y)} & 0 \\ 0 & \frac{1}{\beta_j(y)} \end{pmatrix} \nabla s_j = 0 \quad \text{in } \Omega \tag{3.12}$$

and the mappings

$$g_j(y) = s_j(y) + \sqrt{-1} t_j(y) \tag{3.13}$$

are K-quasiregular mappings ([1]).

By G-convergence (3.8) and (3.9) we have

$$s_j(y) + \sqrt{-1} t_j(y) \rightharpoonup s(y) + \sqrt{-1} t(y)$$

weakly in $W^{1,2}$ and $s(y)$, $t(y)$ satisfy the limit equations

$$\begin{cases} \operatorname{div}(A^{-1} \nabla s(y)) = 0 \\ \operatorname{div}(A^{-1} \nabla t(y)) = 0 \end{cases} \quad \text{in } \Omega'. \tag{3.14}$$

According to Theorem 3.2 we may always assume that $g_j \colon \Omega \to \Omega'$

$$g_j = s_j(y) + \sqrt{-1} t_j(y)$$

are homeomorphisms. Hence by Montel's theorem up to a not relabeled subsequence there exists the limit $G(y)$

$$g_j(y) \to G(y) \quad \text{uniformly}$$

and, by a result in [11], defining their inverses

$$\begin{aligned} f_j(x) &= u_j(x) + \sqrt{-1}\, v_j(x) = g_j^{-1}(x) \\ f(x) &= u(x) + \sqrt{-1}\, v(x) = g^{-1}(x) \end{aligned} \tag{3.15}$$

we have also

$$f_j \to f \qquad w - W^{1,2} \text{ and locally uniformly} \tag{3.16}$$

Applying Lemma 2.1 with

$$C(y) = \begin{pmatrix} \beta_j(y) & 0 \\ 0 & \beta_j(y) \end{pmatrix}, \quad s = s_j(y) \quad \text{and} \quad t = t_j(y) \tag{3.17}$$

we deduce that the components $u_j(x), v_j(x)$ of the inverse mapping $f_j(x) = g_j^{-1}(x)$ satisfy the elliptic equations

$$\operatorname{div} \begin{pmatrix} \frac{1}{\beta_j(f_j(x))} & 0 \\ 0 & \beta_j(f_j(x)) \end{pmatrix} \nabla u_j = 0 \tag{3.18}$$

$$\operatorname{div} \begin{pmatrix} \frac{1}{\beta_j(f_j(x))} & 0 \\ 0 & \beta_j(f_j(x)) \end{pmatrix} \nabla v_j = 0 \tag{3.19}$$

On the other hand, since $\det A = 1$ and $s(y), t(y)$ solve (3.14) by Lemma 2.2 we deduce that $u(x), v(x)$ solve

$$\begin{cases} \operatorname{div}(A\nabla u(x)) = 0 \\ \operatorname{div}(A\nabla v(x)) = 0 \end{cases} \quad \text{in } \Omega' \tag{3.20}$$

Let us now observe that for any $j \in \mathbb{N}$ the diagonal matrix

$$\mathcal{B}_j(x) = \begin{pmatrix} \frac{1}{\beta_j(f_j(x))} & 0 \\ 0 & \beta_j(f_j(x)) \end{pmatrix} \tag{3.21}$$

coincides with the Beltrami matrix associated to the K-quasiconformal mapping f_j, i.e. to the symmetric matrix with determinat equal to one

$$\mathcal{A}(f_j)(x) \overset{\text{def}}{=} \left[\frac{D^t f_j(x) D f_j(x)}{J(x, f_j)} \right]^{-1} \tag{3.22}$$

if $J(x, f_j) > 0$, otherwise we set $\mathcal{A}(f_j)(x) = (\delta_{i,j}) = \mathbf{I}$ the identity matrix. It is immediate that (see [14])

$$\mathcal{A}(f_j)(x) = \frac{1}{J(x, f_j)} \begin{pmatrix} \left(\frac{\partial u_j}{\partial x_2}\right)^2 + \left(\frac{\partial v_j}{\partial x_2}\right)^2 & -\frac{\partial u_j}{\partial x_1}\frac{\partial u_j}{\partial x_2} - \frac{\partial v_j}{\partial x_1}\frac{\partial v_j}{\partial x_2} \\ -\frac{\partial u_j}{\partial x_1}\frac{\partial u_j}{\partial x_2} - \frac{\partial v_j}{\partial x_1}\frac{\partial v_j}{\partial x_2} & \left(\frac{\partial u_j}{\partial x_1}\right)^2 + \left(\frac{\partial v_j}{\partial x_1}\right)^2 \end{pmatrix} \tag{3.23}$$

Hence, by (3.18) we deduce

$$\frac{\partial u_j}{\partial x_1} = \beta_j(f_j(x))\frac{\partial v_j}{\partial x_2}$$
$$\frac{\partial u_j}{\partial x_2} = -\frac{1}{\beta_j(f_j(x))}\frac{\partial v_j}{\partial x_1}$$

and, by simple calculations

$$\mathcal{B}_j(x) = \mathcal{A}(f_j)(x) \quad \text{a.e.} \tag{3.24}$$

Since $f_j \to f$ locally uniformly, we deduce by a Theorem of S. Spagnolo ([23]), that

$$\mathcal{A}(f_j) \xrightarrow{G} \mathcal{A}(f) \tag{3.25}$$

It remains to prove that

$$\mathcal{A}(f)(x) = A \tag{3.26}$$

because then (3.21), (3.24), (3.25) and (3.26) will give us the approximation result. Recall that the components u, v of $f = u + \sqrt{-1}v$ satisfy (3.20) and, equivalently, the system $A = (a_{ij})$ ($\det A = 1$, $a_{12} = a_{21}$)

$$\begin{cases} -u_{x_2} = a_{11}v_{x_1} + a_{12}v_{x_2} \\ u_{x_1} = a_{12}v_{x_1} + a_{22}v_{x_2} \end{cases} \tag{3.27}$$

If we set $(\alpha_{ik}(x)) = \mathcal{A}(f)$, it is worth verifying that the following

$$\begin{cases} -u_{x_2} = \alpha_{11}v_{x_1} + \alpha_{12}v_{x_2} \\ u_{x_1} = \alpha_{12}v_{x_1} + \alpha_{22}v_{x_2} \end{cases} \tag{3.28}$$

we conclude $a_{ij} = \alpha_{ij}(x)$ thanks to (3.27), (3.28) and the following elementary lemma of linear algebra ([6]). ∎

Lemma 3.1. *For given vectors $E = (E_1, E_2)$, $B = (B_1, B_2)$ of $\mathbb{R}^2$ satisfying $\langle E, B\rangle > 0$ there exists a unique symmetric 2×2 matrix $\mathcal{A} = \mathcal{A}[E, B]$ such that*

$$\begin{cases} \det \mathcal{A} = 1 \\ \mathcal{A}E = B \end{cases}$$

If we set

$$\mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}$$

we have

$$\alpha_{11} = \frac{B_1^2 + E_2^2}{\langle E, B\rangle}, \qquad \alpha_{22} = \frac{B_2^2 + E_1^2}{\langle E, B\rangle}, \qquad \alpha_{12} = \frac{B_1B_2 - E_1E_2}{\langle E, B\rangle}.$$

Remark. *Changing the real into the imaginary part in* $g_j(y) = s_j(y) + \sqrt{-1}t_j(y)$ *and working with* $\tilde{g}_j(y) = t_j(y) + \sqrt{-1}s_j(y)$ *it is possible to relate the inverse of* $\tilde{g}_j(y)$*, which we denote by*

$$\tilde{f}_j(y) = v_j(y) + \sqrt{-1}u_j(y)$$

to the matrix

$$\tilde{\mathcal{B}}_j(x) = \begin{pmatrix} \frac{1}{\beta_j(\tilde{f}_j(x))} & 0 \\ 0 & \beta_j(\tilde{f}_j(x)) \end{pmatrix}$$

which is similarly seen to G-converge to A.

Remark. *It is possible to show that no approximation of A by diagonal matrices* $B_j(x)$ *with* $\det B_j = 1$ *can be performed with* $B_j = B_j(x_1)$ *depending only on* x_1*, unless A itself is diagonal*

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