

NONLINEAR RIEMANN–HILBERT PROBLEMS FOR GENERALIZED ANALYTIC FUNCTIONS

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Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday.

Abstract: In this study we deal with the nonlinear Riemann–Hilbert problem (in short (RHP)) for generalized analytic functions in multiply connected domains. Using a similarity principle for multiply connected domains (presented here for the first time), we reduce the nonlinear RHP for generalized analytic functions to a corresponding nonlinear RHP for holomorphic functions with Hölder continuous boundary data. Then the Newton–Kantorovič method combined with a continuation procedure as well as a new existence theorem for holomorphic solutions, which is based on topological degree arguments, leads to existence of at least two topologically different generalized analytic functions solving the nonlinear RHP.

Keywords: Similarity principle for multiply connected domains, nonlinear Riemann–Hilbert problems, generalized analytic functions, topological degree

1. Introduction

In this article we are concerned with nonlinear Riemann–Hilbert problems for generalized analytic functions in multiply connected plane domains. Let $\Omega_m \subset \mathbb{C}$ be a given $(m + 1)$ -connected bounded domain with the boundary $\partial\Omega_m = \Gamma = \bigcup_{j=0}^m \Gamma_j$ consisting of $m + 1$ separated, smooth (at least $\Gamma \in C^2$), closed Jordan curves, and where Γ_0 is the exterior boundary. The closed curves Γ_j have 1-periodic parametric representations $\Gamma_j : z = t_j(s)$ with $s \in \mathbb{R}$, $j = 0, \dots, m$.

The nonlinear Riemann–Hilbert boundary value problem for generalized analytic functions in Ω_m reads:

Find $u(x, y)$ and $v(x, y)$ in the Hölder space $C^\beta(\overline{\Omega}_m)$, $0 < \beta < 1$ such that $w := u + iv$ satisfies the Vekua system

$$\frac{\partial w}{\partial \bar{z}} - Aw - B\bar{w} = 0 \quad \text{in } \Omega_m \quad (1.1)$$

and a parameter $\omega \in \mathbb{R}^{m-1}$ such that the nonlinear boundary conditions

$$F_j(\omega; z, u, v)|_{\Gamma_j} = 0 \quad \text{for } z = t_j(s) \in \Gamma_j, \quad j = 0, 1, \dots, m \quad (1.2)$$

are satisfied and

$$\int_{\Gamma_j} d \arg w = 0 \quad \text{for } j = 0, 1, \dots, m, \tag{1.3}$$

where F_j are given real-valued functions on $\mathbb{R}^{m-1} \times \mathbb{C} \times \mathbb{R}^2$.

The differentiation $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ in (1.1) is the weak derivative in Sobolev's sense [15, S. 24ff.]. For the given coefficient functions in (1.1) we assume $A, B \in L_q(\Omega_m)$ with $q > 4$.

In order to use the geometric and topological properties of the boundary conditions (1.2), we reformulate them as follows.

For any fixed values of $\omega \in \mathbb{R}^{m-1}$ and $z \in \Gamma$, the real-valued functions F_j define by

$$\gamma_{\omega, z, j} := \{w = u + iv \in \mathbb{C}_w \mid F_j(\omega; z, u, v) = 0, \quad z \in \Gamma_j\}, \tag{1.4}$$

a family of $m+1$ curves in the complex state plane \mathbb{C}_w . Correspondingly, the problems (1.1)–(1.3) can also be formulated as follows:

Find $w \in C^\beta(\bar{\Omega}_m)$ satisfying (1.1) in Ω_m and (1.3), along with the parameters $\omega \in \mathbb{R}^{m-1}$ such that the geometric boundary conditions

$$w(z) \in \gamma_{\omega, z, j} \quad \text{for } z \in \Gamma_j, \quad j = 0, 1, \dots, m \tag{1.5}$$

are satisfied.

The conditions on the family of these curves will be further specified after an appropriate family of nonlinear Riemann–Hilbert problems for holomorphic functions will be associated with (1.1)–(1.5). This will be obtained by employing the similarity principle in a specific form in Section 2.

In previous works we considered Cauchy–Riemann systems in plane domains with nonlinear boundary conditions (see, e.g., [9]). As for the first order systems (1.1), the main contributions were achieved in the 50's by I.N. Vekua and W. Haack and their coworkers (see the monographs [15] and [11]). Some parts of I.N. Vekua's book [15] are in particular due to Bogdan Bojarski. There one of the basic ideas, for tackling linear boundary value problems for (1.1), is the similarity principle due to Bers [2] and Vekua [16], which allows to reduce the boundary value problems (1.1) and (1.2) to corresponding ones for holomorphic functions. This approach was successfully used for both, linear as well as nonlinear boundary value problems of the form (1.1) and (1.2) in simply connected domains (see e.g. [1, 12, 18, 19] and the comprehensive monograph by Wegert [17] and references therein). Most general strongly elliptic systems of first order were considered by Bogdan Bojarski and Tadeusz Iwaniec in [4, 5], and reduced to a nonlinear normal form which allows the treatment of Riemann–Hilbert boundary value problems.

In all these papers, the complex function theory in simply connected domains – as the similarity principle, the parameterization of holomorphic functions on the boundary via their real parts employing the Hilbert transform leading to asso-

ciated, in general nonlinear, singular integral equations on the boundary – played a fundamental role. Since the authors investigated the nonlinear Riemann–Hilbert problems for holomorphic functions in multiply connected domains in [7, 8, 9] the first of the authors was asking experts, in particular Bogdan Bojarski, for similarity principles in multiply connected domains needed for studying corresponding Riemann–Hilbert problems for generalized analytic functions, but at that time for multiply connected domains an appropriate formulation of the similarity principle was not known, although the second author could provide the desired formulation for a long time (see [18]). Hence, in this paper, our analysis is based on a representation formula for generalized analytic functions together with the similarity principle for multiply connected domains, which we present here for the first time; an appropriate parameterization of holomorphic functions, a new existence theorem for holomorphic solutions of nonlinear Riemann–Hilbert problems and a new Newton–Kantorovič method for non invertible operators possessing only a right inverse of their Frechét derivative (see [8, Appendix]), which is due to the multiple connectivity of Ω_m for $m \geq 2$.

The paper is organized as follows. Section 2 is devoted to the similarity principle for multiply connected domains. In Section 3 we employ the similarity principle for the reduction of the boundary value problem (1.1) and (1.2) to nonlinear Riemann–Hilbert problems for holomorphic functions. The corresponding boundary conditions (1.5) for the holomorphic functions, however, are multiplied by functions $\varrho_j(s)$, which are only Hölder continuous. Therefore, we need an extension of the method by Reissig and Wegert [12] from their simply connected to our multiply connected case to find an appropriate a priori estimate for the solutions in a Slobodeckii space and a Hölder space. For the case $\varrho_j \equiv 1$ we apply a new topological degree theory for quasiruled Fredholm maps on quasicylindrical domains to nonlinear singular integral equations on Γ associated with (1.1) and (1.2) and prove global solvability for holomorphic functions presented in [9]. There we lift the problem to one on the universal covering which is a perturbation of linear problems and we need (1.3). Then the linearized Frechét derivative problems have Fredholm index $1 - m$, which in general are not solvable; therefore additional unknown constants are introduced and the boundary conditions are modified correspondingly (see [6]). In Section 4, the Newton–Kantorovič method is combined with an embedding continuation method generalizing an approach in [20], which, together with the a priori estimate of subsequent solutions, provides us a path of homotopy. Finally, Schauder’s fixed point theorem guarantees the existence of solutions to (1.1) and (1.2).

2. The similarity principle

Solutions of the homogeneous equation (1.1) are also called “generalized analytic functions” since they are closely related to complex analytic functions by the so-called *similarity principle* which goes back to L. Bers [2] and I.N. Vekua [16] and can be formulated in the form of the following two theorems.

Theorem 2.1 (S1). *Let $w \in C^\beta(\overline{\Omega}_m)$ with $0 < \beta < 1$ be a generalized solution of the Vekua system*

$$\frac{\partial w}{\partial \bar{z}} = Aw + B\bar{w} \quad \text{in } \Omega_m \tag{2.1}$$

with coefficients $A, B \in L_q(\Omega_m)$ where $q > 4$. Then there exists a function $\tilde{w} = \tilde{u} + i\tilde{v} \in C^\alpha(\overline{\Omega}_m)$ with any positive $\alpha \leq 1 - 2/q$ and satisfying

$$\tilde{v}|_{\Gamma_0} = 0, \quad \tilde{v}|_{\Gamma_j} = c_j = -4 \operatorname{Im} \iint_{\Omega} \left(A + \frac{\bar{w}}{w} B \right) \sum_{k=1}^N c_{jk} \frac{\partial e_k}{\partial z} d\Omega, \quad j = 1, \dots, m,$$

such that the function $f := w \exp(-\tilde{w}) \in C^\alpha(\overline{\Omega}_m)$ is holomorphic in Ω_m :

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{2.2}$$

In fact, \tilde{w} is given explicitly as

$$\tilde{u}(\xi, \eta) = 4 \operatorname{Re} \iint_{\Omega_m} \left(A + \frac{\bar{w}}{w} B \right) \left(\frac{\partial}{\partial z} \Gamma^{II}(z, \zeta) \right) d\Omega, \tag{2.3}$$

$$\tilde{v}(\xi, \eta) = 4 \operatorname{Im} \iint_{\Omega_m} \left(A + \frac{\bar{w}}{w} B \right) \left(\frac{\partial}{\partial z} \Gamma^I(z, \zeta) \right) d\Omega \tag{2.4}$$

where $\zeta = \xi + i\eta \in \Omega_m$ and $\frac{\bar{w}}{w}$ is set zero wherever w vanishes.

The derivative $\frac{\partial}{\partial z}$ is defined by $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$.

Theorem 2.2 (S2). *Let $f \in C^\beta(\overline{\Omega}_m)$ be holomorphic in Ω_m , $0 < \beta < 1$. Then there exists $\tilde{w} \in C^\alpha(\overline{\Omega}_m)$, $\alpha < 1 - 2/q$ given as a solution of the nonlinear integral equations*

$$\begin{aligned} \tilde{v}(\xi, \eta) = 2 \iint_{\Omega_m} \{ & -(\operatorname{Re} A + |B| \cos(p - 2\tilde{v})) \Gamma_y^I(z, \zeta) \\ & + ((\operatorname{Im} A + |B| \sin(p - 2\tilde{v})) \Gamma_x^I(z, \zeta)) \} d\Omega(z), \end{aligned} \tag{2.5}$$

$$\begin{aligned} \tilde{u}(\xi, \eta) = 2 \iint_{\Omega_m} \{ & (\operatorname{Re} A + |B| \cos(p - 2\tilde{v})) \Gamma_x^{II}(z, \zeta) \\ & + ((\operatorname{Im} A + |B| \sin(p - 2\tilde{v})) \Gamma_y^{II}(z, \zeta)) \} d\Omega(z) \end{aligned} \tag{2.6}$$

where

$$e^{ip} = \frac{B}{|B|} \frac{\bar{f}}{f} \quad \text{for } B \neq 0 \quad \text{and} \quad p = 0 \quad \text{if } B = 0; \tag{2.7}$$

such that $w := f(z)e^{\tilde{w}}$ is a solution of (2.1) in Ω_m . Moreover, for this specific

solution we have

$$\tilde{v}|_{\Gamma_0} = 0$$

and

$$\begin{aligned} \tilde{v}|_{\Gamma_j} = & 2 \sum_{k=1}^m c_{kj} \iint_{\Omega_m} \{(\operatorname{Re} A + |B| \cos(p - 2\tilde{v}))e_{ky} \\ & - (\operatorname{Im} A + |B| \sin(p - 2\tilde{v}))e_{kx}\} d\Omega|_{\Gamma_j} \quad \text{where } j = 1, \dots, m. \end{aligned} \tag{2.8}$$

The harmonic functions e_j and the modified Green’s and Neumann functions Γ^I and Γ^{II} , respectively, are defined in the Appendix A1, Section 4.

The proofs of these two theorems will be presented in the Appendix A2, Section 4. Note that in general, the solution $\tilde{w} = \tilde{u} + i\tilde{v}$ might not be unique for $m \geq 1$. However, one of those solutions is uniquely determined as the limit obtained by the continuation method [20] in combination with the Newton–Kantorovič iteration (4.49), Theorem 4.8, at $t = 1$, along the path of problems

$$\begin{aligned} \frac{\partial \tilde{w}^t}{\partial \bar{z}} &= t\{A + |B|e^{i(p-2\tilde{v}^t)}\} \quad \text{in } \Omega_m, \\ \tilde{v}^t|_{\Gamma_0} &= 0, \quad \tilde{v}^t|_{\Gamma_j} = 2t \sum_{k=1}^m c_{kj} \iint_{\Omega_m} \{(\operatorname{Re} A + |B| \cos(p - 2\tilde{v}^t))e_{ky} \\ &\quad - (\operatorname{Im} A + |B| \sin(p - 2\tilde{v}^t))e_{kx}\} d\Omega|_{\Gamma_j}, \quad j = 1, \dots, m, \end{aligned} \tag{2.9}$$

where $t \in [0, 1]$, since for $t = 0$, the solution of (2.9) is unique. Then the mapping $f \mapsto \tilde{w}^1 =: \tilde{w}$ is well defined which then will be used later on.

3. The associated Riemann–Hilbert problems

Now let $w \in C^\alpha(\bar{\Omega}_m)$ be a solution of the nonlinear Riemann–Hilbert problem (1.1)–(1.3) where $0 < \alpha < 1 - 2/q$. Then we find with Theorem S1 the function $\tilde{w} = \tilde{u} + i\tilde{v} \in C^\alpha(\bar{\Omega}_m)$ given by (2.3), (2.4) such that

$$f := we^{-\tilde{w}} \in C^\alpha(\bar{\Omega}_m) \quad \text{is holomorphic in } \Omega_m. \tag{3.1}$$

For \tilde{u} given by (2.3) we find on Γ_j the estimates

$$|\tilde{u}|_{\Gamma_j} \leq 4 \max_{\zeta \in \Gamma_j} \iint_{\Omega_m} (|A| + |B|) \left| \frac{\partial}{\partial z} \Gamma^{II}(z, \zeta) \right| d\Omega(z) =: K_j, \quad j = 0, \dots, m \tag{3.2}$$

and from (2.4) for

$$\tilde{v}|_{\Gamma_j} = c_j := \sum_{k=1}^m c_{kj} 4 \operatorname{Im} \iint \left(A + \frac{\bar{w}}{w} B \right) \left(\frac{d}{dz} e_k(z) \right) d\Omega(z)|_{\Gamma_j}, \tag{3.3}$$

$$|c_j| \leq \sum_{k=1}^m c_{kj} 4 \operatorname{Im} \iint (|A| + |B|) \left| \frac{d}{dz} e_k(z) \right| d\Omega(z)|_{\Gamma_j} =: k_j, \tag{3.4}$$

$j = 1, \dots, m$ and $\tilde{v}|_{\Gamma_0} = 0$. Hence, the holomorphic function f satisfies the nonlinear boundary conditions

$$f(z) \in \tilde{\gamma}_{\omega,z,j} \quad \text{for } z \in \Gamma_j, \quad j = 0, \dots, m \tag{3.5}$$

where $\tilde{\gamma}_{\omega,z,1} = e^{-\tilde{u}(\tau)}|_{\Gamma_0} \gamma_{\omega,z,0}$ and $\tilde{\gamma}_{\omega,z,j} = e^{-\tilde{u}(\tau)}|_{\Gamma_j} e^{-ic_j} \gamma_{\omega,z,j}$, $j = 1, \dots, m$

According to (3.5) we now introduce the following families of closed curves characterizing the boundary conditions for f :

$$\begin{aligned} \tilde{\gamma}_{\omega,z,0} &= \{ \tilde{\gamma}_{\omega,z,0} = \varrho_0(z) \gamma_{\omega,z,0} \mid e^{-K_0} \leq \varrho_0(z) \leq e^{K_0}, \\ &\quad z \in \Gamma_0 \wedge \varrho_0 \in C^\alpha(\Gamma_0) \wedge \|\varrho_0\|_{C^{\alpha_1}} \leq C_1 \}, \\ \tilde{\gamma}_{\omega,z,j} &= \{ \tilde{\gamma}_{\omega,z,j} = \varrho_j(z) e^{-ic_j} \gamma_{\omega,z,j} \mid e^{-K_j} \leq \varrho_j(z) \leq e^{K_j}, \\ &\quad |c_j| \leq k_j, z \in \Gamma_j \wedge \varrho_j \in C^{\alpha_1}(\Gamma_j) \wedge \|\varrho_j\|_{C^{\alpha_1}(\Gamma_j)} \leq C_1 \}. \end{aligned} \tag{3.6}$$

Then the holomorphic function f defined by the solution w given by (3.1) with \tilde{w} will satisfy

$$f(z) \in \tilde{\gamma}_{\omega,z,j} \quad \text{for } z \in \Gamma_j, \quad j = 0, 1, \dots, m \tag{3.7}$$

for the curves $\tilde{\gamma}_{\omega,z,j} \in \tilde{\gamma}_{\omega,z,j}$ in those families, together with

$$\int_{\Gamma_j} d \arg f = 0 \quad \text{for } j = 0, \dots, m. \tag{3.8}$$

Now we are in the position to formulate the basic assumptions on our boundary conditions.

Basic Assumptions: (see also (BCG) in [9])

- (a) The curves of the families $\gamma_{\omega,z,j}$ with $\omega \in \mathbb{R}^{m-1}$ and $z \in \Gamma_j$ for every $j = 0, 1, \dots, m$ are at least twice continuously differentiable, closed, non-selfintersecting in the complex state plane \mathbb{C}_w and will be parametrized as $z = t_j(s_j) \in \Gamma_j$, where $0 \leq s_j < 2\pi$. These curves we denote by $\gamma_{\omega,s_j,j}$ and $\tilde{\gamma}_{\omega,s_j,j}$, respectively.
- (b) Let $W_1 < W_2 < \dots < W_{2m+1}$ with $W_{m+1} = 0$ be given points on the real axis of the state space \mathbb{C}_w and let W_{2m+2} be the point at infinity of its 1–point compactification. Then the family of curves must satisfy the following topological conditions:
 For each $j = 0, \dots, m$ we assume that all curves in $\tilde{\gamma}_{\omega,s_j,j}$ for $z \in \Gamma_j$ and $\omega \in \mathbb{R}^{m-1}$ circumvent the closed interval $[W_{j+1}, W_{j+m+1}]$ in its exterior but do not include any points of $[W_{j+m+2}, W_{j+2m+2} \bmod (2m+2)]$.

Recalling Conditions (BCG) in [9], these assumptions are also given as

$$\begin{aligned}
 W_1 &\in \text{int } \tilde{\gamma}_{\omega, s_0, 0} \cap \text{ext } \tilde{\gamma}_{\omega, s_j, j} && \text{for all } (\omega, s_0) \text{ and } (\omega, s_j), \\
 &&& j = 1, \dots, m; \\
 W_k &\in \text{int } \tilde{\gamma}_{\omega, s_\ell, \ell} \cap \text{ext } \tilde{\gamma}_{\omega, s_j, j} && \text{for all } (\omega, s_\ell), \quad \ell = 0, \dots, k-1; \\
 &&& \text{and } j = k, \dots, m; \quad \text{for } k = 2, \dots, m; \\
 W_{m+1} &\in \text{int } \tilde{\gamma}_{\omega, s_\ell, \ell} && \text{for all } (\omega, s_\ell), \quad \ell = 0, \dots, m; \\
 W_{m+1+k} &\in \text{int } \tilde{\gamma}_{\omega, s_\ell, \ell} \cap \text{ext } \tilde{\gamma}_{\omega, s_j, j} && \text{for all } (\omega, s_\ell), \quad \ell = k, \dots, m \\
 &&& \text{and } (\omega, s_j), \quad j = 0, \dots, k-1; \\
 &&& \text{for } k = 1, \dots, m; \\
 W_{2m+2} &\in \text{ext } \tilde{\gamma}_{\omega, s_j, j} && \text{for all } (\omega, s_j), \quad j = 0, \dots, m.
 \end{aligned}$$

By $\text{int } \gamma$ we denote the bounded component of $\mathbb{C}_w \setminus \gamma$. Correspondingly, $\text{ext } \gamma$ denotes the unbounded component of $\mathbb{C}_w \setminus \gamma$. We further introduce the real intervals $\Delta_\ell := [W_\ell, W_{\ell+1}]$ for $\ell = 1, \dots, 2m+1$ and $\Delta_0 := [-W_{2m+2}, W_1]$, and corresponding segments in \mathbb{C}_w . In addition, we require

$$\Delta_\ell \cap \tilde{\gamma}_{\omega, s_j, j} = \emptyset \quad \text{and} \quad \Delta_{m+1+\ell} \cap \tilde{\gamma}_{\omega, s_j, j} = \emptyset \quad \text{for all } (\omega, s_j) \in \mathbb{R} \times [0, 2\pi),$$

where $\ell \neq j$ and $j, \ell = 0, 1, \dots, m$.

- (c) The curves $\tilde{\gamma}_{\omega, s_j, j}$ lie for every (ω, s_j) in a closed ring-shaped strip $\Pi_{\omega, j} := \bigcup_{s_j} \tilde{\gamma}_{\omega, s_j, j}$ where $j = 0, 1, \dots, m$, respectively, and each of the strips $\Pi_{\omega, j}$, depends continuously on $\omega \in \mathbb{R}^{m-1}$ and is homeomorphic to a circular ring. Moreover, we will assume that for every fixed ω the strips $\Pi_{\omega, j}$ also satisfy the conditions (b). This means that every simple, closed smooth curve in one of the strips satisfies the conditions (b) correspondingly.

To the point $w \in \mathbb{C}_w$ we assign the index $\vec{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_m)$ where $\alpha_j \in \{0, 1\}$ with $\alpha_j = 0$ if the point $w \in \text{ext } \tilde{\gamma}_{\omega, s_j, j}$ and $\alpha_j = 1$ if the point $w \in \text{int } \tilde{\gamma}_{\omega, s_j, j}$ for $j = 0, \dots, m$. Correspondingly, for given j and $\omega \in \mathbb{R}^{m-1}$, we define the indices for the strips $\Pi_{\omega, j}$ given as follows:

$$w \in \text{ext } \Pi_{\omega, j} := \bigcap \{ \text{ext } \tilde{\gamma}_{\omega, s_j, j} \mid 0 \leq s_j \leq 2\pi \}$$

or

$$w \in \text{int } \Pi_{\omega, j} := \bigcap \{ \text{int } \tilde{\gamma}_{\omega, s_j, j} \mid 0 \leq s_j \leq 2\pi \}$$

respectively. We say that a domain \mathfrak{G} of the complex \mathbb{C}_w -plane has the index $\vec{\alpha}$ if all points in \mathfrak{G} have the index $\vec{\alpha}$. In order to characterize the possible geometric situations we now require for the binary vector function $\vec{\alpha}$ in the \mathbb{C}_w -plane the following properties.

- (d) For $\omega = (\omega_0, \dots, \omega_{m-2})$ with $|\omega| < 1$, we assume $\text{rank } \left(\frac{\partial F_j}{\partial \omega_\ell} \right)_{\ell=0, \dots, m-2}^{j=0, \dots, m} = m-1$, and that the domain $\mathbb{C}_w \setminus \left(\bigcup_{j=0}^m \tilde{\gamma}_{\omega, s_j, j} \right)$ in the \mathbb{C}_w -plane consists of $2m+1$ components which have the indices $\{\vec{\alpha}_1, \dots, \vec{\alpha}_{2m+1}\}$.

- (e) For ω with $|\omega| > 2$, we assume that the curves $\tilde{\gamma}_{\omega,s_j,j}$ do not depend on ω anymore and that the domain $\mathbb{C}_w \setminus (\bigcup_{j=0}^m \tilde{\gamma}_{\omega,s_j,j})$ consists of components having the indices $\{\vec{\beta}_1, \dots, \vec{\beta}_{2m+1}\} \neq \{\vec{\alpha}_1, \dots, \vec{\alpha}_{2m+1}\}$.

Note, then for $|\omega| > 2$ no liftable (see Def. 3.1) holomorphic solution can exist due to the open mapping property for holomorphic functions.

For the three-fold connected case $m = 2$, the authors give an explicit example for these conditions in [9].

Definition 3.1. *The solution $(f(z), \omega)$ with $\omega \in \mathbb{R}^{m-1}$ of the holomorphic Riemann–Hilbert problem in Ω_m subject to the nonlinear boundary conditions (3.5) is called “liftable” if*

$$f(\Omega_m) \subset \bigcup \{ \text{int } \tilde{\gamma}_{\omega,s_j,j} \mid \omega \in \mathbb{R}, \quad |\omega| < 1, \quad 0 \leq s_j \leq 2\pi \text{ and } j = 0, \dots, m \},$$

$$f(z) : \Omega_m \rightarrow \mathbb{S}^2 \setminus \{W_1, \dots, W_{2m+2}\} \text{ is homotopic to a constant solution,}$$

where \mathbb{S}^2 denotes the standard Riemannian sphere and W_j are the corresponding points on \mathbb{S}^2 .

The conditions (a)–(e) become clearly visible on the Riemann sphere \mathbb{S}^2 , if the assumptions are considered on \mathbb{S}^2 with the images of the intervals and of $\tilde{\gamma}_{\omega,s_j,j}$ for $z \in \Gamma_j, \omega \in \mathbb{R}^{m-1}, j = 0, \dots, m$.

The main result of our paper is now formulated in the following theorem.

Theorem 3.1. *For $q > 4$ and under the Basic Assumptions (a)–(e), the nonlinear Riemann–Hilbert problem (1.1)–(1.3) for the Vekua system has at least two different solutions (w_+, ω_+) and $(w_-, \omega_-) \in C^\beta(\bar{\Omega}_m) \times \mathbb{R}^{m-1}$ and $0 < \beta < 1 - \frac{4}{q}$.*

4. A priori estimates

As we have seen in Theorem S1, if $w \in C^\alpha(\bar{\Omega}_m)$ is a solution of (1.1)–(1.3) then we find a holomorphic function $f \in C^\alpha(\bar{\Omega}_m), \frac{1}{2} < \alpha < 1 - \frac{2}{q}$ and functions $\varrho_0 \in \Gamma_0, \dots, \varrho_m \in \Gamma_m, \{\varrho_0, \varrho_1, \dots, \varrho_m; c_1, \dots, c_m\} \in \Sigma_\alpha$ where

$$\Sigma_\alpha := \{ \{ \varrho_0, \varrho_1, \dots, \varrho_m; c_1, \dots, c_m \} \in \prod_{j=0}^m C^\alpha(\Gamma_j) \times \mathbb{R}^m \mid e^{-K_j} \leq \varrho_j \leq e^{K_j} \wedge c_0 = 1 \} \quad (4.1)$$

$$\wedge |c_j| \leq k_j, \quad \|\varrho_j\|_{C^\alpha} \leq C_1, \quad j = 0, \dots, m\}.$$

Theorem 4.1. *Let f be the holomorphic solution of the nonlinear Riemann–Hilbert problem*

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{in } \Omega_m \quad (4.2)$$

and

$$f(z) \in \varrho_j(z)e^{-ic_j}\gamma_{\omega,z,j} \quad \text{for } z \in \Gamma_j, \quad j = 0, \dots, m \tag{4.3}$$

where $c_0 = 0$ and $(\varrho, \mathbf{c}) := \{\varrho_0, \dots, \varrho_m; c_1, \dots, c_m\} \in \Sigma_{\alpha_0}$ with $\frac{1}{2} < \alpha_0 < 1$. Then there exists a constant $C_0 = C_0(\alpha_0)$ such that

$$\|f\|_{W_p^{1-1/p}(\Gamma)} \leq C_0 \quad \text{and} \quad \|f\|_{C^\beta(\Gamma)} \leq C_0 \tag{4.4}$$

with

$$2 < p < (1 - \alpha_0)^{-1}, \quad 0 < \beta < 2\alpha_0 - 1,$$

and

$$\|f\|_{W_p^{1-1/p}(\Gamma)} := \|f\|_{L^p(\Gamma)} + \left\{ \int_{\Gamma} \int_{\Gamma} \frac{|f(t) - f(s)|^p}{|t - s|^p} dt ds \right\}^{\frac{1}{p}}$$

The proof will be given in Appendix A3.

Now, for every choice of $\{\varrho_0, \dots, \varrho_m; c_1, \dots, c_m\} \in \Sigma_{\alpha_0}$ we have to solve a nonlinear Riemann–Hilbert problem (4.2), (4.3). These problems have been investigated in [9] where we have shown that there exist at least two different solutions (f_+, ω_+) and (f_-, ω_-) by lifting the problem to the universal covering on the unit disk based on the conformal mapping of two Riemann sheets of the slit complex \mathbb{C}_w plane with the real slits

$$[W_1, W_2], [W_3, W_4], \dots, [W_{2m+1}, W_{2m+2}]$$

to circular $(m + 1)$ -polygons and their reflections on the unit disc.

For every chosen $(\varrho, \mathbf{c}) = \{\varrho_0, \dots, \varrho_m; c_1, \dots, c_m\} \in \Sigma_{\alpha}$ we obtain at least two different solutions which define two set valued mappings

$$\Sigma_{\alpha_0} \ni (\varrho, \mathbf{c}) \mapsto f_+ =: \mathfrak{A}_+(\varrho, \mathbf{c}) : \Sigma_{\alpha_0} \mapsto C^\beta(\overline{\Omega}_m), \tag{4.5}$$

$$\Sigma_{\alpha_0} \ni (\varrho, \mathbf{c}) \mapsto f_- =: \mathfrak{A}_-(\varrho, \mathbf{c}) : \Sigma_{\alpha_0} \mapsto C^\beta(\overline{\Omega}_m). \tag{4.6}$$

With the holomorphic function f_+ or f_- , respectively, we now apply Theorem 2.2 and obtain to f_+ generalized analytic functions $w_+(f_+)$, functions \tilde{w}_+ from (2.5), (2.6) and $(\varrho_+, \mathbf{c}_+)$ from (3.5), (3.6). Thus we can define the set valued mapping

$$\begin{aligned} \mathfrak{B}_+ : (\varrho, \mathbf{c}) &\mapsto f_+ \mapsto \tilde{w}_+(\varrho_+, \mathbf{c}_+), \quad \text{i.e.}, \\ \mathfrak{B}_+ : \Sigma_{\alpha_0} &\rightarrow \Sigma_{\alpha_2} \Subset \Sigma_{\alpha_0} \end{aligned} \tag{4.7}$$

where $\alpha_2 > \alpha_0$.

Correspondingly, we also define

$$\begin{aligned} \mathfrak{B}_- : (\varrho, \mathbf{c}) &\mapsto f_- \mapsto \tilde{w}_- \mapsto (\varrho_-, \mathbf{c}_-), \quad \text{i.e.}, \\ \mathfrak{B}_- : \Sigma_{\alpha_0} &\rightarrow \Sigma_{\alpha_2} \Subset \Sigma_{\alpha_0}. \end{aligned} \tag{4.8}$$

Theorem 4.2. *Each of the mappings \mathfrak{B}_+ and \mathfrak{B}_- possesses at least one fixed point $(\varrho_+^0, \mathbf{c}_+^0) \in \mathfrak{B}_+(\varrho_+^0, \mathbf{c}_+^0) \in \Sigma_{\alpha_0}$ and at least one fixed point $(\varrho_-, \mathbf{c}_-) \in \mathfrak{B}_-(\varrho_-^0, \mathbf{c}_-^0) \in \Sigma_{\alpha_0}$ where $\frac{1}{2} < \alpha_1 < \alpha_0 < \alpha_2 < 1 - 2/q$.*

For the proof we show the following Lemma.

Lemma 4.3. *For any given $(\varrho, \mathbf{c}) \in \Sigma_{\alpha_0}$ there exist two holomorphic functions f_+ with $\omega_+ \in \mathbb{R}^{m-1}$ and f_- with $\omega_- \in \mathbb{R}^{m-1}$, which are solutions of the nonlinear Riemann–Hilbert problem*

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{in } \Omega_m \tag{4.9}$$

and

$$f(z) \in \tilde{\gamma}_{\omega, z, j} \quad \text{for } z \in \Gamma_j, \quad j = 0, 1, \dots, m. \tag{4.10}$$

Proof of Lemma 4.3. Let us now consider a whole family of nonlinear Riemann–Hilbert problems (4.9), (4.10) where we associate with any chosen $(\varrho, \mathbf{c}) \in \Sigma_{\alpha_0}$ the family

$$\begin{aligned} \varrho_j(z; \tau) &:= 1 + (\varrho_j(z) - 1)\tau, & c_j(\tau) &:= \tau c_j \\ &\text{for } 0 \leq \tau \leq 1, \quad z \in \Gamma_j, \quad j = 0, \dots, m; \end{aligned} \tag{4.11}$$

which defines a corresponding family of boundary conditions

$$\begin{aligned} f(z) \in \tilde{\gamma}_{\omega, z, j, \tau} &= \{\varrho_j(z; \tau)e^{-ic_j\tau}\gamma_{\omega, z, j}\} \\ &\text{for } 0 \leq \tau \leq 1, \quad z \in \Gamma_j, \quad j = 0, \dots, m \end{aligned} \tag{4.12}$$

that depends continuously on τ . For each $\tau \in [0, 1]$, (4.11), (4.12) defines a nonlinear Riemann–Hilbert problem for holomorphic functions.

However, except for $\tau = 0$ (when $\varrho_j(z; 0) = 1$ for $j = 0, \dots, m$), we cannot apply the arguments as in our paper [9] based on the degree theory for quasiruled Fredholm maps since there we need smooth dependence of the curves

$$\tilde{\gamma}_{\omega, t_j(s), j} = \varrho_j(t_j(s); \tau)e^{-ic_j(\tau)}\gamma_{\omega, t_j(s), j} \tag{4.13}$$

on the parameter $s \in \mathbb{R}$, whereas for $0 < \tau \leq 1$, we only have Hölder continuity of $\varrho_j(t_j(s); \tau)$ available.

For every ω_j and s fixed, we parameterize the smooth, simply closed curve $\gamma_{\omega, t_j(s), j}$ in the complex state plane by the parameter $\sigma \in I$ as the arc length on this regular curve divided by the total length of $\gamma_{\omega, t_j(s), j}$. Varying now s , the curve varies and varying both parameters, s and σ we obtain a regular, doubly periodic C^2 –manifold, the **restriction manifold** (see [17, Sec. 1.8]),

$$\mathcal{M}_j(\tau, \omega) : X = U_j(\tau, \omega; s, \sigma), \quad Y = V_j(\tau, \omega; s, \sigma) \quad \text{for } (s, \sigma) \in I \times I \tag{4.14}$$

for each $j = 0, \dots, m$. Since for fixed s, τ, j, ω , (4.14) describes a simple, closed plane curve $\gamma_{\omega, t_j(s), j}$ and U_j, V_j are 2π –periodic in s , the manifolds $\mathcal{M}_j(\tau, \omega)$ can

be embedded into the ambient space \mathbb{R}^3 forming $m+1$ tori for every fixed $\tau \in [0, 1]$ and $\omega \in \mathbb{R}^{m-1}$; e.g. by the parametric representations

$$\begin{aligned} \mathcal{M}_j(\tau, \omega) : x_1 &= (R + U_j(\tau, \omega; s, \sigma)) \cos s, & x_2 &= (R + U_j(\tau, \omega; s, \sigma)) \sin s, \\ x_3 &= V_j(\tau, \omega; s, \sigma), \end{aligned}$$

where $R > 0$ is chosen large enough to avoid selfintersecting.

Hence, if $f(z)$ is a holomorphic solution in Ω_m satisfying (4.10), then on each of the restriction manifolds we find the point $(s, \sigma_j(s))$ such that

$$f(t_j(s)) = U_j(\tau, \omega; s, \sigma_j(s)) + iV_j(\tau, \omega; s, \sigma_j(s)) \tag{4.15}$$

for $s \in I, \quad j = 0, \dots, m.$

Moreover, if $f(t_j(\bullet)) \in C^\beta(I)$ and, for ω and τ fixed, $\varrho_j(t_j(\bullet); \tau) \in C^\beta(I)$ then $\sigma_j \in C^\beta(I)$ as well (see Lemma 1.8.2 in [17]). Now we associate with (4.15), (4.9) and (4.10) a nonlinear operator equation for the functions $\sigma_0, \dots, \sigma_m$, based on an appropriate representation of $f(z) = u + iv$.

In order to express v on the boundary component Γ_j , we employ the modified Neumann function Γ^{II} and Green’s theorem to find the representation

$$v(\xi, \eta) = \int_{\Gamma} u \, d\Gamma^{II} + \int_{\Gamma} v\theta ds \quad \text{for } \zeta = \xi + i\eta \in \Omega_m$$

which holds up to the boundary Γ . In particular, for $\zeta \in \Gamma_j$ we have

$$\int_{\Gamma} u \, d\Gamma^{II}(\zeta) = -\mathcal{H}_j u|_{\Gamma_j} + \int_{\Gamma} u K_j(\zeta, z(s)) ds,$$

where

$$\mathcal{H}_j u_j = \frac{1}{\pi} \int_{\Gamma_j} u_j(s) \frac{d}{ds} \log |\zeta - z(s)| ds$$

is Hilbert’s singular integral operator, and where the integral is defined as Cauchy’s principal value. The functions K_j are in $C^2(\Gamma \times \Gamma)$ (at least). Since each of the closed curves Γ_j is 1–periodically parameterized, and on Γ_j the traces of functions are 1–periodic, we replace \mathcal{H}_j by the Hilbert transform

$$\mathcal{H}u(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} u(s) \cot(\pi(s-t)) ds, \tag{4.16}$$

and the difference of operators $\mathcal{H}_j - \mathcal{H}_0$ then has a smooth integral operator kernel (at least in $C^2(\Gamma \times \Gamma)$ as well). Hence, with $u_j = u|_{\Gamma_j}$,

$$v|_{\Gamma_j} = \mathcal{H}u_j + T u|_{\Gamma_j} + \int_{\Gamma_0} v\theta ds, \tag{4.17}$$

where $T : L_p(\Gamma) \rightarrow C^2(\Gamma)$ is a continuous linear mapping. Hence, solving the nonlinear Riemann–Hilbert problem, (4.9) and (4.10), equation (4.15) with holomorphic f is equivalent to finding the functions $\sigma_j \in C^\beta(I)$ and $\omega \in \mathbb{R}^{m-1}$ as the solutions of the nonlinear equations

$$\begin{aligned}
 &V_j(\tau, \omega; s, \sigma_j(s)) - \int_{-\frac{1}{2}}^{\frac{1}{2}} \cot(\pi(s' - s)) U_j(\tau, \omega; s', \sigma_j(s')) ds' - TU|_{\Gamma_j} \\
 &\quad - \int_{\Gamma_0} V_0(\tau, \omega; s', \sigma_0(s)) \theta(s') ds' = 0 \quad \text{for } s \in I, \quad j = 0, \dots, m,
 \end{aligned} \tag{4.18}$$

together with

$$\int_{\Gamma_j} V_j(\tau, \omega; s'; \sigma_j(s')) \frac{\partial e_j}{\partial n} ds' = 0 \quad \text{for } j = 1, \dots, m, \tag{4.19}$$

where $U = (U_0, \dots, U_m)$.

For $\tau = 0$, these equations are equivalent to finding the holomorphic solution of (4.9) and (4.10) for the case $\varrho_j \equiv 1$ and $c_j \equiv 0$. This case was considered in [9] where we obtained at least two topologically different holomorphic solutions $f_+^0(z)$ and $f_-^0(z)$ with $f_+^0, f_-^0 \in C^\beta(\bar{\Omega}_m)$.

Let us return to the equation (4.18) written in short as

$$A_\tau(\omega; \boldsymbol{\sigma}) = 0 \tag{4.20}$$

for $\boldsymbol{\sigma} = (\sigma_0, \dots, \sigma_m) \in C^\beta(I)$ and $\omega \in \mathbb{R}^{m-1}$. If we take the linearization with the Frechét derivative about a given solution $(\boldsymbol{\sigma}, \omega)$, then it becomes a system of linear singular integral operators together with a linear mapping in \mathbb{R}^{m-1} ,

$$\begin{aligned}
 &(A'_\tau(\omega, \boldsymbol{\sigma})(\tilde{\omega}, \tilde{\boldsymbol{\sigma}}))|_{\Gamma_j} \\
 &= \begin{cases} \left(\left(\frac{\partial}{\partial \sigma_j} V_j \right) (\tau, \omega; s, \sigma_j(s)) \right) \tilde{\sigma}_j(s) - \left(\mathcal{H}|_{\Gamma_j} \left(\frac{\partial}{\partial \sigma_j} U_j \right) \tilde{\sigma}_j \right) (s) \\ - \sum_{k=0}^m \left(T \frac{\partial U}{\partial \sigma} \right) \tilde{\sigma}_k - \delta_{0k} \int_{\Gamma_0} \frac{\partial V_0}{\partial \sigma_0} \tilde{\sigma}_0(s') \sigma(s') ds'; \\ \sum_{k=1}^m \int_{\Gamma_j} \frac{\partial V_j}{\partial n} ds \tilde{\omega}_k \end{cases} \tag{4.21}
 \end{aligned}$$

of Fredholm index 0 because of (3.8), operating on $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\omega}$. Note that the coefficients are in $C^\beta(I)$, and that the last two terms define linear compact operators. Moreover, the right inverse $B_\tau(\omega, \boldsymbol{\sigma})$ to $A'_\tau(\omega, \boldsymbol{\sigma})$ exists for every $\tau \in [0, 1]$ and $\omega \in \mathbb{R}^{m-1}$ as will be shown in Appendix A4. The dependence on τ is uniformly continuous and defines a homotopic path of operators A_τ connecting the case $\tau = 0$ with $\tau = 1$ on which we can use the a priori estimate (4.4) that guarantees the

uniform boundedness of $B_\tau(\omega, \sigma)$ as well. Hence, we can employ the Newton–Kantorovič method in [8, Appendix], (Appendix A4, Theorems 4.7 and 4.8) which implies that the set $\mathcal{T} := \{\tau \in [0, 1] \text{ where (4.20) has a solution } \sigma \in C^\beta(\Gamma), \omega \in \mathbb{R}^{m-1}\}$ is open, and, because of the continuity of A_τ , is closed in $[0, 1]$ as well. Hence, $\mathcal{T} = [0, 1]$. Since for $\tau = 0$, we have a solution $f_-^0(z)$, we find also a solution $f_-(z)$ for $\tau = 1$, that is of (4.2) and (4.3) which is uniquely associated with $f_-^0(z)$. If we start with $f_+^0(z)$, we find a second solution $f_+(z)$ (see [9, Theorem 7.2]).

In view of Theorem 4.1, we then have $f_+, f_- \in C^\beta(\overline{\Omega}_m)$ with $0 < \beta < 2\alpha_0 - 1$. ■

Note that Lemma 4.3 implies the existence of the two mappings \mathfrak{A}_+ and \mathfrak{A}_- in (4.6), respectively (4.5).

Proof of Theorem 4.2. The mapping

$$\begin{aligned} \mathfrak{B}_+ : (\varrho, \mathbf{c}) &\mapsto f_+ := \mathfrak{A}_+(\varrho, \mathbf{c}) \xrightarrow{Th.2,2} \tilde{w} \xrightarrow{Th.2,1} (\tilde{\varrho}, \tilde{\mathbf{c}}), \\ \mathfrak{B}_+ : \Sigma_{\alpha_0} &\rightarrow C^\beta(\overline{\Omega}_m) \xrightarrow{Th.2,2} C^\beta(\overline{\Omega}_m) \rightarrow C^\beta(\overline{\Omega}_m) \xrightarrow{Th.2,1} \Sigma_{\alpha_2} \hookrightarrow \Sigma_{\alpha_0} \end{aligned} \tag{4.22}$$

with $\alpha_2 > \alpha_0$ is compact because of $\Sigma_{\alpha_2} \Subset \Sigma_{\alpha_0}$; and Σ_{α_0} is closed and convex in $\prod_{j=0}^m C^{\alpha_0}(\Gamma_j) \times \mathbb{R}^{m-1}$. Hence, we apply Schauder’s fixed point principle to find a fixed point

$$(\varrho_+^0, \mathbf{c}_+^0) \in \mathfrak{B}_+(\varrho_+^0, \mathbf{c}_+^0)$$

of \mathfrak{B}_+ .

In the same manner we find a fixed point to \mathfrak{B}_- corresponding to \mathfrak{A}_- as proposed. ■

Appendix A1. The modified Green and Neumann functions

Based on the results in [15, Section IV] by Bogdan Bojarski and [18], we now define the modified Green’s function

$$G^I(z, \zeta) = \frac{1}{2\pi} \ln \frac{1}{|z - \zeta|} + g^I(z, \zeta)$$

of the Laplacian for Ω_m with homogeneous Dirichlet conditions, i.e.

$$G^I(z, \zeta)|_{\zeta \in \partial\Omega_m} = 0 \quad \text{and} \quad z \in \Omega_m.$$

Let $e_k(z)$ be the harmonic basis defined by the following properties:

$$\begin{aligned} \Delta e_k &= 0 \quad \text{in } \Omega_m, \quad k = 1, 2, \dots, m \\ \text{with } e_k|_{\Gamma_j} &= \delta_{k,j}, \quad j = 0, 1, 2, \dots, m. \end{aligned}$$

Then $\oint_{\Gamma_k} \frac{\partial}{\partial n} e_k ds \neq 0$ and

$$a_{jk} := \int_{\partial\Omega_n} e_j \frac{\partial}{\partial n} e_k ds \quad \text{for } j, k \in \{1, 2, \dots, m\}$$

defines a positive definite symmetric capacitance matrix. Its inverse c_{jk} with

$$\sum_{k=1}^m c_{jk} a_{k\ell} = \delta_{j\ell}, \quad j, \ell = 1, 2, \dots, m$$

is also positive definite.

Next, we define the **modified Green's function** by

$$\Gamma^I(z, \zeta) := G^I(z, \zeta) - \sum_{j,k=1}^m c_{jk} e_j(z) e_k(\zeta) \tag{4.23}$$

which now satisfies the boundary conditions

$$\begin{aligned} \Gamma^I(z, \zeta)|_{\zeta \in \Gamma_\ell} &= - \sum_{j=1}^m c_{j\ell} e_j(z) \quad \text{for } \ell = 1, \dots, m, \\ \Gamma^I(z, \zeta)|_{\zeta \in \Gamma_0} &= 0. \end{aligned} \tag{4.24}$$

For any given $F \in C^\alpha(\bar{\Omega}_m)$ and $\varphi \in C^0(\partial\Omega_m)$, the function

$$u := - \int_{\partial\Omega_m} \varphi \frac{\partial}{\partial n} \Gamma^I ds - \iint_{\Omega_m} F \Gamma^I d\Omega$$

then satisfies

$$\begin{aligned} \Delta u &= F \quad \text{in } \Omega_N, \\ u|_{\Gamma_\ell} &= \varphi - \sum_{j,k=1}^m c_{jk} \left\{ \int_{\partial\Omega_m} \varphi \frac{\partial e_j}{\partial n} ds + \iint_{\Omega_m} F e_j d\Omega \right\} \delta_{k\ell}, \quad \ell = 0, \dots, m, \end{aligned}$$

and

$$\int_{\Gamma_\ell} \frac{\partial u}{\partial n} ds = 0.$$

Hence, u admits a harmonic conjugate.

Let $\Gamma^{II}(z, \zeta)$ denote the **modified harmonic Neumann's function**, given by

$$\Gamma^{II}(z, \zeta) = \frac{1}{2\pi} \ln \frac{1}{|z - \zeta|} + g^{II}(z, \zeta) \tag{4.25}$$

satisfying

$$\frac{\partial \Gamma^{II}}{\partial n} \Big|_{\partial\Omega_m} = -\theta(s) \quad \text{with} \quad \int_{\partial\Omega_m} \theta \Gamma^{II} ds = 0 \tag{4.26}$$

and

$$\theta|_{\Gamma_0} > 0, \quad \theta|_{\Gamma_j} = 0, \quad j = 1, \dots, m, \quad \int_{\partial\Omega_m} \theta ds = \int_{\Gamma_0} \theta ds = 1,$$

which is uniquely determined.

In relation with the Cauchy–Riemann operator one has the following well known lemma, which we quote without proof (see [11, Section 9.2]).

Lemma 4.4. *Let $c \in C^\alpha(\overline{\Omega}_m)$ and $\psi \in C^\alpha(\Gamma)$ with $0 < \alpha < 1$.*

(i) *Then $w(\zeta) = u(\zeta) + iv(\zeta)$ given by*

$$\begin{aligned} u(\zeta) &= - \oint_{\Gamma} \psi \frac{\partial \Gamma^I}{\partial n} ds + 4 \operatorname{Re} \iint_{\Omega_m} c \frac{\partial \Gamma^I}{\partial z} d\Omega, \\ v(\zeta) &= \oint_{\Gamma} \psi d\Gamma^II + \kappa + 4 \operatorname{Im} \iint_{\Omega_m} c \frac{\partial \Gamma^II}{\partial z} d\Omega \end{aligned} \tag{4.27}$$

belongs to $C^\alpha(\overline{\Omega}_m)$ for every $\kappa \in \mathbb{R}$ and satisfies

$$\frac{\partial w}{\partial \bar{z}} \quad \text{in } \Omega, \tag{4.28}$$

$$\begin{aligned} u|_{\Gamma_0} &= \psi|_{\Gamma_0} \\ u|_{\Gamma_\ell} &= \psi|_{\Gamma_\ell} + \sum_{k=1}^m c_{k\ell} \left\{ \oint_{\Gamma_k} \psi \frac{\partial e_k}{\partial n} ds - 4 \operatorname{Re} \iint_{\Omega} c \frac{\partial e_k}{\partial z} d\Omega \right\} \end{aligned} \tag{4.29}$$

for $\ell = 1, \dots, m$.

(ii) *If*

$$\oint_{\Gamma_k} \psi \frac{\partial e_k}{\partial n} ds - 4 \operatorname{Re} \iint_{\Omega} c \frac{\partial e_k}{\partial z} d\Omega = 0 \quad \text{for } k = 1, \dots, m,$$

then, for every $\kappa \in \mathbb{R}$, the function w in (4.27) is the solution of (4.28) in Ω and $\operatorname{Re} w|_{\Gamma} = \psi$ on Γ .

Appendix A2. The similarity principle

The Theorem 2.1 (S1) of the similarity principle follows with the choice $\psi = 0$ and $\kappa = 0$ from the following theorem in [18], which goes back to Bers and Nirenberg [2, 3].

Theorem 4.5. *Let $w \in C^\beta(\overline{\Omega}_m)$ with $0 < \beta < 1$ be a solution of (2.1) in Ω_m with coefficients $A, B \in L_q(\Omega_m)$ where $q > 4$. Then for every function $\tilde{w} = \tilde{u} + i\tilde{v}$ given for $\zeta = \xi + i\eta \in \Omega_m$ by*

$$\tilde{u}(\xi, \eta) = - \oint_{\Gamma} \psi d\Gamma^II + \kappa_0 + 4 \operatorname{Re} \iint_{\Omega_m} g(x, y) \frac{\partial \Gamma^II}{\partial z}(x, y; \xi, \eta) d\Omega, \tag{4.30}$$

$$\tilde{v}(\xi, \eta) = - \oint_{\Gamma} \psi \frac{\partial}{\partial n} \Gamma^I ds + 4 \operatorname{Im} \iint_{\Omega_m} g(x, y) \frac{\partial \Gamma^I}{\partial z}(x, y; \xi, \eta) d\Omega \tag{4.31}$$

with

$$g := \begin{cases} A + B\frac{\bar{w}}{w} & \text{where } w \neq 0, \\ 0 & \text{where } w = 0, \end{cases} \tag{4.32}$$

and any chosen $\psi \in C^\alpha(\Gamma)$ with $0 < \alpha \leq 1 - 2/q$ and $\kappa \in \mathbb{R}$, the function

$$f(z) := w(z)e^{-\tilde{w}(z)} \tag{4.33}$$

will be holomorphic in Ω_m and $f \in C^{\beta'}(\overline{\Omega}_m)$, where $\beta' = \min\{\beta, \alpha\}$. Moreover, $\tilde{w} \in C^\alpha(\overline{\Omega}_m)$, and on Γ we have

$$\begin{aligned} \tilde{v}|_{\Gamma_\ell} &= \psi|_{\Gamma_\ell} + \sum_{k=1}^m c_{k\ell} \left\{ \oint_{\Gamma_k} \psi \frac{\partial e_k}{\partial n} ds - 4 \operatorname{Im} \iint_{\Omega_m} g \frac{\partial e_k}{\partial z} d\Omega \right\} \quad \text{for } \ell = 1, \dots, m; \\ \tilde{v}|_{\Gamma_0} &= \psi|_{\Gamma_0}. \end{aligned} \tag{4.34}$$

If, in addition, ψ is absolutely continuous, one has

$$\frac{\partial \tilde{u}}{\partial n} \Big|_{\Gamma} ds = d\psi|_{\Gamma} \quad \text{and} \quad \kappa_0 = \oint_{\Gamma_0} \tilde{u} \theta ds.$$

Proof. Set $W := i\tilde{w}$. Then (4.30) and (4.31) is equivalent to the relation (4.27) for W ,

$$\begin{aligned} W(\zeta) &= - \oint_{\Gamma} \psi \left(\frac{\partial \Gamma^I}{\partial n} - i \frac{\partial \Gamma^{II}}{\partial s} \right) ds - i\kappa \\ &\quad + 2 \iint_{\Omega_m} \left\{ (ig)(\Gamma_z^I + \Gamma_z^{II}) + \overline{(ig)}(\Gamma_{\bar{z}}^I - \Gamma_{\bar{z}}^{II}) \right\} d\Omega, \end{aligned}$$

which is the equation (1.1.23) in [21] with $a = b = 0$ and $c = ig$. Hence,

$$\frac{\partial W}{\partial \bar{z}} = ig,$$

and for f defined in (4.33), we get with (4.32)

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial w}{\partial \bar{z}} e^{-\tilde{w}} - w e^{-\tilde{w}} \frac{\partial \tilde{w}}{\partial \bar{z}} = e^{-\tilde{w}} \{Aw + B\bar{w} - wg\} = 0 \quad \text{in } \Omega,$$

so, f is holomorphic. As for the Hölder continuity of \tilde{w} , since for $w \in C^\beta(\overline{\Omega}_m)$, the function g is measurable and in L_q for $q > 4$, we find from [15] that $\tilde{w} \in C^\alpha(\overline{\Omega}_m)$ with $0 < \alpha \leq 1 - \frac{2}{q}$. The boundary conditions (4.34) are a consequence of the generalized Green functions' properties. ■

The proof of the existence of \tilde{w} in Theorem 2.2 is the same as in [11, Chap.9] for a simply connected domain Ω except for its uniqueness. To this end, we consider

the mapping

$$\begin{aligned}
 T(v) := & 2 \iint_{\Omega_m} \left\{ -(\operatorname{Re} A + |B| \cos(p - 2v)) \Gamma_y^I(z, \zeta) \right. \\
 & \left. + (\operatorname{Im} A + |B| \sin(p - 2v)) \Gamma_x^I(z, \zeta) \right\} d\Omega
 \end{aligned} \tag{4.35}$$

for continuous functions $v \in C^0(\overline{\Omega}_m)$.

Lemma 4.6. *The mapping $T(v)$ has a fixed point $\tilde{v} \in C^\alpha(\overline{\Omega}_m)$, $0 < \alpha < 1 - \frac{2}{q} < 1$: i.e.,*

$$\tilde{v} = T(\tilde{v}). \tag{4.36}$$

Proof. Note that for all $v \in C^0(\overline{\Omega}_m)$ the functions $((\operatorname{Re} A + |B| \cos(p - 2v))$ and $(\operatorname{Im} A + |B| \sin(p - 2v))$ belong to $L_q(\Omega_m)$ and, hence,

$$\|T(v)\|_{C^\alpha(\overline{\Omega}_m)} \leq c_\alpha$$

with a constant c_α not depending on v , and $0 < \alpha < 1 - \frac{2}{q}$ (see [15]). Hence, the convex and compact subset

$$\mathcal{B}_\alpha := \{v \in C^\alpha(\overline{\Omega}_m) \mid \|v\|_{C^\alpha(\overline{\Omega}_m)} \leq c_\alpha\}$$

of the Banach space $C^0(\overline{\Omega}_m)$ equipped with the maximum norm is mapped continuously by T into \mathcal{B}_α . Hence, due to Michael’s selection theorem for multivalued mappings, we can use Schauder’s fixed point theorem for multivalued maps (see [22, Theorem 9.G and Corollary 2.13]), which provides us with a solution $\tilde{v} \in C^\alpha(\overline{\Omega}_m)$, i.e. a solution of Equation (2.5). ■

In contrast to the case of a simply connected domain, where \tilde{v} is uniquely determined, here we use Schauder’s fixed point theorem for multivalued mappings. But then we specify among those fixed points the one obtained by applying (2.9) for $t \in [0, 1]$, the continuation method in combination with the Newton–Kantorovič iteration (4.49) and get a well defined mapping $f \mapsto \tilde{w}$.

With $\tilde{v} \in C^\alpha(\overline{\Omega}_m)$ we now compute $\tilde{u} \in C^\alpha(\overline{\Omega}_m)$ from (2.6). Then Lemma 4.4 with $W = i\tilde{w}$ implies that

$$\frac{\partial \tilde{w}}{\partial \bar{z}} = A + |B|e^{i(p-2\tilde{v})},$$

and for $w := f(z)e^{\tilde{w}}$ we find

$$\frac{\partial w}{\partial \bar{z}} = f(z)e^{\tilde{w}} \frac{\partial \tilde{w}}{\partial \bar{z}} = f(z)e^{\tilde{w}} \{A + |B|e^{ip-2i\tilde{v}}\} = Aw + B\bar{w},$$

i.e., w satisfies (1.1) in Ω_m .

The boundary relations (2.8) follow from (4.29) for $W = i\tilde{w}$ and (2.5). This completes the proof of Theorem 2.2.

Appendix A3: Proof of Theorem 4.1

Following [8], by using conformal mapping, we can assume that for each $t_0 \in [0, 1]$ and $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for $|t - t_0| < \delta(\varepsilon)$ the boundary condition $F_j(\omega, z, u, v) = 0$ for $z \in \Gamma_j$ for $j = 0, \dots, m$ has the form

$$u = g_j(t, \omega, v),$$

where the function $g_j(t, \omega, v)$ is Hölder continuous with respect to t , i.e. in C^α , and differentiable with respect to v , satisfying

$$\left| \frac{\partial g_j}{\partial v} \right| = C_{g_j} \varepsilon \quad \text{and} \quad \frac{|g_j(t_1, \omega, v) - g_j(t_2, \omega, v)|}{|t_1 - t_2|^\alpha} \leq C_0 \tag{4.37}$$

for every fixed ω . Now, the remaining part of the proof of the Zygmund estimates follows closely the presentation in [12]. Due to the maximum principle, for any holomorphic $f(z) = u(z) + iv(z)$ satisfying (4.3) we have

$$\|f\|_{C^0(\Gamma)} \leq c_p \tag{4.38}$$

with the constant c_p given by the maximum of $|\varrho_j e^{-ic_j} \gamma_{\omega, z, j}|$ for $z \in \Gamma_j$, $j = 0, \dots, m$, and $(\varrho, c) \in \Sigma_\alpha$.

To prove (4.4), we use the localization principle mentioned above, as well as the representation (4.17) of holomorphic functions on multiply-connected domains Ω_m . Let φ and ψ in $C^\infty(\Gamma_j)$ be supported on Γ_j . Then for fixed ω and all $t_j^{(1)} = t_j(t_1)$ and $t_j^{(2)} = t_j(t_2)$ with $t_j^{(1)} \neq t_j^{(2)}$ we have

$$\begin{aligned} \varphi(t_1)\psi(t_2) \frac{u_j(t_1) - u_j(t_2)}{t_1 - t_2} &= \varphi(t_1)\psi(t_2) \frac{g_j(t_1, v(t_1)) - g_j(t_2, v(t_1))}{t_1 - t_2} \\ &+ \varphi(t_1)\psi(t_2) \frac{g_j(t_2, v(t_1)) - g_j(t_2, v(t_2))}{v(t_1) - v(t_2)} \cdot \frac{v(t_1) - v(t_2)}{t_1 - t_2}. \end{aligned} \tag{4.39}$$

In view of (4.39), we define the following operators (according to [12]),

$$\begin{aligned} PU(t_1, t_2) &:= \varphi(t_1)\psi(t_2)U(t_1, t_2), \\ Qu(t_1, t_2) &:= \frac{u(t_1) - u(t_2)}{t_1 - t_2}, \\ (B(v))(t_1, t_2) &:= \frac{g_j(t_2, v(t_1)) - g_j(t_2, v(t_2))}{t_1 - t_2}, \\ (A(v))(t_1, t_2) &:= \frac{g_j(t_1, v(t_1)) - g_j(t_2, v(t_1))}{t_1 - t_2}. \end{aligned}$$

Then (4.39) can be written as

$$PQu = PA(v) - PB(v) \cdot \frac{v(t_1) - v(t_2)}{t_1 - t_2}.$$

Since P is defined by multiplication with C^∞ -functions, P is a bounded linear operator $P : L^p(I \times I) \rightarrow L^p(I \times I)$ where $I = [0, 1]$ is the periodicity interval. By the definition of the $W_p^{1-1/p}(I)$ -norm on I , the operator Q acts continuously from $W_p^{1-1/p}(I)$ into $L^p(I \times I)$. Further, we have from (4.37) the estimates

$$|B(v)(t_1, t_2)| \leq \varepsilon \quad \text{and} \quad |A(v)(t_1, t_2)| \leq c_0|t_1 - t_2|^{\alpha_0-1}. \tag{4.40}$$

So, we obtain for $2 < p < \frac{1}{1-\alpha_0}$ together with (4.38)

$$\|PA(v)\|_{L^p(I \times I)} \leq c_p \|A(v)\|_{L^p(I \times I)} \leq c_p c_1;$$

since then the space $W_p^{1-1/p}(I) \hookrightarrow C^\beta(I)$ is continuously imbedded into the Hölder space $C^\beta(I)$ for $\beta < 1 - 2/q$ (see [14, Remark 2, p.71]); finally,

$$\|PQu\|_{L^p(I \times I)} \leq c_p c_1 + \varepsilon c_{g_j} \left| \frac{v(t_1) - v(t_2)}{t_1 - t_2} \right|_{L^p(I \times I)}. \tag{4.41}$$

For the last term in (4.41), we now employ the relation (4.17) and find on Γ_j :

$$\frac{v(t_1) - v(t_2)}{t_1 - t_2} \Big|_I = \frac{1}{t_1 - t_2} (\mathcal{H}u_j(t_1) - \mathcal{H}u_j(t_2)) + \frac{1}{t_1 - t_2} (Tu|_I(t_1) - Tu|_I(t_2)).$$

For the first term on the right-hand side with the Hilbert transform we now exploit 1-periodicity and follow [12], by writing just u for $u|_{\Gamma_j}$:

$$\frac{1}{t_1 - t_2} (\mathcal{H}u(t_1) - \mathcal{H}u(t_2)) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u(s + t_1) - u(s + t_2)}{(s + t_1) - (s + t_2)} \cot(\pi s) ds =: GU(t_1, t_2),$$

where $GU := Q\mathcal{H}u$. With $W(s, t) := U(t, s + t)$ we have

$$GU(t, r + t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} W(r, s + t) \cot(\pi s) ds,$$

and L^p continuity of \mathcal{H} implies

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |GU(t, r + t)|^p dt \leq c \int_{-\frac{1}{2}}^{\frac{1}{2}} |W(r, t)|^p dt = \int_0^1 |U(t, r + t)|^p dt.$$

Hence,

$$\|GU\|_{L^p(I \times I)}^p \leq \|U\|_{L^p(I \times I)}^p.$$

Finally, we localize again,

$$U = PU + (I - P)U = PQu + (I - P)Qu,$$

and $(I - P)Q$ is continuous from $L_1(I)$ into $L_\infty(I \times I)$ (see [12]) which yields

$$\begin{aligned} \|QV\|_{L_p(I \times I)}^p &\leq c_1 \|PQu\|_{L_p(I \times I)}^p + c_1 \|(I - P)Qu\|_{L_p(I \times I)}^p \\ &\quad + c_1 \|QTu\|_{L_p(I \times I)}^p + \left| \int_{\Gamma} v \sigma ds \right|^p \\ &\leq c_1 \|PQu\|_{L_p(I \times I)}^p + c_2, \end{aligned} \tag{4.42}$$

where c_2 depends on c_0 and on ε , whereas c_1 is independent of ε .

Now we insert (4.42) into (4.41) and obtain

$$\|PQu\|_{L_p(I \times I)}^p \leq \varepsilon c_{g_j} c_1 \|PQu\|_{L_p(I \times I)}^p + c_p c_1 + \varepsilon c_{g_j} c_2.$$

Finally, by choosing $\varepsilon > 0$ small enough to satisfy $\varepsilon c_{g_j} c_1 \leq \frac{1}{2}$, we obtain the first estimate in (4.4). Now, if $0 < \beta < 2\alpha_0 - 1$ and $2 < p < (1 - \alpha_0)$, then $\beta < 1 - 2/p$ and $W_p^{1-1/p}(I)$ is continuously imbedded into $C^\beta(I)$ (see [14, (1.3.10), Remark 2 and (2.2.11) on p.71], which yields the second proposition in (4.4). Since $\frac{1}{2} < \alpha_0 < 1 - \frac{2}{q}$ (cf. Theorem 4.2) we find $0 < \beta < 1 - \frac{4}{q}$. ■

Appendix A4. The Newton–Kantorovič method

From [8] we quote the following results.

Theorem 4.7. *Let $\mathcal{A} : X \rightarrow Y$ be a continuous mapping where X and Y are Banach spaces, and let \mathcal{A} be Frechét differentiable with Frechét derivative $\mathcal{A}'_x \in \mathcal{L}(X, Y)$, which is Lipschitz continuous:*

$$\|\mathcal{A}'_{x_0} - \mathcal{A}'_x\|_{\mathcal{L}(X, Y)} \leq \lambda_2 \|x_0 - x\|_X \tag{4.43}$$

with $\lambda_2 \geq 0$.

Let \mathcal{A}'_{x_0} possess a bounded right inverse

$$\mathcal{B}_{x_0} : Y \rightarrow \tilde{X} := \mathcal{B}_{x_0} Y \subset X \quad \text{with } \lambda_1 := \|\mathcal{B}_{x_0}\|_{\mathcal{L}(Y, X)}. \tag{4.44}$$

Then there exists $\varepsilon > 0$ such that for every x with $\|x - x_0\| < \varepsilon$ the Frechét derivatives \mathcal{A}'_x have also bounded right inverses \mathcal{B}_x and

$$\|\mathcal{B}_x\|_{\mathcal{L}(Y, X)} \leq \frac{\lambda_1}{1 - \lambda_1 \lambda_2 \|x_0 - x\|_X}. \tag{4.45}$$

Proof. For the restricted mapping

$$\tilde{\mathcal{A}}'_{x_0} := \mathcal{A}'_{x_0}|_{\tilde{X}}$$

we have

$$\tilde{\mathcal{A}}'_{x_0} \mathcal{B}_{x_0} y = y \quad \text{for all } y \in Y \quad \text{and} \quad \mathcal{B}_{x_0} \tilde{\mathcal{A}}'_{x_0} x = x \quad \text{for all } x \in \tilde{X}.$$

Since for

$$\|\mathcal{B}_{x_0}(\mathcal{A}'_{x_0} - \mathcal{A}'_x)\|_{\mathcal{L}(X,X)} \leq \lambda_1 \lambda_2 \|x_0 - x\|_X < 1$$

the inverse

$$[I - \mathcal{B}_{x_0}(\mathcal{A}'_{x_0} - \mathcal{A}'_x)]^{-1} = \sum_{\ell=0}^{\infty} (\mathcal{B}_{x_0}(\mathcal{A}'_{x_0} - \mathcal{A}'_x))^\ell$$

exists, we find for the restricted operators

$$\tilde{\mathcal{A}}_x|_{\tilde{X}} : \tilde{X} \rightarrow Y$$

the inverse $\mathcal{B}_x := (\tilde{\mathcal{A}}'_x)^{-1}$ as

$$\mathcal{B}_x := \sum_{\ell=0}^{\infty} (\mathcal{B}_{x_0}(\tilde{\mathcal{A}}'_{x_0} - \tilde{\mathcal{A}}'_x))^\ell \mathcal{B}_{x_0},$$

which defines a right inverse to \mathcal{A}'_x . So, if \mathcal{B}_{x_0} is the whole bundle of right inverses, we obtain with the family of spaces \tilde{X} also the bundle of right inverses \mathcal{B}_x . Moreover,

$$\|\mathcal{B}_x\|_{\mathcal{L}(Y,X)} \leq \frac{\lambda_1}{1 - \lambda_1 \lambda_2 \|x - x_0\|}$$

for $\lambda_1 \lambda_2 \|x - x_0\| < 1$ and $\varepsilon := (\lambda_1 \lambda_2)^{-1}$. If $\|x_0 - x\| < \{2(\lambda_1 + 1)(\lambda_2 + 1)\}^{-1}$ then uniformly

$$\|\mathcal{B}_x\|_{\mathcal{L}(Y,X)} < 2\lambda_1. \tag{4.46}$$

■

Theorem 4.8. *Let $x_0 \in X$ and \mathcal{A} have the properties as in Theorem 4.7. Then if*

$$y = \mathcal{A}(x_0) \in Y$$

with

$$\|y\|_Y < \lambda_3 := \min \{ (4\lambda_1^2 \lambda_2)^{-1}, (2(\lambda_1 + 1)(\lambda_2 + 1))^{-1} \}, \tag{4.47}$$

there exists a solution $x \in X$ of

$$\mathcal{A}(x) = 0 \tag{4.48}$$

with

$$\|x - x_0\|_X < \{2(\lambda_1 + 1)(\lambda_2 + 1)\}^{-1}.$$

This solution is uniquely determined by Newton's method:

$$x_{k+1} := x_k - \mathcal{B}_{x_k} f_k \quad \text{and} \quad f_{k+1} := \mathcal{A}(x_{k+1}), \tag{4.49}$$

where $f_0 := -y$ and $x = \lim_{k \rightarrow \infty} x_k$. Hence, for y satisfying (4.47), the mapping $y \mapsto x$ is well defined and continuous.

Proof. From (4.49) we obtain

$$x_{k+1} = x_k + h_k \quad \text{with } h_k = -\mathcal{B}_{x_k} f_k,$$

and from

$$\begin{aligned} \mathcal{A}(x_k + h_k) &= \mathcal{A}(x_k) + \int_0^1 \mathcal{A}'_{x_k + th_k} h_k dt = \mathcal{A}(x_k) + \mathcal{A}'_{x_k} h_k + R, \\ R &= \int_0^1 (\mathcal{A}'_{x_k + th_k} - \mathcal{A}'_{x_k}) h_k dt \end{aligned}$$

the relation

$$f_{k+1} = f_k - \mathcal{A}'_{x_k} \mathcal{B}_{x_k} f_k + R = R.$$

Hence, with (4.43)

$$\|f_{k+1}\| = \|R\| \leq \lambda_2 \int_0^1 t \|h_k\|^2 dt = \frac{1}{2} \lambda_2 \|h_k\|^2.$$

If we guarantee that $\|x_k - x_0\| < \{2(\lambda_1 + 1)(\lambda_2 + 1)\}^{-1}$ then we obtain from (4.46)

$$\|f_{k+1}\| \leq 2\lambda_1^2 \lambda_2 \|f_k\|^2 \leq 2\lambda_1^2 \lambda_2 \lambda_3 \|f_k\|$$

if $\|f_k\| \leq \lambda_3$. So, with $q := 2\lambda_1^2 \lambda_2 \lambda_3$,

$$\|f_k\| \leq q^k \|f_0\| \leq q^k \lambda_3.$$

Hence, with (4.46),

$$\begin{aligned} \|x_{k+1} - x_k\| &= \|h_k\| = \|\mathcal{B}_{x_k} f_k\| \leq 2\lambda_1 \|f_k\| \leq 2\lambda_1 \lambda_3 q^k, \\ \|x_k - x_0\| &\leq \frac{2\lambda_1 \lambda_3}{1 - q}. \end{aligned} \tag{4.50}$$

Therefore, we choose $\lambda_3 > 0$ such that

$$\lambda_3 = \min \{ (4\lambda_1^2 \lambda_2)^{-1}, \{2(\lambda_1 + 1)(\lambda_2 + 1)\}^{-1} \}.$$

Then $q = \frac{1}{2}$, $\lim_{k \rightarrow \infty} x_k = x$ exists, and for every k we have $\|x_k - x_0\| < \{2(\lambda_1 + 1) \times (\lambda_2 + 1)\}^{-1}$, and x satisfies (4.48). ■

Now we are in the position to apply the continuation method to (4.20) with the family A_τ of operators with $\tau \in \mathcal{T} \subseteq [0, 1]$ and $X := \mathbb{R}^{m-1} \times C^\beta(\Gamma) = Y$. The operators depend continuously on τ and for every $\tau \in [0, 1]$ the Fréchet derivative (4.21) exists. For $\tau_0 = 0$, (4.20) has a solution, and the Fréchet derivative possesses a right inverse. Theorems 4.7 and 4.8 show that \mathcal{T} is open since if $A_{\tau_0}(\omega_{\tau_0}, \sigma_{\tau_0}) = 0$ has a solution and $A'_{\tau_0}(\omega_{\tau_0}, \sigma_{\tau_0})$ has a right inverse, then $A_{\tau_1}(\omega_{\tau_1}, \sigma_{\tau_1}) = 0$ will also admit a solution, and $A'_{\tau_1}(\omega_{\tau_1}, \sigma_{\tau_1}) = 0$ has a right inverse if $y = A_{\tau_1}(\omega_{\tau_0}, \sigma_{\tau_0})$, which is the case for $0 < |\tau_1 - \tau_0|$ small enough. Because of (4.50), if $\tau_1 \rightarrow \tau_0$, also $(\omega_{\tau_0}, \sigma_{\tau_1}) \rightarrow (\omega_{\tau_0}, \sigma_{\tau_0})$ in X since $y \rightarrow 0$. So, \mathcal{T} is also closed. Consequently, $\mathcal{T} = [0, 1]$ as proposed in Appendix A4.

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