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A TEST FOR THE RIEMANN HYPOTHESIS

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Abstract: We prove that the Riemann Hypothesis holds if and only if

$$I = \int_1^{+\infty} \big\{\Pi(x) - \mathrm{Li}(x)\big\}^2 x^{-2}\, dx < +\infty$$

with $I = J$, where J is some definite, computable real number $(1.266 < J < 1.273)$. This provides us with a numerical test for the Riemann Hypothesis.

The main interest of our test lies in the fact that it can also supply a *goal*. Namely, having computed $J(a) := \int_1^a \big\{\Pi(x) - \mathrm{Li}(x)\big\}^2 x^{-2}\, dx < J$ for a number of values of $a = a_n$, we can estimate a value a for which, within our precision, we will have $J(a) \approx J$.
Keywords: Riemann hypothesis, prime numbers, Fourier Transform

1. Introduction

The main result of this paper is that the integral

$$I = \int_1^{+\infty} \big\{\Pi(x) - \mathrm{Li}(x)\big\}^2 x^{-2}\, dx,$$

is finite if and only if the Riemann Hypothesis (RH, for short) is true. Moreover, if $I < +\infty$, then I is equal to a computable real number J given by the integral

$$J = \frac{1}{\pi} \int_0^{+\infty} \Big| \frac{\log\{(-1/2 + it)\zeta(1/2 + it)\}}{1/2 + it} \Big|^2\, dt,$$

where the log must be understood in a definite sense that we explain in section 2.

As usual, we denote $\Pi(x) = \sum_{p^m \le x} \frac{1}{m}$.

By a test for the Riemann Hypothesis we understand here a proposition equivalent to the Riemann Hypothesis, which is of the form $(\forall x > 0)P(x)$ (or $(\forall n \in \mathbf{N})P(n)$), where $P(x)$ is a condition that can be verified numerically, for every x (or n), by a halting algorithm. Thus, the algorithm must yield, in a finite amount of time, a decision about the truth or falsity of $P(x)$.

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Our test will be

$$(\forall x > 1) \quad \int_1^x \bigl\{\Pi(y) - \mathrm{Li}(y)\bigr\}^2 y^{-2}\,dy < J.$$

Strictly $(\forall x)(I(x) < J)$ is not a test in the sense defined above. If the RH is not true then there exists a finite y such that $I(y) = J$, but for this value any algorithm will have problems deciding whether $I(y) < J$ or not. However, this is not a real problem. For example, we can write the test in the form $(\forall x)[(I(x) < J)$ or $(I(x+1) < J)]$. (The same remark applies to the tests by Schoenfeld (1.1), Robin (1.2), and Xian-Jin Li (1.3) which we will recall on page 2.)

There are many equivalent formulations of the Riemann Hypothesis (see [10], [3], [14], [1], [8], [9]). But most of them cannot be written in the form of a test as described above. From those which can be rephrased as a test the most representative are:

Schoenfeld's [13]

$$(\forall x > 2657) \quad |\pi(x) - \mathrm{Li}(x)| \le \frac{\sqrt{x}\,\log x}{8\pi}, \tag{1.1}$$

Robin's [11]

$$(\forall n > 5040) \quad \sigma(n) < e^\gamma n \log\log n, \tag{1.2}$$

Xian–Jin Li's [7]

$$(\forall n \ge 1) \quad \lambda_n \ge 0 \quad \text{where} \quad \lambda_n = \frac{1}{n}\sum_\rho \Bigl[1 - \Bigl(\frac{\rho}{1-\rho}\Bigr)^n\Bigr], \tag{1.3}$$

and that of Nicolas [9]

$$(\forall k) \quad \frac{N_k}{\varphi(N_k)} > e^\gamma \log\log N_k, \tag{1.4}$$

where N_k denotes the product of the first k primes.

In all these cases, even assuming that we have checked these tests for all $y \le x_0$, there is no clue about which value of y will have a good chance of yielding a counterexample to the Riemann Hypothesis. This is why our test is more interesting. If we compute an approximation J_0 to the number J, and then compute the integral $I(x)$ in our test for $x \le x_0$, we can give a value $a > x_0$ where we expect $I(a) \approx J_0$. We explain this in Section 4. We expect this to happen for an a between 10^{16} and 10^{31}. (This broad range could be narrowed by computing J with more precision.) Since we can compute $\pi(x)$ to 10^{20}, the computation of $I(10^{20})$ is feasible.

The above considerations do not mean that our test gives more weight to the opinion that the Riemann Hypothesis is false. On the contrary, the computations will probably give the answer $I(a) < J$, and point to a new $b > a$, and so on.

It would be interesting to find a connection between a hypothetical a with $I(a) > J$ and the ordinate of the first zero off the critical line. But we have not pursued this any further.

2. The main Theorem

The function $\mathrm{Li}(x)$ is defined as

$$\mathrm{Li}(x) = \mathrm{P.\,V.}\int_0^x \frac{dt}{\log t}, \qquad x > 1.$$

The following lemmas are well known but we prove them for the sake of completeness.

Lemma 2.1. *For every $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$ we have*

$$s\int_0^{+\infty} \mathrm{Li}(e^t)e^{-st}\,dt = -\log(s-1) \tag{2.1}$$

where the integral is absolutely convergent.

Proof. In [6] we find that

$$\mathrm{Li}(x) = \gamma + \log\log x + \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!\,k}, \qquad x > 1.$$

Assuming $s > 1$ we have

$$\int_0^{+\infty} \mathrm{Li}(e^t)e^{-st}\,dt = \gamma\int_0^{+\infty} e^{-st}\,dt + \int_0^{+\infty} (\log t)e^{-st}\,dt + \int_0^{+\infty}\sum_{k=1}^{\infty} \frac{t^k}{k!\,k}e^{-st}\,dt =$$

$$= \frac{\gamma}{s} + \frac{1}{s}\int_0^{+\infty}\Bigl(\log\frac{y}{s}\Bigr)e^{-y}\,dy + \frac{1}{s}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{s^k} = -\frac{1}{s}\log s - \frac{1}{s}\log(1-1/s).$$

Therefore, (2.1) is true for every real $s > 1$. Since both sides of the identity are holomorphic functions on $\mathrm{Re}\,s > 1$, the assertion follows by the principle of analytic continuation. ■

Lemma 2.2. *For every complex number s with $\mathrm{Re}(s) > 1$ we have*

$$\frac{\log\{(s-1)\zeta(s)\}}{s} = \int_1^{+\infty} \{\Pi(x) - \mathrm{Li}(x)\}\,x^{-s-1}\,dx. \tag{2.2}$$

Proof. In [2, p. 22] it is proved that for every $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$

$$\log\zeta(s) = s\int_0^{+\infty} \Pi(x)x^{-s-1}\,dx.$$

Since $\Pi(x) = 0$ for $0 < x < 2$, the integral may be restricted to the interval $(1, +\infty)$. Combining this with the previous lemma we obtain (2.2). ■

Let Ω be the complex plane $\mathbf{C}$ with a cut along the negative real axis and additional cuts along the half-lines L_ρ starting at each non-trivial zero ρ of $\zeta(s)$ in the direction of the negative real axis. (That is to say, $L_\rho = \{(\sigma + it\colon \sigma \le \beta, t = \gamma\})$. The function $(s-1)\zeta(s)$ is analytic on Ω and does not vanish there. Since Ω is simply connected there exists a branch of the logarithm $\log\{(s-1)\zeta(s)\}$. We fix the unique branch of this logarithm which is positive for real $s > 2$. Now define for $s \in \Omega$

$$H(s) = \frac{\log\{(s-1)\zeta(s)\}}{s}.$$

Observe that for every $\sigma \ge \frac{1}{2}$, the function $t \mapsto H(\sigma + it)$ is defined for all $t \in \mathbf{R}$ unless t coincides with the ordinate of a non-trivial zero of $\zeta(s)$.

Remark. Notice that in the following Proposition we do not assume the Riemann Hypothesis.

Proposition 2.1. *There exists a constant $C < +\infty$ such that for all $\sigma \ge \frac{1}{2}$ we have*

$$\int_{-\infty}^{+\infty} |H(\sigma + it)|^2\,dt < C.$$

Proof. First assume that $\frac{1}{2} \le \sigma \le 2$. By Theorem 9.6 (B) of Titchmarsh [15] we have, uniformly in $-1 \le \sigma \le 2$,

$$\log\zeta(s) = \sum_{|t-\gamma|<1} \log(s-\rho) + \boldsymbol{\mathcal{O}}(\log t)$$

where $\log\zeta(s)$ has its usual meaning and $|\operatorname{Im}\log(s-\rho)| \le \pi$. The number of γ satisfying $|t-\gamma| < 1$ is less than $c\log(2+|t|)$ (cf. Titchmarsh, Theorem 9.2). It follows that $\arg\zeta(s) = \boldsymbol{\mathcal{O}}(\log t)$, with an implicit constant independent of $\sigma \in (\frac{1}{2}, 2]$.

Since we can write

$$H(\sigma+it) = \frac{\log\bigl\{((\sigma-1)^2+t^2)^{1/2}|\zeta(\sigma+it)|\bigr\} + i(\arg\zeta(\sigma+it) + \arg(\sigma-1+it))}{\sigma+it}$$

we only have to show that

$$(\sigma+it)^{-1}\log|\zeta(\sigma+it)|$$

is uniformly bounded in $\mathcal{L}^2(\mathbf{R})$ for $\sigma \in (\frac{1}{2}, 2]$. Again, by (2),

$$\log|\zeta(\sigma+it)| = \sum_{|t-\gamma|<1} \log|\sigma-\beta+i(t-\gamma)| + \boldsymbol{\mathcal{O}}(\log t)$$

and we have to show that

$$R(t) := (\sigma+it)^{-1} \sum_{|t-\gamma|<1} \log|\sigma-\beta+i(t-\gamma)|$$

is uniformly bounded in $\mathcal{L}^2(\mathbf{R})$.

Observing that

$$(t-\gamma)^2 \le (\sigma-\beta)^2 + (t-\gamma)^2 \le 5$$

we find that (taking into account that $|t-\gamma| < 1$)

$$\big|\log|\sigma-\beta+i(t-\gamma)|\big| \le \frac{1}{2}\log 5 - \log|t-\gamma|.$$

Since the number of γ satisfying $|t-\gamma| < 1$ is less than $c\log(2+|t|)$, and we assume $\frac{1}{2} < \sigma \le 2$, we get

$$|R(t)| \le |1/2+it|^{-1}\Big\{C\log(2+|t|) - \sum_{|t-\gamma|<1} \log|t-\gamma|\Big\}.$$

Therefore, we only have to show that

$$(1/2+it)^{-1}\sum_{|t-\gamma|<1}\log|t-\gamma| =: \frac{U(t)}{1/2+it}$$

is in $\mathcal{L}^2(\mathbf{R})$. Here $U(t)$ is defined by

$$U(t) := \sum_{|t-\gamma|<1}\log|t-\gamma| = \sum_\gamma \log|t-\gamma| I_\gamma(t),$$

where I_γ is the characteristic function of the interval $(\gamma-1,\gamma+1)$.

With this notation we have

$$\begin{aligned} U(t)^2 &= \Big(\sum_\gamma\big\{\log|t-\gamma|\big\}I_\gamma\Big)^2 \\ &= \sum_\gamma\big\{\log|t-\gamma|\big\}^2 I_\gamma + \sum_{\gamma\ne\gamma'}\big\{\log|t-\gamma|\big\}\big\{\log|t-\gamma'|\big\}I_\gamma I_{\gamma'}. \end{aligned}$$

To every term of the last sum we apply the Cauchy-Schwarz inequality, obtaining

$$|U(t)|^2 \le \sum_\gamma N(\gamma+2,\gamma-2)\big\{\log|t-\gamma|\big\}^2 I_\gamma,$$

where for every t we denote by $N(t+2,t-2)$ the number of zeros $\alpha = \beta + i\gamma$ of $\zeta(s)$ with $0 < \beta < 1$ and $t-2 < \gamma < t+2$. The number $N(\gamma+2,\gamma-2)$ then denotes the number of γ' such that $I_\gamma I_{\gamma'} \ne 0$.

Now we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{U(t)^2}{1/4+t^2}\,dt &\le \sum_{\gamma} N(\gamma+2,\gamma-2)\int_{I_\gamma}\frac{\bigl\{\log|t-\gamma|\bigr\}^2}{1/4+t^2}\,dt\\ &\le 2\sum_{\gamma>0} N(\gamma+2,\gamma-2)\int_{I_\gamma}\frac{\bigl\{\log|t-\gamma|\bigr\}^2}{1/4+t^2}\,dt\\ &\le 2\sum_{\gamma>0}\frac{N(\gamma+2,\gamma-2)}{1/4+(\gamma-1)^2}\int_{I_\gamma}\bigl\{\log|t-\gamma|\bigr\}^2\,dt\\ &\le C\sum_{\gamma>0}\frac{\log(2+|\gamma|)}{1/4+(\gamma-1)^2}<+\infty\end{aligned}$$

where we have applied the trivial fact that for every γ

$$\int_{I_\gamma}\bigl\{\log|t-\gamma|\bigr\}^2\,dt=4$$

and that $\gamma>14$ for every $\gamma>0$. The convergence of the series follows from $\gamma_n\sim 2\pi n/\log n$ (cf. Titchmarsh p. 214).

For $\sigma\ge 2$, we have $|\zeta(s)-1|<0.65$, so that $|\log\zeta(s)|$ is bounded. Hence, in this case the bound of the integral is elementary. ■

Following Proposition 2.1 we put

$$\begin{aligned}J:&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Bigl|\frac{\log\{(-1/2+it)\zeta(1/2+it)\}}{1/2+it}\Bigr|^2\,dt\\ &=\frac{1}{\pi}\int_{0}^{+\infty}\Bigl|\frac{\log\{(-1/2+it)\zeta(1/2+it)\}}{1/2+it}\Bigr|^2\,dt<+\infty.\end{aligned}$$

We can now formulate our main result.

Theorem 2.1. *There are only two possibilities for the integral*

$$I:=\int_{1}^{+\infty}\Bigl|\frac{\Pi(x)-\mathrm{Li}(x)}{x}\Bigr|^2\,dx:$$

(a) $I=J<+\infty$, and then the Riemann Hypothesis is true.
(b) $I=+\infty$, and then the Riemann Hypothesis is false.

Proof. First assume that the integral I is finite. Then the integral in the right hand side of (2.2) is absolutely convergent for $\operatorname{Re}s>\frac12$. The right hand side will then be holomorphic on $\operatorname{Re}s>\frac12$, but this implies that $\zeta(s)$ does not vanish for $\operatorname{Re}s>\frac12$. Hence, $I<+\infty$ implies the Riemann Hypothesis.

Now assume the Riemann Hypothesis holds. By Proposition 2.1 we can apply the Paley–Wiener Theorem [12, Theorem 19.2] to the function $H(s)$ on the half-plane $\operatorname{Re}s>\frac12$. We thus find that there exists a function $F\in\mathcal{L}^2(0,+\infty)$ such that for $\operatorname{Re} s > \frac{1}{2}$

$$H(s) = \int_0^{+\infty} F(u) e^{-u(s-\frac{1}{2})}\, du.$$

But, for $\operatorname{Re} s > 1$, by (2.2)

$$H(s) = \int_1^{+\infty} \{\Pi(x) - \operatorname{Li}(x)\}\, x^{-s-1}\, dx = \int_0^{+\infty} \{\Pi(e^u) - \operatorname{Li}(e^u)\}\, e^{-us}\, du,$$

and it follows that

$$F(u) = \{\Pi(e^u) - \operatorname{Li}(e^u)\}\, e^{-u/2}.$$

It is easily seen that

$$\int_0^{+\infty} |F(u)|^2\, du = \int_1^{+\infty} \left| \frac{\Pi(x) - \operatorname{Li}(x)}{x} \right|^2 dx.$$

Since $F \in \mathcal{L}^2(0, +\infty)$, we have proved that the Riemann Hypothesis implies that $I < +\infty$.

Assuming the Riemann Hypothesis holds, the conditions of the Paley–Wiener Theorem are satisfied, and in this case the Fourier transform of $F(u)$ equals the pointwise limit $\lim_{\sigma \to 1/2} H(\sigma + 2\pi i t)$. Hence, by Plancherel's theorem

$$I = \int_{-\infty}^{+\infty} |F(u)|^2\, du = \int_{-\infty}^{+\infty} |H(\tfrac{1}{2} + 2\pi i t)|^2\, dt = J.$$

So, we have proved that $I < +\infty$ if and only if the Riemann Hypothesis is true, and furthermore in this case $I = J$. This completes the proof of Theorem 2.1. ■

Remark. The proof above gives something more. Namely, assuming the Riemann Hypothesis holds, the functions

$$\{\Pi(e^x) - \operatorname{Li}(e^x)\}\, e^{-x/2} \quad \text{and} \quad \frac{\log\{(-1/2 + 2\pi i t)\zeta(1/2 + 2\pi i t)\}}{1/2 + 2\pi i t}$$

are in $\mathcal{L}^2(\mathbf{R})$ and the second is the Fourier transform of the first. If we do not assume the Riemann Hypothesis, only the second function is in $\mathcal{L}^2(\mathbf{R})$.

3. Computational results

In order to obtain approximations of I and J we have performed extensive computations. The results are presented in this section.

First we establish an upper bound for the integral

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{\log\{(-1/2 + it)\zeta(1/2 + it)\}}{1/2 + it} \right|^2 dt$$
$$= \frac{1}{\pi} \int_0^{+\infty} \left| \frac{\log\{(-1/2 + it)\zeta(1/2 + it)\}}{1/2 + it} \right|^2 dt.$$

Table 1: Values of $J(T)$.

T	$J(T)$	T	$J(T)$	T	$J(T)$
10 000	1.2631401	130 000	1.2662640	250 000	1.2664425
20 000	1.2646766	140 000	1.2662883	260 000	1.2664504
30 000	1.2652488	150 000	1.2663118	270 000	1.2664577
40 000	1.2655542	160 000	1.2663316	280 000	1.2664646
50 000	1.2657463	170 000	1.2663492	290 000	1.2664710
60 000	1.2658790	180 000	1.2663650	300 000	1.2664770
70 000	1.2659766	190 000	1.2663793	310 000	1.2664827
80 000	1.2660517	200 000	1.2663922	320 000	1.2664881
90 000	1.2661113	210 000	1.2664040	330 000	1.2664931
100 000	1.2661600	220 000	1.2664148	340 000	1.2664979
110 000	1.2662004	230 000	1.2664248	350 000	1.2665024
120 000	1.2662346	240 000	1.2664340	360 000	1.2665067

To this end we split the integral as follows

$$J = \frac{1}{\pi}\int_0^T \cdots \quad + \frac{1}{\pi}\int_T^{+\infty} \cdots\,,$$

and compute

$$J(T) := \frac{1}{\pi}\int_0^T \Bigl|\frac{\log\{(-1/2+it)\zeta(1/2+it)\}}{1/2+it}\Bigr|^2\,dt.$$

It will be convenient to take $T=\gamma_N$, the height of a zero of the zeta function. We rewrite the integral as follows

$$\frac{1}{\pi}\int_0^T \cdots = \frac{1}{\pi}\int_0^{\gamma_1}\cdots + \sum_{n=1}^{N-1}\frac{1}{\pi}\int_{\gamma_n}^{\gamma_{n+1}}\cdots$$

(To this end we have computed the ordinates of the first 10^6 zeros of $\zeta(s)$ with more than 100 D. But we only used 40 significant digits of the first 300000 zeros).

The values obtained for $J(T)$ are tabulated in Table 1.

We have also obtained rigorous upper bounds for the rest. Finally we prove

$$1.2663935\ldots \le J = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Bigl|\frac{\log\{(-1/2+it)\zeta(1/2+it)\}}{1/2+it}\Bigr|^2\,dt \le 1.2723669\ldots$$

With respect to the integral I, we compute

$$I(y) = \int_1^y \Bigl(\frac{\Pi(u)-\mathrm{Li}(u)}{u}\Bigr)^2\,du.$$

Table 2: Values of $I(y)$.

$y/10^6$	$I(y)$	$y/10^6$	$I(y)$	$y/10^6$	$I(y)$
10	1.2638110	180	1.2642554	350	1.2643224
20	1.2639254	190	1.2642704	360	1.2643233
30	1.2639572	200	1.2642754	370	1.2643286
40	1.2640642	210	1.2642781	380	1.2643358
50	1.2640987	220	1.2642818	390	1.2643415
60	1.2641128	230	1.2642899	400	1.2643422
70	1.2641313	240	1.2642960	410	1.2643427
80	1.2641482	250	1.2642988	420	1.2643432
90	1.2641583	260	1.2643019	430	1.2643440
100	1.2641708	270	1.2643071	440	1.2643445
110	1.2641863	280	1.2643088	450	1.2643456
120	1.2642043	290	1.2643109	460	1.2643468
130	1.2642149	300	1.2643116	470	1.2643488
140	1.2642192	310	1.2643136	480	1.2643508
150	1.2642269	320	1.2643174	490	1.2643555
160	1.2642341	330	1.2643181	500	1.2643607
170	1.2642406	340	1.2643214	510	1.2643622

We split this integral into integrals over the intervals (a_n, a_{n+1}), where $(a_n)_n$ is the increasing sequence consisting of primes and prime-powers. The integrals are easy to compute, and the results are contained in Table 2.

In order to see how the function $I(y)$ increases, we have plotted the obtained values on a logarithmic scale for y. We observe that the increase is slow. It seems that the Riemann Hypothesis is not threatened at all by these values of $I(y)$. But recall that, even assuming the Riemann Hypothesis, $I(y)$ is eventually greater than the goal 1.2663935.

4. Setting of the goal

Now we try to compute a value of x for which we think that $I(x)$ will be approaching or exceeding J. Of course, if the Riemann Hypothesis is true, then the function $I(x)$ will always be smaller than J. So we will take the position of the nonbelievers.

First, by Theorem 2.1 there would be a real number $a \in (1, +\infty)$ such that

$$\int_1^a \big\{\Pi(x) - \mathrm{Li}(x)\big\} x^{-2}\, dx = J.$$

Thus, there would exist a finite set of prime powers

$$Q = \{p^k : p^k \leq a\}$$

whose mere existence would contradict the Riemann Hypothesis. How large would a be? Answering this question led us to the following theorem.

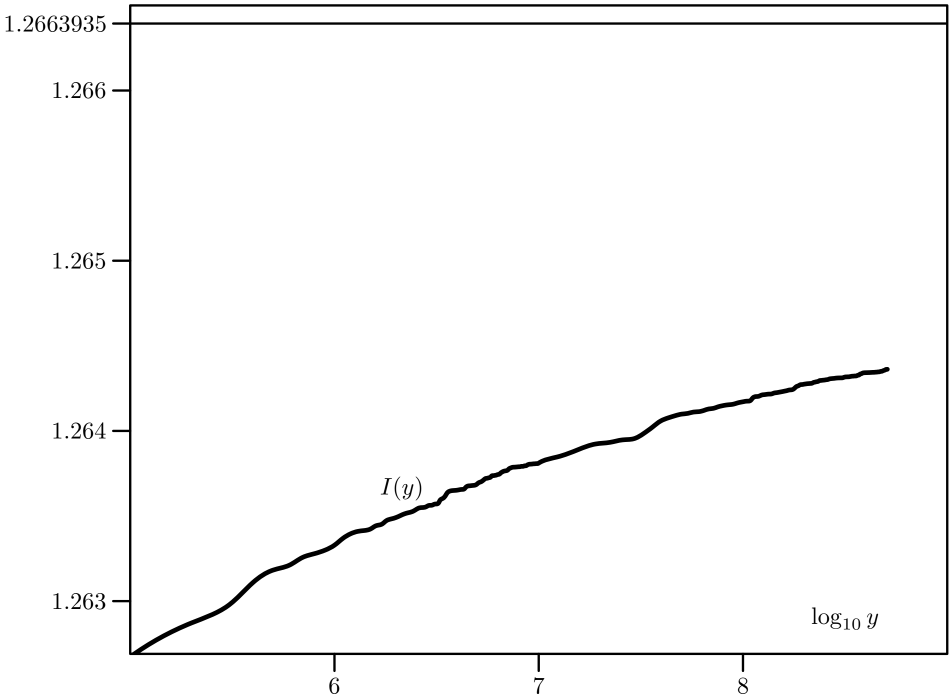

Figure 1.1: Growth of $I(y)$.

Theorem 4.1. *Assuming that there are some constants $A > 0$ and $0 < \theta \le 1$ such that for every $x > 1$*

$$|\{t : 1 \le t \le x, |\Pi(t) - \mathrm{Li}(t)| > A\sqrt{t}\}| \ge \theta(x-1) \tag{4.1}$$

then the Riemann Hypothesis is not true. (Here $|E|$ denotes the Lebesgue measure of the set E.)

Also, there exists $x_0 = x_0(A, \theta)$ such that if inequality (4.1) is true for $x < x_0$, then the Riemann Hypothesis is false.

Proof. Let $b > 0$ be such that $\theta b > 2$ and denote by E_n the set

$$E_n := \{t : 1 \le t \le b^n, |\Pi(t) - \mathrm{Li}(t)| > A\sqrt{t}\}.$$

By hypothesis, E_n is a subset of $[1, b^n]$ such that $|E_n| \ge \theta(b^n - 1)$. Therefore,

$$\{t : b^n < t \le b^{n+1}, |\Pi(t) - \mathrm{Li}(t)| > A\sqrt{t}\} = E_{n+1} \smallsetminus E_n$$

has measure $|E_{n+1} \smallsetminus E_n| \ge \theta(b^{n+1} - 1) - (b^n - 1) > b^n$.

Hence, we have

$$\int_{b^n}^{b^{n+1}} \Big|\frac{\Pi(t) - \mathrm{Li}(t)}{t}\Big|^2 dt \ge A^2 \int_{E_{n+1} \smallsetminus E_n} \frac{dt}{t} \ge A^2 \int_{b^{n+1} - b^n}^{b^{n+1}} \frac{dt}{t} = A^2 \log \frac{b}{b-1}.$$

Thus, with the notation of Theorem 2.1, we have

$$I > \int_1^{b^n} \Big|\frac{\Pi(t) - \mathrm{Li}(t)}{t}\Big|^2 \,dt \geq nA^2 \log\frac{b}{b-1}.$$

Choose n_0 such that

$$n_0 A^2 \log\frac{b}{b-1} > J.$$

Then, if (4.1) is true for $x < x_0 = b^{n_0}$, the Riemann Hypothesis will not be true since we would have $I > J$. ∎

A look at the values in Table 2 suggests that

$$\int_{2^n 10^7}^{2^{n+1}10^7} \Big|\frac{\Pi(t) - \mathrm{Li}(t)}{t}\Big|^2 \,dt$$

are greater than a fixed constant. In fact the values for $n = 0$, 1, 2, 3 and 4 are

$$0.0001144, \quad 0.0001388, \quad 0.000084, \quad 0.0000859, \quad 0.0000833, \dots$$

Thus, assuming these integrals are always ≥ 0.00008, we have to determine n_0 such that

$$\int_1^{32\cdot 10^7} \Big|\frac{\Pi(t) - \mathrm{Li}(t)}{t}\Big|^2 \,dt + 0.00008(n_0 - 5) = 1.2643173 + 0.00008(n_0 - 5) = J.$$

Our computations imply that $1.2663935 \leq J \leq 1.2723669$. Therefore we get two limits $n_0 = 31$ and $n_0 = 106$. Hence the possible value of a lies between 10^{16} and 10^{39}.

5. Some other equivalences for the Riemann Hypothesis

A glance at the values of $\Pi(x)$ and $\mathrm{Li}(x)$ reveals that in fact the approximation is very good. So, it seems that Theorem 4.1 is not very striking. But we can write the theorem in terms of the function $\pi(x)$ and here the result is more striking, because the known values of this function make one believe that usually $|\pi(x) - \mathrm{Li}(x)|$ is of the size of $\sqrt{x}$.

Corollary 5.1 (to Theorem 2.1). *The Riemann Hypothesis is true if and only if*

$$\int_1^{+\infty} \Big|\frac{\pi(x) - \mathrm{Li}(x)}{x}\Big|^2 \,dx < +\infty. \tag{5.1}$$

Proof. We know that $\Pi(x) = \pi(x) + \mathcal{O}(x^{1/2}/\log x)$ (see [4, p. 104]). Since $x^{-1/2}/\log(1+x) \in \mathcal{L}^2(1, +\infty)$, we have that $(\Pi(x) - \mathrm{Li}(x))x^{-1}$ is in $\mathcal{L}^2(1, +\infty)$ if and only if $(\pi(x) - \mathrm{Li}(x))x^{-1}$ is in $\mathcal{L}^2(1, +\infty)$. ∎

Theorem 5.1. *Assuming that there are some constants $A > 0$ and $0 < \theta \leq 1$ such that for every $x > 1$*

$$|\{t : 1 \leq t \leq x, |\pi(t) - \mathrm{Li}(t)| > A\sqrt{t}\}| \geq \theta(x-1) \tag{5.2}$$

then the Riemann Hypothesis is not true. (Here $|E|$ denotes the Lebesgue measure of the set E).

Also, there exists $x_0 = x_0(A, \theta)$, such that if inequality (5.2) is true for $x < x_0$, then the Riemann Hypothesis is not true.

Proof. Similar to that of Theorem 4.1. Refining the proof of the above corollary we could even obtain an explicit limes superior for the integral in (5.1). ■

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