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STABILITY OF ISOMETRIES IN *P*-BANACH SPACES

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Abstract: It is known that the isometry equation is stable in Banach spaces. In this paper we investigate stability of isometries in real p-Banach spaces, that is Frechet spaces with p-homogenous norms, where $p \in (0,1]$.

Let X, Y be p-Banach spaces and let $f : X \to Y$ be an ε-*isometry*, that is a function such that $|\|f(x) - f(y)\| - \|x - y\|| \leqslant \varepsilon$ for all $x, y \in X$. We show that if f is a surjective then there exists an affine surjective isometry $U : X \to Y$ and a constant C_p such that

$$\|f(x) - U(x)\| \leqslant C_p(\varepsilon + \varepsilon^p \|x\|^{(1-p)}) \quad \text{for} \quad x \in X.$$

We also show that in general the above estimation cannot be improved.
Keywords: p-homogeneous Frechet space, approximate isometry, Hyers-Ulam stability.

1. Introduction

Let X, Y be metric spaces and $\varepsilon \geqslant 0$. A mapping $f : X \to Y$ is called an ε-*isometry* if

$$|d(f(x), f(y)) - d(x, y)| \leqslant \varepsilon \quad \text{for} \quad x, y \in X.$$

A mapping which is an ε-isometry with a certain $\varepsilon > 0$ we call an approximate isometry. The problem of stability of isometries was posed by S. Ulam in 1940 [9]:

Ulam's Problem. *Does for every $\varepsilon > 0$ there exist a $\delta > 0$ such that for each δ-isometry $f : X \to Y$ there exists an isometry $U : X \to Y$ satisfying the inequality*

$$d(f(x), U(x)) \leqslant \varepsilon \quad \textit{for} \quad x \in X\,?$$

The first step in solving this problem was done by D. H. Hyers and S. Ulam [4] who noticed that surjectivity is an essential assumption and gave (under this additional assumption) a positive answer in Hilbert spaces. For Banach spaces

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the problem was affirmatively solved by J. Gevirtz [3]. His result was improved by M. Omladič and P. Šemrl [5]. More information about the general stability of isometries can be found in [2].

The latest improvement was obtained by P. Šemrl and J. Väisälä in [8]. To formulate it we need the notion of *nearsurjectivity.* By $B(x,r)$ we denote the closed ball centered at x with radius r. Let X, Y be metric spaces and $\delta \geqslant 0$. We say that $f: X \to Y$ is δ-nearsurjective if

$$Y \subset \bigcup_{x \in X} B(f(x), \delta).$$

The function f is called nearsurjective if it is δ-nearsurjective with a certain $\delta > 0$.

Theorem SV [8, Theorem 3.5] *Suppose that $f: X \to Y$ is a nearsurjective ε-isometry between Banach spaces satisfying $f(0) = 0$. Then there exists a bijective linear isometry $U: X \to Y$ such that*

$$\|f(x) - U(x)\| \leqslant 2\varepsilon \quad \textit{for } x \in X.$$

Hence f is 2ε-nearsurjective. The constant 2 is the best possible.

Since the original problem was posed for metric spaces, it is natural to investigate the stability of isometries in Frechet spaces. In fact our investigation is restricted to a certain important class of Frechet spaces, namely p-normed spaces. From now on we consider only real vector spaces.

For the convenience of the reader we recall the definition of p-normed space [1, 6].

Definition 1.1. Let X be a vector space and $p \in (0,1]$. A function $\|\cdot\|: X \to \mathbb{R}_+$ satisfying

(i) $\|x\| = 0 \Leftrightarrow x = 0$,
(ii) $\|\alpha x\| = |\alpha|^p \|x\|$,
(iii) $\|x + y\| \leqslant \|x\| + \|y\|$,

is called a *p-norm* and the pair $(X, \|\cdot\|)$ is called a *p-normed space.* By a *p-Banach space* we understand a complete p-normed space.

Let us briefly describe the contents of the paper. In the next section we show that the isometry equation is not stable in the Hyers-Ulam sense in p-Banach spaces, with $p < 1$. In Sections 3 and 4 we show the best estimation of an approximate isometry by an isometry is of the type $K(1 + \|x\|^{1-p})$. More precisely, there exists a constant $C_p > 0$ such that for every $f: X \to Y$, a surjective ε-isometry between p-Banach spaces X, Y, there exists a linear isometry $U: X \to Y$ satisfying

$$\|f(x) - f(y) - U(x-y)\| \leqslant L_p(\varepsilon + \varepsilon^p \|x - y\|^{1-p}) \quad \text{for } x, y \in X.$$

We will repeatedly use the following elementary inequalities:

$$(a+b)^p \leqslant a^p + b^p \qquad \text{for } a, b \geqslant 0, \quad p \in (0,1], \tag{1}$$

$$a^p - b^p \leqslant 2\frac{a-b}{a^{1-p} + b^{1-p}} \qquad \text{for } a \geqslant b > 0, \quad p \in (0,1]. \tag{2}$$

By $\mathbb{N}$ we denote the set of positive integers and by 0^0 we understand 1.

2. Lack of stability

We are going to show what kind of stability we may expect (as we will see, it is not possible to get the classical Hyers-Ulam stability). To do so we need the following observation [10, Proposition 2] concerning stability of the isometry equation in Banach spaces.

Proposition 2.1. *Let X, Y be Banach spaces and let $U, V : X \to Y$, $U(0) = V(0) = 0$, be isometries. Assume additionally that V is a linear bijection and that*

$$\lim_{x:\|x\|\to\infty} \frac{\|U(x) - V(x)\|}{\|x\|} = 0. \tag{3}$$

Then $U = V$ and consequently U is a linear bijective isometry.

As it is mentioned by the author of [10], the proof of the above proposition is in fact an easy modification of a part of the proof of Theorem SV. Before proceeding further we would also like to mention that it is not known if the above result holds for p-normed spaces (the idea of the proof does not seem to be adaptable to p-normed spaces).

Now we may state a refinement of Theorem SV.

Theorem 2.1. *Let $f : X \to Y$ be a nearsurjective ε-isometry between Banach spaces satisfying $f(0) = 0$. Then there exists a unique isometry $U : X \to Y$, $U(0) = 0$, such that*

$$K := \sup_{x\in X} \|f(x) - U(x)\| < \infty. \tag{4}$$

Moreover, U is a linear bijection and $K \leqslant 2\varepsilon$.

Proof. By Theorem SV there exists a linear bijective isometry V such that

$$\|f(x) - V(x)\| \leqslant 2\varepsilon \quad \text{for } x \in X.$$

Now, if U is as in the theorem, this and (4) imply that

$$\|U(x) - V(x)\| \leqslant (K + 2\varepsilon) \quad \text{for } x \in X.$$

Thus by Proposition 2.1 $U = V$. ■

We will need the following result

Lemma 2.1. *Let X be a Banach space and $p \in (0, 1]$. Let $f : X \to X$ be defined by*

$$f(x) := x + \frac{1}{\|x\|^p} x \quad \textit{for } x \neq 0, f(0) = 0.$$

Then

$$|\|f(x) - f(y)\|^p - \|x - y\|^p| \leqslant 12 \quad \textit{for } x, y \in X.$$

Proof. Without loss of generality we may assume that $\|x\| \geqslant \|y\|$, $x \neq y$.

Let us first discuss the case $y = 0$. Then

$$|\|f(x) - f(y)\|^p - \|x - y\|^p| = |\|f(x)\|^p - \|x\|^p| \overset{\text{by (2)}}{\leqslant} 2\frac{|\|f(x)\| - \|x\||}{\|f(x)\|^{1-p} + \|x\|^{1-p}}$$
$$\leqslant 2\frac{\|f(x) - x\|}{\|x\|^{1-p}} = 2\frac{\|\frac{x}{\|x\|^p}\|}{\|x\|^{1-p}} = 2.$$

Assume now that $x, y \neq 0$. Then we have

$$\begin{aligned}
&|\|f(x) - f(y)\|^p - \|x - y\|^p| \\
&\overset{\text{by (2)}}{\leqslant} 2\frac{|\|f(x) - f(y)\| - \|x - y\||}{|\|f(x) - f(y)\|^{1-p} + \|x - y\|^{1-p}|} \\
&\leqslant 2\frac{\|(f(x) - x) - (f(y) - y)\|}{\|x - y\|^{1-p}} = \frac{2}{\|x\|^p\|y\|^p} \cdot \frac{\|\|y\|^p x - \|x\|^p y\|}{\|x - y\|^{1-p}} \\
&= \frac{2}{\|x\|^p\|y\|^p} \cdot \frac{\|\|y\|^p(x - y) + (\|y\|^p - \|x\|^p)y\|}{\|x - y\|^{1-p}} \\
&\leqslant \frac{2\|x - y\|^p}{\|x\|^p} + \frac{2(\|x\|^p - \|y\|^p)\|y\|^{1-p}}{\|x - y\|^{1-p} \cdot \|x\|^p} \\
&\leqslant \frac{2\|x - y\|^p}{\|x\|^p} + \frac{2(\|x\| - \|y\|)}{\|x - y\|^{1-p} \cdot \|x\|^p} \leqslant \frac{2\|x - y\|^p}{\|x\|^p} + \frac{4\|x - y\|}{\|x - y\|^{1-p} \cdot \|x\|^p} \\
&= 6\frac{\|x - y\|^p}{\|x\|^p} \leqslant 6\frac{(\|x\| + \|y\|)^p}{\|x\|^p} \leqslant 6\frac{(\|x\| + \|x\|)^p}{\|x\|^p} \leqslant 12.
\end{aligned}$$

■

Now we are ready to show that in general there is no stability of isometries in p-normed spaces.

Example 2.1. Let $(X, \|\cdot\|)$ be a nontrivial Banach space. Let $p \in (0, 1]$ and let $\|\cdot\|_p := \|\cdot\|^p$. Then X is a complete p-normed space with p-norm $\|\cdot\|_p$.

Let $F : X \to X$ be defined by

$$F(x) := x + \frac{x}{\|x\|_p} \quad \text{for } x \neq 0, F(0) = 0.$$

By Lemma 2.1 we obtain that

$$|\|F(x) - F(y)\|_p - \|x - y\|_p| \leqslant 12 \quad \text{for } x, y \in X,$$

which means that F is an approximate isometry in $(X, \|\cdot\|_p)$.

Let us check that F is a bijection. First we observe that $F(\mathbb{R}x) \subset \mathbb{R}x$ for every $x \in X$. Moreover,

$$F(rx) = (r + \operatorname{sgn}(r)|r|^{1-p}\frac{1}{\|x\|_p}) \cdot x \quad \text{for } r \in \mathbb{R}, x \in X \setminus \{0\}.$$

Since for $x \in X \setminus \{0\}$ the function $F_x : \mathbb{R} \ni r \to (r + \mathrm{sgn}(r) \cdot |r|^{1-p}/\|x\|_p)$ is an increasing continuous function and $F_x(-\infty) = -\infty$, $F_x(\infty) = \infty$, we obtain that F_x is a bijection. Consequently F is also a bijection.

Suppose that there exists an isometry $U : (X, \|\cdot\|_p) \to (X, \|\cdot\|_p)$ and $K > 0$, $r \in [0, 1-p)$ such that

$$\|F(x) - U(x)\|_p \leqslant K(1 + \|x\|_p^r) \quad \text{for } x \in X. \tag{5}$$

Let $U_0(x) := U(x) - U(0)$. Then U_0 is an isometry, $U_0(0) = 0$ and by (5) we obtain

$$\|F(x) - U_0(x)\|_p \leqslant K_0(1 + \|x\|_p^r) \quad \text{for } x \in X, \tag{6}$$

where $K_0 := K + \|U(0)\|$. Then

$$\|x - U_0(x)\|_p \leqslant \|x - F(x)\|_p + \|F(x) - U_0(x)\|_p \leqslant \|x\|_p^{1-p} + K_0(1 + \|x\|_p^r).$$

Consequently

$$\|x - U_0(x)\| \leqslant (\|x\|^{p(1-p)} + K_0 + K_0\|x\|^{pr}))^{1/p}.$$

As $U_0 : (X, \|\cdot\|) \to (X, \|\cdot\|)$ is an isometry, the assumptions of Proposition 2.1 are satisfied for $V = \mathrm{id}$, and therefore $U_0 = \mathrm{id}$. Then we get

$$\|F(x) - U_0(x)\| = \|F(x) - x\| = \|x\|^{1-p},$$

i.e.

$$\|F(x) - U_0(x)\|_p = \|x\|_p^{1-p}.$$

We obtain a contradiction with (6) and the fact that $r < (1-p)$.

In the next example we briefly discuss the situation in L^p spaces, since they are a usual model for p-normed spaces (Example 2.1 is slightly artificial). We are only able to prove that there is no stability in the Hyers-Ulam sense if we restrict to the case when the approximating isometry is assumed to be linear.

Example 2.2. Let (Ω, ν) be a measure space, $\nu(\Omega) = 1$, and let $p \in (0, 1)$ be fixed. We are going to construct an approximate isometry in the space $L^p(\Omega)$ with the p-norm $\|\cdot\|_p$ defined by $\|f\|_p = \int_\Omega |f(x)|^p d\nu(x)$.

We define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(r) := r + \mathrm{sgn}(r)|r|^{1-p}.$$

By Lemma 2.1 we obtain that

$$|\,|\phi(r) - \phi(s)|^p - |r - s|^p\,| \leqslant 12 \quad \text{for } r, s \in \mathbb{R}. \tag{7}$$

One can easily see that ϕ is a continuous odd increasing bijection such that

$$r \leqslant \phi(r) \leqslant 2r + 1 \quad \text{for } r \in \mathbb{R}_+. \tag{8}$$

This yields in particular that ϕ^{-1} is an odd increasing continuous bijection and

$$0 \leqslant \phi^{-1}(r) \leqslant r \quad \text{for} \quad r \in \mathbb{R}_+. \tag{9}$$

We define a function $\Phi : L^p(\Omega) \to L^p(\Omega)$ by the formula

$$\Phi(x)(\omega) := \phi(x(\omega)) \quad \text{for} \quad x \in L^p(\Omega), \omega \in \Omega.$$

It follows from (8) that Φ is well-defined, i.e. that $\Phi(x) \in L^p(\Omega)$ for $x \in L^p(\Omega)$.

Let us verify that Φ is an approximate isometry. For arbitrary $x_1, x_2 \in L^p(\Omega)$ we have

$$\begin{aligned}
&|\|x_1 - x_2\|_p - \|\Phi(x_1) - \Phi(x_2)\|_p| \\
&= \left| \int_\Omega |x_1(\omega) - x_2(\omega)|^p - |\phi(x_1(\omega)) - \phi(x_2(\omega))|^p d\nu(\omega) \right| \\
&\leqslant \int_\Omega \big| |x_1(\omega) - x_2(\omega)|^p - |\phi(x_1(\omega)) - \phi(x_2(\omega))|^p \big| d\nu(\omega) \overset{\text{by (7)}}{\leqslant} \int_\Omega 12 d\nu(\omega) = 12.
\end{aligned}$$

The fact that Φ is a bijection follows from (9) and the fact that ϕ is a bijection.

Suppose that there exists a linear isometry U such that

$$\sup_{x \in L^p(\Omega)} \|\Phi(x) - U(x)\|_p < \infty.$$

Then by the linearity of U we trivially obtain that

$$U(x) = \lim_{n \to \infty} \frac{\Phi(nx)}{n} = x.$$

However, $\sup_{x \in L^p(\Omega)} \|\Phi(x) - x\|_p = \infty$, which yields a contradiction.

3. Jensen equation

As is the case for the stability of isometries in Banach spaces, the basic role is played by the fact that approximate isometries satisfy approximate Jensen equation.

Lemma 3.1. *Let $p \in (0,1)$, $\varepsilon > 0$ and let X, Y be p-normed spaces. Let $f : X \to Y$ be a surjective ε-isometry. Then there exists a constant $C_p > 0$ (depending only on p) such that*

$$\|f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2}\| \leqslant 2\|x-y\| \cdot 2^{-pn} + C_p \cdot 2^{(1-p)n}\varepsilon$$

for $x, y \in X, n \in \mathbb{N}$.

Proof. The proof in the case when $n \geqslant 2$ is a routine p-norm modification of the classical argument (see for example [2, Lemma 15.3] or [5]) and therefore we skip it.

One can easily verify that the estimation for $n = 1$ follows from the estimation for $n = 2$ (we simply enlarge the constant C_p two times). ■

We will need the following simple observation

Lemma 3.2. *Let $p \in (0,1)$ and $w \in \mathbb{R}_+$ be fixed and let $f_w : \mathbb{N} \to \mathbb{R}$ be defined by*

$$f_w(k) := w \cdot 2^{-pk} + 2^{(1-p)k} \quad \text{for } k \in \mathbb{N}.$$

Then

$$\inf_{k \in \mathbb{N}} f_w(k) \leqslant 4w^{1-p} + \frac{2}{p}.$$

Proof. If $w = 0$ then the assertion of the lemma holds trivially. Assume that $w > 0$.

Let $F_w : \mathbb{R} \to \mathbb{R}$ be defined by $F_w(x) := w \cdot 2^{-px} + 2^{(1-p)x}$. Let $x_w := \log_2(\frac{pw}{1-p})$. One can easily check that F_w is decreasing on the interval $(-\infty, x_w)$ and increasing on (x_w, ∞). Consequently, we have

$$\inf_{x \in \mathbb{R}} F_w(x) = F(x_w) = \frac{w^{1-p}}{p^p (1-p)^{1-p}}. \tag{10}$$

If $x_w < 0$ we put $k_w = 1$, else $k_w = \lfloor x_w \rfloor + 1$, where $\lfloor x \rfloor$ denotes the entire part of x. If $x_w < 0$, then $w \leqslant (1-p)/p$ which yields

$$f_w(k_w) = f_w(1) = w 2^{-p} + 2^{(1-p)} \leqslant \frac{2}{p}.$$

If $x_w \geqslant 0$, then $x_w \leqslant k_w \leqslant x_w + 1$. Applying monotonicity of F_w on the interval $(x_w, +\infty)$ and (10) we get

$$f_w(k_w) = w \cdot 2^{-k_w p} + 2^{k_w(1-p)} \leqslant w \cdot 2^{-px_w} + 2 \cdot 2^{(1-p)x_w} \leqslant 2 \cdot \frac{w^{1-p}}{p^p (1-p)^{1-p}}.$$

However, the infimum of the function $(0,1) \ni p \to p^p(1-p)^{1-p}$ is attained at $p = 1/2$ and is equal to $1/2$, and therefore $f_w(k_w) \leqslant 4w^{1-p}$. ■

Now we are ready to prove the main result of this section, which shows that approximate isometries are approximate Jensen functions.

Theorem 3.1. *Let $p \in (0,1)$, $\varepsilon > 0$. Then there exists a constant K_p such that for every p-normed spaces X, Y and a surjective ε-isometry $f : X \to Y$ we have*

$$\|f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2}\| \leqslant K_p(\varepsilon + \varepsilon^p \|x-y\|^{1-p}) \quad \text{for } x, y \in X.$$

Proof. Directly from Lemma 3.1 we get

$$\|f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2}\| \leqslant 2\frac{\|x-y\|}{2^{pk}} + 2C_p \cdot 2^{(1-p)k}\varepsilon$$

for $x, y \in X, k \in \mathbb{N}$. Now by Lemma 3.2 we obtain

$$\inf_{k\in\mathbb{N}} \left(2\frac{\|x-y\|}{2^{pk}} + 2C_p \cdot 2^{(1-p)k}\varepsilon \right) = 2C_p\varepsilon \cdot \inf_{k\in\mathbb{N}} \left(\frac{\|x-y\|}{C_p \cdot \varepsilon} 2^{-pk} + 2^{(1-p)k} \right)$$
$$\leqslant 2C_p\varepsilon \cdot \left(4\frac{\|x-y\|^{1-p}}{(C_p)^{1-p}\cdot\varepsilon^{1-p}} + \frac{2}{p} \right) \leqslant K_p(\varepsilon^p\|x-y\|^{1-p} + \varepsilon),$$

with $K_p = \max\{8(C_p)^p, \frac{4C_p}{p}\}$. ■

4. Stability

As we have seen, the investigation of stability of isometries have led us naturally to the Jensen equation. Thus we first discuss the stability of the Jensen equation.

Proposition 4.1. *Let X, Y be p-Banach spaces, $\varepsilon > 0$ and $p \in (0,1)$ and let $f : X \to Y$ be such that*

$$\|f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2}\| \leqslant K(\varepsilon + \varepsilon^p\|x-y\|^{1-p}) \quad \text{for } x, y \in X. \tag{11}$$

Then the function $a : X \to Y$ defined by

$$a(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \tag{12}$$

is a well-defined additive function and

$$\|f(x) - f(y) - a(x-y)\| \leqslant \frac{K\varepsilon}{1-2^{-p}} + K\varepsilon^p \frac{2^{p-p^2}}{1-2^{-p^2}} \|x-y\|^{1-p}$$

for $x, y \in X$.

As the proof follows the classical Hyers method we omit it.

Now we are ready to proceed to the proof of our main result on the stability of isometries in p-normed spaces.

Theorem 4.1. *Let $\varepsilon > 0$, $p \in (0,1)$ be arbitrary. Then there exists a constant $C_p > 0$ such that for every p-Banach spaces X, Y and every surjective ε-isometry $f : X \to Y$ there exists a linear isometry $U : X \to Y$ satisfying*

$$\|f(x) - f(y) - U(x-y)\| \leqslant C_p(\varepsilon + \varepsilon^p \cdot \|x-y\|^{1-p}) \quad \text{for } x, y \in X. \tag{13}$$

Proof. By Theorem 3.1 and Proposition 4.1 we obtain that the function

$$U(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{for } x \in X,$$

is a well-defined additive function such that

$$\|f(x)-f(y)-U(x-y)\| \leqslant C_p(\varepsilon+\varepsilon^p\cdot\|x-y\|^{1-p}) \quad \text{for } x\in X, \tag{14}$$

with a certain constant C_p depending only on p.

We have

$$\|U(x)-U(y)\| = \lim_{n\to\infty}\frac{1}{n^p}\|U(nx-ny)\| \overset{\text{by (14)}}{=} \lim_{n\to\infty}\frac{1}{n^p}\|f(nx)-f(ny)\|$$

Since f is an ε-isometry $|\|f(nx)-f(ny)\|-\|nx-ny\||\leqslant\varepsilon$, which implies that

$$\lim_{n\to\infty}\frac{1}{n^p}\|f(nx)-f(ny)\| = \lim_{n\to\infty}\frac{1}{n^p}\|nx-ny\| = \|x-y\|.$$

Thus U is an isometry. As a continuous additive function U is linear.

To complete the proof it remains to prove that U is surjective. Consider an arbitrary $w\in Y$. Since f is surjective we can find a sequence $(x_n)\subset X$ such that $f(nx_n)-f(0)=nw$.

As f is an ε-isometry

$$|\,\|f(nx_n)-f(0)\|-\|nx_n\|\,|\leqslant\varepsilon.$$

Whence

$$\|nx_n\|\leqslant\|f(nx_n)-f(0)\|+\varepsilon\leqslant n^p\|w\|+\varepsilon. \tag{15}$$

By the additivity of U we obtain

$$\begin{aligned}\lim_{n\to\infty}\|w-U(x_n)\| &= \lim_{n\to\infty}\frac{1}{n^p}\|nw-U(nx_n)\|\\ &= \lim_{n\to\infty}\frac{1}{n^p}\|f(nx_n)-f(0)-U(nx_n)\|\\ &\overset{\text{by (13)}}{\leqslant} \lim_{n\to\infty}\frac{1}{n^p}C_p(\varepsilon+\varepsilon^p\cdot\|nx_n\|^{1-p})\\ &\overset{\text{by (15)}}{\leqslant} \lim_{n\to\infty}\frac{1}{n^p}C_p(\varepsilon+\varepsilon^p\cdot(n^p\|w\|+\varepsilon)^{1-p})=0.\end{aligned}$$

We conclude that $w=\lim_{n\to\infty}U(x_n)$. But U is an isometry and therefore $U(X)$ is complete, since X is. Thus $U(X)$ is closed, and therefore $w\in U(X)$. ∎

As an immediate corollary we get

Corollary 4.1. *Let X,Y be p-Banach spaces such that there exists a surjective ε-isometry F between them. Then there exists a surjective linear isometry U : $X\to Y$ such that*

$$\|F(x)-U(x)\|=o(\|x\|) \text{ as } \|x\|\to\infty.$$

This implies in particular that X and Y are linearly isometric.

Remark 4.1. In this remark we would like to explain why the knowledge of approximate isometries in the case of p-normed spaces (for $p \in (0,1)$) is less precise that in the case of Banach spaces.

Let X, Y be Banach space and let $F : X \to Y$ be surjective ε-isometry. Then, as we have mentioned in the introduction, there exists an affine isometry U such that

$$\|F(x) - U(x)\| \leqslant 2\varepsilon.$$

On the other hand, if F is an arbitrary surjective function satisfying the above inequality for a certain affine isometry U, then automatically F is a $K\varepsilon$-approximate isometry, with $K = 4$.

However, the situation for p-normed spaces is different. If X, Y are p-Banach spaces and $F : X \to Y$ is a surjective ε-isometry, then by our main result, there exist $C_p > 0$ and a linear isometry U such that

$$\|F(x) - F(y) - U(x-y)\| \leqslant C_p(\varepsilon + \varepsilon^p \cdot \|x-y\|^{1-p}).$$

But, as one can easily notice, in general the assumption that F is a surjective function which satisfies the above inequality, where U is a linear isometry, does not guarantee that F is a $K\varepsilon$-isometry for a certain $K > 0$ (see the following example).

Example 4.1. Let $p \in (0,1)$ and let X be a p-normed space such that for some $e, f \in X$, $\|e\| = \|f\| = 1$, the following condition holds

$$\|\alpha e + \beta f\| = \|\alpha e\| + \|\beta f\| \quad \text{for } \alpha, \beta \in \mathbb{R}.$$

This holds in particular for every at least two dimensional p-normed space $L^p(\Omega)$ defined on a measure space (Ω, Σ, ν), $\nu(\Omega) = 1$ (then there exists $A \in \Sigma$ with $\nu(A) \in (0,1)$ and we put $e = \nu(A)^{-1/p}\chi_A$, $f = \nu(\Omega \setminus A)^{-1/p}\chi_{\Omega\setminus A}$).

We take an arbitrary $r \in (0,1)$ with $rp < 1-p$ and put

$$F(x) = x + \|x\|^r e \quad \text{for } x \in X.$$

One can easily check that

$$\|F(x) - F(y) - (x-y)\| \leqslant \|x-y\|^{rp} \leqslant 1 + \|x-y\|^{1-p}.$$

For $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} |\|F(\alpha f) - F(0)\| - \|\alpha f - 0\|| &= |\|\alpha f + \|\alpha f\|^r e\| - \|\alpha f\|| \\ &= |\|\|\alpha f\|^r e\| = \|\alpha f\|^{pr} = |\alpha|^{p^2 r} \to \infty \text{ as } \alpha \to \infty, \end{aligned}$$

hence F is not a $K\varepsilon$-isometry.

Problem 4.1. At the end of the paper we would like to pose two problems worth, in our opinion, investigation:

– is the isometry equation unstable (in the Hyers-Ulam sense) in all p-Banach spaces (we construct examples only in special spaces)?

– is Theorem SV valid for p-Banach spaces?

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