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SOME ESTIMATES FOR THE AVERAGE OF THE ERROR TERM OF THE MERTENS PRODUCT FOR ARITHMETIC PROGRESSIONS

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Abstract: We give estimates for the error term of the Mertens product over primes in arithmetic progressions of the Bombieri–Vinogradov and Barban–Davenport–Halberstam type.
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1. Introduction

Recall that γ denotes the Euler constant. In our paper [2] we proved a generalization to primes belonging to arithmetic progressions of the famous Mertens formula

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + \mathcal{O}\left(\frac{1}{\log^2 x}\right) \quad \text{as } x \to +\infty,$$

which is uniform with respect to the modulus. This generalized and strengthened a previous result due to Williams [3] that dealt with a *fixed* arithmetic progression. Let $q \geqslant 1$ and a be integers with $(a, q) = 1$, and define

$$P(x; q, a) = \prod_{\substack{p \leqslant x \\ p \equiv a \bmod q}} \left(1 - \frac{1}{p}\right) \tag{1}$$

and

$$M(x; q, a) = \frac{C(q, a)}{(\log x)^{1/\varphi(q)}},$$

where φ is the Euler totient function. Here $C(q, a)$ is real and positive and satisfies

$$C(q, a)^{\varphi(q)} = e^{-\gamma} \prod_{p} \left(1 - \frac{1}{p}\right)^{\alpha(p;q,a)},$$

where $\alpha(p; q, a) = \varphi(q) - 1$ if $p \equiv a \bmod q$ and $\alpha(p; q, a) = -1$ otherwise.

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In [2] we proved an asymptotic formula for the product in (1) of the form

$$P(x;q,a) = M(x;q,a)(1 + \mathcal{O}(\text{ErrorTerm})) \tag{2}$$

where both the size of error term and the range of uniformity for q depend on the existence of the "exceptional zero" (or "Siegel zero") for a suitable set of Dirichlet L-functions: see Lemma 1 of [2] for an accurate description of this phenomenon, and Theorem 1 there for the precise statement.

Our aim here is to prove that, on average over q, the error term in (2) is small and that its order of magnitude is the one that can be obtained assuming the Generalized Riemann Hypothesis (GRH). In fact, Theorem 4 of [2] shows that the GRH implies the bound

$$P(x;q,a) = M(x;q,a)\left(1 + \mathcal{O}\left((\log x)x^{-1/2}\right)\right)$$

as $x \to +\infty$, uniformly for every $q \leqslant x$ and any integer a with $(a,q) = 1$.

Our first result can be considered as an analogue of the Bombieri–Vinogradov theorem for primes in arithmetic progressions (see *e.g.* §28 of Davenport [1]) and its proof is based on it.

Theorem 1. *For every $A > 0$ there exists a constant $B = B(A) > 0$ such that*

$$\sum_{q \leqslant Q} \max_{\substack{a=1,\dots,q \\ (a,q)=1}} \left| \log \frac{P(x;q,a)}{M(x;q,a)} \right| \ll (\log x)^{-A}$$

as $x \to +\infty$, where $Q = x^{1/2}(\log x)^{-B}$.

The proof shows that we may take $B = A + 4$. We also study two different but related averages of the same quantity.

Corollary 1. *For every $A > 0$ there exists a constant $B = B(A) > 0$ such that*

(i)
$$\sum_{q \leqslant Q} \max_{\substack{a=1,\dots,q \\ (a,q)=1}} \left| \frac{P(x;q,a)}{M(x;q,a)} - 1 \right| \ll (\log x)^{-A}$$

(ii)
$$\sum_{q \leqslant Q} \max_{\substack{a=1,\dots,q \\ (a,q)=1}} |P(x;q,a) - M(x;q,a)| \ll (\log x)^{-A}$$

as $x \to +\infty$ where, in both cases, $Q = x^{1/2}(\log x)^{-B}$.

Our second result can be considered as an analogue of the Barban–Davenport--Halberstam theorem for primes in arithmetic progressions (see *e.g.* §29 of Davenport [1]) and its proof is based on it.

Theorem 2. *For every $A > 0$ there exists a constant $B = B(A) > 0$ such that*

$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\log\frac{P(x;q,a)}{M(x;q,a)}\right)^2 \ll (\log x)^{-A}$$

as $x \to +\infty$, where $Q = x(\log x)^{-B}$.

Corollary 2. *For every $A > 0$ there exists a constant $B = B(A) > 0$ such that*

(i)
$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\frac{P(x;q,a)}{M(x;q,a)}-1\right)^2 \ll (\log x)^{-A}$$

(ii)
$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(P(x;q,a)-M(x;q,a)\right)^2 \ll (\log x)^{-A}$$

as $x \to +\infty$ where, in both cases, $Q = x(\log x)^{-B}$.

2. Proof of Theorem 1

Let $L(x) = \exp\big((\log x)^{3/5}(\log\log x)^{-1/5}\big)$. The proof is based on the identity

$$\log\frac{P(x;q,a)}{M(x;q,a)} = -\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\overline{\chi}(a)\sum_{p>x}\chi(p)\log\Big(1-\frac{1}{p}\Big) + R(x) \tag{3}$$

where

$$R(x) = \frac{1}{\varphi(q)}\Big(\gamma + \log\log x + \sum_{p\leqslant x}\log\Big(1-\frac{1}{p}\Big)\Big). \tag{4}$$

Identity (3) is proved combining (10) and Lemma 6 in [2]. In fact, using Williams' expression for $C(q,a)$ in the statement of his Theorem 1 we have

$$\log M(x;q,a) = \frac{1}{\varphi(q)}\Big(-\gamma + \log\frac{q}{\varphi(q)} + \sum_{\chi\neq\chi_0}\sum_{p}\chi(p)\log\Big(1-\frac{1}{p}\Big) - \log\log x\Big),$$

while (10) and Lemma 6 from [2] imply that

$$\begin{aligned}\log P(x;q,a) &= \frac{1}{\varphi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\sum_{p\leqslant x}\chi(p)\log\Big(1-\frac{1}{p}\Big)\\ &= \frac{1}{\varphi(q)}\Big(\log\frac{q}{\varphi(q)} + \sum_{p\leqslant x}\log\Big(1-\frac{1}{p}\Big) + \sum_{\chi\neq\chi_0}\overline{\chi}(a)\sum_{p\leqslant x}\chi(p)\log\Big(1-\frac{1}{p}\Big)\Big)\end{aligned}$$

and relations (3) and (4) follow at once.

Since $R(x) \ll L(x)^{-c}\varphi(q)^{-1}$ for some positive c by Lemma 5 in [2], its total contribution is $\ll L(x)^{-c}\log Q$ and therefore it is negligible. For $\chi \neq \chi_0$ let

$$S(x,\chi) = \sum_{p>x} \chi(p)\log\Bigl(1-\frac{1}{p}\Bigr) = -\sum_{p>x}\frac{\chi(p)}{p} + \mathcal{O}\bigl(x^{-1}\bigr). \tag{5}$$

The total contribution of the error term is $\ll Qx^{-1}$ and we may neglect it as well. For brevity, let

$$\theta(x,\chi) = \sum_{p\leqslant x}\chi(p)\log p \qquad \text{and} \qquad \Theta(x;q,a) = \frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\overline{\chi}(a)\theta(x,\chi).$$

The presence of $\overline{\chi}(a)$ in the definition of Θ implies that we may drop the condition $(a,q)=1$. By equation (9) of [2] we have

$$\begin{aligned}&\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\max_a\Bigl|\sum_{\chi\neq\chi_0}\overline{\chi}(a)\sum_{p>x}\frac{\chi(p)}{p}\Bigr| \\ &= \sum_{q\leqslant Q}\max_a\Bigl|\frac{\Theta(x;q,a)}{x\log x} - \int_x^{+\infty}\Theta(t;q,a)\,\frac{\log t+1}{t^2(\log t)^2}\,\mathrm{d}t\Bigr|.\end{aligned} \tag{6}$$

After a transition to primitive characters as on page 163 of Davenport [1], we see that

$$|\Theta(x;q,a)| \ll \log q + \frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}|\theta(x,\chi_1)|,$$

where χ_1 denotes the primitive character that induces χ. The total contribution of $\log q \leqslant \log Q$ is $\ll Q\log Q(x\log x)^{-1}$ which is negligible. We also notice that $\theta(x,\chi) = \psi(x,\chi) + \mathcal{O}\bigl(x^{1/2}\bigr)$, and that the total error term arising here is $\ll Qx^{-1/2}$. The triangle inequality now shows that, up to "small" error terms, the right hand side of (6) is

$$\begin{aligned}&\leqslant \frac{1}{x\log x}\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}|\psi(x,\chi_1)| \\ &\quad + \int_x^{+\infty}\Bigl(\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}|\psi(t,\chi_1)|\Bigr)\frac{\log t+1}{t^2(\log t)^2}\,\mathrm{d}t + \mathcal{O}\Bigl(Qx^{-1/2}\Bigr).\end{aligned}$$

Arguing again as in page 163 of [1], we get

$$\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}|\psi(t,\chi_1)| \ll \log x\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sideset{}{^*}\sum_{\chi\neq\chi_0}|\psi(t,\chi)|$$

and we conclude with $B = A+4$ by an appeal to the following inequality, which is (3) in Chapter 28 of [1],

$$\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\leqslant x}|\psi'(y,\chi)| \ll x^{1/2}Q(\log x)^4,$$

where $\psi'(y,\chi) = \psi(y,\chi)$ if $\chi\neq\chi_0$ and $\psi'(y,\chi_0) = \psi(y,\chi_0) - y$.

3. Proof of Theorem 2

Recalling the inequality $|a+b|^2 \leqslant 2|a|^2 + 2|b|^2$ and using again (3) with $R(x) \ll L(x)^{-c}\varphi(q)^{-1}$ as above, we have

$$\begin{aligned}
&\sum_{q\leqslant Q} \sideset{}{^*}\sum_{a=1}^{q} \Big|\log \frac{P(x;q,a)}{M(x;q,a)}\Big|^2 \\
&\leqslant 2\sum_{q\leqslant Q} \sideset{}{^*}\sum_{a=1}^{q} \frac{1}{\varphi(q)^2} \sum_{\substack{\chi_1\neq\chi_0\\ \chi_2\neq\chi_0}} \chi_1(a)\overline{\chi}_2(a) S(x,\chi_1)S(x,\overline{\chi}_2) + \mathcal{O}\Big(\frac{\log Q}{L(x)^{2c}}\Big) \\
&= 2\sum_{q\leqslant Q} \frac{1}{\varphi(q)^2} \sum_{\substack{\chi_1\neq\chi_0\\ \chi_2\neq\chi_0}} S(x,\chi_1)S(x,\overline{\chi}_2) \sideset{}{^*}\sum_{a=1}^{q} \chi_1(a)\overline{\chi}_2(a) + \mathcal{O}\Big(\frac{\log Q}{L(x)^{2c}}\Big) \\
&= 2\sum_{q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} |S(x,\chi)|^2 + \mathcal{O}\Big(\frac{\log Q}{L(x)^{2c}}\Big),
\end{aligned}$$

where $S(x,\chi)$ is defined in (5). The contribution of the error term x^{-1} in (5) has size $\ll Qx^{-2}$. Hence, we need to prove the bound

$$\sum_{q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \Big|\sum_{p>x} \frac{\chi(p)}{p}\Big|^2 \ll \frac{Q}{x}. \tag{7}$$

Arguing as in (6) and using again the inequality $|a+b|^2 \leqslant 2|a|^2 + 2|b|^2$, we see that the left hand side above is

$$\ll \sum_{q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \Big(\frac{|\theta(x,\chi)|^2}{(x\log x)^2} + \Big|\int_x^{+\infty} \theta(t,\chi)\,\frac{(\log t+1)\,\mathrm{d}t}{(t\log t)^2}\Big|^2\Big). \tag{8}$$

For the second summand, the Cauchy inequality shows that

$$\Big|\int_x^{+\infty} \theta(t,\chi)\,\frac{(\log t+1)\,\mathrm{d}t}{(t\log t)^2}\Big|^2 \leqslant \int_x^{+\infty} \frac{|\theta(t,\chi)|^2}{t^3}\,\mathrm{d}t \int_x^{+\infty} \frac{(\log t+1)^2\,\mathrm{d}t}{t(\log t)^4}.$$

It is easy to see that the second integral is $\ll (\log x)^{-1}$. The contribution of the second term in (8) is therefore

$$\ll (\log x)^{-1} \int_x^{+\infty} \Big(\sum_{q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} |\theta(t,\chi)|^2\Big)\frac{\mathrm{d}t}{t^3}.$$

After a transition to primitive characters as on page 163 of Davenport [1], we see that

$$\frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} |\theta(t,\chi)|^2 \ll \log^2 q + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} |\theta(x,\chi_1)|^2,$$

where χ_1 denotes the primitive character that induces χ. The total contribution of $\log^2 q \leqslant \log^2 Q$ is $\ll Q \log^2 Q (x^2 \log x)^{-1}$ which is negligible. Hence we have to prove that

$$(\log x)^{-1} \int_x^{+\infty} \Bigl(\sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} |\theta(t, \chi_1)|^2 \Bigr) \frac{\mathrm{d}t}{t^3} \ll \frac{Q}{x}. \tag{9}$$

Recalling that $\theta(x, \chi) = \psi(x, \chi) + \mathcal{O}\bigl(x^{1/2}\bigr)$, the total error term arising here is $\ll Q(x \log x)^{-1}$. An appeal to the following inequality, which is the equation at line -7 of page 170 in Chapter 29 of [1],

$$\sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi \bmod q} |\psi'(x, \chi_1)|^2 \ll xQ \log x,$$

where $\psi'(x, \chi) = \psi(x, \chi)$ if $\chi \neq \chi_0$ and $\psi'(x, \chi_0) = \psi(x, \chi_0) - x$, allows us to prove (9).

The first summand in (8) is treated analogously and its total contribution is $\ll Q(x \log x)^{-1}$. Hence (7) holds and so we can conclude that Theorem 2 holds with $B = A$.

4. Proof of Corollaries 1 and 2

The proofs of these Corollaries are similar. The proof of point (i) is straightforward, since it depends on the fact that $e^u - 1 \ll |u|$ for bounded u. Here u is the left hand side of (3) and, to prove (i) of Corollary 1, it is enough to remark that it is $\ll (\log x)^{-A}$, uniformly for $Q = x^{1/2} (\log x)^{-B}$, by Theorem 1. For Corollary 2, it is $\ll (\log x)^{-A/2}$ uniformly for $Q = x (\log x)^{-B}$, by Theorem 2. We remark that, in both cases, u is obviously much smaller.

For the other points, equation (3) shows that

$$M(x; q, a) = P(x; q, a) \exp\Bigl\{ \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(a) S(x, \chi) - R(x) \Bigr\},$$

where $R(x)$ is defined in (4) and $S(x, \chi)$ is defined in (5). Thus

$$\begin{aligned} M(x; q, a) - P(x; q, a) &= P(x; q, a) \Bigl(\exp\Bigl\{ \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(a) S(x, \chi) - R(x) \Bigr\} - 1 \Bigr) \\ &\ll \Bigl| \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(a) S(x, \chi) - R(x) \Bigr|, \end{aligned}$$

by the same argument as above, since, obviously, $P(x; q, a) \leqslant 1$. This is enough to prove (ii) of Corollary 1. Squaring out both sides of the previous equation the second point of Corollary 2 follows.

References

[1] H. Davenport, *Multiplicative Number Theory*, Springer-Verlag, third edition, 2000.

[2] A. Languasco and A. Zaccagnini, A note on Mertens' formula for arithmetic progressions, *Journal of Number Theory* 127 (2007), 37–46.

[3] K.S. Williams, Mertens' theorem for arithmetic progressions, *J. Number Theory* 6 (1974), 353–359.

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