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MULTIPOINT METHOD FOR GENERALIZED EQUATIONS UNDER MILD DIFFERENTIABILITY CONDITIONS

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Abstract: We are concerned with the problem of approximating a locally unique solution of a generalized equation using a multipoint method in a Banach spaces.In [9]–[11] the authors showed that the previous method is superquadratically (or cubically) convergent when the second Fréchet derivative satisfies the usual Hölder continuity condition (or center–Hölder continuity condition). Here, we weaken these conditions by using ω–condition (or σ–condition) on the second derivative introduced by us [1]–[4], [22] (for nonlinear equations), with ω and σ a non–decreasing continuous real functions. We provide also an improvement of the ratio of our algorithm under some ω–center–condition (or σ–center–condition) and less computational cost.
Keywords: Banach space, local convergence, multipoint method, generalized equation, Aubin continuity, Lipschitz condition, set-valued map, ω–condition, radius of convergence.

1. Introduction

There are many interesting scientific problems based of the solution of generalized equations introduced by Robinson [19, 20]. These equations are an abstract model of a wide variety of variational including systems of inequalities, variational inequalities (for example first–order necessary conditions for nonlinear programming), linear and nonlinear complementary problems, systems of nonlinear equations. Generalized equations may characterize optimality or equilibrium and then have several applications economics and engineering (see for example [14]).

Our notation is basically standard (see [4], [17]). X and Y are arbitrary Banach spaces with the norms denoted by $\parallel . \parallel$. The distance from a point x and a subset A of X will be denoted by $\operatorname{dist}(x, A) = \inf\limits_{a \in A} \parallel x - a \parallel$. The excess e from A to the set $C \subset X$ is given by $e(C, A) = \sup \{\operatorname{dist}(x, A),\ x \in C\}$. We denote by $\mathbb{B}_r(x)$ the closed ball centered at x with radius r. A set–valued mapping Λ from X to Y is indicated by $\Lambda : X \longrightarrow 2^Y$, its graph is the set $\operatorname{gph} \Lambda := \{(x, y) \in X \times Y,\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y) = \{x \in X,\ y \in \Lambda(x)\}$. From now on $F : X \to Y$ denotes a Fréchet differentiable function while $G : X \longrightarrow 2^Y$ stands for a set–valued mapping with closed graph.

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We are concerned with the problem of approximating a solution x^* of the generalized equation in the form

$$0 \in F(x) + G(x). \tag{1.1}$$

We consider the following iterative method for solving (1.1):

$$0 \in A(x_{k+1}, x_k) + G(x_{k+1}), \tag{1.2}$$

a_i and β_i are real numbers for $i = 1, 2, \cdots, M$ with M is a fixed integer in $\mathbb{N}$,

$$A(y,x) = F(x) + \sum_{i=1}^{i=M} a_i \nabla F(x + \beta_i\,(y-x))\,(y-x),\ \forall x, y \in X, \tag{1.3}$$

$$\sum_{i=1}^{i=M} a_i = 1 \qquad \text{and} \qquad \sum_{i=1}^{i=M} a_i\,(1-\beta_i) = \frac{1}{2}. \tag{1.4}$$

Algorithm (1.2) is based on some multipoint iteration formula for approximating the solution of nonlinear equations. The cubically convergence of method (1.2) is presented in [10] when the second Fréchet derivative is L-Lipschitz in some neighborhood V of x^*:

$$\| \nabla^2 F(x) - \nabla^2 F(y) \| \leqslant L \ \| x - y \|,\ x, y \in V, \tag{1.5}$$

and the set–valued mapping $[A(\cdot, x^*) + G(\cdot)]^{-1}$ is Aubin continuous around $(0, x^*)$ (or pseudo–Lipschitz at $(0, x^*)$). The same hypotheses are used in [15] to study of the convergence and the stability of some method based on the second–degree Taylor polynomial expansion of F. Recall that a set–valued map $\Gamma : Y \longrightarrow 2^X$ is pseudo–Lipschitz at $(y^\star, x^\star) \in \mathrm{gph}\Gamma$ if there exist constants a, b, M such that for every $y_1, y_2 \in \mathbb{B}_b(y^\star)$ and for every $x_1 \in \Gamma(y_1) \cap \mathbb{B}_a(x^\star)$ there exists $x_2 \in \Gamma(y_2)$ with

$$\| x_1 - x_2 \| \leqslant M \| y_1 - y_2 \| .$$

The pseudo–lipschitzian property is introduced in [8] and is tied to the concept of metric regularity; actually, the Aubin continuity of Γ around $(y^\star, x^\star)$ is equivalent to the metric regularity of the inverse Γ^{-1} of Γ at $x^\star$ for $y^\star$, i.e., $y^\star \in \Gamma^{-1}(x^\star)$ and there exists $\kappa \in [0, \infty[$ along with neighborhoods U of $x^\star$ and V of $y^\star$ such that

$$\text{dist } (x, \Gamma(y)) \leqslant \kappa \text{ dist } (y, \Gamma^{-1}(x)),\ \ \forall x \in U,\ y \in V.$$

The infimum of such moduli κ is called the exact regularity bound of Γ^{-1} around $(x^\star,\ y^\star)$. For more details on these topics one can refer to the books [8, 17, 18, 21] and the references therein. Cabuzel in [9] showed that the sequence (1.2) is locally superquadratic convergent to the solution x^* whenever $\nabla^2 F$ satisfies α–Hölder–type condition on some neighborhood V of x^* with constant K ($\alpha \in (0, 1],\ L > 0$):

$$\| \nabla^2 F(x) - \nabla^2 F(y) \| \leqslant L \ \| x - y \|^\alpha,\ x, y \in V. \tag{1.6}$$

In [11] the authors provided a finer local superquadratic convergence of algorithm (1.2) using α–center–Hölder condition on some neighborhood V of x^* with constant K_0 ($\alpha \in (0,1]$, $L_0 > 0$):

$$\| \nabla^2 F(x) - \nabla^2 F(x^*) \| \leqslant L_0 \| x - x^* \|^\alpha, \; x \in V. \tag{1.7}$$

In this paper, we relax these usual Lipschitz, α–Hölder and α–center–Hölder conditions (see assumptions (1.5), (1.6) and (1.7)) by ω-condition and Ptak-type condition [2] on the second Fréchet derivative. The main conditions required are

$$\| \nabla^2 F(x) - \nabla^2 F(y) \| \leqslant \omega(\| x - y \|), \text{ for } x, y \text{ in } V, \tag{1.8}$$

and

$$\begin{gathered} \| \nabla^2 F(x) - \nabla^2 F(y) \| \leqslant \sigma(\| x - y \|) \; \| x - y \|^\theta, \\ \text{for all } x, y \text{ in } V \text{ and } \theta \text{ is fixed in } (0,1], \end{gathered} \tag{1.9}$$

where $\omega, \sigma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are a continuous nondecreasing functions. When the condition (1.8) is satisfied, we say that $\nabla^2 F$ is ω–conditioned. The conditions (1.8) and (1.9) are used in [1, 2, 13, 16] to study of Newton's method for solving nonlinear equations ($G = \{0\}$ in (1.1)). Some part of our goal is also to provide a finer local convergence of algorithm (1.2) by using a center–type conditions of (1.8) and (1.9) as follows

$$\| \nabla^2 F(x) - \nabla^2 F(x^*) \| \leqslant \overline{\omega}(\| x - x^* \|), \text{ for } x \text{ in } V, \tag{1.10}$$

and

$$\| \nabla^2 F(x) - \nabla^2 F(x^*) \| \leqslant \overline{\sigma}(\| x - x^* \|) \; \| x - x^* \|^\theta, \text{ for } x \text{ in } V, \tag{1.11}$$

where $\overline{\omega}, \overline{\sigma} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous nondecreasing functions. Similar conditions to (1.8) – (1.11) are used in [5]–[7] to study Newton's methods, the superquadratic algorithm and Hummel–Seebeck–type method for solving (1.1). The rest of this paper is organized as follows. In section 2 we have collected a fixed point theorem [12] and a number of necessary results, needed in our local analysis. In section 3, we give some convergence results of algorithm (1.2) under the different assumptions. Finally, we provide in section 4 an improvement of the ratio of this algorithm under a center–conditioned second Fréchet derivative and we give some remarks on our method using some ideas related to nonlinear equations [1]–[4].

2. Background material and assumptions

Let us begin with some basic results that will be used throughout this paper. By the second order Taylor expansion of F at $y \in V$ with the remainder is given by integral form, the following Lemma is obtained directely (see [7]).

Lemma 2.1. *The following assertions are checked:*

1. *If the assumption (1.8) is satisfied on a convex neighborhood* V *, then for all* x *and* y *in* V *we have the following*

$$\| F(x) - F(y) - \nabla F(y)\,(x-y) - \frac{1}{2}\nabla^2 F(y)\,(x-y)^2 \|$$
$$\| x-y \|^2 \int_0^1 (1-t)\,\omega(t \| x-y \|)\,dt.$$

2. *If the assumption (1.9) is satisfied on a convex neighborhood* V *, then for all* x *and* y *in* V *we have the following*

$$\| F(x) - F(y) - \nabla F(y)\,(x-y) - \frac{1}{2}\nabla^2 F(y)\,(x-y)^2 \|$$
$$\| x-y \|^{2+\theta} \int_0^1 t^\theta\,(1-t)\,\sigma(t \| x-y \|)\,dt.$$

The second tool in our analysis is the fixed point theorem for set–valued maps proved by Dontchev and Hager [12].

Lemma 2.2. (see [12]) *Let* ϕ *a set–valued map from* X *into the closed subsets of* X*, let* $\eta_0 \in X$ *and let* r *and* λ *be such that* $0 \leqslant \lambda < 1$ *and the following conditions hold:*

(a) dist $(\eta_0, \phi(\eta_0)) \leqslant r(1-\lambda)$.

(b) $e(\phi(x_1) \cap \mathbb{B}_r(\eta_0), \phi(x_2)) \leqslant \lambda \;\| x_1 - x_2 \|, \;\; \forall x_1, x_2 \in \mathbb{B}_r(\eta_0)$.

Then ϕ *has a fixed–point in* $\mathbb{B}_r(\eta_0)$*. That is, there exists* $x \in \mathbb{B}_r(\eta_0)$ *such that* $x \in \phi(x)$*. If* ϕ *is single–valued, then* x *is the unique fixed point of* ϕ *in* $\mathbb{B}_r(\eta_0)$*.*

We need also to introduce some notations before stating the main results on this study. First, for $k \in \mathbb{N}$ and (x_k) defined in (1.2), let us define the set–valued mappings $Q : X \longrightarrow 2^Y$ and $\psi_k : X \longrightarrow 2^X$ by the following

$$Q(.) := A(., x^*) + G(.); \quad \phi_k(.) := Q^{-1}(Z_k(.)), \tag{2.1}$$

where Z_k is defined from X to Y by

$$Z_k(x) := A(x, x^*) - A(x, x_k). \tag{2.2}$$

Let us note that x_1 is a fixed point of ϕ_0 if and only if $0 \in A(x_1, x_0) + G(x_1)$. We will make the following assumptions in a open convex neighborhood V of x^*:

($\mathcal{H}0$) ∇F is K–Lipschitz on V with $K > 0$.

($\mathcal{H}1$) The condition (1.8) is satisfied on V.

($\mathcal{H}1$)* The condition (1.9) is satisfied on V.

($\mathcal{H}2$) The set–valued map $[A(., x^*) + G(.)]^{-1}$ is pseudo–Lipschitz around $(0, x^*)$ with constants M, a and b (these constants are given by definition of Aubin continuity).

Finally, we consider the following constants:

$$\kappa_1 = \int_0^1 t^\theta \,(1-t)\; \sigma(t\,a)\; dt, \tag{2.3}$$

$$\kappa_2 = \sum_{i=1}^{i=M} \left(|a_i|\,|1-\beta_i|^{1+\theta} \int_0^1 (1-t)^\theta \;\sigma(|1-\beta_i|\,(1-t)\,a)\; dt \right), \tag{2.4}$$

and

$$\kappa_3 = \sqrt{\frac{2\,b}{3\,K \displaystyle\sum_{i=1}^{i=M} |a_i|\,(1+2\,|\beta_i|)}}. \tag{2.5}$$

3. Convergence analysis

The main theorem of this study read as follows:

Theorem 3.1. *Let x^* be a solution of* (1.1). *We suppose that assumptions ($\mathcal{H}0$), ($\mathcal{H}1$)* and ($\mathcal{H}2$) are satisfied. For every $C > M\,(\kappa_1+\kappa_2)$, where κ_1 and κ_2 are given respectively by* (2.3) *and* (2.4), *there exist $\delta > 0$ such that for every starting point $x_0 \in \mathbb{B}_\delta(x^*)$, and a sequence (x_k) for* (1.1), *defined by* (1.2), *which satisfies*

$$\parallel x_{k+1} - x^* \parallel \leqslant C \parallel x_k - x^* \parallel^{2+\theta}. \tag{3.1}$$

In other words, (1.2) *generates (x_k) with superquadratic convergence.*

Theorem 3.1 is showed as follows. Once x_k is computed, we show that the function ϕ_k has a fixed point x_{k+1} in X. This process allows us to prove the existence of a sequence (x_k) satisfying (1.2). Now, we state a result which is the starting point of our algorithm. It will be very usefull to prove Theorem 3.1 and reads as follows:

Proposition 3.2. *Under the hypotheses of Theorem* 3.1, *there exists $\delta > 0$ such that for all $x_0 \in \mathbb{B}_\delta(x^*)$ ($x_0 \neq x^*$), the map ϕ_0 has a fixed point x_1 in $B_\delta(x^*)$ satisfying $\parallel x_1 - x^* \parallel \leqslant C \parallel x_0 - x^* \parallel^{2+\theta}$, where the constant C is given by Theorem* 3.1.

Proof of Proposition 3.2. Fix $\delta > 0$ such that

$$\delta < \min\left\{ a,\ \left(\frac{1}{C}\right)^{\frac{1}{1+\theta}},\ \left(\frac{b}{\kappa_1+\kappa_2}\right)^{\frac{1}{2+\theta}},\ \kappa_3 \right\}. \tag{3.2}$$

By hypothesis ($\mathcal{H}2$) we have

$$e(Q^{-1}(y') \cap \mathbb{B}_a(x^*), Q^{-1}(y'')) \leqslant M \parallel y' - y'' \parallel,\ \forall y', y'' \in \mathbb{B}_b(0). \tag{3.3}$$

We apply Lemma 2.2 to map ϕ_0 by choosing

$$\eta_0 := x^* \qquad \text{and} \qquad r := r_0 = C \parallel x^* - x_0 \parallel^{2+\theta},$$

r and λ defined in 2.2 are some numbers to be set for ϕ_0. By (2.2) we have

$$\begin{aligned}\parallel Z_0(x^*) \parallel &= \parallel A(x^*, x^*) - A(x_0, x^*) \parallel \\ &= \parallel F(x^*) - F(x_0) - \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i\,(x^* - x_0))\ (x^* - x_0) \parallel \\ &\leqslant A_1 + A_2, \end{aligned} \tag{3.4}$$

where

$$A_1 = \parallel F(x_0) - F(x^*) - \nabla F(x^*)\,(x_0 - x^*) - \frac{1}{2} \nabla^2 F(x^*)\,(x_0 - x^*)^2 \parallel, \tag{3.5}$$

and

$$\begin{aligned} A_2 = \parallel & \nabla F(x^*)\,(x_0 - x^*) + \\ & \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i\,(x^* - x_0))\ (x^* - x_0) + \frac{1}{2} \nabla^2 F(x^*)\,(x_0 - x^*)^2 \parallel . \end{aligned} \tag{3.6}$$

By Lemma 2.1 and for $x_0 \in \mathbb{B}_\delta(x^*)$ we obtain

$$A_1 \leqslant \kappa_1 \parallel x_0 - x^* \parallel^{2+\theta} . \tag{3.7}$$

By (1.4) we can re–write A_2 given by (3.6) in the following form

$$\begin{aligned} & A_2 \\ &= \parallel \sum_{i=1}^{i=M} a_i (\nabla F(x^*) - \nabla F(x_0 + \beta_i\,(x^* - x_0)))\ (x^* - x_0) - \frac{1}{2} \nabla^2 F(x^*)\,(x_0 - x^*)^2 \parallel . \end{aligned} \tag{3.8}$$

By the Mean Value Theorem (integral representation) we can write

$$\begin{aligned} & \nabla F(x^*) - \nabla F(x_0 + \beta_i\,(x^* - x_0)) \\ & = (1 - \beta_i) \int_0^1 \nabla^2 F(x_0 + (\beta_i + t\,(1 - \beta_i))\,(x^* - x_0))\,dt\,(x^* - x_0). \end{aligned} \tag{3.9}$$

For $x_0 \in \mathbb{B}_\delta(x^*)$ and using (1.4), (3.8), (3.9) and the assumption $(\mathcal{H}1)^\star$ we can estimate A_2 by

$$\begin{aligned} A_2 \leqslant & \sum_{i=1}^{i=M} |a_i|\,|1 - \beta_i| \int_0^1 \parallel \nabla^2 F(x_0 + (\beta_i + t\,(1 - \beta_i))\,(x^* - x_0)) - \nabla^2 F(x^*) \parallel\ dt \\ & \parallel x_0 - x^* \parallel^2 \\ \leqslant & \sum_{i=1}^{i=M} |a_i|\,|1 - \beta_i| \int_0^1 \sigma(|1 - t|\,|1 - \beta_i| \parallel x_0 - x^* \parallel)\,(1 - t)^\theta\,|1 - \beta_i|^\theta\,dt \\ & \parallel x_0 - x^* \parallel^{2+\theta} \\ \leqslant & \kappa_2 \parallel x_0 - x^* \parallel^{2+\theta} . \end{aligned} \tag{3.10}$$

Moreover, for all $x_0 \in B_\delta(x^*)$ such that $x_0 \neq x^*$ we have by (3.4), (3.7) and (3.10) the following estimate

$$\| Z_0(x^*) \| \leqslant (\kappa_1+\kappa_2) \ \| x_0 - x^* \|^{2+\theta} . \tag{3.11}$$

Then (3.2) yields, $\| Z_0(x^*) \| < b$. Hence from (3.3) one has

$$e\Big(Q^{-1}(0)\cap\mathbb{B}_\delta(x^*), \phi_0(x^*)\Big) = e\Big(Q^{-1}(0)\cap\mathbb{B}_\delta(x^*), Q^{-1}[Z_0(x^*)]\Big) \leqslant M\beta \| x^* - x_0 \|^2 .$$

According to the definition of the excess, we get

$$\operatorname{dist}(x^*, \phi_0(x^*)) \leqslant M\,(\kappa_1+\kappa_2) \ \| x^* - x_0 \|^{2+\theta} . \tag{3.12}$$

Since $C > M\,(\kappa_1+\kappa_2)$ there exists $\lambda \in]0,1[$ such that $C(1-\lambda) \geqslant M\,(\kappa_1+\kappa_2)$. Hence,

$$\operatorname{dist}(x^*, \phi_0(x^*)) \leqslant C\,(1-\lambda) \ \| x_0 - x^* \|^{2+\theta} . \tag{3.13}$$

We can deduce from (3.‘3) that assertion (a) in Lemma 2.2 is satisfied. Now, we show that condition (b) of Lemma 2.2 is satisfied. By (3.2) and $\| x^* - x_0 \| \leqslant \delta$, we have $r_0 \leqslant \delta \leqslant a$. Moreover we can write for $x \in \mathbb{B}_\delta(x^*)$

$$\begin{aligned} \| Z_0(x) \| =& \| F(x^*) + \sum_{i=1}^{i=M} a_i \nabla F(x^* + \beta_i\,(x-x^*))\ (x-x^*) \\ & - F(x_0) - \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i\,(x-x_0))\ (x - x^* + x^* - x_0) \| \\ & \leqslant B_1 + B_2, \end{aligned} \tag{3.14}$$

where

$$B_1 = \| F(x^*) - F(x_0) - \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i\,(x-x_0))\ (x^* - x_0) \|, \tag{3.15}$$

and

$$B_2 = \| \sum_{i=1}^{i=M} a_i \Big(\nabla F(x^* + \beta_i\,(x-x^*)) - \nabla F(x_0 + \beta_i\,(x-x_0)) \Big) \| \, \| x - x^* \| . \tag{3.16}$$

By the Mean Value Theorem and assumption $(\mathcal{H}0)$ we obtain

$$\begin{aligned} B_1 \leqslant & \sum_{i=1}^{i=M} |a_i| \int_0^1 \| \nabla F(x_0 + \beta_i\,(x^* - x_0)) - \nabla F(x_0 + \beta_i\,(x-x_0)) \| \ dt \\ & \| x_0 - x^* \| \\ & \leqslant K \sum_{i=1}^{i=M} |a_i| \int_0^1 \| t\,(x^* - x_0) - \beta_i\,(x - x_0)) \| \ dt \ \| x_0 - x^* \| \\ & \leqslant \frac{K\,\delta^2}{2} \sum_{i=1}^{i=M} |a_i|\,(1 + 4\,|\beta_i|) \end{aligned} \tag{3.17}$$

and

$$B_2 \leqslant K \sum_{i=1}^{i=M} |a_i| \ \| (x^* - x_0) + \beta_i \, (x - x^* - (x - x_0)) \| \, \| \, x - x^* \, \|$$
$$\leqslant K \, \delta^2 \sum_{i=1}^{i=M} |a_i| \, (1 + |\beta_i|). \tag{3.18}$$

Using (3.17) and (3.18), the inequality (3.14) becomes

$$\| \, Z_0(x) \, \| \leqslant \frac{3 \, K \, \delta^2}{2} \sum_{i=1}^{i=M} |a_i| \, (1 + 2 \, |\beta_i|) \tag{3.19}$$

Then by (3.2) we deduce that for all $x \in \mathbb{B}_\delta(x^*)$, $Z_0(x) \in \mathbb{B}_b(0)$. Then it follows that for all $x', x'' \in \mathbb{B}_{r_0}(x^*)$, we have

$$I = e(\phi_0(x') \cap \mathbb{B}_{r_0}(x^*), \phi_0(x'')) \leqslant e(\phi_0(x') \cap \mathbb{B}_\delta(x^*), \phi_0(x'')), \tag{3.20}$$

which yields by (3.3)

$$I \leqslant M \ \| \, Z_0(x') - Z_0(x'') \, \| \, . \tag{3.21}$$

By Assumption $(\mathcal{H}0)$ and (3.2) we deduce that

$$\begin{aligned}
I \leqslant M \ \| & \sum_{i=1}^{i=M} a_i \nabla F(x^* + \beta_i \, (x' - x^*)) \, (x' - x'' + x'' - x^*) \\
& - \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i \, (x' - x_0)) \, (x' - x'' + x'' - x_0) \\
& - \sum_{i=1}^{i=M} a_i \nabla F(x^* + \beta_i \, (x'' - x^*)) \, (x'' - x^*) \\
& + \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i \, (x'' - x_0)) \, (x'' - x_0) \, \| \\
\leqslant M \Bigg(& \| \sum_{i=1}^{i=M} a_i \nabla F(x^* + \beta_i \, (x' - x^*)) \\
& - \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i \, (x' - x_0)) \, \| \, \| \, x'' - x' \, \| \qquad (3.22) \\
& + \| \sum_{i=1}^{i=M} a_i \nabla F(x^* + \beta_i \, (x' - x^*)) - \sum_{i=1}^{i=M} a_i \nabla F(x^* + \beta_i \, (x'' - x^*)) \, \| \| \, x'' - x^* \, \| \\
& + \| \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i \, (x'' - x_0)) - \sum_{i=1}^{i=M} a_i \nabla F(x_0 + \beta_i \, (x' - x_0)) \, \| \| \, x'' - x_0 \, \| \Bigg) \\
\leqslant M \, K \, \delta & \sum_{i=1}^{i=M} |a_i| \, (1 + 4 \, |\beta_i|) \ \ \| \, x'' - x' \, \| \, .
\end{aligned}$$

By choosing λ such that $\delta\, M\, K \sum_{i=1}^{i=M} |a_i|\,(1+4\,|\beta_i|) < \lambda$, the condition (b) of Lemma 2.2 is satisfied. We can deduce the existence of a fixed point $x_1 \in \mathbb{B}_{r_0}(x^*)$ for the map ϕ_0. The proof of Proposition 3.2 is completed. ■

Now that we proved Proposition 3.2, the proof of Theorem 3.1 is straightforward as it is shown below.

Proof of Theorem 3.1. Proceeding by induction, keeping $\eta_0 = x^*$ and setting $r_k = C \parallel x_k - x^* \parallel^{2+\theta}$, the application of Proposition 3.2 to the map ϕ_k respectively gives the desired result. ■

When the second Fréchet derivative satisfies w–condition given by (1.8), we obtain the following result involving a quadratic convergence of algorithm (1.2).

Proposition 3.3. *Let x^* be a solution of* (1.1). *We suppose that assumptions ($\mathcal{H}0$), ($\mathcal{H}1$) and ($\mathcal{H}2$) are satisfied. For every $C' > M\,(\kappa_1' + \kappa_2')$, where κ_1' and κ_2' are given respectively by*

$$\kappa_1' = \int_0^1 (1-t)\ \omega(t\,a)\ dt, \tag{3.23}$$

and

$$\kappa_2' = \sum_{i=1}^{i=M} \left(|a_i|\,|1-\beta_i| \int_0^1\ \omega(|1-\beta_i|\,(1-t)\,a)\ dt \right), \tag{3.24}$$

there exist $\gamma > 0$ with

$$\gamma < \min\left\{ a,\ \frac{1}{C'},\ \left(\frac{b}{\kappa_1' + \kappa_2'}\right)^{\frac{1}{2}},\ \kappa_3 \right\}, \tag{3.25}$$

such that for every starting point $x_0 \in \mathbb{B}_\gamma(x^*)$, and a sequence (x_k) for (1.1), *defined by* (1.2), *which satisfies*

$$\parallel x_{k+1} - x^* \parallel \leqslant C' \parallel x_k - x^* \parallel^2 . \tag{3.26}$$

4. An improved local convergence and remarks

In this section, we show by using more precise estimates that under less computational cost, and weaker hypotheses (see (1.10) and (1.11)): the ratio of convergence of method (1.2) is improved and the radius of convergence is enlarged. The idea from the works on nonlinear equations [3], [4]. We consider the following constants:

$$\overline{\kappa_1} = \int_0^1 t^\theta\,(1-t)\ \overline{\sigma}(t\,a)\ dt, \tag{4.1}$$

$$\overline{\kappa_2} = \sum_{i=1}^{i=M} \left(|a_i| \, |1-\beta_i|^{1+\theta} \int_0^1 (1-t)^\theta \, \overline{\sigma}(|1-\beta_i| \, (1-t) \, a) \, dt \right), \tag{4.2}$$

$$\overline{\kappa_1'} = \int_0^1 (1-t) \, \overline{\omega}(t \, a) \, dt, \tag{4.3}$$

and

$$\overline{\kappa_2'} = \sum_{i=1}^{i=M} \left(|a_i| \, |1-\beta_i| \int_0^1 \overline{\omega}(|1-\beta_i| \, (1-t) \, a) \, dt \right). \tag{4.4}$$

The following results improve Theorem 3.1 and Proposition 3.3.

Proposition 4.1. *We suppose that assumptions* ($\mathcal{H}0$), (1.11) *and* ($\mathcal{H}2$) *are satisfied. For every* $\overline{C} > M\,(\overline{\kappa_1} + \overline{\kappa_2})$, *where* $\overline{\kappa_1}$ *and* $\overline{\kappa_2}$ *are given respectively by* (4.1) *and* (4.2), *there exist* $\overline{\delta} > 0$ *such that for every starting point* $x_0 \in \mathbb{B}_{\overline{\delta}}(x^*)$, *and a sequence* (x_k) *for* (1.1), *defined by* (1.2), *which satisfies*

$$\| \, x_{k+1} - x^* \, \| \leqslant \overline{C} \, \| \, x_k - x^* \, \|^{2+\theta} \, . \tag{4.5}$$

Proposition 4.2. *We suppose that assumptions* ($\mathcal{H}0$), (1.10) *and* ($\mathcal{H}2$) *are satisfied. For every* $\overline{C'} > M\,(\overline{\kappa_1'} + \overline{\kappa_2'})$, *where* $\overline{\kappa_1'}$ *and* $\overline{\kappa_2'}$ *are given respectively by* (4.3) *and* (4.4), *there exist* $\overline{\gamma} > 0$ *such that for every starting point* $x_0 \in \mathbb{B}_{\overline{\gamma}}(x^*)$, *and a sequence* (x_k) *for* (1.1), *defined by* (1.2), *which satisfies*

$$\| \, x_{k+1} - x^* \, \| \leqslant \overline{C'} \, \| \, x_k - x^* \, \|^{2} \, . \tag{4.6}$$

The proof of Proposition 4.1 and Proposition 4.2 is the same one as that of the proof of Proposition 3.2. It is enough to make some modifications by choosing the constants $\overline{\delta}$ and $\overline{\gamma}$ in Proposition 4.1 and Proposition 4.2 respectively such that

$$\overline{\delta} < \min \left\{ a, \, \left(\frac{1}{\overline{\overline{C}}} \right)^{\frac{1}{1+\theta}}, \, \left(\frac{b}{\overline{\kappa_1} + \overline{\kappa_2}} \right)^{\frac{1}{2+\theta}}, \, \kappa_3 \right\}, \tag{4.7}$$

and

$$\overline{\gamma} < \min \left\{ a, \, \frac{1}{\overline{\overline{C'}}}, \, \left(\frac{b}{\overline{\kappa_1'} + \overline{\kappa_2'}} \right)^{\frac{1}{2}}, \, \kappa_3 \right\}. \tag{4.8}$$

Remark 4.3. In general, ω and σ given in (1.8) and (1.9) are not easy to compute. This is our motivation in this section for introducing weaker hypotheses (1.10) and (1.11).

Note that in general

$$\overline{\omega} \leqslant \omega, \tag{4.9}$$
$$\overline{\sigma} \leqslant \sigma, \tag{4.10}$$
$$\overline{\kappa_1} \leqslant \kappa_1, \tag{4.11}$$
$$\overline{\kappa_2} \leqslant \kappa_2, \tag{4.12}$$
$$\overline{\kappa_1'} \leqslant \kappa_1', \tag{4.13}$$

and

$$\overline{\kappa_2'} \leqslant \kappa_2', \tag{4.14}$$

holds, and $\dfrac{\omega}{\overline{\omega}}$, $\dfrac{\sigma}{\overline{\sigma}}$, $\dfrac{\kappa_1+\kappa_2}{\overline{\kappa_1}+\overline{\kappa_2}}$, $\dfrac{\kappa_1'+\kappa_2'}{\overline{\kappa_1'}+\overline{\kappa_2'}}$ can be arbitrarily large [3], [4]. It then follows from the definitions of C, C', $\overline{C}$, $\overline{C'}$, (3.2), (4.7), (3.25) and (4.8) that

$$\overline{C} \leqslant C, \tag{4.15}$$
$$\overline{C'} \leqslant C', \tag{4.16}$$
$$\delta \leqslant \overline{\delta}, \tag{4.17}$$

and

$$\gamma \leqslant \overline{\gamma}. \tag{4.18}$$

Note that parameters $\overline{\omega}$ and $\overline{\sigma}$ are easier to determine than ω and σ. Using the above observations we have provided in this section under weaker hypotheses and less computational cost a local convergence analysis with the following advantages:

(i) A larger radius of convergence which allows a larger choice of initial guesses x_0.

(ii) A finer error estimates on the distances $\| x_n - x^* \|$ $(n \geqslant 0)$. These observations are very important in computational mathematics [3], [4].

Remark 4.4. In order for us to compare our results with the corresponding ones in [9]–[11], let us mention that stronger conditions (1.5)–(1.7) used in [9]–[11] to prove a result similar to Theorem 3.1 are a particular cases of our hypotheses. Our assumption (1.8) extends the condition (1.6) used in [9] by considering ω in the form $\omega(t) = L\, t^{\theta}$ and if $\theta = 1$ then $\omega(t) = L\, t$ corresponds to the condition (1.5) used in [10]. We can also notice that the condition (1.7) used in [11] is reduced to the particular case of our condition (1.10) by considering $\overline{\omega}(t) = L_0\, t^{\theta}$.

Define also parameters $\overline{\overline{\delta}}$ and $\overline{\overline{C}}$ used in [9] by

$$\overline{\overline{\delta}} < \min\left\{ a,\ \left(\frac{1}{\overline{\overline{C}}}\right)^{\frac{1}{1+\theta}}, \left(\frac{b\,(1+\theta)\,(2+\theta)}{K\,(1+(2+\theta)\sum_{i=1}^{i=M} |a_i|\,|1-\beta_i|^{1+\theta})}\right)^{\frac{1}{2+\theta}}, \kappa_3 \right\}, \tag{4.19}$$

and

$$\overline{\overline{C}} = \frac{M\,L}{(1+\theta)\,(2+\theta)}\;(1+(2+\theta)\sum_{i=1}^{i=M}|a_i|\,|1-\beta_i|^{1+\theta}). \tag{4.20}$$

Clearly, if we suppose that σ in (1.9) is the constant function equal to L then a simple computation of our constants κ_1 and κ_2 given by (2.3) and (2.4) respectively allows us to check that:

$$\overline{\overline{C}} = M\,(\kappa_1+\kappa_2), \tag{4.21}$$

and

$$\overline{\overline{\delta}} = \delta. \tag{4.22}$$

Hence, the claims made in the introduction have been justified.

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