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ON EXPONENTIAL SUMS WITH HECKE SERIES AT CENTRAL POINTS

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Dedicated to Jean-Marc Deshouillers
on the occasion of his 60th birthday

Abstract: Upper bound estimates for the exponential sum

$$\sum_{K<\kappa_j\leqslant K'<2K} \alpha_j H_j^3(\tfrac12)\cos\left(\kappa_j\log\left(\frac{4\mathrm{e}T}{\kappa_j}\right)\right) \qquad (T^\varepsilon \leqslant K \leqslant T^{1/2-\varepsilon})$$

are considered, where $\alpha_j = |\rho_j(1)|^2(\cosh\pi\kappa_j)^{-1}$, and $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue $\lambda_j = \kappa_j^2+\frac14$ to which the Hecke series $H_j(s)$ is attached. The problem is transformed to the estimation of a classical exponential sum involving the binary additive divisor problem. The analogous exponential sums with $H_j(\frac12)$ or $H_j^2(\frac12)$ replacing $H_j^3(\frac12)$ are also considered. The above sum is conjectured to be $\ll_\varepsilon K^{3/2+\varepsilon}$, which is proved to be true in the mean square sense.
Keywords: Hecke series, Riemann zeta-function, hypergeometric function, exponential sums.

1. Introduction

The main purpose of this paper is to transform and estimate exponential sums of Hecke series at central points, namely the sums

$$S(K) = S(K;K',T) := \sum_{K<\kappa_j\leqslant K'} \alpha_j H_j^3(\tfrac12)\cos\left(\kappa_j\log\frac{4\mathrm{e}T}{\kappa_j}\right), \tag{1.1}$$

under the condition

$$T^\varepsilon \leqslant K < K' \leqslant 2K \leqslant T^{1/2-\varepsilon}. \tag{1.2}$$

Sums of this form are important in the theory of the Riemann zeta-function $\zeta(s)$; see e.g., (2.5) and (2.8) for more details. Here and later $\varepsilon>0$ denotes arbitrarily small constants, not necessarily the same ones at each occurrence. The quantities

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$\alpha_j, H_j(\frac12)$ and κ_j are connected with the spectral theory of the non-Euclidean Laplacian. For a comprehensive account of spectral theory the reader is referred to Y. Motohashi's monograph [23], and here we only briefly explain some basic notions.

Let $\{\lambda_j = \kappa_j^2 + \frac14\}_{j=1}^\infty \cup \{0\}$ be the eigenvalues (discrete spectrum) of the hyperbolic Laplacian

$$\Delta = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)$$

acting over the Hilbert space composed of all Γ-automorphic functions which are square integrable with respect to the hyperbolic measure ($\Gamma = \mathrm{PSL}(2,\mathbb{Z})$). Let $\{\psi_j\}_{j=1}^\infty$ be a maximal orthonormal system such that $\Delta\psi_j = \lambda_j\psi_j$ for each $j \geqslant 1$ and $T(n)\psi_j = t_j(n)\psi_j$ for each integer $n \in \mathbb{N}$, where

$$\big(T(n)f\big)(z) \;=\; \frac{1}{\sqrt{n}}\sum_{ad=n}\sum_{b=1}^{d} f\left(\frac{az+b}{d}\right)$$

is the Hecke operator. We shall further assume that $\psi_j(-\bar z) = \epsilon_j\psi_j(z)$ with the parity sign $\epsilon_j = \pm 1$. We then define ($s = \sigma + it$ will denote a complex variable)

$$H_j(s) \;=\; \sum_{n=1}^\infty t_j(n)n^{-s} \qquad (\sigma > 1),$$

which we call the Hecke series associated with the Maass wave form $\psi_j(z)$, and which can be continued analytically to an entire function over $\mathbb{C}$. It is known that $H_j(\frac12) \geqslant 0$ (see Katok–Sarnak [15]), and that

$$\sum_{\kappa_j\leqslant K}\alpha_j H_j^3(\tfrac12) = K^2\sum_{j=0}^{3} d_j \log^j K + O(K^{5/4}\log^{37/4} K) \tag{1.3}$$

with suitable constants d_j, proved by the author in [9]. Here as usual we insert in the sum over κ_j the normalizing factor

$$\alpha_j = |\rho_j(1)|^2(\cosh \pi\kappa_j)^{-1},$$

where $\rho_j(1)$ is the first Fourier coefficient of $\psi_j(z)$. We also have (see the author's paper [7])

$$\sum_{K-G\leqslant\kappa_j\leqslant K+G}\alpha_j H_j^3(\tfrac12) \;\ll_\varepsilon\; GK^{1+\varepsilon} \tag{1.4}$$

for

$$K^\varepsilon \;\leqslant\; G \;\leqslant\; K. \tag{1.5}$$

In view of $H_j(\frac{1}{2}) \geqslant 0$ we obtain from (1.4) the convexity-breaking bound $H_j(\frac{1}{2}) \ll_\varepsilon \kappa_j^{1/3+\varepsilon}$, which is hitherto the sharpest one.

Note that by (1.3) and trivial estimation we obtain

$$S(K) \ll K^2 \log^3 K, \tag{1.6}$$

and our wish is to try to decrease the exponent of K in (1.6). It was conjectured in [8] that

$$\sum_{K-1\leqslant\kappa_j\leqslant K+1} \alpha_j H_j^3(\tfrac{1}{2}) \exp\left(i\kappa_j \log\left(\frac{\tau}{\kappa_j}\right)\right) \ll_\varepsilon K^{1/2+\varepsilon} \tag{1.7}$$

holds for

$$\tau^\delta \ll K \ll \tau^{1+\delta} \qquad (0<\delta<1). \tag{1.8}$$

This gives

$$S(K) \ll_\varepsilon K^{3/2+\varepsilon}, \tag{1.9}$$

thereby improving (1.6) by essentially a factor of $\sqrt{K}$. The conjecture (1.7)–(1.8) is deep, and is certainly out of reach at present. Heuristic reasons that it is best possible are given in [8]. It was also shown there that its truth would imply essentially the best possible bounds for the eighth moment of $|\zeta(\frac{1}{2}+it)|$, and for the error term (see (2.2)) in the fourth moment formula for $|\zeta(\frac{1}{2}+it)|$.

2. Statement of results

If $d(k)$ is the number of divisors of k, then we have

Theorem 1. *If $S(K)$ is defined by (1.1) and (1.2) holds, then for some constants $0 < C_1 < C_2, c_\ell$ and $L \in \mathbb{N}$, all of which may be effectively evaluated, we have*

$$\begin{aligned} S(K) = \Re\mathrm{e}\Bigg[\sum_{f\leqslant 3K} f^{\frac{1}{2}} \sum_{C_1TK^{-1}f\leqslant m\leqslant C_2TK^{-1}f} & m^{-\frac{3}{2}} d(m)d(m+f)\mathrm{e}^{i\frac{Tf}{m}}\Big\{c_0+ \\ &+\sum_{l=1}^{L} c_\ell\varphi_\ell(K,T;m,f)\Big\}\Bigg] + O_\varepsilon(K^{\frac{3}{2}+\varepsilon}). \end{aligned} \tag{2.1}$$

The functions $\varphi_\ell(K,T;m,f)$ may be also explicitly evaluated, and they are all $o(1)$ as $K\to\infty$ and (1.2) holds.

The explicit shape of the functions $\varphi_\ell(K,T;m,f)$ will transpire during the proof, and a discussion on their precise shape is given at the end of Section 5. Essentially they are (positive or negative) powers in each variable. Thus they are non-oscillating and, as stated, all $o(1)$ as $K\to\infty$ and (1.2) holds. The important fact is that they do not affect the oscillating factor $\mathrm{e}^{i\frac{Tf}{m}}$ in (2.1), and in fact can be removed conveniently by partial summation techniques. For these reasons it seemed more expedient to formulate Theorem 1 in the form given by (2.1), than to write down explicitly all the functions $\varphi_\ell(K,T;m,f)$. The number L is a (large) constant, arising in (3.5) (and later in a similar context). It comes from cutting the tails of a suitable series in such a way that the tails in question make a negligible contribution. By "negligible contribution" we shall mean, here and later, a contribution which is $\ll K_0^{-A}$ (or $\ll T^{-A}$) for any fixed $A > 0$.

To abbreviate notation, sometimes in the proof we shall write expressions similar to (2.1) as $A \asymp B + O_\varepsilon(K^{\frac{3}{2}+\varepsilon})$. Namely $A \asymp B$ will mean, here and later, that A is a multiple of B, plus a finite number of sums (terms), each of which gives a bound not larger than the bound for B, with some non-oscillating functions $\varphi_\ell(K,T;m,f)$, as in (2.1).

The importance of the sum $S(K)$ comes primarily from its connection with the function $E_2(T)$, the error term in the asymptotic formula for the fourth moment of $|\zeta(\frac{1}{2}+it)|$. This formula is customarily written as

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t \;=\; TP_4(\log T) \;+\; E_2(T), \quad P_4(x) \;=\; \sum_{j=0}^{4} a_j x^j. \tag{2.2}$$

It was proved by A.E. Ingham that $a_4 = 1/(2\pi^2)$ (see e.g., [2, Chapter 5]), and much later by D.R. Heath-Brown [1] that

$$a_3 \;=\; 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2},$$

who also produced more complicated expressions for a_0, a_1 and a_2 in (2.3) ($\gamma = 0.577\ldots$ is Euler's constant). For an explicit evaluation of the a_j's the reader is referred to [4].

In recent years, due to the application of powerful methods of spectral theory, much advance has been made in connection with $E_2(T)$. We refer the reader to the works [3], [5], [6], [11]–[13], [20] and [21]–[24]. Thus N.I. Zavorotnyi [24] proved that $E_2(T) = O_\varepsilon(T^{2/3+\varepsilon})$, and it is known now that

$$E_2(T) \;=\; O(T^{2/3}\log^{C_1} T), \quad E_2(T) \;=\; \Omega(T^{1/2}), \tag{2.3}$$

and

$$\int_0^T E_2(t)\,\mathrm{d}t \;=\; O(T^{3/2}), \quad \int_0^T E_2^2(t)\,\mathrm{d}t \;=\; O(T^2\log^{C_2} T), \tag{2.4}$$

with effective constants $C_1, C_2 > 0$ (the values $C_1 = 8, C_2 = 22$ are worked out in [23]). The above results were proved by Y. Motohashi and the author (see [3], [11], [12] and [21]). The omega-result in (2.3) ($f = \Omega(g)$ means that $f = o(g)$ does not hold, $f = \Omega_\pm(g)$ means that $\limsup f/g > 0$ and that $\liminf f/g < 0$) was improved to $E_2(T) = \Omega_\pm(T^{1/2})$ by Y. Motohashi [22]. There is an obvious discrepancy between the O–result and Ω–result in (2.3). It was already mentioned that the conjecture $E_2(T) = O_\varepsilon(T^{1/2+\varepsilon})$ holds if the conjecture (1.7)–(1.8) is true. It would imply (by (2.9)) the hitherto unproved bound $\zeta(\frac{1}{2}+it) \ll_\varepsilon t^{1/8+\varepsilon}$.

Y. Motohashi proved (see [3, Chapter 6] and [23])

$$\begin{aligned}&\frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty}|\zeta(\tfrac12+iT+it)|^4\exp(-(t/G)^2)\,\mathrm{d}t \qquad (2.5)\\ &=\frac{\pi}{\sqrt{2T}}\sum_{j=1}^{\infty}\alpha_jH_j^3(\tfrac12)\kappa_j^{-\frac12}\sin\Big(\kappa_j\log\frac{\kappa_j}{4\mathrm{e}T}\Big)\exp\Big(-\tfrac14\big(\frac{G\kappa_j}{T}\big)^2\Big)+O(\log^{3D+9}T),\end{aligned}$$

if $T^{1/2}\log^{-D}T\leqslant G\leqslant T/\log T$ for an arbitrary, fixed constant $D>0$, and

$$\begin{aligned}&\frac{1}{\sqrt{\pi}G}\int_0^V\int_{-\infty}^{\infty}|\zeta(\tfrac12+iT+it)|^4\exp(-(t/G)^2)\,\mathrm{d}t\,\mathrm{d}T \qquad (2.6)\\ &=VP_4(\log V)+\pi\sqrt{\tfrac12V}\sum_{j=1}^{\infty}\alpha_jH_j^3(\tfrac12)\kappa_j^{-\frac32}\cos\Big(\kappa_j\log\frac{\kappa_j}{4\mathrm{e}V}\Big)\exp\Big(-\tfrac14\big(\frac{G\kappa_j}{V}\big)^2\Big)\\ &\quad+O(V^{1/2}\log^CV)+O(G\log^5V),\end{aligned}$$

for $V^{1/2}\log^{-A}V\leqslant G\leqslant V\exp(-\sqrt{\log V})$, $C=C(A)\ (>0)$ for any arbitrary, fixed constant $A>0$, where P_4 is given by (2.2). Then we have, as proved in [3, Lemma 5.1],

$$\begin{aligned}E_2(2T)-E_2(T)\leqslant S(2T+\Delta\log T,\Delta)-S(T-\Delta\log T,\Delta)&\\ +O(\Delta\log^5T)+O(T^{1/2}\log^CT)&\end{aligned}\qquad (2.7)$$

with $T^{1/2}\leqslant\Delta\leqslant T^{1-\varepsilon}$ and

$$S(T,\Delta):=\pi\sqrt{\tfrac12T}\sum_{j=1}^{\infty}\alpha_jH_j^3(\tfrac12)\kappa_j^{-\frac32}\cos\Big(\kappa_j\log\frac{\kappa_j}{4\mathrm{e}T}\Big)\exp\Big(-\tfrac14\big(\frac{\Delta\kappa_j}{T}\big)^2\Big). \qquad (2.8)$$

A lower bound analogous to (2.7) holds also for $E_2(2T)-E_2(T)$, and the estimation of $\zeta(\frac12+it)$ is derived from [3, Lemma 4.1], namely

$$\zeta(\tfrac12+iT)\ll\log^{5/4}T+\Big(\log T\max_{t\in[T-1,T+1]}|E_2(t)|\Big)^{1/4}. \qquad (2.9)$$

The upper bound in (2.3) follows from (2.7)–(2.8) and trivial estimation, namely (1.6), since the innocuous factors $\kappa_j^{-\frac32}$ and $\exp\Big(-\frac14\big(\frac{\Delta\kappa_j}{T}\big)^2\Big)$ can be conveniently removed by partial summation from (2.8). Thus the problem of the estimation of $E_2(T)$ (and hence also $\zeta(\frac12+it)$) is reduced to the estimation of our sum $S(K)$. The Lindelöf exponent $\mu(\frac12)$ is therefore seen not to exceed one fourth of the exponent in the bound for $E_2(T)$ where, as usual, we define the Lindelöf exponent as

$$\mu(\sigma)\ =\ \limsup_{t\to\infty}\frac{\log|\zeta(\sigma+it)|}{\log t}\qquad(\sigma\in\mathbb{R}). \qquad (2.10)$$

The famous, yet unsettled, Lindelöf hypothesis is that $\mu(\frac12) = 0$, or equivalently that $\mu(\sigma) = 0$ for $\sigma \geqslant \frac12$.

The prominent feature of (2.1) is that the right-hand side contains no quantities from spectral theory, but only classical exponential sums with the divisor function $d(n) = \sum_{\delta|n} 1$. In fact, the sum in question can be considered as an exponential sum attached to the so-called binary additive divisor problem (the evaluation and estimation of $\sum_{m\leqslant x} d(m)d(m+f)$, where f is not fixed). Averages for $E(x;f)$, the error term in the asymptotic formula for this sum, have been obtained by Y. Motohashi and the author [13]. The techniques developed in this work could be applied here, since the problem reduces to the evaluation of the sum ($X \asymp Y$ means that $C_1X \leqslant Y \leqslant C_2X$ holds for some constants $0 < C_1 < C_2$)

$$\sum_{F<f\leqslant 2F} \int_N^{2N} \mathrm{e}^{i\frac{Tf}{x}} E(x;f)\,\mathrm{d}x \qquad \left(F \ll K,\, N \asymp \frac{TF}{K}\right). \tag{2.11}$$

Also the sum in (2.11) could be, at least in principle, evaluated by Motohashi's formula [21] for the sum $\sum_{n=1}^\infty d(n)d(n+f)W(n/f)$, where W is a suitable smooth function. Unfortunately, it appears that after the application of these procedures one will eventually wind up with a sum of the same type as $S(K)$ in (1.1), plus some manageable error terms. The mechanism is technically quite involved, and for this reason it will not be discussed here in detail. However, it can be seen heuristically from (4.4)–(4.7) of [13]. Namely the major contribution to $E(x;f)$ comes from

$$\Re\mathrm{e}\left\{\tfrac12 x^{1/2} \sum_{\kappa_j\leqslant Q} \alpha_j t_j(f) H_j^2(\tfrac12)(f/x)^{i\kappa_j} v(\kappa_j)\right\}, \tag{2.12}$$

where $v(x) \ll x^{-3/2}$ and Q is a parameter satisfying certain conditions. Inserting (2.12) expression in (2.11) we obtain exponential integrals with the saddle point at $x_0 \asymp TF/\kappa_j$, hence $\kappa_j \asymp K$ is the relevant range for κ_j. After the evaluation of the integral by the saddle point method (see e.g., [2, Chapter 2]) we replace sums of $t_j(f)f^{-1/2}$ with $H_j(\frac12)$ plus (small) error, to arrive at sums of the type $S(K)$ in (1.1), i.e., our original sum.

This type of impasse is well known from the estimation of classical exponential sums (of the van der Corput type), where the so-called B-process (essentially Poisson summation), when applied twice, leads to the original exponential sum plus some (usually manageable) error terms. It vitiates our efforts to attain a satisfactory estimate via the application of binary additive problem techniques. Naturally, one may try other methods to obtain from (2.1) a non-trivial bound, even if conditional estimates such as the Lindelöf hypothesis are assumed. However, at present this seems difficult. One can separate the variables in (2.1) by setting $n = m+f$ and letting f lie in intervals of the form $[F, 2F]$ with $F \ll K$. Then the sum is majorized by $O(\log T)$ subsums of the form

$$\left|\sum_{C_1TK^{-1}F\leqslant m\leqslant C_2TK^{-1}F} d(m)m^{-3/2} \sum_{m+F<n\leqslant m+2F} (n-m)^{1/2} d(n)\mathrm{e}^{iTn/m}\right|.$$

The factor $(n-m)^{1/2}$ can be conveniently removed by partial summation. After that, one can apply the Voronoï summation formula (see e.g., [2, Chapter 3]) to the sum over n. The main difficulty is that the sum over n is "short", in the sense that F is much smaller than m, and even after the application of the Voronoï summation formula to both sums, nothing better than the final trivial estimate $\ll_\varepsilon T^{1/2+\varepsilon}K^{3/2}$ seems to come out. This is no surprise, since even the trivial bound

$$\sum_{x<n\leqslant x+h} d(n) \ll_\varepsilon hx^\varepsilon \qquad (1 \ll h \ll x)$$

cannot be obtained yet by the Voronoï summation formula. Other methods, such as the use of J.R. Wilton's approximate functional equation and related transformations (see M. Jutila [14]) can be also applied to the sum over n, but the problem remains a very difficult one.

Instead of the sum $S(K)$ in (1.1) we may consider the analogous sums when $H_j^3(\frac12)$ is replaced by $H_j(\frac12)$ or $H_j^2(\frac12)$. The problem becomes then considerably less difficult. On the other hand the exponential sums in question do not seem to have immediate applications such as $S(K)$ does. As we saw, $S(K)$ is crucial in the estimation of $E_2(T)$ and $\zeta(\frac12+it)$, which is our primary motivation. We shall prove

Theorem 2. *If* (1.2) *holds, then*

$$\begin{aligned} \sum_{K<\kappa_j\leqslant K'<2K} \alpha_j H_j^2(\tfrac12)\cos\left(\kappa_j\log\left(\frac{4\mathrm{e}T}{\kappa_j}\right)\right) &\ll_\varepsilon T^{1/2+\varepsilon}K^{1/2},\\ \sum_{K<\kappa_j\leqslant K'<2K} \alpha_j H_j(\tfrac12)\cos\left(\kappa_j\log\left(\frac{4\mathrm{e}T}{\kappa_j}\right)\right) &\ll_\varepsilon T^{1/2+\varepsilon}K^{1/4}. \end{aligned} \tag{2.13}$$

Therefore we see that the first bound improves the trivial bound (see Y. Motohashi [23]) $O(K^2\log K)$ in the range $T^{1/3+\varepsilon}\leqslant K\leqslant T^{1/2-\varepsilon}$. The trivial bound for the second sum in (2.13) is $O(K^2)$ (see Ivić–Jutila [10]), and it is improved for K satisfying $T^{2/7+\varepsilon}\leqslant K\leqslant T^{1/2-\varepsilon}$. Clearly the method of proof of Theorem 1 and Theorem 2 can be used to estimate certain other exponential sums of a similar nature.

Similarly to the conjecture (1.9), one may conjecture that the sums on the right-hand side of (2.13) are both $\ll_\varepsilon K^{3/2+\varepsilon}$. This conjecture, like (1.9), is supported by the following mean square result. This is

Theorem 3. *Let, for* $m\in\mathbb{N}$ *and* $1\ll K<K'\leqslant 2K\ll T, T\leqslant t\leqslant 2T$,

$$S_m(K;K',t) := \sum_{K<\kappa_j\leqslant K'<2K} \alpha_j H_j^m(\tfrac12)\cos\left(\kappa_j\log\left(\frac{4\mathrm{e}t}{\kappa_j}\right)\right). \tag{2.14}$$

Then, for $m=1,2,3$,

$$\int_T^{2T} \left(S_m(K;K',t)\right)^2 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}K^3. \tag{2.15}$$

Corollary. *We have*

$$\int_0^T E_2^2(t)\,\mathrm{d}t \ll_\varepsilon T^{2+\varepsilon}. \tag{2.16}$$

Note that (2.16) is a slightly weakened form of the second bound in (2.4), obtained by Ivić–Motohashi [11], and it is essentially best possible (see the author's paper [6]). The proof in [11] was based on a large values estimate for $E_2(T)$, whose derivation employed the spectral large sieve inequality. The new proof of (2.16) is simpler, being a direct consequence of (2.15) with $m = 3$.

The plan of the paper is as follows. In Section 3 we make the technical preparation for the proof. Instead of the "long" sum (1.1), we shall use the transformation formulas involving $H_j(\frac12)$ for suitable (smooth) "short" sums. Then we integrate over the parameter to recover eventually the desired "long" sum. The necessary tool, which transforms our problem into a problem of the estimation of the double exponential sum (cf. (2.1)) with two divisor functions, is Motohashi's formula. It it presented in Section 4. The principal part of the proof of Theorem 1 is contained in Section 5, and the remaining part will be given in Section 6. Finally Theorem 2 is proved in Section 7, while Theorem 3 is proved in Section 8.

3. Technical preparation for the proof

The basic idea of the proof of Theorem 1 is, as with the proof of (1.4)-(1.5) in [7], to use the transformation formula of Y. Motohashi (see [19] and [23, Chapter Chapter 3]) for bilinear forms of Hecke L-functions. Unfortunately, the shape (1.1) of the fundamental sum $S(K)$ is not suited for the direct application of the transformation formula. Before we can apply it, we have to transform $S(K)$ into a suitable form. Although this is primarily a technical problem, it is not obvious how one should tackle it, and therefore the details will be given in this section.

We begin by considering, under the condition (1.2), the expression

$$\begin{aligned} &\frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}\sum_{j=1}^\infty \alpha_j H_j^3(\tfrac12)\exp\left(i\kappa_j\log\frac{4\mathrm{e}T}{\kappa_j}-(\kappa_j-K)^2G^{-2}\right)\mathrm{d}K\\ &\qquad = \frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}\sum(K;G)\,\mathrm{d}K, \end{aligned} \tag{3.1}$$

say, where $G = G(K_0)$ is a parameter satisfying

$$K_0^\varepsilon \leqslant G \leqslant K_0^{1/2-\varepsilon}, \quad K_0 \leqslant K \leqslant K_0' \leqslant 2K_0. \tag{3.2}$$

Exchanging the order of summation and integration in (3.1) we have, in view of

(1.1), that

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}G}\Re\mathrm{e}\left\{\int_{K_0}^{K_0'}\sum(K;G)\,\mathrm{d}K\right\}\\
&=\frac{1}{\sqrt{\pi}G}\Re\mathrm{e}\left\{\sum_{K_0-G\log K_0\leqslant\kappa_j\leqslant K_0'+G\log K_0}\alpha_j H_j^3(\tfrac12)\exp\left(i\kappa_j\log\frac{4\mathrm{e}T}{\kappa_j}\right)\times\right.\\
&\quad\left.\times\int_{\kappa_j-G\log K_0}^{\kappa_j+G\log K_0}\mathrm{e}^{-(\kappa_j-K)^2G^{-2}}\,\mathrm{d}K\right\}+O_\varepsilon(K_0^\varepsilon)\\
&=\Re\mathrm{e}\left\{\sum_{K_0-G\log K_0\leqslant\kappa_j\leqslant K_0'+G\log K_0}\alpha_j H_j^3(\tfrac12)\exp\left(i\kappa_j\log\frac{4\mathrm{e}T}{\kappa_j}\right)\times\right.\\
&\quad\left.\times\frac{1}{\sqrt{\pi}}\int_{-\log K_0}^{\log K_0}\mathrm{e}^{-u^2}\,\mathrm{d}u\right\}+O_\varepsilon(K_0^\varepsilon)\\
&=S(K_0;K_0',T)+O_\varepsilon(K_0^{1+\varepsilon}G),
\end{aligned}
\tag{3.3}
$$

where we used (1.4) to estimate the contribution from κ_j lying in the intervals $[K_0-G\log K_0,\ K_0]$ and $[K_0',\ K_0'+G\log K_0]$. On the other hand we have

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}\sum(K;G)\,\mathrm{d}K=O_\varepsilon(K_0^\varepsilon)+\\
&+\frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}\sum_{|\kappa_j-K|\leqslant G\log K_0}\alpha_j H_j^3(\tfrac12)\exp\left(i\kappa_j\log\frac{4\mathrm{e}T}{\kappa_j}-(\kappa_j-K)^2G^{-2}\right)\mathrm{d}K.
\end{aligned}
\tag{3.4}
$$

Note that, for $|\kappa_j-K|\leqslant G\log K_0$ and $K_0\leqslant K\leqslant 2K_0$, we have

$$
\begin{aligned}
&\kappa_j\log\frac{4\mathrm{e}T}{\kappa_j}-K-\kappa_j\log\frac{4T}{K}=\kappa_j-K+\kappa_j\log\frac{K}{\kappa_j}\\
&=\kappa_j-K+\kappa_j\log\left(1+\frac{K-\kappa_j}{\kappa_j}\right)\\
&=\sum_{\ell=2}^{L}\frac{(-1)^{\ell-1}}{\ell}\kappa_j\left(\frac{K-\kappa_j}{\kappa_j}\right)^\ell+O\left(\frac{G^{L+1}\log^{L+1}K_0}{K_0^L}\right)
\end{aligned}
\tag{3.5}
$$

for any fixed integer $L\geqslant 2$. But as, for $\ell\in\mathbb{N}$, $|\kappa_j-K|\leqslant G\log K_0$,

$$
\kappa_j^{-\ell}=K^{-\ell}\left(1+\frac{\kappa_j-K}{K}\right)^{-\ell}=K^{-\ell}\left\{1+\sum_{j=1}^{\infty}\binom{-\ell}{j}\left(\frac{\kappa_j-K}{K}\right)^\ell\right\},
$$

we obtain

$$
\begin{aligned}
\exp\Bigl(i\kappa_j\log\frac{4\mathrm{e}T}{\kappa_j}\Bigr) &= \exp\Biggl\{iK + i\kappa_j\log\frac{4T}{K} \\
&\quad + i\sum_{\ell=2}^{L}\frac{(-1)^{\ell-1}}{\ell}\kappa_j\Bigl(\frac{K-\kappa_j}{\kappa_j}\Bigr)^{\ell} + O\left(\frac{G^{L+1}\log^{L+1}K_0}{K_0^L}\right)\Biggr\} \\
&= \mathrm{e}^{iK}\exp\left(i\kappa_j\log\frac{4T}{K}\right)\cdot\Biggl\{1+\sum_{\ell=2}^{L}a_\ell K^{1-\ell}(K-\kappa_j)^{\ell} \\
&\quad + O\left(\frac{G^{L+1}\log^{L+1}K_0}{K_0^L}\right)\Biggr\}
\end{aligned}
\tag{3.6}
$$

with suitable constants a_ℓ. In view of (3.2) we can choose $L\,(\geqslant 2)$ so large that the error term in (3.6), when inserted in (3.4), will make a contribution which is negligible (i.e., $\ll K_0^{-A}$ for any given $A>0$).

The remaining terms in (3.6) have the property that the summands in the sum over ℓ are of decreasing order of magnitude, since for $|K-\kappa_j|\leqslant G\log K_0$ and $K_0\leqslant K\leqslant 2K_0$, we have

$$|K-\kappa_j|K^{-1}\ll GK_0^{-1}\log K_0 \ll_\varepsilon K_0^{-\varepsilon-1/2}\log K_0.$$

Therefore we can write

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}\sum_{|\kappa_j-K|\leqslant G\log K_0}\alpha_j H_j^3(\tfrac12)\mathrm{e}^{iK}\exp\left(i\kappa_j\log\frac{4T}{K}\right)\mathrm{e}^{-(\kappa_j-K)^2G^{-2}}\,\mathrm{d}K \\
&= \frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}R_0(K;T,G)\mathrm{e}^{iK}\cdot\mathrm{d}K \\
&\quad + \sum_{\ell=2}^{L}a_\ell\frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}R_\ell(K;T,G)\mathrm{e}^{iK}\cdot\mathrm{d}K + O_\varepsilon(K_0^\varepsilon),
\end{aligned}
\tag{3.7}
$$

say, where for $\ell=0,1,2,\ldots$ we have set

$$R_\ell(K;T,G) := \sum_{j=1}^{\infty}\alpha_j H_j^3(\tfrac12)h_\ell(\kappa_j;T,K,G), \tag{3.8}$$

and the function h_ℓ is defined as follows. For a fixed $N\in\mathbb{N}$ we set

$$q_N(r) := \frac{\left(r^2+\frac14\right)\left(r^2+\frac94\right)\cdots\left(r^2+\frac{(2N-1)^2}{4}\right)}{(r^2+100N^2)^N}, \tag{3.9}$$

and then define

$$h_\ell(r;T,K,G) := q_N(r)\left(L_\ell(r;T,K,G) + L_\ell(-r;T,K,G)\right),$$
$$L_\ell(r;T,K,G) := K^{1-\ell}(K-r)^\ell\left(\frac{4T}{K}\right)^{ir}\mathrm{e}^{-(r-K)^2G^{-2}}, \tag{3.10}$$

so that h_ℓ is an even function of r. From (3.3) and (3.5)–(3.10) it follows that

$$\begin{aligned} S(K_0;K_0',T) &= \Re\mathrm{e}\left\{\frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'} R_0(K;T,G)\mathrm{e}^{iK}\cdot \mathrm{d}K\right\} \\ &+ \Re\mathrm{e}\left\{\sum_{\ell=2}^{L} a_\ell \frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'} R_\ell(K;T,G)\mathrm{e}^{iK}\cdot \mathrm{d}K\right\} + O_\varepsilon(K_0^{1+\varepsilon}G), \end{aligned} \tag{3.11}$$

and clearly the main contribution to our sum (cf. (1.1)) $S(K_0;K_0',T)$ comes from the integral with R_0.

The function $h_\ell(r;T,K,G)$, defined by (3.10), is a modified Gaussian weight function in r, which is regular in the horizontal strip $|\Im\mathrm{m}\, r| \leqslant N+1$. Moreover it is even, satisfies $h_\ell(\pm\frac{1}{2}ij;T,K,G) = 0$ for $j = 1,3,\ldots,\frac{1}{2}(N-1)$ and every ℓ and the decay condition

$$h_\ell(r;T,K,G) \ll_{\ell,T,K,G} \exp(-c|r|^2) \qquad (c>0) \tag{3.12}$$

in the above strip. Thus it satisfies all the conditions necessary for the application of Motohashi's transformation formula, which will be discussed in the next section. This ends the technical preparation for the proofs.

4. Motohashi's transformation formula

The basic idea of the transformation formula is to transform the expression, for a suitable weight function $h_0(r)$,

$$\mathcal{C}(K,G) := \sum_{j=1}^{\infty} \alpha_j H_j^3(\tfrac{1}{2}) h_0(\kappa_j) \tag{4.1}$$

into a sum of terms which do not contain quantities from the spectral theory of the non-Euclidean Laplacian. In this way the problem of the evaluation or estimation of $\mathcal{C}(K,G)$ is transformed into a problem of classical Analytic Number Theory. The function $\mathcal{C}(K,G)$ will be actually $R_\ell(K;T,G)$ from (3.8). For the function $h_0(r)$, which is regular in a (large) fixed horizontal strip, it is sufficient to assume that it is even and decays in the strip like

$$h_0(r) \ll \mathrm{e}^{-c|r|^2} \qquad (c>0). \tag{4.2}$$

We set $\lambda = C\log K$ $(C>0)$ and note that one has (this is Y. Motohashi [23, eq. (3.4.18)], with the extraneous factor $(1-(\kappa_j/K)^2)^\nu$ omitted)

$$\begin{aligned}\mathcal{C}(K,G) = &\sum_{f\leqslant 3K} f^{-\frac12}\exp\Bigl(-\bigl(\frac{f}{K}\bigr)^{\lambda}\Bigr)\mathcal{H}(f;h_0)\\ &-\sum_{\nu=0}^{N_1}\sum_{f\leqslant 3K} f^{-\frac12}U_\nu(fK)\mathcal{H}(f;h_\nu)+O(1),\end{aligned}\tag{4.3}$$

with

$$\begin{aligned}h_\nu(r) &= h_0(r)\left(1-\Bigl(\frac{r}{K}\Bigr)^2\right)^\nu,\\ \mathcal{H}(f;h) &= \sum_{\nu=1}^{7}\mathcal{H}_\nu(f;h),\end{aligned}\tag{4.4}$$

$$\begin{aligned}\mathcal{H}_1(f;h) &= -2\pi^{-3}i\left\{(\gamma-\log(2\pi\sqrt{f}))(\hat h)'(\tfrac12)+\tfrac14(\hat h)''(\tfrac12)\right\}d(f)f^{-\frac12},\\ \mathcal{H}_2(f;h) &= \pi^{-3}\sum_{m=1}^{\infty} m^{-\frac12}d(m)d(m+f)\Psi^+\Bigl(\frac{m}{f};h\Bigr),\\ \mathcal{H}_3(f;h) &= \pi^{-3}\sum_{m=1}^{\infty}(m+f)^{-\frac12}d(m)d(m+f)\Psi^-\Bigl(1+\frac{m}{f};h\Bigr),\\ \mathcal{H}_4(f;h) &= \pi^{-3}\sum_{m=1}^{f-1} m^{-\frac12}d(m)d(f-m)\Psi^-\Bigl(\frac{m}{f};h\Bigr),\\ \mathcal{H}_5(f;h) &= -(2\pi^3)^{-1}f^{-\frac12}d(f)\Psi^-(1;h),\\ \mathcal{H}_6(f;h) &= -12\pi^{-2}i\sigma_{-1}(f)f^{\frac12}h'(-\tfrac12 i),\\ \mathcal{H}_7(f;h) &= -\pi^{-1}\int_{-\infty}^{\infty}\frac{|\zeta(\frac12+ir)|^4}{|\zeta(1+2ir)|^2}\sigma_{2ir}(f)f^{-ir}h(r)\,\mathrm{d}r\quad\Bigl(\sigma_a(f)=\sum_{d\mid f}d^a\Bigr),\end{aligned}\tag{4.5}$$

where

$$\hat h(s) = \int_{-\infty}^{\infty} rh(r)\frac{\Gamma(s+ir)}{\Gamma(1-s+ir)}\,\mathrm{d}r,\tag{4.6}$$

$$\Psi^+(x;h) = \int_{(\beta)}\Gamma^2(\tfrac12-s)\tan(\pi s)\hat h(s)x^s\,\mathrm{d}s,\tag{4.7}$$

$\int_{(\beta)}$ denotes integration over the line $\Re\mathrm{e}\, s=\beta$,

$$\Psi^-(x;h) = \int_{(\beta)}\Gamma^2(\tfrac12-s)\frac{\hat h(s)}{\cos(\pi s)}x^s\,\mathrm{d}s\tag{4.8}$$

with $-\frac32<\beta<\frac12$, N_1 is a sufficiently large integer,

$$U_\nu(x) = \frac{1}{2\pi i\lambda}\int_{(-\lambda^{-1})}(4\pi^2K^{-2}x)^w u_\nu(w)\Gamma(\frac{w}{\lambda})\,\mathrm{d}w \ll \Bigl(\frac{x}{K^2}\Bigr)^{-\frac{C}{\log K}}\log^2 K,$$

where $u_\nu(w)$ is a polynomial in w of degree $\leqslant 2N_1$, whose coefficients are bounded. As already mentioned, the prominent feature of Motohashi's explicit expression for $\mathcal{C}(K,G)$ is that it contains series and integrals with the classical divisor function $d(n)$ only, with no quantities from spectral theory.

5. Proof of Theorem 1

We need, in view of (3.11), to transform and estimate the functions $R_\ell(K;T,G)$ in (3.8). To this end we shall employ (4.1), where $h_0(r)$ equals

$$h_\ell(r;T,K,G)\biggl(1-\Bigl(\frac{r}{K}\Bigr)^2\biggr)^\nu \qquad (\nu=0,1,2,\dots;\ell=0,1,2,\dots). \tag{5.1}$$

All the functions of the form (5.1) are treated analogously. Therefore it is sufficient to consider in detail only the case $\nu=\ell=0$, when for simplicity the function in (5.1) will be again denoted by $h(r)$. It is clearly this case which will furnish eventually the largest contribution to (2.1).

In the sequel we shall repeatedly use the classical formula

$$\int_{-\infty}^{\infty} \mathrm{e}^{Au-Bu^2}\,\mathrm{d}u = \sqrt{\frac{\pi}{B}}\exp\left(\frac{A^2}{4B}\right) \qquad (\Re\mathrm{e}\, B>0). \tag{5.2}$$

By taking $B=1$ and then differentiating (5.2) as the function of A, we also obtain

$$\int_{-\infty}^{\infty} u^j\mathrm{e}^{Au-u^2}\,\mathrm{d}u = P_j(A)\mathrm{e}^{\frac14 A^2} \qquad (j=0,1,2,\dots,\ P_0(A)=\sqrt{\pi}\,), \tag{5.3}$$

where $P_j(z)$ is a polynomial in z of degree j, which may be explicitly evaluated. The basic idea is that the factor $(4T/K)^{\pm ir}$ (cf. (3.10)) is the dominating oscillating factor which in most cases, after the use of (5.2) or (5.3), will produce exponential functions of fast decay which will make a negligible contribution. We recall that a "negligible contribution" is one which is $\ll K_0^{-A}$ (or $\ll T^{-A}$) for any fixed $A>0$.

This is precisely what happens with the contribution of $\mathcal{H}_1(f;h)$, which we shall first show to be negligible. Namely from (4.6) we find that

$$(\hat h)'(\tfrac12) = 2\int_{-\infty}^{\infty} rh(r)\frac{\Gamma'}{\Gamma}(\tfrac12+ir)\,\mathrm{d}r. \tag{5.4}$$

But (see e.g., [18])

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|}\right), \tag{5.5}$$

where the O-term admits an asymptotic expansion. The non-negligible contribution in (5.4) is for the range $|r \pm K| \leqslant G \log K$. We make the change of variable $r \pm K = Gu$ and use Taylor's formula to simplify the integrand. After this we may use (5.2) and (5.3), which will produce exponential factors of the form $\exp(-\frac14 G^2(\log\frac{4T}{K})^2)$, which will make a negligible contribution. The O-term in (5.5), by trivial estimation, will make a total contribution of $O_\varepsilon(K^{3/2+\varepsilon})$. The contribution of $(\hat h)''(\frac12)$ is estimated analogously, and we see that the total contribution of $\mathcal{H}_1(f;h)$ is $O_\varepsilon(K_0^{3/2+\varepsilon})$.

Next we note that

$$\mathcal{H}_6(f;h) = -12\pi^{-2} i\sigma_{-1}(f) f^{\frac12} h'(-\tfrac12 i) \ll \sigma_{-1}(f) f^{\frac12}\exp(-\tfrac12 K^2 G^{-2}),$$

hence summation over f in (4.3) yields a contribution which is negligible.

The total contribution of

$$\mathcal{H}_5(f;h) = -(2\pi^3)^{-1} f^{-\frac12} d(f)\Psi^-(1;h) \tag{5.6}$$

is also negligible. This follows from [23, eq. (3.3.44)], in view of the presence of $\sinh\pi r/(\cosh\pi r)^2$, which decays like $\exp(-\pi|r|)$.

The total contribution of

$$\mathcal{H}_3(f;h) = \pi^{-3}\sum_{m=1}^\infty (m+f)^{-\frac12} d(m)d(m+f)\Psi^-\left(1+\frac{m}{f};h\right) \tag{5.7}$$

is also negligible, but this is somewhat more involved than the contribution of $\mathcal{H}_5(f;h)$. We need the representation (this is [23, eq. (3.3.43)])

$$\Psi^-(x;h) = 2\pi i\int_0^1 (y(1-y)(1-y/x))^{-1/2}\int_{-\infty}^\infty \frac{rh(r)}{\cosh(\pi r)}\left\{\frac{y(1-y)}{x-y}\right\}^{ir} \mathrm{d}r\,\mathrm{d}y, \tag{5.8}$$

which is valid for $x > 1$. Motohashi derived (5.8) for a somewhat different weight function $h(r)$, essentially without the factor $(4T/K)^{\pm ir}$, but it is clear by following his proof that (5.8) will hold for the present function $h(r)$ as well. The same remark holds for other forms of the functions $\Psi^\pm(x;h)$ which will be needed in the sequel. To deal with the series over m in (5.7) we need to have a good bound in m. This is achieved, as in [23], by shifting the line of integration (in the integral over r) in (5.8) to $\Im\mathrm{m}\, r = -1$. In this process use is made of the fact that $h(-\frac12 i) = 0$, since this zero at $-\frac12 i$ cancels with the zero of $\cosh\pi r$. We then note that, in the relevant range for r, $1/\cosh(\pi r) \ll \exp(-\frac12\pi K)$. Thus, for $x = 1 + m/f \geqslant 3$, we obtain by trivial estimation

$$\Psi^-\left(1+\frac{m}{f};h\right) \ll fm^{-1}TG\exp(-\tfrac12\pi K) \qquad (m \geqslant 2f).$$

This is more than sufficient to render the total contribution of $m \geqslant 2f$ negligible, and the same follows for the contribution of the remaining m's if we use the trivial estimate (coming directly from (5.8))

$$\Psi^-\left(1+\frac{m}{f};h\right) \ll KG\exp(-\tfrac{1}{2}\pi K) \qquad (m \leqslant 2f).$$

To deal with

$$\mathcal{H}_7(f;h) = -\pi^{-1}\int_{-\infty}^{\infty}\frac{|\zeta(\frac{1}{2}+ir)|^4}{|\zeta(1+2ir)|^2}\sigma_{2ir}(f)f^{-ir}h(r)\,\mathrm{d}r,$$

note that we have $1/\zeta(1+ir) \ll \log(|r|+1)$ and

$$\sum_{n=1}^{\infty}\sigma_{2ir}(n)n^{-ir-s} = \zeta(s-ir)\zeta(s+ir) \qquad (r\in\mathbb{R},\ \Re\mathrm{e}\, s>1).$$

Consequently by the Perron inversion formula (see e.g., [2, eq. (A.10)])

$$\sum_{f\leqslant 3K}\sigma_{2ir}(f)f^{-\frac{1}{2}-ir} \ll_\varepsilon K^{2\mu(\frac{1}{2})+\varepsilon} \ll_\varepsilon K^{\frac{1}{3}+\varepsilon} \qquad (K \ll |r| \ll K), \tag{5.9}$$

where $\mu(\sigma)$ is given by (2.10), and we used the classical bound $\mu(\frac{1}{2}) \leqslant 1/6$. Since the relevant range of r in $\mathcal{H}_7(f;h)$ is $|r \pm K| \leqslant G\log K_0$, it follows by using (5.9) that

$$\begin{aligned}
&G^{-1}\int_{K_0}^{K_0'}\sum_{f\leqslant 3K} f^{-1/2}\mathcal{H}_7(f;h)\,\mathrm{d}K\\
&\ll_\varepsilon 1 + K_0^{1/3+\varepsilon}G^{-1}\int_{K_0}^{K_0'}\int_{K-G\log K_0}^{K+G\log K_0}|\zeta(\tfrac{1}{2}+ir)|^4\,\mathrm{d}r\;\mathrm{d}K\\
&\ll_\varepsilon K_0^{1/3+\varepsilon}G^{-1}\int_{K_0-G\log K_0}^{K_0'+G\log K_0}|\zeta(\tfrac{1}{2}+ir)|^4\int_{r-G\log K_0}^{r+G\log K_0}\mathrm{d}K\cdot\mathrm{d}r\\
&\ll_\varepsilon K_0^{4/3+\varepsilon},
\end{aligned}$$

hence this is the total contribution of $\mathcal{H}_7(f;h)$ to the right-hand side of (3.7).

It remains yet to deal with

$$\mathcal{H}_2(f;h) = \pi^{-3}\sum_{m=1}^{\infty} m^{-\frac{1}{2}}d(m)d(m+f)\Psi^+\left(\frac{m}{f};h\right) \tag{5.10}$$

and $\mathcal{H}_4(f;h)$, which will be done in Section 6. The contribution of $\mathcal{H}_2(f;h)$ is the principal one. It is estimated according to the range of m in (5.10).

We shall show first that the contribution of $m \geqslant fTK^{\varepsilon-1}$ in (5.10) is negligible. We use the representation (this is [23, eq. (3.3.41)])

$$\Psi^+(x;h) = \tag{5.11}$$
$$2\pi\int_0^1 \{y(1-y)(1+y/x)\}^{-1/2}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\left\{\frac{y(1-y)}{x+y}\right\}^{ir}\,\mathrm{d}r\,\mathrm{d}y$$

with $x = m/f \geqslant K^\varepsilon$, and shift the line of integration in the inner integral to $\Im\mathrm{m}\, r = -N$. This is permissible, since by (3.9) and (3.10) the function $h(r)$ is regular for $|\Im\mathrm{m}\, r| \leqslant N+1$. Then the inner integral in (5.11) becomes

$$\int_{-\infty}^{\infty}(r-Ni)h(r-Ni)\tanh(\pi r)\left\{\frac{y(1-y)}{x+y}\right\}^{ir}\left\{\frac{y(1-y)}{x+y}\right\}^{N}\mathrm{d}r$$
$$\ll KG(y(1-y))^N\left(\frac{Tf}{mK}\right)^N.$$

Since $N\ (=N(\varepsilon))$ can be taken arbitrarily large, it follows that the total contribution of $m/f \geqslant TK^{\varepsilon-1}$ in (5.10) is negligible.

We shall show that the contribution of $m/f \leqslant TK^{-\varepsilon-1}$ is also negligible. We make the change of variable $r = \pm K + Gu$ in the r-integral in (5.11), and note that

$$\tanh(\pi r) = \mathrm{sgn(r)} + \mathrm{O}(\mathrm{e}^{-2\pi|\mathrm{r}|}) \qquad (\mathrm{r}\in\mathbb{R}). \tag{5.12}$$

After the application of (5.2) there will appear the exponential factors

$$\exp\left(-\tfrac14 G^2\log^2\left(\frac{4T}{K}\cdot\frac{y(1-y)}{x+y}\right)\right)$$

and

$$\exp\left(-\tfrac14 G^2\log^2\left(\frac{4T}{K}\cdot\frac{x+y}{y(1-y)}\right)\right).$$

Since, in view of (1.2),

$$\frac{4T}{K}\cdot\frac{x+y}{y(1-y)} \geqslant \frac{4Tx}{K} = \frac{4Tm}{fK} \geqslant \frac{4T}{3K^2} \gg T^{\varepsilon/2},$$

the contribution of the latter is negligible. The contribution of the former is also negligible if

$$\frac{4T}{K}\cdot\frac{y(1-y)}{x+y} \leqslant 1 - G^{-1}\log T \quad \text{or} \quad \frac{4T}{K}\cdot\frac{y(1-y)}{x+y} \geqslant 1 + G^{-1}\log T.$$

If this condition is not satisfied, then

$$y\in[y_1,\,y_2],\ y_1 \approx Kx/T \ll K^{-\varepsilon},\ y_1 - y_2 \approx \frac{Kx\log T}{TG}.$$

In the y-integral in (5.11) over $[y_1,\, y_2]$ we integrate by parts the factor $y^{ir-\frac12}$ a large number of times. Each time the exponent of y will increase by unity, while the order of the r-integral will remain unchanged. Trivial estimation shows then that the contribution of $m/f \leqslant TK^{-\varepsilon-1}$ is indeed negligible.

Thus the critical range in the estimation of $\mathcal{H}_2(f;h)$ is (since $K_0 \leqslant K \leqslant 2K_0$)

$$fTK_0^{-1-\varepsilon} \leqslant m \leqslant fTK_0^{-1+\varepsilon}. \tag{5.13}$$

For the range (5.13) we shall use the representation which follows from [23, eq. (3.3.39)] and the formula after it, with $x = m/f \to \infty$ (as $K_0 \to \infty$), namely

$$\Psi^+(x;h) = \tag{5.14}$$
$$2\pi\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\Re\mathrm{e}\left\{\frac{\Gamma^2(\frac12+ir)}{\Gamma(1+2ir)}F\left(\tfrac12+ir,\tfrac12+ir;1+2ir;-\frac{1}{x}\right)x^{-ir}\right\}\,\mathrm{d}r,$$

where F is the hypergeometric function. We could use the asymptotic formula, valid for $y \geqslant y_0 > 1$ and $r \to \infty$,

$$\begin{aligned} &F\left(\tfrac12+ir,\tfrac12+ir;1+2ir;-\frac{1}{y^2}\right) = O(y^{-4}r^{-2})\\ &+(2y)^{2ir}(y+\sqrt{1+y^2})^{-2ir}\left(\frac{y^2}{1+y^2}\right)^{1/4}\left(1-\frac{1}{8ir}\cdot\frac{2y^2+1}{2y\sqrt{1+y^2}}\right), \end{aligned} \tag{5.15}$$

which yields directly the main term. This formula is to be found in the work of N.I. Zavorotnyi [24]. A sketch of proof is given by N.V. Kuznetsov [17], where the asymptotics are given by means of a solution of a certain second-order differential equation (see his work [16]). One can avoid the use of (5.15) by appealing to the classical quadratic transformation formula (see [18, eq. (9.6.12)]) for the hypergeometric function, as was done by the author [7] in his work on sums of $\alpha_j H_j^3(\frac12)$ in short intervals. This is

$$F(\alpha,\beta;2\beta;z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(\alpha,\alpha-\beta+\tfrac12;\beta+\tfrac12;\left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right),$$

and then one can develop the resulting hypergeometric function into a convergent power series, of which the main contribution will come from the leading term, namely 1. The main term in (2.1) (the summand with c_0) will be in both cases the same, of course, and the latter approach yields the remaining summands with φ_ℓ.

In (5.14) the relevant ranges of integration are $[-K-G\log K_0, -K+G\log K_0]$ and $[K-G\log K_0, K+G\log K_0]$. We recall that (5.12) holds, and in the first range of integration we change r to $-r$. Then we obtain that the critical expression in question is

$$\begin{aligned}
&\frac{4\sqrt{\pi}}{G}\int_{K_0}^{K_0'} \mathrm{e}^{iK} \sum_{f\leqslant 3K_0} f^{-1/2} \sum_{TK_0^{-1-\varepsilon}f\leqslant m\leqslant TK_0^{-1+\varepsilon}f} m^{-1/2}d(m)d(m+f)\times\\
&\int_{K-G\log K_0}^{K+G\log K_0} r\left(\frac{4T}{K}\right)^{ir} \mathrm{e}^{-(r-K)^2G^{-2}}\times \qquad (5.16)\\
&\Re\mathrm{e}\left\{\frac{\Gamma^2(\frac12+ir)}{\Gamma(1+2ir)}F\left(\tfrac12+ir,\tfrac12+ir;1+2ir;-\frac{1}{x}\right)x^{-ir}\right\}\,\mathrm{d}r\,\mathrm{d}K.
\end{aligned}$$

To (5.16) we shall apply (5.15) with $y=\sqrt{x}=\sqrt{m/f}$, under (5.13). The gamma-factors are simplified by Stirling's formula, namely that for $t\geqslant t_0>0$

$$\Gamma(s)=\sqrt{2\pi}\,t^{\sigma-\frac12}\exp\left(-\tfrac12\pi t+it\log t-it+\tfrac12\pi i(\sigma-\tfrac12)\right)\cdot\left(1+O_\sigma\left(t^{-1}\right)\right), \quad (5.17)$$

with the understanding that the O–term in (5.17) admits an asymptotic expansion in terms of negative powers of t. Hence using the symbol $\asymp$ (defined after the formulation of Theorem 1) the expression in (5.16) is ($x=m/f$)

$$\begin{aligned}
&\asymp \frac{1}{G}\int_{K_0}^{K_0'} \mathrm{e}^{iK}\sum_f\sum_m\cdots\int_{K-G\log K_0}^{K+G\log K_0} r\left(\frac{4T}{K}\right)^{ir}\mathrm{e}^{-(r-K)^2G^{-2}}\times\\
&\Re\mathrm{e}\left\{r^{-1/2}\mathrm{e}^{-2ir\log 2}x^{-ir}2^{2ir}x^{ir}(\sqrt{x}+\sqrt{1+x})^{-2ir}\,\mathrm{d}r\right\}\,\mathrm{d}K\\
&\asymp \frac{1}{G}\int_{K_0}^{K_0'} K^{1/2}\mathrm{e}^{iK}\sum_f\sum_m\cdots\times \qquad (5.18)\\
&\times\int_{K-G\log K_0}^{K+G\log K_0}\left(\frac{4T}{K}\right)^{ir}\cos(2r\log(\sqrt{x}+\sqrt{1+x}))\mathrm{e}^{-(r-K)^2G^{-2}}\,\mathrm{d}r\,\mathrm{d}K.
\end{aligned}$$

The cosine is written as the sum of exponentials, after which the change of variable $r=K+Gu$ is made in the r-integral. The inner integral in (5.18) thus reduces to

$$G\int_{-\log K_0}^{\log K_0}\mathrm{e}^{-u^2}\exp\left\{(iK+iGu)\left(\log\frac{4T}{K}\pm\log(\sqrt{x}+\sqrt{1+x})^2\right)\right\}\,\mathrm{d}u, \quad (5.19)$$

after which we restore the integration to the whole real line, making a negligible error. Then we apply (5.2), noting that the integral with the $+$-sign makes a negligible contribution. The integral with the $-$-sign equals

$$\sqrt{\pi}G\exp\left\{iK\log\left(\frac{4T}{K(\sqrt{x}+\sqrt{1+x})^2}\right)-\frac14G^2\log^2\left(\frac{4T}{K(\sqrt{x}+\sqrt{1+x})^2}\right)\right\}.$$

It follows that (5.18) is

$$\begin{aligned}
&\asymp\sum_f\sum_m\cdots\int_{K_0}^{K_0'}K^{1/2}\exp\left\{iK\log\Bigl(\frac{4\mathrm{e}T}{K(\sqrt{x}+\sqrt{1+x})^2}\Bigr)\right\}\\
&\times\exp\left\{-\tfrac14G^2\log^2\Bigl(\frac{4T}{K(\sqrt{x}+\sqrt{1+x})^2}\Bigr)\right\}\,\mathrm{d}K. \qquad (5.20)
\end{aligned}$$

The last exponential factor yields that only the range $m/f \approx T/K_0$ makes a non-negligible contribution. More precisely, we have

$$\frac{4T}{K(\sqrt{x}+\sqrt{1+x})^2} = \frac{T}{K\left(x + \sum\limits_{j=0}^{\infty} b_j x^{-j}\right)} \qquad (x = m/f > 1)$$

with suitable coefficients b_j. Therefore the second exponential factor in (5.20) is negligibly small, unless

$$K = \frac{T}{x + \sum\limits_{j=0}^{\infty} b_j x^{-j}}\left(1 + O\left(\frac{\log T}{G}\right)\right). \tag{5.21}$$

This means that the relevant interval of integration over K in (5.20), for fixed f and m, has length $\ll Tf\log T/(mG)$.

The integral in (5.20) is an exponential integral of the form

$$\int_{K_0}^{K_0'} g(K)\mathrm{e}^{if(K)}\,\mathrm{d}K, \quad f(K) := K\log\Big(\frac{4\mathrm{e}T}{K(\sqrt{x}+\sqrt{1+x})^2}\Big).$$
$$g(K) := K^{1/2}\exp\left\{-\tfrac14 G^2\log^2\Big(\frac{4T}{K(\sqrt{x}+\sqrt{1+x})^2}\Big)\right\}.$$

The saddle point K_1 (the root of $f'(K) = 0$) is given by

$$K_1 = \frac{4T}{(\sqrt{x}+\sqrt{1+x})^2}. \tag{5.22}$$

Since $f''(K) = -1/K$, it follows by the saddle point method (see e.g., [2, Chapter 2]) that (5.20) is ($0 < C_1 < C_2$ are suitable constants, $x = m/f$)

$$\asymp T\sum_{f\leqslant 3K_0} f^{-\frac12}\sum_{\frac{C_1Tf}{K_0}\leqslant m\leqslant\frac{C_2Tf}{K_0}} m^{-\frac12}\frac{d(m)d(m+f)}{(\sqrt{x}+\sqrt{1+x})^2}\exp\left(\frac{4iT}{(\sqrt{x}+\sqrt{1+x})^2}\right),$$

plus an error term which is certainly $\ll_\varepsilon K_0^{3/2+\varepsilon}$. But since

$$\frac{4iT}{(\sqrt{x}+\sqrt{1+x})^2} = \frac{iT}{x}\left(1 + \sum_{j=1}^{\infty} c_j x^{-j}\right) \tag{5.23}$$

with suitable constants c_j and $Tx^{-2} \ll K^2/T \ll T^{-\varepsilon}$ in view of (1.2), it follows that (5.20) is

$$\asymp T\sum_{f\leqslant 3K_0} f^{\frac12}\sum_{\frac{C_1Tf}{K_0}\leqslant m\leqslant\frac{C_2Tf}{K_0}} m^{-\frac32}d(m)d(m+f)\exp\left(\frac{iTf}{m}\right) + O_\varepsilon(K_0^{3/2+\varepsilon}). \tag{5.24}$$

Therefore the proof of Theorem 1 will be complete after we show that the contribution of $\mathcal{H}_4(f;h)$ is negligible, and choose $G = K_0^{1/2-\varepsilon}$. Note that trivial estimation gives that the expression in (5.24) is

$$\ll_\varepsilon T^{1/2+\varepsilon} K_0^{3/2},$$

which is worse that the trivial estimation of $S(K)$, since (1.2) holds. Likewise the use of the range of integration (5.21) gives also a poor bound.

We shall conclude with a discussion on the shape of the functions $\varphi_\ell(K,T; m,f)$, which appear in (2.1). We note that (see (3.9)) we have

$$q_N(r) = 1 + \sum_{\ell=1}^{L} b_\ell r^{-2\ell} + O_{N,L}(r^{-2L-2}) \tag{5.25}$$

with effectively computable constants b_ℓ, where (as before) L is taken so large that the error term makes, in the appropriate expressions, a negligible contribution. Each factor $r^{-2\ell}$ in (5.25) becomes, after change of variable in the integral in (5.19),

$$(K+Gu)^{-2\ell} = K^{-2\ell}\left\{1 + \sum_{j=1}^{L} d_\ell (Gu/K)^j + O_{\ell,L}\left((Gu/K)^{L+1}\right)\right\},$$

which is then evaluated by (5.3), furnishing a sum containing powers of G and K.

In what concerns the factors $K^{1-\ell}(K-r)^\ell$ in (3.10), note that $(K-r)^\ell$ introduces the factor $(Gu)^\ell$ in (5.19), and then the corresponding integral is again evaluated by (5.3), producing eventually a suitable power of G. The factor $K^{1-\ell}$, after the saddle point method is applied, in view of (5.22) leads to

$$K_1^{1-\ell} = (4T)^{1-\ell}(\sqrt{x} + \sqrt{1+x})^{2\ell-2} \qquad (x = m/f),$$

and we have the power expansion (5.23). When this is all put together, we get terms of the type $\varphi_\ell(K,T;m,f)$, which are power functions in each of the variable, all of which are certainly $o(1)$ (as $K \to \infty$ and (2.1) holds).

6. Completion of proof of Theorem 1

To complete the proof of Theorem 1 we shall show that

$$\mathcal{H}_4(f;h) = \pi^{-3} \sum_{m=1}^{f-1} m^{-\frac{1}{2}} d(m) d(f-m) \Psi^-\left(\frac{m}{f}; h\right) \tag{6.1}$$

makes a negligible contribution to (4.3). We use the representation (this is [23, eq. (3.3.45)]), valid for $x = m/f < 1$ and $-1 < \beta < -\frac{1}{2}$,

$$\Psi^-(x;h) \tag{6.2}$$
$$= \int_0^\infty \Big\{ \int_{(\beta)} x^s (y(1+y))^{s-1} \frac{\Gamma^2(\frac{1}{2}-s)\,\mathrm{d}s}{\Gamma(1-2s)\cos(\pi s)} \Big\} \int_{-\infty}^{\infty} rh(r) \left(\frac{y}{1+y}\right)^{ir} \mathrm{d}r\,\mathrm{d}y,$$

where the triple integral converges absolutely. The function (6.2) can be compared to the representation (5.11) for $\Psi^+(x;h)$: the function $\Psi^-(x;h)$ is easier to deal with because of the factor $\cos(\pi s)$ in the denominator, and summation over m in (6.1) is finite. On the other hand, it has the drawback that the integral over y is not finite, and there is an additional integration over s. As before, it will suffice to consider the contribution of $|r \pm K| \leqslant G\log K$. Namely if $|r \pm K| \geqslant G\log K$ we interchange the order of integration, and in the y integral we integrate by parts the subintegral over $(0, 1]$ to obtain that the contribution is $\ll x^\beta \exp(-\frac{1}{2}\log^2 K)$. For $|r - K| \leqslant G\log K$ (the case of the '+' sign is analogous) we make the change of variable $r = K + Gu$ to obtain that the dominant contribution of the r-integral will be

$$GK\mathrm{e}^{iK\log\frac{y}{1+y}}\mathrm{e}^{iK\log\frac{4T}{K}} \int_{-\log K}^{\log K} \exp\Big(-u^2 \pm iGu\log\frac{4T}{K} + iGu\log\frac{y}{1+y}\Big)\,\mathrm{d}u. \tag{6.3}$$

Using (5.2) it follows that (6.3) becomes, up to a negligible error, a multiple of

$$\begin{aligned} &GK\exp\left(iK\log\left(\frac{y}{1+y}\cdot\frac{4T}{K}\right)\right)\exp\left(-\tfrac{1}{4}G^2\left(\log\Big(\frac{y}{1+y}\cdot\frac{4T}{K}\Big)\right)^2\right)\\ &+GK\exp\left(iK\log\left(\frac{y}{1+y}\cdot\frac{K}{4T}\right)\right)\exp\left(-\tfrac{1}{4}G^2\left(\log\Big(\frac{y}{1+y}\cdot\frac{K}{4T}\Big)\right)^2\right). \end{aligned} \tag{6.4}$$

Since

$$\left(\log\Big(\frac{y}{1+y}\cdot\frac{K}{4T}\Big)\right)^2 \geqslant \log^2\left(\frac{4T}{K}\right) \qquad (y>0),$$

this means that the contribution of the second exponential factor above will be negligible, and the same holds for the first exponential factor, if $y \geqslant 1$. In view of Stirling's formula (see (5.17)) and

$$|\cos(x+iy)| = \sqrt{\cos^2 x + \sinh^2 y} \qquad (x\in\mathbb{R},\ y\in\mathbb{R}),$$

it follows that the contribution of $|\Im\mathrm{m}\, s| = |t| > \log^2 K$ in (6.1) will be negligibly small. If $0 \leqslant y \leqslant 1$ and

$$\frac{y}{1+y}\cdot\frac{4T}{K} \leqslant 1 - \frac{\log T}{G} \tag{6.5}$$

or

$$\frac{y}{1+y}\cdot\frac{4T}{K} \geqslant 1 + \frac{\log T}{G}, \tag{6.6}$$

the total contribution is again negligible. If (6.5) and (6.6) both fail, then y lies in an interval of length $\approx (K\log T)/(TG)$. But then we may integrate by parts the factor y^{ir} in the integral, each time increasing the exponent of y by unity. If this is done sufficiently many times, then trivial estimation shows that the total contribution of (6.1) is negligibly small, and Theorem 1 is proved, if we take $G = K_0^{1/2-\varepsilon}$ in (3.11) and (5.9) and replace K_0 by K.

7. The proof of Theorem 2

The proof of the first bound in (2.13) is straightforward. Namely Motohashi derived the transformation formula for (4.1) by writing $H_j^3(\frac12) = H_j^2(\frac12)\cdot H_j(\frac12)$, and then by expressing $H_j(\frac12)$ as a partial sum of $t_j(f)f^{-1/2}$ (see [23, Lemma 3.9] or (7.7)) to which the transformation formula for the bilinear sum of Hecke series is applied. Therefore our problem reduces essentially to the evaluation and estimation of Theorem 1 in the case $f = 1$. We obtain

$$\begin{aligned}
&\sum_{K<\kappa_j\leqslant K'<2K} \alpha_j H_j^2(\tfrac12)\cos\left(\kappa_j\log\left(\frac{4\mathrm{e}T}{\kappa_j}\right)\right)\\
&\asymp T \sum_{C_1TK^{-1}\leqslant m\leqslant C_2TK^{-1}} m^{-\frac32} d(m)d(m+1)\mathrm{e}^{i\frac{T}{m}} + O_\varepsilon(K^{3/2+\varepsilon})\\
&\ll_\varepsilon T^{1/2+\varepsilon}K^{1/2} + K^{3/2+\varepsilon} \ll_\varepsilon T^{1/2+\varepsilon}K^{1/2},
\end{aligned} \tag{7.1}$$

since (1.2) holds. We remark, similarly as in the discussion concerning Theorem 1, that the sum over m in (7.1) could be treated by the techniques of [12]–[13] involving the binary additive divisor problem, but it seems that the result that would be obtained in this fashion does not improve the above (trivial) bound.

For the proof of the second bound in (2.13) we proceed analogously to the proof of

$$\sum_{\kappa_j\leqslant T} \alpha_j H_j(\tfrac12) = \left(\frac{T}{\pi}\right)^2 - BT\log T + O(T(\log T)^{1/2}) \qquad (B>0), \tag{7.2}$$

given by M. Jutila and the author in [10]. The proof of (7.2) rested on the use of (see e.g., [23] for a proof)

Lemma 1. (The first Bruggeman-Kuznetsov trace formula) *Let $f(r)$ be an even, regular function for $|\Im\mathrm{m}\, r| \leqslant \frac12$ such that $f(r) \ll (1+|r|)^{-2-\delta}$ for some $\delta > 0$.*

Then

$$\begin{aligned}&\sum_{j=1}^{\infty}\alpha_j t_j(m)t_j(n)f(\kappa_j)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}f(r)\,\mathrm{d}r\\&=\frac{1}{\pi^2}\delta_{m,n}\int_{-\infty}^{\infty}r\tanh(\pi r)f(r)\,\mathrm{d}r+\sum_{\ell=1}^{\infty}\frac{1}{\ell}S(m,n;\ell)f_+\left(\frac{4\pi\sqrt{mn}}{\ell}\right),\end{aligned}\tag{7.3}$$

where $\delta_{m,n} = 1$ *if* $m = n$ *and zero otherwise* $(m, n > 0)$, $\sigma_a(d) = \sum_{d|n} d^a$, $S(m, n; \ell)$ *is the Kloosterman sum and*

$$f_+(x)=\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{r}{\cosh(\pi r)}J_{2ir}(x)f(r)\,\mathrm{d}r.\tag{7.4}$$

In this formula one takes $n = 1$ and $f(r) \equiv h_\ell(r; T, K, G)$, as given by (3.10), and follows the scheme of proof of Theorem 1. This consists of evaluating

$$\begin{aligned}&\frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}\sum_{|\kappa_j-K|\leqslant G\log K_0}\alpha_j H_j(\tfrac12)\mathrm{e}^{iK}\exp\left(i\kappa_j\log\frac{4T}{K}\right)\mathrm{e}^{-(\kappa_j-K)^2G^{-2}}\,\mathrm{d}K\\&=\frac{1}{\sqrt{\pi}G}\int_{K_0}^{K_0'}\sum\nolimits_0(K;T,G)\mathrm{e}^{iK}\,\mathrm{d}K+O(1),\end{aligned}\tag{7.5}$$

where G satisfies (3.2) and

$$\sum\nolimits_0(K;T,G):=\sum_{j=1}^{\infty}\alpha_j H_j(\tfrac12)h(\kappa_j;T,K,G).\tag{7.6}$$

To obtain the expression for (7.6) one multiplies (7.3) by $m^{-1/2}$, since (see [10] for proof) we have

Lemma 2. *Let* $\kappa_j = (1 + o(1))K$, $r = (1 + o(1))K$ $(r \in \mathbb{R})$ *as* $K \to \infty$, $Y = (1+\delta)\frac{K^2}{4\pi^2}$, *with* $\delta > 0$ *a given constant. Then, for any fixed positive constant* $A > 0$, *there exists a constant* $C = C(A, \delta) > 0$ *such that, for* $h = C \log K$, *we have*

$$H_j(\tfrac12)=\sum_{m\leqslant(1+\delta)Y}t_j(m)m^{-1/2}\mathrm{e}^{-(m/Y)^h}+O(K^{-A}),\tag{7.7}$$

and

$$\zeta(\tfrac12+ir)\zeta(\tfrac12-ir)=\sum_{m\leqslant(1+\delta)Y}\sigma_{2ir}(m)m^{-\frac12-ir}\mathrm{e}^{-(m/Y)^h}+O(K^{-A}).\tag{7.8}$$

In the proof of (7.2) the main term came from the integral

$$\int_{-\infty}^{\infty} r\tanh(\pi r) f(r)\,\mathrm{d}r \tag{7.9}$$

in (7.3). However, now in the function $f(r)$ we shall have the additional oscillating factor $(4T/K)^{\pm ir}$. Because of this, when we make the change of variable $r = \pm K + Gu$, we shall eventually wind up with exponential factors of the form

$$\exp\left\{-\tfrac14 G^2\left(\log\frac{4T}{K}\right)^2\right\},$$

which make a negligible contribution. The total contribution of the continuous spectrum (the integral on the left-hand side of (7.3)) is easily seen to be $\ll_\varepsilon K_0^{1+\varepsilon}$. The only delicate part is the Kloosterman-sum contribution, coming from the right-hand side of (7.3). However, this presents no major problem, since the estimation is analogous to the one made in [10] for the proof of (7.2). We shift the line of integration in the integral defining f_+ to $\Im\mathrm{m}\, r = -1$ and use the power series representation

$$J_{2+ix}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{2+ix+2k}}{\Gamma(k+1)\Gamma(k+2+ix+1)} \qquad (z = 4\pi\sqrt{m}/\ell \ll K^{1-B}),$$

which shows that the contribution of $\ell > K^B$ is $\ll K^{-A}$ for any fixed $A > 0$, provided that $B = B(A)$ is sufficiently large. The only difference from [10] is that, in making the shift, the factor $(4T/K)^{ir}$ will make now a contribution of $O(T/K)$, which is harmless if B is sufficiently large. In the remaining sum, we substitute (see e.g., [18, p. 139])

$$J_{2ir}(x) - J_{-2ir}(x) = \frac{2i}{\pi}\sinh(\pi r)\int_{-\infty}^{\infty}\cos(x\cosh u)\cos(2ru)\,\mathrm{d}u.$$

Integration by parts shows that, for $x > 0$ and $r \geqslant 0$,

$$\begin{aligned} J_{2ir}(x) - J_{-2ir}(x) = {} & \frac{2i}{\pi}\sinh(\pi r)\int_{-\log^2 K}^{\log^2 K}\cos(x\cosh u)\cos(2ru)\,\mathrm{d}u \\ & + O\left(x^{-1}(r+1)\exp(\pi r - \tfrac12\log^2 K)\right). \end{aligned} \tag{7.10}$$

The error term in (7.10) clearly makes a negligible contribution. The main term in (7.10) will contribute to f_+

$$-\frac{4}{\pi^2}\int_{-\log^2 K}^{\log^2 K}\cos(x\cosh u)\int_0^{\infty} r f(r,K)\tanh(\pi r)\cos(2ru)\,\mathrm{d}r\,\mathrm{d}u, \tag{7.11}$$

where

$$x \;=\; 4\pi\frac{\sqrt{m}}{\ell} \leqslant 2(1+\delta)K. \tag{7.12}$$

In the inner integral we use (5.12) and make the change of variable $r = K + Gv$. In the ensuing v-integral the non-negligible contribution will be from the range $|v| \leqslant \log K$. Since $f(r)$ contains the factor $(4T/K)^{ir}$, it follows by (5.2) and (5.3) that the contribution of f_+ is

$$\asymp \Re\mathrm{e}\left\{GK\int_{-\log^2 K}^{\log^2 K}\cos(x\cosh u)\exp\left(-\frac{G^2}{4}\left(\log\frac{4T}{K}\pm 2u\right)^2 \pm 2iKu\right)\mathrm{d}u\right\}. \tag{7.13}$$

The relevant exponential factor will be of the form

$$\exp(ig(u)),\ g(u) = x\cosh u \pm 2Ku,\ g'(u) = x\sinh u \pm 2K.$$

The saddle point u_1 is (here the solution of $g'(u_1) = 0$ with the plus sign is treated, since the other case is similar)

$$u_1 = \log\left(\frac{2K}{x} + \sqrt{\frac{4K^2}{x^2}+1}\right),$$

and we have

$$g''(u_1) = x\cosh(u_1) \gg K.$$

Since $K/x \gg 1$ in view of (7.12), it follows by the saddle point method that the main contribution to (7.11) is

$$\asymp\int_{K_0}^{K_0'}\mathrm{e}^{\pm iK+iH(K)}K^{1/2}\exp\left(-\frac{G^2}{4}\left(\log\frac{4T/K}{2K/x+\sqrt{(2K/x)^2+1}}\right)^2\right)\mathrm{d}K, \tag{7.14}$$

plus an error term which does not exceed $O(T^{1/2+\varepsilon}K^{1/4})$, where

$$H(K) := g(u_1), \quad |H'(K)| = \log\left(\frac{2K}{x} + \sqrt{\frac{4K^2}{x^2}+1}\right) + O(1),$$

and the contribution is negligible unless

$$\frac{C_1T}{K_0^2}\sqrt{m} \leqslant \ell \leqslant \frac{C_2T}{K_0^2}\sqrt{m} \qquad (0 < C_1 < C_2). \tag{7.15}$$

Thus by the first derivative test the integral in (7.14) is $\ll K_0^{1/2}\log K_0$. If we use Weil's classical bound $|S(m,n;\ell)| \leqslant (m,n,\ell)^{1/2}d(\ell)\ell^{1/2}$, then we see that the total contribution of the Kloosterman sum term in (7.3) is

$$\begin{aligned}
&\ll_\varepsilon K_0^{1/2+\varepsilon} \sum_{m\ll K} m^{-1/2} \sum_{\ell\approx \frac{T}{K_0^2}\sqrt{m}} \frac{1}{\ell}|S(m,1;\ell)| \\
&\ll_\varepsilon K_0^{1/2+\varepsilon} \sum_{m\ll K_0} m^{-1/2} \sum_{\ell\approx \frac{T}{K_0^2}\sqrt{m}} d(\ell)\ell^{-1/2} \\
&\ll_\varepsilon K_0^{\varepsilon-1/2}T^{1/2} \sum_{m\ll K_0} m^{-1/4} \\
&\ll_\varepsilon T^{1/2+\varepsilon}K_0^{1/4}.
\end{aligned}$$

We take $G = K_0^\varepsilon$, note that $K_0 \ll T^{1/2}K_0^{1/4}$ in view of (1.2) and finally replace K_0 by K. Then the second bound in (2.13) follows and the proof of Theorem 2 is complete.

8. Proof of Theorem 3

Suppose that the hypotheses of Theorem 3 hold. We start from

$$\int_T^{2T} \big(S_m(K;K',t)\big)^2\,\mathrm{d}t \leqslant \int_{T/2}^{5T/2} \varphi(t)\big(S_m(K;K',t)\big)^2\,\mathrm{d}t, \tag{8.1}$$

where $\varphi(t)$ is a non-negative, smooth function supported in $[T/2,\, 5T/2]$ such that $\varphi(t) = 1$ for $T \leqslant t \leqslant 2T$. We assume that $m = 3$, as this is the most interesting case. The proof of the cases $m = 1, 2$ is analogous, only instead of (1.4)–(1.5) we shall need the corresponding bounds with $H_j^2(\frac12)$ (see [23, eq. (3.4.4)]) or $H_j(\frac12)$ (see [10]). If the cosine is written as a sum of exponentials, then for $m = 3$ the right-hand side of (8.1) becomes, after integration by parts,

$$\begin{aligned}
&\ll \int\limits_{T/2}^{5T/2} \varphi(t) \sum_{K<\kappa_j,\kappa_\ell\leqslant K'} \alpha_j\alpha\ell H_j^3(\tfrac12)H_\ell^3(\tfrac12)\mathrm{e}^{i(\kappa_\ell\log\kappa_\ell-\kappa_j\log\kappa_j)}\,(4\mathrm{e}t)^{i\kappa_j-i\kappa_\ell}\,\mathrm{d}t \\
&= -\sum_{K<\kappa_j,\kappa_\ell\leqslant K'} \alpha_j\alpha\ell H_j^3(\tfrac12)H_\ell^3(\tfrac12)\mathrm{e}^{i(\kappa_\ell\log\kappa_\ell-\kappa_j\log\kappa_j)} \\
&\quad\times \int\limits_{T/2}^{5T/2} \frac{\varphi'(t)}{i\kappa_j - i\kappa_\ell + 1}\,(4\mathrm{e})^{i\kappa_j-i\kappa_\ell}\,t^{i\kappa_j-i\kappa_\ell+1}\,\mathrm{d}t.
\end{aligned} \tag{8.2}$$

In (8.2) we may continue to integrate by parts, noting that

$$\varphi^{(r)}(T/2) = \varphi^{(r)}(5T/2) = 0, \quad \varphi^{(r)}(t) \ll_r T^{-r} \qquad (r = 0,1,2,\dots). \tag{8.3}$$

Therefore taking $r = r(A, \varepsilon)$ sufficiently large, it follows from (8.3) that the contribution of κ_j, κ_ℓ such that $|\kappa_j - \kappa_\ell| > T^\varepsilon$ is $\ll T^{-A}$ for any given, large $A > 0$. The contribution of the remaining pairs κ_j, κ_ℓ is estimated trivially by the use of (1.3)–(1.5) as

$$\ll \int\limits_{T/2}^{5T/2} \varphi(t) \sum_{K<\kappa_j\leqslant K'} \alpha_j H_j^3(\tfrac{1}{2}) \sum_{|\kappa_j-\kappa_\ell|\leqslant T^\varepsilon} \alpha\ell H_\ell^3(\tfrac{1}{2})\,\mathrm{d}t$$

$$\ll_\varepsilon T^\varepsilon K \int\limits_{T/2}^{5T/2} \varphi(t) \sum_{K<\kappa_j\leqslant K'} \alpha_j H_j^3(\tfrac{1}{2})\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}K^3,$$

and this is asserted by (2.15). If the conjectural (1.7)–(1.8) holds, then obviously (2.15) can be improved (for $m = 3$) to

$$\int_T^{2T} \big(S(K;K',t)\big)^2\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}K^{5/2}.$$

Also by direct integration we have

$$\int_T^{2T} S(K;K',t)\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}K, \tag{8.4}$$

while the integral in (8.4) is $\ll_\varepsilon T^{1+\varepsilon}K^{1/2}$ if (1.7)–(1.8) holds.

Finally we sketch the proof of (2.16) of the Corollary. We start from

$$\int_T^{2T} (E_2(2t) - E_2(t))^2\,\mathrm{d}t \leqslant \int_{T/2}^{5T/2} \varphi(t)(E_2(2t) - E_2(t))^2\,\mathrm{d}t, \tag{8.5}$$

where $\varphi(t)$ is as in (8.1). Then we use (2.7)–(2.8), truncating the series in (2.8) at $T\Delta^{-1}\log T$ with a negligible error. After this, we remove the monotonic coefficients $\kappa_j^{-3/2}$ and $\exp\Big(-\frac{1}{4}\big(\frac{\Delta\kappa_j}{T}\big)^2\Big)$ by partial summation. Then we obtain the sum $S_m(K;K',t)$ with $m = 3$ and t replaced by $2t + \Delta\log T$ or $t - \Delta\log T$, which does not cause any trouble. Hence the integral on the left-hand side of (8.5) is essentially majorized by $\ll_\varepsilon T^\varepsilon$ integrals of the type

$$T\int_{T/2}^{5T/2} \varphi(t)\big(K^{-3/2}S_m(K;K',t)\big)^2\,\mathrm{d}t \ll_\varepsilon T^{2+\varepsilon},$$

and (2.16) follows on replacing t by $t2^{-j}$ in the integrand in (8.5), and summing up the corresponding bounds over $j = 1, 2, \dots$.

It may be remarked that the method of proof of Theorem 3 gives also, for $1 \ll K < K' \leqslant 2K \ll T$,

$$\int_T^{2T} \big(S_m(K;K',t)\big)^4\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}K^7 \qquad (m = 1, 2, 3),$$

which means that, in the mean fourth sense, the sum $S_m(K;K',t)$ is $\ll_\varepsilon K^{7/4+\varepsilon}$.

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