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ESTIMATES OF THE APPROXIMATION ERROR FOR ABSTRACT SAMPLING TYPE OPERATORS IN ORLICZ SPACES

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Abstract: We get some inequalities concerning the modular distance $I^{\varphi}_G[Tf-f]$ for bounded functions $f: G \to \mathbb{R}$. Here G is a locally compact Hausdorff topological space provided with a regular and σ-finite measure μ_G, I^{φ}_G is the modular functional generating the Orlicz spaces $L^{\varphi}(G)$ and T is a nonlinear integral operator of the form

$$(Tf)(s) = \int_H K(s,t,f(t))d\mu_H(t),$$

where H is a closed subset of G endowed with another regular and σ-finite measure μ_H. As a consequence we obtain a convergence theorem for a net of such operators. Some applications to discrete operators are given.
Keywords: Sampling operators, discrete operators, Orlicz spaces, moduli of continuity.

1. Introduction

In [7] we consider a net of nonlinear integral operators of type

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))d\mu_w(t), \quad w > 0, \; s \in G,$$

where G is a Hausdorff locally compact topological space provided with a regular and σ-finite measure μ_G and $(H_w)_{w>0}$ is a family of nonempty closed subspaces of G such that $\overline{\bigcup_{w>0} H_w} = G$ and μ_w is a regular and σ-finite measure on H_w.

The main result of [7] states a global modular convergence theorem in an Orlicz space $L^{\varphi}(G)$ for the family of operators $(T_w)_{w>0}$ and various applications to sampling type series and other kinds of discrete operators are discussed.

The method used in [7] is based on a modular density theorem which states that $\overline{C_c(G)} = L^{\varphi}(G)$, being $C_c(G)$ the subspace of all the continuous functions with compact support (see [3]). This approach did not give explicitely an estimate

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of the error of modular approximation $I^\varphi_G[T_w f - f]$, because we did not involve a modulus of continuity. Here I^φ_G is the modular functional generating the Orlicz space $L^\varphi(G)$ defined by

$$I^\varphi_G[f] = \int_G \varphi(|f(s)|)d\mu_G(s),$$

for $f \in L^0(G)$, being $L^0(G)$ the space of all μ_G-measurable function defined on G.

Here we consider some kinds of moduli of continuity in Orlicz spaces and we get some inequalities which give bounds for the error $I^\varphi_G[T_w f - f]$ in terms of these moduli. The complexity of these estimates has the advantage to be applied to convergence theorems with a direct approach without using density results and without boundedness assumptions on the family $(\mu_w)_{w>0}$ as used in [7].

We furnish various applications to a wide range of integral operators, from Urysohn type operators to discrete operators, both in linear and nonlinear case.

In particular, in nonlinear frame we give explicit results for Mellin type operators ([11], [9]) and in linear case we give detailed applications for generalized sampling operators ([19], [13], [14]) and Szász-Mirak'jan operator ([1]).

2. Preliminaries

Let G be a locally compact Hausdorff topological space provided with its family of Borel sets $\mathcal{B}$. Let μ_G be a regular and σ-finite measure defined on $\mathcal{B}$. Moreover let $H \subset G$ be a nonempty closed (measurable) subset of G, and let μ_H be another regular and σ-finite measure on the Borel σ-algebra generated by the family $\{A \cap H : A \text{ open subset of } G\}$. We will assume that the topology of G is uniformizable, i.e. there is a uniform structure $\mathcal{U} \subset G \times G$ which generates the topology of G (see [20]). For every $U \in \mathcal{U}$, we put $U_s = \{t \in G : (s,t) \in U\}$. By local compactness, we assume that for every $s \in G$, the base $\{U_s : U \in \mathcal{U}\}$ contains compact sets.

By $L^0(G)$ we denote the vector space of all real-valued μ_G-measurable functions $f : G \to \mathbb{R}$ provided with equality a.e., by $C(G)$ the subspace of all uniformly continuous and bounded functions and by $C_c(G)$ the subspace of all continuous functions with compact support.

Let Ψ be the class of all functions $\psi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that the function ψ is continuous and non decreasing, with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.

For a given $\psi \in \Psi$, let $\mathcal{K}_\psi$ be the class of all functions $K : G \times H \times \mathbb{R} \to \mathbb{R}$, such that

K.1) $K(\cdot,\cdot,u)$ is globally measurable on $G \times H$ for every $u \in \mathbb{R}$ and $K(s,t,0) = 0$, for every $(s,t) \in G \times H$.

K.2) K is (L,ψ)-Lipschitz i.e. there is a function $L : G \times H \to \mathbb{R}^+_0$, globally measurable on $G \times H$, such that

$$|K(s,t,u) - K(s,t,v)| \leqslant L(s,t)\psi(|u-v|)$$

for every $s \in G$, $t \in H$, $u,v \in \mathbb{R}$.

Now for $K \in \mathcal{K}_\psi$ we take into consideration the following nonlinear integral operator T of the form

$$(Tf)(s) = \int_H K(s,t,f(t))d\mu_H(t),$$

$s \in G,\ f \in DomT$, where $DomT$ is the subset of $L^0(G)$ on which Tf is well defined as a μ_G-measurable function of $s \in G$. In [6], it is proved that $L^\infty(G)$ is contained in $DomT$.

Let Φ be the class of all functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

i) φ is continuous, non decreasing,

ii) $\varphi(0) = 0,\ \varphi(u) > 0$ for $u > 0$ and $\lim_{u\to+\infty} \varphi(u) = +\infty$.

Moreover we denote by $\widetilde{\Phi}$ the subspace of Φ whose elements are convex functions.

Now, for $\varphi \in \Phi$, we define the functional

$$I_G^\varphi[f] = \int_G \varphi(|f(s)|)d\mu_G(s)$$

for every $f \in L^0(G)$.

As it is well known, I_G^φ is a modular on $L^0(G)$ and the subspace

$$L^\varphi(G) = \{f \in L^0(G) : I_G^\varphi[\lambda f] < +\infty \text{ for some } \lambda > 0\}$$

is the Orlicz space generated by φ, (see [17]).

We point out that if $\varphi \in \widetilde{\Phi}$, then I_G^φ is a convex modular.

Given $\varphi,\ \eta \in \Phi$ and $\psi \in \Psi$ we will say that the triple (φ, ψ, η) is properly directed if for every $\lambda > 0$ there exists $C_\lambda \in]0,1[$ satisfying

$$\varphi(C_\lambda \psi(u)) \leqslant \eta(\lambda u), \tag{1}$$

for every $u \in \mathbb{R}_0^+$, (see [9]).

Moreover, given $\eta \in \Phi$, we will denote by I_H^η the modular

$$I_H^\eta[f] = \int_H \eta(|f(t)|)d\mu_H(t)$$

for any $f \in X(H)$, the space of all real valued μ_H-measurable functions on H.

Here with $L^\eta(H)$, for $\eta \in \Phi$, we denote the space of all functions $f \in L^0(G)$ such that the restriction $f_{|H}$ is an element of the Orlicz space generated by the modular I_H^η.

If H is a discrete set, for example $H = \mathbb{N}$ or $H = \boldsymbol{Z}$, we will denote by $\ell^\varphi(H)$ the corresponding Orlicz space, called the sequence Orlicz space.

By using the modular I_G^φ, with $\varphi \in \widetilde{\Phi}$, we define the Luxemburg norm in $L^\varphi(G)$ in the following way (see [17])

$$\|f\|_{\varphi,G} = \inf\{u > 0 : I_G^\varphi[\frac{f}{u}] \leqslant 1\}.$$

Analogously we define the norm $\|f\|_{\varphi,H}$ when the base space is H.

The following inequality was proved in [8]

Theorem 1. *Let $\psi \in \Psi$, $\varphi \in \widetilde{\Phi}$ and $\eta \in \Phi$ be functions satisfying property (1) and $K \in \mathcal{K}_\psi$. Let us assume, furtherly, that there exists a constant $D > 0$ such that*

$$\int_H L(s,t)d\mu_H(t) \leqslant D \quad \text{and} \quad \int_G L(s,t)d\mu_G(s) \leqslant D$$

for every $s \in G$ and $t \in H$ respectively. Then for every $f \in DomT$ and for every $\lambda > 0$ there exists $\alpha > 0$ such that

$$I_G^\varphi[\alpha T f] \leqslant I_H^\eta[\lambda f].$$

In particular, $Tf \in L^\varphi(G)$ whenever $f \in L^\eta(H)$.

3. Estimates for the modular error of approximation in $L^\infty(G)$

3.1. Estimates in terms of pointwise modulus of continuity. Let $K \in \mathcal{K}_\psi$ for a fixed $\psi \in \Psi$. We shall apply the following notations for every $s \in G$ and $f \in L^0(G)$

$$R(f,s) := \left| \int_H K(s,t,f(s))d\mu_H(t) - f(s) \right|,$$

$$R_n(s) := \sup_{\frac{1}{n} \leqslant |u| \leqslant n} \left| \int_H K(s,t,u)d\mu_H(t) - u \right|,$$

for every $n \in \mathbb{N}^+$ and

$$R(s) = \sup_{n \in \mathbb{N}} R_n(s).$$

For every $s \in G$ and $U \in \mathcal{U}$, let U_s be a neighbourhood of $s \in G$. We put

$$\omega_\eta(f, U_s) := \sup_{t \in U_s} \eta(|f(t) - f(s)|),$$

and for any compact subset $B \subset G$

$$\omega_\eta(f, B, U) := \sup_{s \in B} \omega_\eta(f, U_s)$$

for $f \in L^0(G)$.

We need the following definition: we will say that the family $(L(\cdot,t))_{t \in H}$ satisfies property $(*)$ if for every compact $C \subset G$ and for every $\varepsilon > 0$ there exists a compact subset $B \subset G$ such that

$$\int_{G \setminus B} L(s,t)d\mu_G(s) < \varepsilon$$

for every $t \in H \cap C$.

We prove the following theorem

Theorem 2. *Let us suppose that the assumptions of Theorem 1 hold and the family $(L(\cdot,t))_{t\in H}$ satisfies property $(*)$. Let $f \in L^\infty(G)\cap L^{\varphi+\eta}(G)\cap L^\eta(H)$ be fixed. Let $C \subset G$ be an arbitrary nonempty compact subset. Then for every ε, $\lambda > 0$ and $U \in \mathfrak{U}$, there exist $\alpha > 0$ and a compact subset $B \subset G$, depending only on C and ε, such that*

$$
\begin{aligned}
&I_G^\varphi[\alpha(Tf-f)] \leqslant \frac{1}{4}\omega_\eta(\lambda f, B, U)\mu_G(B)\\
&+\frac{\eta(2\lambda\|f\|_\infty)}{4D}\int_B\left[\int_{H\setminus U_s} L(s,t)d\mu_H(t)\right]d\mu_G(s)\\
&+\frac{\varepsilon}{2D}I_H^\eta[2\lambda f]+\frac{1}{2}I_H^\eta[2\lambda f\chi_{H\setminus C}]+\frac{3}{4}I_G^\eta[2\lambda f\chi_{G\setminus B}]+\frac{1}{4}I_G^\varphi[4\alpha f\chi_{G\setminus B}]\\
&+\frac{1}{2}\int_B\varphi(2\alpha R(f,s))d\mu_G(s).
\end{aligned}
$$

Proof. Let $\lambda > 0$ be fixed and let $\alpha > 0$ be such that $4\alpha D \leqslant C_\lambda$. We have

$$
\begin{aligned}
&|(Tf)(s)-f(s)|\\
&\leqslant \int_H L(s,t)\psi(|f(t)-f(s)|)d\mu_H(t)+\left|\int_H K(s,t,f(s))d\mu_H(t)-f(s)\right|.
\end{aligned}
$$

Thus by convexity of φ,

$$
\begin{aligned}
&I_G^\varphi[\alpha(Tf-f)] \leqslant \frac{1}{2}\int_G\varphi\left(2\alpha\int_H L(s,t)\psi(|f(t)-f(s)|)d\mu_H(t)\right)d\mu_G(s)\\
&+\frac{1}{2}\int_G\varphi\left(2\alpha\left|\int_H K(s,t,f(s))d\mu_H(t)-f(s)\right|\right)d\mu_G(s)=J_1+J_2.
\end{aligned}
$$

Now we consider J_1. We fix, arbitrarily, a compact subset $C \subset G$ and $\varepsilon > 0$. Let B be the compact subset in the assumption $(*)$. Then we have

$$
\begin{aligned}
J_1 &= \frac{1}{2}\left(\int_B+\int_{G\setminus B}\right)\varphi\left(2\alpha\int_H L(s,t)\psi(|f(t)-f(s)|)d\mu_H(t)\right)d\mu_G(s)\\
&= J_1^1+J_1^2.
\end{aligned}
$$

Using Jensen inequality and (1), we obtain

$$
\begin{aligned}
J_1^1 &\leqslant \frac{1}{4}\int_B\varphi\left(4\alpha\int_{H\cap U_s}L(s,t)\psi(|f(t)-f(s)|)d\mu_H(t)\right)d\mu_G(s)\\
&+\frac{1}{4}\int_B\varphi\left(4\alpha\int_{H\setminus U_s}L(s,t)\psi(|f(t)-f(s)|)d\mu_H(t)\right)d\mu_G(s)
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \frac{1}{4D}\int_B\left(\int_{H\cap U_s}\eta(\lambda|f(t)-f(s)|)L(s,t)d\mu_H(t)\right)d\mu_G(s)\\
&\quad+\frac{1}{4D}\int_B\left(\int_{H\setminus U_s}\eta(\lambda|f(t)-f(s)|)L(s,t)d\mu_H(t)\right)d\mu_G(s)\\
&\leqslant \frac{1}{4}\omega_\eta(\lambda f,B,U)\mu_G(B)+\frac{\eta(2\lambda\|f\|_\infty)}{4D}\int_B\left(\int_{H\setminus U_s}L(s,t)d\mu_H(t)\right)d\mu_G(s).
\end{aligned}$$

Now we evaluate J_1^2. Using the same arguments as before and property $(*)$, we have

$$\begin{aligned}
J_1^2 &\leqslant \frac{1}{2D}\int_{G\setminus B}\left[\int_H \eta(\lambda|f(t)-f(s)|)L(s,t)d\mu_H(t)\right]d\mu_G(s)\\
&\leqslant \frac{1}{2D}\int_{G\setminus B}\left[\int_H \eta(2\lambda|f(t)|)L(s,t)d\mu_H(t)\right]d\mu_G(s)\\
&\quad+\frac{1}{2D}\int_{G\setminus B}\left[\int_H \eta(2\lambda|f(s)|)L(s,t)d\mu_H(t)\right]d\mu_G(s)\\
&\leqslant \frac{1}{2D}\int_{H\cap C}\eta(2\lambda|f(t)|)\left[\int_{G\setminus B}L(s,t)d\mu_G(s)\right]d\mu_H(t)\\
&\quad+\frac{1}{2D}\int_{H\setminus C}\eta(2\lambda|f(t)|)\left[\int_{G\setminus B}L(s,t)d\mu_G(s)\right]d\mu_H(t)\\
&\quad+\frac{1}{2}\int_{G\setminus B}\eta(2\lambda|f(s)|)d\mu_G(s)\\
&\leqslant \frac{\varepsilon}{2D}\int_{H\cap C}\eta(2\lambda|f(t)|)d\mu_H(t)+\frac{1}{2}\int_{H\setminus C}\eta(2\lambda|f(t)|)d\mu_H(t)\\
&\quad+\frac{1}{2}\int_{G\setminus B}\eta(2\lambda|f(s)|)d\mu_G(s).
\end{aligned}$$

We consider now J_2. We have

$$\begin{aligned}
J_2 &= \frac{1}{2}\left(\int_B+\int_{G\setminus B}\right)\varphi\left(2\alpha\left|\int_H K(s,t,f(s))d\mu_H(t)-f(s)\right|\right)d\mu_G(s)\\
&= J_2^1+J_2^2.
\end{aligned}$$

Note that

$$J_2^1=\frac{1}{2}\int_B\varphi(2\alpha R(f,s))d\mu_G(s),$$

while

$$
\begin{aligned}
J_2^2 &\leqslant \frac{1}{4D}\int_{G\setminus B}\left[\int_H \varphi(4\alpha D\psi(|f(s)|))L(s,t)d\mu_H(t)\right]d\mu_G(s) \\
&\quad+\frac{1}{4}\int_{G\setminus B}\varphi(4\alpha|f(s)|)d\mu_G(s) \\
&\leqslant \frac{1}{4}\int_{G\setminus B}\eta(\lambda|f(s)|)d\mu_G(s)+\frac{1}{4}\int_{G\setminus B}\varphi(4\alpha|f(s)|)d\mu_G(s).
\end{aligned}
$$

Since by monotonicity of η we have $\eta(\lambda|f(s)|)\leqslant\eta(2\lambda|f(s)|)$, the theorem is completely proved. ■

In particular, given a function f satisfying the assumptions of Theorem 2, we can choose a suitable compact set C such that the following corollary holds

Corollary 1. *Let us suppose that the assumptions of Theorem 1 hold and the family $(L(\cdot,t))_{t\in H}$ satisfies property $(*)$. Let $f\in L^\infty(G)\cap L^{\varphi+\eta}(G)\cap L^\eta(H)$ be fixed. Let $\lambda>0$ be such that $I_H^\eta[2\lambda f]+I_G^\eta[2\lambda f]<+\infty$. Then for every $\varepsilon>0$ and $U\in\mathfrak{U}$ there exists $\alpha>0$ and a compact subset $B\subset G$ such that*

$$
\begin{aligned}
&I_G^\varphi[\alpha(Tf-f)] \\
&\leqslant \frac{1}{4}\omega_\eta(\lambda f,B,U)\mu_G(B)+\frac{\eta(2\lambda\|f\|_\infty)}{4D}\int_B\left[\int_{H\setminus U_s}L(s,t)d\mu_H(t)\right]d\mu_G(s) \\
&\quad+\frac{1}{2}\int_B\varphi(2\alpha R(f,s))d\mu_G(s)+\varepsilon.
\end{aligned}
$$

Proof. Let $\lambda>0$ be fixed as in the assumptions and let $\alpha>0$ be such that $4\alpha D\leqslant C_\lambda$ and $I_G^\varphi[4\alpha f]<+\infty$. We can assume that $I_H^\eta[2\lambda f]$ is positive. Let $\varepsilon>0$ be fixed. There exists a compact $C=C_\varepsilon$ such that

$$\frac{1}{2}I_H^\eta[2\lambda f\chi_{H\setminus C}]\leqslant\frac{\varepsilon}{3}.$$

Now, using property $(*)$ there exists a compact $B=B_\varepsilon\subset G$ such that

$$\int_{G\setminus B}L(s,t)d\mu_G(s)<\frac{2D\varepsilon}{3I_H^\eta[2\lambda f]}$$

for every $t\in H\cap C$. We can choose B in such a way that

$$\frac{3}{4}I_G^\eta[2\lambda f\chi_{G\setminus B}]+\frac{1}{4}I_G^\varphi[4\alpha f\chi_{G\setminus B}]<\frac{\varepsilon}{3}.$$

The assertion comes from Theorem 2. ■

Now we can obtain a similar inequality in terms of $R_n(s)$ for $n \in \mathbb{N}$.

Corollary 2. *Under the assumptions of Theorem 2, for a given compact subset $C \subset G$ we have that for every $\varepsilon,\ \lambda > 0$ and $U \in \mathfrak{U}$ there exists $\alpha > 0$ and a compact subset $B \subset G$, depending only on C and ε such that*

$$
\begin{aligned}
I_G^{\varphi}[\alpha(Tf-f)] &\leqslant \frac{1}{4}\omega_\eta(\lambda f, B, U)\mu_G(B) \\
&+ \frac{\eta(2\lambda\|f\|_\infty)}{4D}\int_B\left[\int_{H\setminus U_s} L(s,t)d\mu_H(t)\right]d\mu_G(s) \\
&+ \frac{\varepsilon}{2D}I_H^{\eta}[2\lambda f] + \frac{1}{2}I_H^{\eta}[2\lambda f\chi_{H\setminus C}] + \frac{3}{4}I_G^{\eta}[2\lambda f\chi_{G\setminus B}] + \frac{1}{4}I_G^{\varphi}[4\alpha f\chi_{G\setminus B}] \\
&+ \frac{1}{4}\mu_G(B)[\eta(\frac{\lambda}{n}) + \varphi(\frac{4\alpha}{n})] + \frac{1}{2}\int_B \varphi(2\alpha R_n(s))d\mu_G(s),
\end{aligned}
$$

for sufficiently large $n \in \mathbb{N}$.

Proof. Using the same methods and notations of the proof of Theorem 2, we have only to estimate the term

$$
\begin{aligned}
J_2 &= \frac{1}{2}\left(\int_B + \int_{G\setminus B}\right)\varphi\left(2\alpha\left|\int_H K(s,t,f(s))d\mu_H(t) - f(s)\right|\right)d\mu_G(s) \\
&= J_2^1 + J_2^2.
\end{aligned}
$$

We estimate now J_2^1. Since $f \in L^\infty(G)$ there exists an integer $\overline{n}$ such that $|f(s)| \leqslant \overline{n}$ for every $s \in G$. For $n \geqslant \overline{n}$, we put $A_n = \{s \in G : 0 \leqslant |f(s)| \leqslant \frac{1}{n}\}$ and we obtain

$$
\begin{aligned}
J_2^1 &= \frac{1}{2}\int_B \varphi\left(2\alpha\left|\int_H K(s,t,f(s))d\mu_H(t) - f(s)\right|\right)d\mu_G(s) \\
&= \frac{1}{2}\int_B \varphi\left(2\alpha\left|\int_H K(s,t,f(s)\chi_{A_n}(s))d\mu_H(t) - f(s)\chi_{A_n}(s)\right|\right)d\mu_G(s) \\
&+ \frac{1}{2}\int_B \varphi\left(2\alpha\left|\int_H K(s,t,f(s)\chi_{G\setminus A_n}(s))d\mu_H(t) - f(s)\chi_{G\setminus A_n}(s)\right|\right)d\mu_G(s) \\
&= J_2^{1,1} + J_2^{1,2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
J_2^{1,1} &= \frac{1}{2}\int_B \varphi\left(2\alpha\left|\int_H K(s,t,f(s)\chi_{A_n}(s))d\mu_H(t) - f(s)\chi_{A_n}(s)\right|\right)d\mu_G(s) \\
&\leqslant \frac{1}{4}\int_B \varphi\left(4\alpha\int_H |K(s,t,f(s)\chi_{A_n}(s))|d\mu_H(t)\right)d\mu_G(s) \\
&\quad + \frac{1}{4}\int_B \varphi(4\alpha|f(s)\chi_{A_n}(s)|)d\mu_G(s)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{1}{4}\int_B \varphi\left(4\alpha\int_H L(s,t)\psi(\frac{1}{n})d\mu_H(t)\right)d\mu_G(s)+\frac{1}{4}\int_B \varphi(\frac{4\alpha}{n})d\mu_G(s)\\
&\leqslant \frac{1}{4D}\int_B\left(\int_H \varphi(4\alpha D\psi(\frac{1}{n}))L(s,t)d\mu_H(t)\right)d\mu_G(s)+\frac{1}{4}\varphi(\frac{4\alpha}{n})\mu_G(B)\\
&\leqslant \frac{1}{4}\varphi(4\alpha D\psi(\frac{1}{n}))\mu_G(B)+\frac{1}{4}\varphi(\frac{4\alpha}{n})\mu_G(B)\\
&\leqslant \frac{1}{4}\eta(\frac{\lambda}{n})\mu_G(B)+\frac{1}{4}\varphi(\frac{4\alpha}{n})\mu_G(B)\\
&=\frac{1}{4}\mu_G(B)[\eta(\frac{\lambda}{n})+\varphi(\frac{4\alpha}{n})].
\end{aligned}
$$

Moreover

$$
\begin{aligned}
J_2^{1,2}&=\frac{1}{2}\int_B \varphi\left(2\alpha\left|\int_H K(s,t,f(s)\chi_{G\setminus A_n}(s))d\mu_H(t)-f(s)\chi_{G\setminus A_n}(s)\right|\right)d\mu_G(s)\\
&\leqslant \frac{1}{2}\int_B \varphi(2\alpha R_n(s))d\mu_G(s).
\end{aligned}
$$

Finally J_2^2 is estimated exactly as in Theorem 2. ■

Remarks.

1.1. Under the assumptions of Theorem 2, if $R(s)<+\infty$ for every $s\in G$, then the inequality given in this theorem becomes

$$
\begin{aligned}
I_G^{\varphi}[\alpha(Tf-f)]&\leqslant \frac{1}{4}\omega_\eta(\lambda f,B,U)\mu_G(B)\\
&+\frac{\eta(2\lambda\|f\|_\infty)}{4D}\int_B\left[\int_{H\setminus U_s} L(s,t)d\mu_H(t)\right]d\mu_G(s)\\
&+\frac{\varepsilon}{2D}I_H^{\eta}[2\lambda f]+\frac{1}{2}I_H^{\eta}[2\lambda f\chi_{H\setminus C}]+I_G^{\eta}[2\lambda f\chi_{G\setminus B}]+\frac{1}{4}I_G^{\varphi}[4\alpha f\chi_{G\setminus B}]\\
&+\frac{1}{2}\int_B \varphi(2\alpha R(s))d\mu_G(s).
\end{aligned}
$$

In the same way the inequality expressed in Corollary 1, becomes

$$
\begin{aligned}
I_G^{\varphi}[\alpha(Tf-f)]&\leqslant \frac{1}{4}\omega_\eta(\lambda f,B,U)\mu_G(B)\\
&+\frac{\eta(2\lambda\|f\|_\infty)}{4D}\int_B\left[\int_{H\setminus U_s} L(s,t)d\mu_H(t)\right]d\mu_G(s)\\
&+\frac{1}{2}\int_B \varphi(2\alpha R(s))d\mu_G(s)+\varepsilon.
\end{aligned}
$$

1.2. Let us observe that if $f \in C(G)$ we can choose $U \in \mathfrak{U}$ such that $\omega_\eta(\lambda f, B, U)\mu_G(B) < 4\varepsilon$. Thus by Corollary 1, we get

$$I_G^\varphi[\alpha(Tf - f)] \leqslant \frac{\eta(2\lambda\|f\|_\infty)}{4D} \int_B \left[\int_{H\setminus U_s} L(s,t)d\mu_H(t)\right] d\mu_G(s) + \frac{1}{2}\int_B \varphi(2\alpha R(f,s))d\mu_G(s) + 2\varepsilon$$

1.3. The inequality in Theorem 2 (see also Corollary 1) represents a non local version of a similar result given in [6] for subsets of finite measure.

3.2. Estimates in terms of a norm modulus of continuity. Let σ, $\gamma \in \widetilde{\Phi}$ be two complementary functions in the sense of Young (see [18]).

For any function $f \in L^\infty(G)$, $U \in \mathfrak{U}$ and a compact subset $B \subset G$, we define the norm modulus of continuity by

$$\widetilde{\omega}_{\gamma\circ\eta}(f,B,U) := \sup_{s\in B} \|\eta(|f(\cdot) - f(s)|)\chi_{U_s}(\cdot)\|_{\gamma,H}.$$

We have the following

Theorem 3. *Let the assumptions of Theorem 2 be satisfied and furtherly* $L(s,\cdot) \in L^\sigma_{loc}(H)$. *Let* $U \in \mathfrak{U}$ *and* $\|L(:,\cdot)\chi_{U_{(:)}}(\cdot)\|_{\sigma,H} \in L^1_{loc}(G)$. *Let* $C \subset G$ *be an arbitrary nonempty compact subset. Then for every* $\varepsilon, \lambda > 0$ *there exist* $\alpha > 0$ *and a compact subset* $B \subset G$, *depending only on* C *and* ε, *such that*

$$\begin{aligned} I_G^\varphi[\alpha(Tf - f)] &\leqslant \frac{1}{D}\widetilde{\omega}_{\gamma\circ\eta}(\lambda f, B, U)\int_B \|L(s,\cdot)\chi_{U_s}(\cdot)\|_{\sigma,H}d\mu_G(s) \\ &+ \frac{\eta(2\lambda\|f\|_\infty)}{2D}\int_B\left[\int_{H\setminus U_s} L(s,t)d\mu_H(t)\right]d\mu_G(s) \\ &+ \frac{\varepsilon}{2D}I_H^\eta[2\lambda f] + \frac{1}{2}I_H^\eta[2\lambda f\chi_{H\setminus C}] + \frac{3}{4}I_G^\eta[2\lambda f\chi_{G\setminus B}] + \frac{1}{4}I_G^\varphi[4\alpha f\chi_{G\setminus B}] \\ &+ \frac{1}{2}\int_B \varphi(2\alpha R(f,s))d\mu_G(s). \end{aligned}$$

Proof. Let $\lambda > 0$ be fixed and let C_λ be the constant in (1). Then, as in Theorem 2, for any $\alpha > 0$ such that $2\alpha D \leqslant C_\lambda$ we have

$$\begin{aligned} I_G^\varphi[\alpha(Tf - f)] &\leqslant \frac{1}{2D}\int_G\left[\int_H L(s,t)\eta(\lambda|f(t) - f(s)|)d\mu_H(t)\right]d\mu_G(s) \\ &+ \frac{1}{2}\int_G \varphi\left(2\alpha\left|\int_H K(s,t,f(s))d\mu_H(t) - f(s)\right|\right)d\mu_G(s) \\ &= J_1 + J_2. \end{aligned}$$

As to concerns J_1 we have

$$J_1 \leqslant \frac{1}{2D}\left(\int_B + \int_{G\setminus B}\right)\left[\int_H L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_H(t)\right]d\mu_G(s) = J_1^1 + J_1^2.$$

In order to evaluate J_1^1, for $U \in \mathcal{U}$ we write

$$\begin{aligned}
&\int_H L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_H(t) = \int_{U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_H(t)\\
&\quad + \int_{H\setminus U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_H(t)\\
&= I_1 + I_2
\end{aligned}$$

Using Proposition 1 in [18], we have

$$I_1 \leqslant 2\widetilde{\omega}_{\gamma\circ\eta}(\lambda f, B, U)\|L(s,\cdot)\chi_{U_s}(\cdot)\|_{\sigma,H}$$

for every $s \in B$. As to I_2 we have

$$I_2 \leqslant \eta(2\lambda\|f\|_\infty)\int_{H\setminus U_s} L(s,t)d\mu_H(t)$$

Thus

$$\begin{aligned}
J_1^1 &\leqslant \frac{\widetilde{\omega}_{\gamma\circ\eta}(\lambda f, B, U)}{D}\int_B \|L(s,\cdot)\chi_{U_s}(\cdot)\|_{\sigma,H}\ d\mu_G(s)\\
&\quad + \frac{\eta(2\lambda\|f\|_\infty)}{2D}\int_B\left[\int_{H\setminus U_s} L(s,t)d\mu_H(t)\right]d\mu_G(s).
\end{aligned}$$

The terms J_1^2 and J_2 can be estimated exactly as in Theorem 2. ■

Remark 2. As in Corollary 2, under the same assumptions of Theorem 3, we can obtain an analogous estimates in terms of $R_n(s)$, for $n \in \mathbb{N}$ in the following way

$$\begin{aligned}
&I_G^\varphi[\alpha(Tf-f)] \leqslant \frac{1}{D}\widetilde{\omega}_{\gamma\circ\eta}(\lambda f, B, U)\int_B \|L(s,\cdot)\chi_{U_s}(\cdot)\|_{\sigma,H}d\mu_G(s)\\
&\quad + \frac{\eta(2\lambda\|f\|_\infty)}{2D}\int_B\left[\int_{H\setminus U_s} L(s,t)d\mu_H(t)\right]d\mu_G(s)\\
&\quad + \frac{\varepsilon}{2D}I_H^\eta[2\lambda f] + \frac{1}{2}I_H^\eta[2\lambda f\chi_{H\setminus C}] + \frac{3}{4}I_G^\eta[2\lambda f\chi_{G\setminus B}] + \frac{1}{4}I_G^\varphi[4\alpha f\chi_{G\setminus B}]\\
&\quad + \frac{1}{4}\mu_G(B)[\eta(\frac{\lambda}{n}) + \varphi(\frac{4\alpha}{n})] + \frac{1}{2}\int_B \varphi(2\alpha R_n(s))d\mu_G(s).
\end{aligned}$$

3.3. Applications to convergence. For every $w > 0$, let H_w be a nonempty closed subset of G with $\overline{\cup_{w>0} H_w} = G$. Let μ_{H_w} be a regular and σ-finite measure on the σ-algebra $\mathcal{B}(H_w)$ of the Borel sets of H_w. In the following we put $\mu_w = \mu_{H_w}$.

Let $\Xi = (\psi_w)_{w>0} \subset \Psi$ be a family of functions such that the net $(\psi_w(u))_{w>0}$ is bounded for every $u \geqslant 0$.

Let us denote by $\mathcal{K}_\Xi$ the class of all the family of functions $\mathbb{K} = (K_w)_{w>0}$ such that for every $w > 0$ we have $K_w \in \mathcal{K}_{\psi_w}$. Let us denote by $\mathbb{L} = (L_w)_{w>0} \subset \mathcal{L}$ the corresponding class of functions such that the Lipschitz condition holds for any $w > 0$ i. e.

$$|K_w(s,t,u) - K_w(s,t,v)| \leqslant L_w(s,t)\psi_w(|u-v|)$$

for every $s \in G,\ t \in H_w,\ u, v \in \mathbb{R}$ and $w > 0$.

For a given $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ we will study the approximation properties of the family of operators $\mathbf{T} = (T_w)_{w>0}$ given by

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))d\mu_w(t), \quad s \in G$$

where $f \in Dom\mathbf{T} = \cap_{w>0} DomT_w$; here $DomT_w$ is the set of all functions $f \in L^0(G)$ for which T_w is well defined as a μ_G-measurable function of $s \in G$.

In the following we will assume that the triple (φ, ψ_w, η) is properly directed for every $w > 0$.

We will say that the family $\mathbb{K}$ is singular if the following conditions hold

1) There is a constant $D > 0$ such that, for every $w > 0,\ s \in G,\ t \in H_w,$ we have

$$\int_{H_w} L_w(s,t)d\mu_w(t) \leqslant D, \qquad \int_G L_w(s,t)d\mu_G(s) \leqslant D.$$

2) For every $s \in G$ and for every $U \in \mathcal{U}$ we have

$$\lim_{w\to+\infty} \int_{H_w \setminus U_s} L_w(s,t)d\mu_w(t) = 0.$$

3) For every $s \in G$ and $u \in \mathbb{R}$ we have

$$\lim_{w\to+\infty} \int_{H_w} K_w(s,t,u)d\mu_w(t) = u.$$

We will say that the family $\mathbb{K}$ is uniformly singular if conditions 2) and 3) are replaced by the following ones

2)' For every $U \in \mathcal{U}$ we have

$$\lim_{w\to+\infty} \int_{H_w \setminus U_s} L_w(s,t)d\mu_w(t) = 0$$

uniformly with respect to $s \in G$.

3)' We have

$$\lim_{w\to+\infty}\int_{H_w} K_w(s,t,u)d\mu_w(t)=u,$$

uniformly with respect to $s\in G$ and $u\in C$, where C is any compact subset of $\mathbb{R}\setminus\{0\}$.

In the following we will write

$$R_w(f,s):=\left|\int_{H_w} K_w(s,t,f(s))d\mu_w(t)-f(s)\right|.$$

For nonlinear integral operators of convolution type a stronger concept of singularity was given in [16] and [9].

Moreover the inequalities in 1) can be satisfied only for sufficiently large $w>0$.

Using the previous notations, we will say that the family $(L_w(\cdot,t))_{t\in H_w,w>0}$ satisfies property $(**)$ if for every compact $C\subset G$ and for every $\varepsilon>0$ there exists a compact subset $B\subset G$ such that

$$\int_{G\setminus B} L_w(s,t)d\mu_G(s)<\varepsilon,$$

for every $t\in H_w\cap C$ and sufficiently large $w>0$.

We will say that a function $f\in L^\eta(H_w)$, uniformly with respect to $w>0$, for a fixed $\eta\in\Phi$, if there exist two constants $\lambda,M>0$ such that $I^\eta_{H_w}[\lambda f]\leqslant M$ for sufficiently large $w>0$, depending only on f.

Remark 3. Let $G=\mathbb{R}$ and, for $w>0$, $H_w=\frac{1}{w}\boldsymbol{Z}$. Let us consider the measures μ_G the Lebesgue measure and $\mu_w=\frac{1}{w}\mu_c$, being μ_c the counting measure. In this instance, we have

$$I^\eta_{H_w}[\lambda f]=\frac{1}{w}\sum_{k=-\infty}^{+\infty}\eta(\lambda|f(\frac{k}{w})|).$$

Note that the above series is a generalized Riemann sum of the generalized integral

$$\int_{-\infty}^{+\infty}\eta(\lambda|f(s)|)ds.$$

So, for example, using the caracterization given in [15], every function f such that $\eta\circ\lambda f$ is of bounded coarse varition (see [15]) is uniformly in $L^\eta(H_w)$.

In the following we put

$$\widetilde{\omega}^w_{\gamma\circ\eta}(f,B,U):=\sup_{s\in B}\|\eta(|f(\cdot)-f(s)|)\chi_{U_s}(\cdot)\|_{\gamma,H_w}.$$

In what follows we denote by $\Upsilon_{\gamma\circ\eta}(H_w)$ the subclass of all the functions $f \in L^\infty(G)$ for which there exists $\lambda > 0$ such that

$$\lim_{U\in\mathfrak{U}} \widetilde{\omega}^w_{\gamma\circ\eta}(\lambda f, B, U) = 0$$

for every compact subset $B \subset G$, uniformly with respect to sufficiently large $w > 0$. In the previous formula we have taken the limit with respect to the filter of sets given by $\mathfrak{U}$. Thus by Theorem 3, we deduce a modular convergence theorem for the class of operators $(T_w)_{w>0}$.

Theorem 4. *Let $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$, $\Xi = (\psi_w)_{w>0} \subset \Psi$ be such that the triple (φ, ψ_w, η) is properly directed for every $w > 0$ and σ, $\gamma \in \widetilde{\Phi}$ be two complementary functions in the sense of Young. Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Assume furtherly that $L_w(s,\cdot) \in L^\sigma_{loc}(H_w)$ and $(L_w(\cdot,t))_{t\in H_w, w>0}$ satisfies property $(**)$. Let us assume that for a given $U \in \mathfrak{U}$, $\|L_w(:,\cdot)\chi_{U_{(:)}}(\cdot)\|_{\sigma,H_w} \in L^1_{loc}(G)$, uniformly with respect to $w > 0$. Let $f \in \Upsilon_{\gamma\circ\eta}(H_w) \cap L^{\varphi+\eta}(G) \cap L^\eta(H_w)$ uniformly with respect to $w > 0$. Then there exists $\alpha > 0$ such that*

$$\lim_{w\to+\infty} I^\varphi_G[\alpha(T_w f - f)] = 0.$$

Proof. We will use the notations of Theorem 3 and its estimate in terms of $R_w(f,s)$. Let $\lambda > 0$ be such that

$$I^\eta_G[2\lambda f] + I^\eta_{H_w}[2\lambda f] + I^\varphi_G[\lambda f] \leqslant P < +\infty,$$

for an absolute constant $P > 0$ and sufficiently large $w > 0$. Let $\varepsilon > 0$ be fixed and let $C \subset G$ be a compact subset such that

$$\frac{1}{2} I^\eta_{H_w}[2\lambda f \chi_{H_w\setminus C}] < \varepsilon,$$

for sufficiently large $w > 0$. By property $(**)$ let B be a compact subset of G such that

$$\int_{G\setminus B} L_w(s,t)\, d\mu_G(s) < \varepsilon,$$

for every $t \in H_w \cap C$ and sufficiently large $w > 0$. We can choose B in such a way that

$$\frac{3}{4} I^\eta_G[2\lambda f \chi_{G\setminus B}] + \frac{1}{4} I^\varphi_G[\lambda f \chi_{G\setminus B}] < \varepsilon.$$

Now by the singularity assumptions, we have

$$\lim_{w\to+\infty} R_w(f,s) = 0$$

for every $s \in G$ and so, since $R_w(f,s) \leqslant \psi_w(\|f\|_\infty)D + \|f\|_\infty$ by the property of the family Ξ and the Lebesgue dominated convergence Theorem, we have

$$\lim_{w\to+\infty} \int_B \varphi(\lambda R_w(f,s))\, d\mu_G(s) = 0.$$

Moreover, by assumptions 1) and 2) of singularity and again the Lebesgue Theorem,

$$\lim_{w\to+\infty}\int_B\left[\int_{H_w\setminus U_s}L_w(s,t)d\mu_w(t)\right]d\mu_G(s)=0,$$

where $U\in\mathfrak{U}$ is fixed. Finally, by assumptions there is a constant $P'>0$ such that

$$\int_B\|L_w(s,\cdot)\chi_{U_s(\cdot)}\|_{\sigma,H_w}d\mu_G(s)\leqslant P'$$

for every $w>0$. Now since $f\in\Upsilon_{\gamma\circ\eta}(H_w)$ we can choose $U\in\mathfrak{U}$ in such a way that

$$\frac{P'}{D}\widetilde{\omega}^w_{\gamma\circ\eta}(\lambda f,B,U)<\varepsilon$$

for sufficiently large $w>0$. Thus, choosing $\alpha>0$ such that $4\alpha\leqslant\lambda$ and $2\alpha D\leqslant C_\lambda$, applying Theorem 3 with $H=H_w$, $\mu_H=\mu_w$, $L=L_w$ and $T=T_w$, we obtain the assertion. ■

In the same way we can obtain a convergence theorem using the pointwise modulus of continuity, according to Theorem 2.

Theorem 5. *Let us suppose that the assumptions of Theorem 1 hold and the family $(L_w(\cdot,t))_{t\in H_w,w>0}$ satisfies property $(**)$. Let $\mathbb{K}=(K_w)_{w>0}\in\mathcal{K}_\Xi$ be singular. Assume that the triple (φ,ψ_w,η) is properly directed for every $w>0$. Let $f\in C(G)\cap L^{\varphi+\eta}(G)\cap L^\eta(H_w)$ uniformly with respect to $w>0$. Then there exists $\alpha>0$ such that*

$$\lim_{w\to+\infty}I^\varphi_G[\alpha(T_wf-f)]=0.$$

4. Applications to Urysohn operators

Here we assume $H=G$ endowed with the same measure μ_G. Thus our integral operator is given now by

$$(Tf)(s)=\int_G K(s,t,f(t))d\mu_G(t),\quad s\in G.$$

In this instance property $(*)$ for the family $(L(\cdot,t))_{t\in G}$ takes the form: for every compact $C\subset G$ and for every $\varepsilon>0$ there exists a compact subset $B\subset G$ such that $B\supset C$ and

$$\int_{G\setminus B}L(s,t)d\mu_G(s)<\varepsilon$$

for every $t\in C$. The results in Section 3 hold in this particular case with obviuos changes. In this frame we can obtain also estimates without the boundedness assumption on function f. Indeed we have the following result.

Theorem 6. *Let us suppose that the assumptions of Theorem 1 hold and the family* $(L(\cdot,t))_{t\in G}$ *satisfies property* $(*)$. *Furtherly assume that* $L(s,\cdot)\in L^{\sigma}_{loc}(G)$. *Let* $U\in\mathfrak{U}$ *and* $\|L(:,\cdot)\chi_{U_{(:)}}(\cdot)\|_{\sigma,G}\in L^1_{loc}(G)$. *Let* $f\in DomT\cap L^{\varphi+\eta}(G)\cap L^{\gamma\circ\eta}(G)$ *be fixed. Let* $C\subset G$ *be an arbitrary nonempty compact subset. Then for every* $\varepsilon,\ \lambda>0$ *there exist* $\alpha>0$ *and a compact subset* $B\subset G$, *depending only on* C *and* ε, *such that*

$$
\begin{aligned}
I^{\varphi}_G[\alpha(Tf-f)] &\leqslant \frac{1}{D}\widetilde{\omega}_{\gamma\circ\eta}(\lambda f,B,U)\int_B \|L(s,\cdot)\chi_{U_s(\cdot)}\|_{\sigma,G}d\mu_G(s)\\
&+\frac{1}{2D}\int_G \eta(2\lambda|f(t)|)\left[\int_{G\setminus U_t} L(s,t)d\mu_G(s)\right]d\mu_G(t)\\
&+\frac{1}{2D}\int_G \eta(2\lambda|f(s)|)\left[\int_{G\setminus U_s} L(s,t)d\mu_G(t)\right]d\mu_G(s)+\frac{\varepsilon}{2D}I^{\eta}_G[2\lambda f]\\
&+\frac{5}{4}I^{\eta}_G[2\lambda f\chi_{G\setminus C}]+\frac{1}{4}I^{\varphi}_G[4\alpha f\chi_{G\setminus B}]+\frac{1}{2}\int_B\varphi(2\alpha R(f,s))d\mu_G(s),
\end{aligned}
$$

Proof. Let $\lambda>0$ and $\alpha>0$ as in the proof of Theorem 3. We have

$$
\begin{aligned}
I^{\varphi}_G[\alpha(Tf-f)] &\leqslant \frac{1}{2D}\int_G\left[\int_G L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_G(t)\right]d\mu_G(s)\\
&+\frac{1}{2}\int_G\varphi\left(2\alpha\left|\int_G K(s,t,f(s))d\mu_G(t)-f(s)\right|\right)d\mu_G(s)=J_1+J_2.
\end{aligned}
$$

Using the same notations of Theorem 3, we can estimate the term J_1 in the following way

$$
\begin{aligned}
J_1 &\leqslant \frac{1}{2D}\int_G\left[\int_{U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_G(t)\right]d\mu_G(s)\\
&+\frac{1}{2D}\int_G\left[\int_{G\setminus U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_G(t)\right]d\mu_G(s)\\
&=J_1^1+J_1^2.
\end{aligned}
$$

Concerning J_1^1 we have

$$
\begin{aligned}
J_1^1 &= \frac{1}{2D}\int_B\left[\int_{U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_G(t)\right]d\mu_G(s)\\
&+\frac{1}{2D}\int_{G\setminus B}\left[\int_{U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_G(t)\right]d\mu_G(s)\\
&\leqslant \frac{\widetilde{\omega}_{\gamma\circ\eta}(\lambda f,B,U)}{D}\int_B\|L(s,\cdot)\chi_{U_s}(\cdot)\|_{\sigma,G}d\mu_G(s)\\
&+\frac{1}{2D}\int_{G\setminus B}\left[\int_{U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_G(t)\right]d\mu_G(s).
\end{aligned}
$$

Now we have

$$\begin{aligned}
&\frac{1}{2D}\int_{G\setminus B}\left[\int_{U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_G(t)\right]d\mu_G(s)\\
&\leqslant \frac{1}{2D}\int_{G\setminus B}\left[\int_{U_s} L(s,t)\eta(2\lambda|f(t)|)d\mu_G(t)\right]d\mu_G(s)\\
&\quad+\frac{1}{2D}\int_{G\setminus B}\left[\int_{U_s} L(s,t)\eta(\lambda|f(s)|)d\mu_G(t)\right]d\mu_G(s)=A_1+A_2.
\end{aligned}$$

Concerning A_1 we have

$$\begin{aligned}
A_1 &\leqslant \frac{1}{2D}\int_G \eta(2\lambda|f(t)|)\left[\int_{G\setminus B} L(s,t)d\mu_G(s)\right]d\mu_G(t)\\
&= \frac{1}{2D}\int_C \eta(2\lambda|f(t)|)\left[\int_{G\setminus B} L(s,t)d\mu_G(s)\right]d\mu_G(t)\\
&\quad+\frac{1}{2D}\int_{G\setminus C} \eta(2\lambda|f(t)|)\left[\int_{G\setminus B} L(s,t)d\mu_G(s)\right]d\mu_G(t)\\
&\leqslant \frac{\varepsilon}{2D}I^\eta_G[2\lambda f]+\frac{1}{2}I^\eta_G[2\lambda f\chi_{G\setminus C}].
\end{aligned}$$

For A_2, taking into account that $B\supset C$, we have

$$A_2\leqslant \frac{1}{2}I^\eta_G[2\lambda f\chi_{G\setminus B}]\leqslant \frac{1}{2}I^\eta_G[2\lambda f\chi_{G\setminus C}].$$

As to J_1^2, noting that $\chi_{G\setminus U_s}(t)=\chi_{G\setminus U_t}(s)$ for every $(s,t)\in G\times G$, we have

$$\begin{aligned}
&\frac{1}{2D}\int_G\left[\int_{G\setminus U_s} L(s,t)\eta(\lambda|f(t)-f(s)|)d\mu_G(t)\right]d\mu_G(s)\\
&\leqslant \frac{1}{2D}\int_G \eta(2\lambda|f(t)|)\left[\int_{G\setminus U_t} L(s,t)d\mu_G(s)\right]d\mu_G(t)\\
&\quad+\frac{1}{2D}\int_G \eta(2\lambda|f(s)|)\left[\int_{G\setminus U_s} L(s,t)d\mu_G(t)\right]d\mu_G(s).
\end{aligned}$$

The remainder J_2 is exactly as in Theorem 3 and so summing up the previous inequalities we obtain the assertion. ■

For what concerns the convergence for a family of Urysohn operators $(T_w)_{w>0}$ let us consider $H_w=G$ for every $w>0$ and $\mu_w=\mu_G$. The notion of singularity for the kernel $\mathbb{K}=(K_w)_{w>0}\in\mathcal{K}_\Xi$ is easily obtained substituting $H_w=G$, $\mu_w=\mu_G$ for every $w>0$ and analogously for the uniform singularity.

Thus, under the assumptions of Theorem 4 with $T = T_w$, $L = L_w$ we can obtain a modular convergence theorem for bounded but not necessarily continuous functions. At the same time we can obtain an analogous convergence theorem for continuous functions using the pointwise modulus of continuity, as in Section 3.3.

For not necessarily bounded functions we can obtain a convergence theorem directly by Theorem 6 modifying property 2) of singularity assuming

For every $s \in G$ and for every $U \in \mathcal{U}$ we have

$$\lim_{w\to+\infty}\int_{G\setminus U_s} L_w(s,t)d\mu_G(t) = 0, \quad \lim_{w\to+\infty}\int_{G\setminus U_t} L_w(s,t)d\mu_G(s) = 0,$$

uniformly with respect to $s, t \in G$ respectively.

Example. (*Nonlinear Mellin-type convolution operator*) Let $G = H = \mathbb{R}^+$ be provided with the measure $\mu_G = \dfrac{dt}{t}$ being dt the Lebesgue measure. Let $\overline{K} : \mathbb{R}^+ \to \mathbb{R}$ be a measurable function in $L^1(\mathbb{R}^+)$ such that

$$\int_0^{+\infty} \overline{K}(z)\frac{dz}{z} = 1.$$

Moreover let $\Gamma : \mathbb{R} \to \mathbb{R}$ be a function satisfying a Lipschitz condition of type

$$|\Gamma(u) - \Gamma(v)| \leqslant \psi(|u - v|),$$

for every $u, v \in \mathbb{R}$. Here $\psi \in \Psi$ is a fixed function. We define now

$$K(s,t,u) = \overline{K}(ts^{-1})\Gamma(u),$$

$(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $u \in \mathbb{R}$. Clearly, the kernel K satisfies the (L,ψ)-Lipschitz condition with $L(s,t) = |\overline{K}(ts^{-1})|$. The corresponding operator has the form

$$(Tf)(s) \equiv (Mf)(s) = \int_0^{+\infty} \overline{K}(ts^{-1})\Gamma(f(t))\frac{dt}{t}.$$

In [4] it is proved that $Dom\, M$ contains every Orlicz spaces. Note that the assumptions of Theorem 1 are satisfied taking $D = \|\overline{K}\|_1$. We prove now that the function $L(s,t)$ satisfies property $(*)$. In order to do that, since $\overline{K} \in L^1$ for a given $\varepsilon > 0$ there exist $\delta > 0$ and $P > 0$ such that

$$\int_A |\overline{K}(t)|\frac{dt}{t} < \frac{\varepsilon}{2}$$

for every measurable set A such that $\mu_G(A) < \delta$ and

$$\int_P^{+\infty} |\overline{K}(t)|\frac{dt}{t} < \frac{\varepsilon}{2}.$$

Let $C = [b, M]$ with $b > 0$ and let $B = [a, N]$ such that $\frac{M}{N} < \delta$ and $\frac{b}{a} > P$. So for $t \in [b, M]$ we have

$$\int_{G \setminus B} L(s,t) d\mu_G(s) = \int_N^{+\infty} |\overline{K}(ts^{-1})| \frac{ds}{s} + \int_0^a |\overline{K}(ts^{-1})| \frac{ds}{s}$$
$$= \int_0^{t/N} |\overline{K}(z)| \frac{dz}{z} + \int_{t/a}^{+\infty} |\overline{K}(z)| \frac{dz}{z}$$
$$\leqslant \int_0^{M/N} |\overline{K}(z)| \frac{dz}{z} + \int_{b/a}^{+\infty} |\overline{K}(z)| \frac{dz}{z} < \varepsilon.$$

As a consequence of Theorem 6, assuming that

$$R(s) \equiv R = \sup_{u \neq 0} |\Gamma(u) - u| < +\infty,$$

we have the following

Corollary 3. *Let us assume that* $\overline{K} \in L^{\sigma}_{loc}(\mathbb{R}^+)$. *Let* $f \in L^{\varphi+\eta}(\mathbb{R}^+) \cap L^{\gamma\circ\eta}(\mathbb{R}^+)$ *be fixed. Let* $\delta > 1$ *be fixed and let* $C = [b, M]$. *Then for every* ε, $\lambda > 0$ *there exist* $\alpha > 0$ *and a compact set* $B = [a, N]$ *such that*

$$I^{\varphi}_{\mathbb{R}^+}[\alpha(Mf - f)] \leqslant \frac{1}{\|\overline{K}\|_1} \widetilde{\omega}_{\gamma\circ\eta}(\lambda f, B, \delta)(N-a) \|\overline{K}\chi_{]1/\delta,\delta[}\|_{\sigma,\mathbb{R}^+}$$
$$+ \frac{1}{\|\overline{K}\|_1} I^{\eta}_{\mathbb{R}^+}[2\lambda f] \int_{\mathbb{R}^+ \setminus]1/\delta,\delta[} |\overline{K}(t)| \frac{dt}{t} + \frac{\varepsilon}{2\|\overline{K}\|_1} I^{\eta}_{\mathbb{R}^+}[2\lambda f]$$
$$+ \frac{5}{4} I^{\eta}_{\mathbb{R}^+}[2\lambda f \chi_{\mathbb{R}^+ \setminus [b,M]}] + \frac{1}{4} I^{\varphi}_{\mathbb{R}^+}[4\alpha f \chi_{\mathbb{R}^+ \setminus [a,N]}] + \frac{\varphi(2\alpha R)}{2}(N-a),$$

Proof. Using the notation of Theorem 6, putting $U_s =]s/\delta, s\delta[$, we have

$$\int_B \|L(s,\cdot)\chi_{U_s(\cdot)}\|_{\sigma,G} d\mu_G(s) = \|\overline{K}\chi_{]1/\delta,\delta[}\|_{\sigma,\mathbb{R}^+}(N-a).$$

Indeed note that, for $u > 0$,

$$\int_{s/\delta}^{s\delta} \sigma\left(\frac{|\overline{K}(ts^{-1})|}{u}\right) \frac{dt}{t} = \int_{1/\delta}^{\delta} \sigma\left(\frac{|\overline{K}(z)|}{u}\right) \frac{dz}{z}.$$

This implies that the Luxemburg norm of $L(s,\cdot)\chi_{U_s(\cdot)}$ is independent of s and it coincides with $\|\overline{K}\chi_{]1/\delta,\delta[}\|_{\sigma,\mathbb{R}^+}$. Moreover, taking $U_t =]t/\delta, t\delta[$,

$$\int_{G \setminus U_t} L(s,t) d\mu_G(s) = \int_0^{t/\delta} |\overline{K}(ts^{-1})| \frac{ds}{s} + \int_{t\delta}^{+\infty} |\overline{K}(ts^{-1})| \frac{ds}{s}$$
$$= \int_{\delta}^{+\infty} |\overline{K}(z)| \frac{dz}{z} + \int_0^{1/\delta} |\overline{K}(z)| \frac{dz}{z}$$

and analogously for

$$\int_{G\setminus U_s} L(s,t)d\mu_G(t).$$

The remainder follows directly by Theorem 6. ∎

Remark 4. If we assume that the function $\overline{K}$ is bounded then clearly the assumption $\overline{K} \in L^{\sigma}_{loc}(\mathbb{R}^+)$ is automatically satisfied. For example, for a fixed $n \in \mathbb{N}$, we can take the kernel

$$\overline{K}(t) = nt^n \chi_{]0,1[}(t) \quad t \in \mathbb{R}^+,$$

which generates the linear or nonlinear moment operators ([21], [2], [4], [9]).

As a consequence of the general theory developed above, it can be obtained a modular convergence theorem for a sequence of Mellin type operators

$$(M_n f)(s) = \int_0^{+\infty} \overline{K}_n(ts^{-1})\Gamma_n(f(t))\frac{dt}{t},$$

where the functions $\overline{K}_n$ and Γ_n satisfy the following assumptions

a) for every $n \in \mathbb{N}$,

$$\int_0^{+\infty} \overline{K}_n(z)\frac{dz}{z} = 1$$

and

$$\|\overline{K}_n\|_1 \leqslant D,$$

b) for every $\delta > 1$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^+\setminus[1/\delta,\delta]} |\overline{K}_n(u)|\frac{du}{u} = 0,$$

c) the family of functions $(\overline{K}_n)$ is uniformly integrable in $\mathbb{R}^+$.

d) for every $n \in \mathbb{N}$ we have

$$|\Gamma_n(u) - \Gamma_n(v)| \leqslant \psi_n(|u-v|)$$

where $\Xi = (\psi_n)_{n\in\mathbb{N}} \subset \Psi$, for every $u, v \in \mathbb{R}$,

e) for every $u \in \mathbb{R}$

$$\lim_{n\to+\infty} \Gamma_n(u) = u.$$

It is easy to show that the above assumptions imply all the conditions used to obtain the convergence theorem. For general Mellin convolution operators in linear case see [11].

5. Application to linear case

Now we study the particular case of a linear operator of the form

$$(Tf)(s) = \int_H \widetilde{K}(s,t) f(t) d\mu_H(t), \quad s \in G.$$

Here $\widetilde{K}(s,t)$ satisfies the following assumptions

$$\int_H \widetilde{K}(s,t) d\mu_H(t) = 1, \quad s \in G$$

and

$$\int_G |\widetilde{K}(s,t)| d\mu_G(s) \leqslant D, \qquad \int_H |\widetilde{K}(s,t)| d\mu_H(t) \leqslant D,$$

for every $t \in H$ and $s \in G$ respectively. In this case we can take $\psi(u) = u,\ u \geqslant 0$ and $L(s,t) = |\widetilde{K}(s,t)|$. Thus, putting $\eta = \varphi$, we obtain the following corollary with respect to pointwise modulus of continuity.

Corollary 4. *Let us suppose that the family $(|\widetilde{K}(\cdot,t)|)_{t\in H}$ satisfies property $(*)$. Let $f \in L^\infty(G) \cap L^\varphi(G) \cap L^\varphi(H)$ be fixed. Let $C \subset G$ be an arbitrary nonempty compact subset. Then for every $\varepsilon > 0$ and $U \in \mathfrak{U}$, there exist $\alpha > 0$ and a compact subset $B \subset G$, depending only on C and ε, such that*

$$\begin{aligned} I_G^\varphi[\alpha(Tf - f)] &\leqslant \omega_\varphi(\alpha D f, B, U)\mu_G(B) \\ &+ \frac{\varphi(2\alpha D\|f\|_\infty)}{D} \int_B \left[\int_{H\setminus U_s} |\widetilde{K}(s,t)| d\mu_H(t) \right] d\mu_G(s) \\ &+ \frac{\varepsilon}{2D} I_H^\varphi[2\alpha D f] + \frac{1}{2} I_H^\varphi[2\alpha D f \chi_{H\setminus C}] + \frac{1}{2} I_G^\varphi[2\alpha D f \chi_{G\setminus B}]. \end{aligned}$$

Proof. By the linearity of the operator, for every $s \in G$ we have

$$|(Tf)(s) - f(s)| \leqslant \int_H |\widetilde{K}(s,t)||f(t) - f(s)| d\mu_H(t)$$

thus

$$\begin{aligned} I_G^\varphi[\alpha(Tf - f)] &\leqslant \frac{1}{D} \int_G \left[\int_H \varphi(\alpha D|f(t) - f(s)|)|\widetilde{K}(s,t)| d\mu_H(t) \right] d\mu_G(s) \\ &= \frac{1}{D} \left(\int_B + \int_{G\setminus B} \right) \left[\int_H \varphi(\alpha D|f(t) - f(s)|)|\widetilde{K}(s,t)| d\mu_H(t) \right] d\mu_G(s) \\ &= J_1 + J_2. \end{aligned}$$

Following analogous reasonings as in the proof of Theorem 2, for a given $U \in \mathfrak{U}$, we have

$$J_1 \leqslant \omega_\varphi(\alpha D f, B, U)\mu_G(B) + \frac{\varphi(2\alpha D\|f\|_\infty)}{D} \int_B \left[\int_{H\setminus U_s} |\widetilde{K}(s,t)| d\mu_H(t) \right] d\mu_G(s).$$

Given $\varepsilon > 0$ and an arbitrary compact subset $C \subset G$, let $B \subset G$ be the compact subset in the assumption $(*)$. Then

$$\begin{aligned}
J_2 &\leqslant \frac{1}{2D}\int_{G\setminus B}\left[\int_H \varphi(2\alpha D|f(t)|)|\widetilde{K}(s,t)|d\mu_H(t)\right]d\mu_G(s)\\
&\quad+\frac{1}{2D}\int_{G\setminus B}\left[\int_H \varphi(2\alpha D|f(s)|)|\widetilde{K}(s,t)|d\mu_H(t)\right]d\mu_G(s)\\
&\leqslant \frac{\varepsilon}{2D}I^{\varphi}_H[2\alpha Df]+\frac{1}{2}I^{\varphi}_H[2\alpha Df\chi_{H\setminus C}]+\frac{1}{2}I^{\varphi}_G[2\alpha Df\chi_{G\setminus B}].
\end{aligned}$$

So the assertion follows. ∎

Using the same arguments of Corollary 1, we get the following

Corollary 5. *Under the assumptions of Corollary 1, let* $\alpha > 0$ *be such that* $I^{\varphi}_H[2\alpha Df] + I^{\varphi}_G[2\alpha Df] < +\infty$, *then for every* $\varepsilon > 0$ *and* $U \in \mathfrak{U}$, *there exists a compact subset* $B \subset G$ *such that*

$$\begin{aligned}
&I^{\varphi}_G[\alpha(Tf-f)] \leqslant \omega_{\varphi}(\alpha Df, B, U)\mu_G(B)\\
&\quad+\frac{\varphi(2\alpha D\|f\|_\infty)}{D}\int_B\left[\int_{H\setminus U_s}|\widetilde{K}(s,t)|d\mu_H(t)\right]d\mu_G(s)+\varepsilon.
\end{aligned}$$

Note that, analogously to Remark 1.2, we can get an estimate in case of functions in $C(G)$.

For what concerns the use of the norm modulus of continuity, under the assumptions of Theorem 3, we get the following modular estimate in linear case

Corollary 6. *Let the assumptions of Theorem 2 be satisfied and furtherly* $|\widetilde{K}(s,\cdot)| \in L^{\sigma}_{loc}(H)$. *Let* $U \in \mathfrak{U}$ *and assume that* $\|\widetilde{K}(:,\cdot)\chi_{U_{(:)}}\|_{\sigma,H} \in L^1_{loc}(G)$. *Let* $C \subset G$ *be an arbitrary nonempty compact subset. Then for every* $\varepsilon > 0$ *there exist* $\alpha > 0$ *and a compact subset* $B \subset G$, *depending only on* C *and* ε, *such that*

$$\begin{aligned}
I^{\varphi}_G[\alpha(Tf-f)] &\leqslant \frac{2\widetilde{\omega}_{\gamma\circ\varphi}(\alpha Df,B,U)}{D}\int_B\|\widetilde{K}(s,\cdot)\chi_{U_s}(\cdot)\|_{\sigma,H}d\mu_G(s)\\
&\quad+\frac{\varphi(2\alpha D\|f\|_\infty)}{D}\int_B\left[\int_{H\setminus U_s}|\widetilde{K}(s,t)|d\mu_H(t)\right]d\mu_G(s)\\
&\quad+\frac{\varepsilon}{2D}I^{\varphi}_H[2\alpha Df]+\frac{1}{2}I^{\varphi}_H[2\alpha Df\chi_{H\setminus C}]+\frac{1}{2}I^{\varphi}_G[2\alpha Df\chi_{G\setminus B}].
\end{aligned}$$

As before, by inequalities given in Corollaries 4 and 6, it is possible to get modular convergence results both in continuous and discontinuous case, using the moduli of continuity ω_{φ} and $\widetilde{\omega}_{\gamma\circ\varphi}$.

6. Applications to discrete operators

In this section we give some applications to sampling type operator and Szász-Mirak'jan operator.

6.1. Generalized sampling series. Let us take $G = \mathbb{R}$ provided with the Lebesgue measure and $H = \boldsymbol{Z}$ provided with the counting measure.

Let us consider the generalized sampling series of a function $f \in L^\infty(\mathbb{R})$ defined by

$$(Sf)(s) = \sum_{k=-\infty}^{+\infty} K(s-k)f(k), \quad s \in \mathbb{R},$$

(see [19] [13], [14]), where $K : \mathbb{R} \to \mathbb{R}$ is a function satisfying the following properties:

for every $s \in \mathbb{R}$

$$\sum_{k=-\infty}^{+\infty} K(s-k) = 1, \qquad \sum_{k=-\infty}^{+\infty} |K(s-k)| \leqslant D$$

and

$$\int_{-\infty}^{+\infty} |K(s)|ds \leqslant D,$$

for an absolute constant $D > 0$.

Let us remark that in this case the kernel

$$\widetilde{K}(s,k) = K(s-k)$$

satisfies all the assumptions of Corollary 4 (see [7]). We obtain the following

Corollary 7. *Let $f \in L^\infty(\mathbb{R}) \cap L^\varphi(\mathbb{R}) \cap \ell^\varphi(\boldsymbol{Z})$ and $M > 0$ be fixed. Then for every $\varepsilon, \delta > 0$ there exist $\alpha > 0$ and $N > 0$ such that*

$$I^\varphi_{\mathbb{R}}[\alpha(Sf - f)] \leqslant 2N\omega_\varphi(\alpha Df, N, \delta) + \frac{\varphi(2\alpha D\|f\|_\infty)}{D} \int_{-N}^{N} \sum_{|s-k|>\delta} |K(s-k)|ds$$
$$+ \frac{\varepsilon}{2D} \sum_{k=-\infty}^{+\infty} \varphi(2\alpha D|f(k)|) + \frac{1}{2} \sum_{|k|>M} \varphi(2\alpha D|f(k)|) + \frac{1}{2} \int_{|s|>N} \varphi(2\alpha D|f(s)|)ds.$$

Proof. The proof follows by Corollary 4 putting $C = [-M, M]$, $U = \{(s,k) \in \mathbb{R} \times \boldsymbol{Z} : |s-k| < \delta\}$, $B = [-N, N]$. ■

More generally, let $w > 0$ be fixed, $G = \mathbb{R}$ provided with the Lebesgue measure and $H_w = \frac{1}{w}\boldsymbol{Z}$ provided with the measure $\mu_w = \frac{1}{w}\mu_c$ being μ_c the counting measure. We take into consideration the sampling series

$$(S_w f)(s) = \sum_{k=-\infty}^{+\infty} K(ws-k)f(\frac{k}{w}), \quad s \in \mathbb{R},$$

where $f \in L^\infty(\mathbb{R})$ and K satisfies the above assumptions.

Let us remark that in this case the kernel

$$\widetilde{K}(s, \frac{k}{w}) = wK(w(s - \frac{k}{w}))$$

satisfies all the previous assumptions. We obtain the following

Corollary 8. *Let* $f \in L^\infty(\mathbb{R}) \cap L^\varphi(\mathbb{R}) \cap \ell^\varphi(\frac{\boldsymbol{Z}}{w})$ *and* $M > 0$ *be fixed. Then for every* $\varepsilon, \delta > 0$ *there exist* $\alpha > 0$ *and* $N > 0$ *such that*

$$I^\varphi_{\mathbb{R}}[\alpha(S_w f - f)] \leqslant 2N\omega_\varphi(\alpha Df, N, \delta) + \frac{\varphi(2\alpha D\|f\|_\infty)}{D}\int_{-N}^{N}\sum_{|ws-k|>w\delta}|K(ws-k)|ds$$

$$+ \frac{\varepsilon}{2Dw}\sum_{k=-\infty}^{+\infty}\varphi(2\alpha D|f(\frac{k}{w})|) + \frac{1}{2w}\sum_{|k|>Mw}\varphi(2\alpha D|f(\frac{k}{w})|)$$

$$+ \frac{1}{2}\int_{|s|>N}\varphi(2\alpha D|f(s)|)ds.$$

Remark 5. Note that under the above assumptions on the function K the family of kernels

$$\widetilde{K}_w(s, \frac{k}{w}) = wK(w(s - \frac{k}{w}))$$

satisfies the singularity assumptions and property $(**)$ used in the general theory (see [7]). Then we can obtain convergence results for the family of sampling operators $(S_w)_{w>0}$.

6.2. Szász-Mirak'jan operator. Let us take $G = \mathbb{R}^+_0$ endowed with the Lebesgue measure and, for every $n \in \mathbb{N}$, $H_n = \frac{1}{n}\mathbb{N} = \{\frac{k}{n},\ k = 0, 1, \ldots\}$ endowed with the measure $\mu_n = \frac{1}{n}\mu_c$. We take into consideration the Szász-Mirak'jan operator given by (see [1], [7])

$$(S_n f)(s) = \sum_{k=0}^{+\infty} f(\frac{k}{n})e^{-ns}\frac{(ns)^k}{k!}, \quad s \in \mathbb{R}^+_0.$$

In this case, for every $n \in \mathbb{N}$, the kernel is given by

$$\widetilde{K}(s,t) = \widetilde{K}(s, \frac{k}{n}) = ne^{-ns}\frac{(ns)^k}{k!}.$$

We will prove that the kernek $\widetilde{K}$ satisfies all the above assumptions. To begin with, note that

$$\int_{H_n}\widetilde{K}(s, \frac{k}{n})d\mu_n(k) = \sum_{k=0}^{+\infty}e^{-ns}\frac{(ns)^k}{k!} = 1.$$

Moreover

$$\int_G|\widetilde{K}(s, \frac{k}{n})|d\mu_G(s) = \frac{n}{k!}\int_0^{+\infty}e^{-ns}\frac{(ns)^k}{k!}ds = 1.$$

So we can obtain the following

Corollary 9. *Let* $f \in L^\infty(\mathbb{R}_0^+) \cap L^\varphi(\mathbb{R}_0^+) \cap \ell^\varphi(\frac{\boldsymbol{Z}}{w})$ *and* $M > 0$ *be fixed. Then for every* $\varepsilon, \delta > 0$ *there exist* $\alpha > 0$ *and* $N > 0$ *such that*

$$I^\varphi_{\mathbb{R}_0^+}[\alpha(S_n f - f)] \leqslant N\omega_\varphi(\alpha f, N, \delta) + \varphi(2\alpha\|f\|_\infty)\int_0^N \sum_{|s-\frac{k}{n}|>\delta} ne^{-ns}\frac{(ns)^k}{k!}ds$$
$$+\frac{\varepsilon}{2n}\sum_{k=0}^{+\infty}\varphi(2\alpha|f(\frac{k}{n})|) + \frac{1}{2n}\sum_{|k|>Mn}\varphi(2\alpha|f(\frac{k}{n})|) + \frac{1}{2}\int_N^{+\infty}\varphi(2\alpha|f(s)|)ds.$$

Proof. The proof follows by Corollary 4 putting $C = [0, M]$, $U = \{(s, \frac{k}{n}) \in \mathbb{R}_0^+ \times \frac{\mathbb{N}}{n} : |s - \frac{k}{n}| < \delta\}$, $B = [0, N]$. ■

As shown in [7], the family of functions $\widetilde{K}_n(s, \frac{k}{n}) = \widetilde{K}(s, \frac{k}{n})$ satisfies property $(**)$ and the singularity assumptions so it is possible to obtain convergence theorems.

Let us remark that in the frame of discrete operators it is meaningful to obtain modular convergence theorem for continuous functions, as a consequence of Corollaries 7, 8 and 9. Indeed the use of the norm modulus of continuity, introduced in Section 2, is not suitable here because it is strictly connected with continuity.

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References

[1] F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications.* de Gruyter Studies in Mathematics, 17. Walter de Gruyter, Berlin, (1994).

[2] F. Barbieri, Approssimazione mediante nuclei momento. *Atti Sem. Mat. Fis. Univ. Modena.* **32** (1983), 308–328.

[3] C. Bardaro and I. Mantellini, A modular convergence theorem for nonlinear integral operators, *Commentationes Math.* **36**, (1996), 27–37.

[4] C. Bardaro and I. Mantellini, Linear integral operators with homogeneous kernel: approximation properties in modular spaces. Applications to Mellin-type convolution operators and to some classes of fractional operators. *Applied Math. Rev. vol I* , World Scientific Publ., Ed. G. Anastassiou, (2000), 45–67.

[5] C. Bardaro and I. Mantellini, On approximation properties of Urysohn integral operators. *Intern. Journal of Pure and Appl. Math..* **3** (2002), 129–148.

[6] C. Bardaro and I. Mantellini, On some estimates for general sampling operators and approximation properties. *Intern. Journal of Math. Sci..* **2** (2003), 289–326.

[7] C. Bardaro and I. Mantellini, On global approximation properties of abstract integral operators in Orlicz spaces and applications. *J. Inequalities in Pure and Applied Math.* (electronic), **6**(4), article 123, (2005).

[8] C. Bardaro and I. Mantellini, Estimates and superposition of general nonlinear sampling operators and applications. *Comm. Appl. Nonlinear Anal.* **13**(2) (2006), 1–36.

[9] C. Bardaro and J. Musielak and G. Vinti, *Nonlinear integral operators and applications* De Gruyter Series in Nonlinear Analysis and Appl., Vol.9, 2003.

[10] P.L. Butzer, A. Fisher and R.L. Stens, Generalized sampling approximation of multivariate signals: theory and applications. *Note di Matematica* **10** Suppl. n.1 (1990), 173–191.

[11] P.L. Butzer and S. Jansche, Mellin transform, the Mellin-Poisson summation formula and the exponential sampling theorem. *Atti Sem. Mat. Fis. Univ. Modena* Suppl. Vol. **46**, a special volume dedicated to Professor Calogero Vinti, (1998), 99–122.

[12] P.L. Butzer and R.J. Nessel, *Fourier Analysis and Approximation, I*. Academic Press, New York-London, 1971.

[13] P.L. Butzer and R.L. Stens, Sampling theory for not necessarily band-limited functions: a historical overview. *SIAM Review* **34** (1992), 40–53.

[14] P.L. Butzer and R.L. Stens, Linear prediction by samples from the past. In: *Advanced Topics in Shannon Sampling and Interpolation Theory,* Marks II, R.J., (eds.), pp. 157–183, Springer-Verlag, New York, 1993.

[15] S. Haber and O. Shisha, Improper integrals, simple integrals and numerical quadrature. *J. Approx. Theory* **11** (1974), 1–15.

[16] I. Mantellini and G. Vinti, Approximation results for nonlinear integral operators in modular spaces and applications. *Ann. Polon. Math.*. **81**(1), (2003), 55–71.

[17] J. Musielak, Orlicz spaces and modular spaces. *Lecture Notes in Math.* **1034**, Springer (1983).

[18] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Monographs Textbooks Pure Appl. Math. 146, Marcel Dekker, Inc., New York 1991.

[19] S. Ries and R.L. Stens, Approximation by generalized sampling series, in *Constructive Theory of Functions*, Sendov, Bl., Petrushev, P., Maleev, R., Tashev, S. (Eds.), Publishing House of the Bulgarian Academy of Sciences, Sofia, (1984), 746–756.

[20] S. Willard, *General Topology*, Addison Wesley Publ. Comp., 1968.

[21] V. Zanelli, Funzioni momento convergenti dal basso in variazione di ordine non intero. *Atti Sem. Mat. Fis. Univ. Modena*. **30**, (1981), 355–369.

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