A New Method to Obtain Series for $1/\pi$ and $1/\pi^2$

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CONTENTS

We give several conjectures that allow us to derive many series for $1/\pi$ and $1/\pi^2$. These series include Ramanujan's series, as well as those associated with the Domb numbers and Apéry numbers. We have checked the conjectures numerically in many examples with a precision of two hundred digits.

1. RAMANUJAN-TYPE SERIES

Let $B(n)$ be any of the followings expressions:

$$B_1(n) = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n^3},$$
$$B_2(n) = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3},$$
$$B_3(n) = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3},$$
$$B_4(n) = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3},$$

where $(a)_n$ is the rising factorial defined by

$$(a)_n = a(a+1)\cdots(a+n-1),$$

or, more generally, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

It is known that for $u = 1$ or $u = -1$, there exist positive algebraic numbers q, a, and b such that

$$\sum_{n=0}^{\infty} u^n B(n) q^n (a+bn) = \frac{1}{\pi}. \qquad (1\text{–}1)$$

These series were discovered by Ramanujan. He found 17 of them, which were published in 1914 in [Ramanujan 14]. Since 1985, many others have been discovered and proved by J. M. Borwein and P. B. Borwein [Borwein and Borwein]; D. V. Chudnovsky and G. V. Chudnovsky [Chudnovsky and Chudnovsky 87, Berndt and Chan 01]; and H. H. Chan, W. C. Liaw, and V. Tan [Chan et al.

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01, Chan et al. 04]. Since Ramanujan was the discoverer of this kind of series, they are called Ramanujan-type series. Some examples are

$$\sum_{n=0}^{\infty} \frac{B_2(n)}{99^{4n}} \left(\frac{2206\sqrt{2}}{9801} + \frac{52780\sqrt{2}}{9801} n \right) = \frac{1}{\pi}, \quad (1\text{–}2)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{B_4(n)}{16^n} \left(\frac{7\sqrt{3}}{36} + \frac{17\sqrt{3}}{12} n \right) = \frac{1}{\pi}, \quad (1\text{–}3)$$

$$\sum_{n=0}^{\infty} (-1)^n B_3(n) \left(\frac{3}{8}\right)^{3n} \left(\frac{15\sqrt{2}}{64} + \frac{77\sqrt{2}}{32} n \right) = \frac{1}{\pi}. \quad (1\text{–}4)$$

The proofs of (1–2) and (1–3) can be found in [Borwein and Borwein] and [Chan et al. 01], respectively, and are based on the theory of modular functions. A WZ-method proof of (1–4) can be found in [Guillera 06a].

Conjecture 1.1. *When (1–1) holds for the positive algebraic numbers a, b, and q, we conjecture the existence of a positive rational number $k = k(a, b, q)$ such that as $x \to 0$,*

$$\sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} [a + b(n+x)] = \frac{1}{\pi} - \frac{k\pi}{2} x^2 + O(x^3). \quad (1\text{–}5)$$

This conjecture is inspired by some series in [Guillera 06b]. We have checked this conjecture numerically in many examples and with a precision of two hundred digits.

1.1 The Coefficient of the Next Term

In some cases we have also been able to identify the coefficient of the next term with the help of the following function:

$$\sigma_2(x) = \Im(\mathrm{Li}_2(e^{ix})) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}.$$

When q, a, and b are rational we find that the coefficient of the next term is a rational multiple of Catalan's constant $G = \sigma_2(\pi/2)$. Two examples are

$$\sum_{n=0}^{\infty} \frac{B_1(n+x)}{64^{n+x}} \left[\frac{5}{16} + \frac{42}{16}(n+x) \right] = \frac{1}{\pi} - 3\pi x^2 + 64 G x^3 + O(x^4)$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{B_2(n+x)}{18^{2n+2x}} \left[\frac{23}{72} + \frac{65}{18}(n+x) \right] = \frac{1}{\pi} - \frac{11\pi}{2} x^2 + 160 G x^3 + O(x^4).$$

When q, $a\sqrt{3}$, and $b\sqrt{3}$ are rational we find that the coefficient of the next term is a rational multiple of the constant $A = \sigma_2(\pi/3)$. Two examples are

$$\sum_{n=0}^{\infty} \frac{B_2(n+x)}{7^{4n+4x}} \left[\frac{59\sqrt{3}}{49} + \frac{120\sqrt{3}}{49}(n+x) \right] = \frac{1}{\pi} - 8\pi x^2 + 120 A x^3 + O(x^4)$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{B_4(n+x)}{16^{n+x}} \left[\frac{7\sqrt{3}}{36} + \frac{17\sqrt{3}}{12}(n+x) \right] = \frac{1}{\pi} - \frac{13\pi}{6} x^2 + 20 A x^3 + O(x^4).$$

When q, $a\sqrt{2}$, and $b\sqrt{2}$ are rational we find that the coefficient of the next term is a rational multiple of the constant $B = \sigma_2(\pi/4) - G/4$. Two examples are

$$\sum_{n=0}^{\infty} (-1)^n \frac{B_1(n+x)}{8^{n+x}} \left[\frac{\sqrt{2}}{4} + \frac{3\sqrt{2}}{2}(n+x) \right] = \frac{1}{\pi} - \frac{3\pi}{2} x^2 + 32 B x^3 + O(x^4)$$

and

$$\sum_{n=0}^{\infty} \frac{B_2(n+x)}{99^{4n+4x}} \left[\frac{2206\sqrt{2}}{9801} + \frac{52780\sqrt{2}}{9801}(n+x) \right] = \frac{1}{\pi} - 28\pi x^2 + 1920 B x^3 + O(x^4).$$

1.2 An Application of Conjecture 1.1

If we define

$$R(x) = \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} [a + b(n+x)],$$

then (1–5) implies that

$$k = \frac{-R''(0)}{\pi},$$

and we can associate this positive rational number k with every Ramanujan-type series. For example, for the series (1–2), (1–3), and (1–4) we get respectively $k = 56$, $k = 13/3$, and $k = 7$. Moreover, we are going to show that u, $B(n)$, and k determine all the parameters of a Ramanujan-type series. To see this, we write

$$\sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} [a + b(n+x)] = a \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} + b \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} (n+x),$$

and define the functions

$$S(x) = \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x}, \qquad (1\text{–}6)$$

$$T(x) = \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x}(n+x), \qquad (1\text{–}7)$$

so that

$$R(x) = aS(x) + bT(x).$$

From (1–5) we obtain the following system:

$$aS(0) + bT(0) = \frac{1}{\pi}, \qquad (1\text{–}8)$$
$$aS'(0) + bT'(0) = 0.$$

Using (1–5) and the values for a and b obtained from system (1–8) we obtain an equation

$$f(q) = k, \qquad (1\text{–}9)$$

relating q and k for every binomial part $B(n)$ and $u = 1$ or $u = -1$, where

$$f(q) = \frac{-aS''(0)}{\pi} + \frac{-bT''(0)}{\pi}.$$

By numerical experimentation we have discovered an algorithm to solve (1–9): First we get a good first approximation q_1 to q if instead of $S(x)$ and $T(x)$ we solve (1–9) using the functions $B(x)q^x$ and $B(x)q^x x$, obtained by taking only the terms $n = 0$ in (1–6) and (1–7). This is a second-degree equation in $\ln(q)$, and one of its solutions for q_1 is invalid because it is greater than unity; the other one is (1–21). Next, we use the formula

$$\frac{\ln q_n}{k_n} = \frac{\ln q_{n+1}}{k},$$

obtained by considering the linear relation between k and $\ln q$. So, to get better approximations of q we can use the recurrence

$$k_n = f(q_n), \qquad q_{n+1} = q_n^{k/k_n}. \qquad (1\text{–}10)$$

Our interest is to guess q when k is a rational number. So, after finding the numerical approximation of q, we try to obtain the algebraic expression of it. The functions *identify* (see [Borwein et al.]) and *minpoly*, implemented in Maple 9, are adequate for this purpose. When we get q, the system (1–8) allows us to obtain numerical approximations of a and b, and again we must guess which algebraic numbers they are. We give an example of the procedure: Take $u = 1$ and $B(n) = B_2(n)$. Then substituting $k = 4$ in (1–9) and using the recurrence (1–10) with the initial value (obtained as explained before)

$$q_1 = 256\ \mathrm{e}^{-\pi\sqrt{6}},$$

we get the following sequence of approximations:

n	q_n	k_n
1	0.1164700015	3.923087536
2	0.1116624439	3.991903391
3	0.1111670393	3.999176679
4	0.1111167760	3.999916585
5	0.1111116848	3.999991552
6	0.1111111692	3.999999144
7	0.1111111169	3.999999913
8	0.1111111117	3.999999991
9	0.1111111111	3.999999999
10	0.1111111111	3.999999999
11	0.1111111111	4.000000000

We easily guess that

$$q = 0.1111111111\ldots = \frac{1}{9}.$$

Substituting this value in (1–8), we obtain

$$a = 4.618802153517 \quad \text{and} \quad b = 0.577350269189,$$

and the function *identify* of Maple 9 (see [Borwein et al.]), recognizes these constants as

$$a = \frac{8}{3}\sqrt{3} \quad \text{and} \quad b = \frac{1}{3}\sqrt{3}. \qquad (1\text{–}11)$$

In the following examples we use the function *identify* to recognize the constants: Take $u = 1$ and $B(n) = B_1(n)$. Then for $k = 4$, we get

$$q = 9 - 4\sqrt{5}, \quad a = \frac{1}{2}\sqrt{10\sqrt{5} - 22}, \quad b = \sqrt{20\sqrt{5} - 40}. \qquad (1\text{–}12)$$

Take $u = -1$ and $B(n) = B_1(n)$. Then for $k = 5$, we get

$$q = 17 - 12\sqrt{2}, \quad a = 2\sqrt{2} - \frac{5}{2}, \quad b = 6\sqrt{2} - 6. \qquad (1\text{–}13)$$

For the alternating series associated with $k = 15$ and corresponding to the binomial parts $B_3(n)$ and $B_4(n)$ we get, respectively, the following parameters

$$q = \frac{1}{512}, \quad a = \frac{25}{192}\sqrt{6}, \quad b = \frac{57}{32}\sqrt{6}, \qquad (1\text{–}14)$$

and

$$q = \frac{1}{3024}, \quad a = \frac{13}{108}\sqrt{7}, \quad b = \frac{55}{36}\sqrt{7}. \qquad (1\text{–}15)$$

1.3 The Number N

From (1–8) we obtain

$$a = \frac{1}{\pi} \frac{T'(0)}{S(0)T'(0) - S'(0)T(0)}, \tag{1–16}$$

$$b = \frac{1}{\pi} \frac{-S'(0)}{S(0)T'(0) - S'(0)T(0)}, \tag{1–17}$$

and substituting these values in (1–9) we obtain

$$\frac{S''(0) + k\pi^2 S(0)}{S'(0)} = \frac{T''(0) + k\pi^2 T(0)}{T'(0)}. \tag{1–18}$$

We now define the number N as

$$N := \left[\frac{S''(0) + k\pi^2 S(0)}{-2\pi S'(0)} \right]^2. \tag{1–19}$$

Numerical experimentation with many examples reveals that N and k are related in a simple way that depends only on the binomial part $B(n)$. For $B_1(n)$, $B_2(n)$, $B_3(n)$, and $B_4(n)$ we have respectively $N = k + 1$, $N = k + 2$, $N = k + 4$, and $N = k + \frac{4}{3}$. For example, for the series (1–2), (1–3), (1–11), (1–12), (1–13), (1–14), (1–15), we get in the same order $N = 58$, $N = 17/3$, $N = 6$, $N = 5$, $N = 6$, $N = 19$, and $N = 49/3$. Direct observation of these values allows us to guess that the number N was used in the modular theory of Ramanujan-type series as a parameter to obtain $a = a(N)$, $b = b(N)$, and $q = q(N)$ (see [Borwein and Borwein], [Chan et al. 04]). From (1–18) and (1–19), we get the following system:

$$S''(0) + k\pi^2 S(0) = -2\pi\sqrt{N} S'(0),$$
$$T''(0) + k\pi^2 T(0) = -2\pi\sqrt{N} T'(0).$$

Differentiating the first equation of the system with respect to q, and simplifying using the second, we get

$$\pi \frac{dk}{dq} S(0) = -\frac{1}{\sqrt{N}} \frac{dN}{dq} S'(0),$$

but for the four cases relating N and k, we have

$$\frac{dN}{dq} = \frac{dk}{dq}.$$

So we obtain

$$S'(0) = -\pi\sqrt{N}\, S(0), \tag{1–20}$$

and instead of (1–9) we can use this simpler equation (it does not involve second derivatives) to get the value of q for a choice of u, $B(n)$, and N. To solve it, we define the partial sum $S_j(x)$ of $S(x)$,

$$S_j(x) = \sum_{n=0}^{j} u^n B(n + x) q^{n+x},$$

and solving

$$S_0'(0) = -\pi\sqrt{N}\, S_0(0),$$

we get a first approximation of q,

$$q_1 = M e^{-\pi\sqrt{N}}, \tag{1–21}$$

where $M = 64$ if $B(n) = B_1(n)$, $M = 256$ if $B(n) = B_2(n)$, $M = 1728$ if $B(n) = B_3(n)$, and $M = 108$ if $B(n) = B_4(n)$. Then we get better and better approximations of q by means of the recurrence

$$N_n = f(q_n), \qquad q_{n+1} = M \left(\frac{q_n}{M} \right)^{\sqrt{N/N_n}},$$

or using the recurrence (1–10), where $f(q)$ is the function (see (1–20))

$$f(q) = \left[\frac{S'(0)}{-\pi S(0)} \right]^2.$$

We can now state the following conjecture.

Conjecture 1.2. *For some rational numbers N, the solution q of (1–20) and the values of a and b obtained by substituting that value of q in (1–16) and (1–17) are positive algebraic numbers that can be used to get an identity of the form (1–1).*

2. SERIES FOR $1/\pi$ INVOLVING SOME SPECIAL SEQUENCES OF NUMBERS

In this section we consider series of the form (1–1) associated with some special sequences of numbers, which are motivated by [Almkvist and Zudilin 03], [Chan 05], and [Yang 04], for example to the Domb numbers

$$B(n) = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2n-2j}{n-j}, \tag{2–1}$$

to the Apéry numbers

$$B(n) = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}^2, \tag{2–2}$$

to the numbers

$$B(n) = \sum_{j=0}^{n} \binom{2j}{j}^2 \binom{2n-2j}{n-j}^2, \tag{2–3}$$

or to the numbers

$$B(n)=\sum_{j=0}^{n}\binom{n}{j}^4. \tag{2–4}$$

If we call $B(n+x)$ the functions obtained by replacing n by $n+x$ except in the summation symbols, then we believe that Conjecture 1.2 is also true for these series. We give the series found by applying it for $N=3$ with $u=1$ and the numbers (2–3), for $N=38/5$ with $u=1$ and the numbers (2–4), and for $N=17/5$ with $u=-1$ and the numbers (2–4):

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{2j}{j}^2\binom{2n-2j}{n-j}^2\left(\frac{2-\sqrt{3}}{64}\right)^n \times\left(\frac{1}{4}+\frac{3+2\sqrt{3}}{4}n\right)=\frac{1}{\pi},$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}^4\frac{1}{76^{2n}}\left(\frac{47\sqrt{95}}{1444}+\frac{102\sqrt{95}}{361}n\right)=\frac{1}{\pi}$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}^4\frac{(-1)^n}{18^{2n}}\left(\frac{4\sqrt{5}}{27}+\frac{68\sqrt{5}}{81}n\right)=\frac{1}{\pi}.$$

Series for $1/\pi$ using the Apéry numbers (2–2) were presented in a talk by T. Sato [Sato 02]. Motivated by these, similar series associated with the Domb numbers (2–1) have been studied and proved in [Chan et al. 04]. Y. Yang has proved similar evaluations using the numbers (2–4) and following essentially the same method [Yang 05].

3. ABOUT SIMILAR SERIES FOR $1/\pi^2$

We now consider series of the form

$$\sum_{n=0}^{\infty}u^nB(n)q^n(a+bn+cn^2)=\frac{1}{\pi^2}, \tag{3–1}$$

where $u=1$ or $u=-1$; a, b, c, q are positive algebraic numbers; and $(1)_n^5B(n)$ is the product of five rising factorials of fractions smaller than unity satisfying the following condition: For every denominator in the fraction of a rising factorial, we have rising factorials with all possible irreducible fractions corresponding to that denominator. Some examples for the binomial part $B(n)$ are

$$B_1(n)=\frac{\left(\frac12\right)_n\left(\frac12\right)_n\left(\frac12\right)_n\left(\frac12\right)_n\left(\frac12\right)_n}{(1)_n^5},$$

$$B_2(n)=\frac{\left(\frac12\right)_n\left(\frac12\right)_n\left(\frac12\right)_n\left(\frac14\right)_n\left(\frac34\right)_n}{(1)_n^5},$$

$$B_3(n)=\frac{\left(\frac12\right)_n\left(\frac14\right)_n\left(\frac34\right)_n\left(\frac16\right)_n\left(\frac56\right)_n}{(1)_n^5},$$

$$B_4(n)=\frac{\left(\frac12\right)_n\left(\frac13\right)_n\left(\frac23\right)_n\left(\frac14\right)_n\left(\frac34\right)_n}{(1)_n^5},$$

$$B_5(n)=\frac{\left(\frac12\right)_n\left(\frac13\right)_n\left(\frac23\right)_n\left(\frac16\right)_n\left(\frac56\right)_n}{(1)_n^5},$$

$$B_6(n)=\frac{\left(\frac12\right)_n\left(\frac18\right)_n\left(\frac38\right)_n\left(\frac58\right)_n\left(\frac78\right)_n}{(1)_n^5}.$$

I proved three identities of the form (3–1) in [Guillera 02] and [Guillera 06a], and inspired by its form I found, without proving them, four more identities of the same kind in [Guillera 03].

Conjecture 3.1. *When (3–1) holds for the positive algebraic numbers a, b, c, and q, we conjecture the existence of a positive rational number $k=k(a,b,c,q)$ such that as $x\to 0$,*

$$\sum_{n=0}^{\infty}u^nB(n+x)q^{n+x}[a+b(n+x)+c(n+x)^2] =\frac{1}{\pi^2}-\frac{k}{2}x^2+O(x^4). \tag{3–2}$$

This conjecture is inspired by some series in [Guillera 06b]. For all the series we have found, we have also been able to recognize the coefficient of the next term as a rational multiple of π^2. One example is

$$\sum_{n=0}^{\infty}(-1)^n\frac{B_4(n+x)}{48^{n+x}}\left(\frac{5}{48}+\frac{21}{16}(n+x)+\frac{21}{4}(n+x)^2\right) =\frac{1}{\pi^2}-\frac{3}{2}x^2+\frac{157\pi^2}{24}x^4+O(x^5). \tag{3–3}$$

3.1 An Application of Conjecture 3.1

If we define

$$R(x)=\sum_{n=0}^{\infty}u^nB(n+x)q^{n+x}[a+b(n+x)+c(n+x)^2],$$

then (3–2) implies that

$$k=-R''(0),$$

and we can associate this positive rational number k with every series of the form (3–1). For example, for the series (2-4) in [Guillera 03],

$$\sum_{n=0}^{\infty}(-1)^n\frac{B_5(n)}{80^{3n}}\left(\frac{29\sqrt{5}}{640}+\frac{693\sqrt{5}}{640}n+\frac{5418\sqrt{5}}{640}n^2\right)=\frac{1}{\pi^2}, \tag{3–4}$$

we get $k = 15$. Moreover, we are going to show that u, $B(n)$, and k determine all the parameters of this kind of series. To see this, we write

$$\sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} [a + b(n+x) + c(n+x)^2]$$
$$= a \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} + b \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} (n+x) + c \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} (n+x)^2,$$

and define the functions

$$S(x) = \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x},$$
$$T(x) = \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} (n+x),$$
$$U(x) = \sum_{n=0}^{\infty} u^n B(n+x) q^{n+x} (n+x)^2,$$

so that

$$R(x) = aS(x) + bT(x) + cU(x).$$

From (3–2) we obtain the following system:

$$\begin{aligned} aS(0) + bT(0) + cU(0) &= 1/\pi^2, \\ aS'(0) + bT'(0) + cU'(0) &= 0, \\ aS'''(0) + bT'''(0) + cU'''(0) &= 0. \end{aligned} \tag{3–5}$$

Using (3–2) and the values for a, b, and c obtained from the system (3–5) we obtain an equation

$$f(q) = k, \tag{3–6}$$

where

$$f(q) = -aS''(0) - bT''(0) - cU''(0),$$

that relates q with k for every binomial part $B(n)$ and $u = 1$ or $u = -1$. Equation (3–6) can be solved in the same way as (1–9), but this time the first approximation for $\ln q_1$ is a solution of a third-degree equation. The recurrence (1–10) can be used again to obtain q numerically when we select a value for k. Our interest is to guess q when k is a rational number. So, after finding the numerical approximation of q, we try to guess its algebraic expression. When we get q, the system (3–5) allows us to obtain the values of a and b, and again we must try to guess these numbers. We give an example of the procedure: Take $u = -1$, $B(n) = B_1(n)$, and $k = 5$. To solve (3–6) we take, as initial value q_1, the only solution smaller than unity of the equation of third degree in $\ln q$ (obtained as explained before)

$$\ln^3 q - 30 \ln 2 \ln^2 q + (300 \ln^2 2 - 20\pi^2) \ln q + [200\pi^2 \ln 2 - 1000 \ln^3 2 - 60\zeta(3)] = 0.$$

Using the recurrence (1–10), we get the following sequence of better and better approximations:

n	q_n	k_n
1	0.000976266984418	5.00027949591
2	0.000976645321010	4.99992168497
3	0.000976539294280	5.00002194447
4	0.000976569002482	4.99999385104
5	0.000976560677972	5.00000172298
6	0.000976563010544	4.99999951721
7	0.000976562356942	5.00000013528
8	0.000976562540086	4.99999996209
9	0.000976562488768	5.00000001062
10	0.000976562503147	4.99999999702
11	0.000976562499118	5.00000000083
12	0.000976562500247	4.99999999977
13	0.000976562499931	5.00000000007
14	0.000976562500019	4.99999999998
15	0.000976562499995	5.00000000001
16	0.000976562500002	5.00000000000
17	0.000976562500000	5.00000000000
18	0.000976562500000	5.00000000000

And we easily guess that

$$q = 0.0009765625 = \frac{1}{1024}.$$

Substituting this value in (3–5), we get

$$a = 0.101562500000, \quad b = 1.406250000000, \quad c = 6.406250000000,$$

and again, we easily guess that

$$a = \frac{13}{128}, \quad b = \frac{180}{128}, \quad c = \frac{820}{128},$$

which correspond to the series (1-1) in [Guillera 03]. We give two more examples: Taking $k = 15$ and using the recurrence (1–10) to solve (3–6), we rediscover the series (3–4). Taking $k = 8$ we rediscover the series (2-5) in [Guillera 03]:

$$\sum_{n=0}^{\infty} \frac{B_6(n)}{7^{4n}} \left(\frac{15\sqrt{7}}{392} + \frac{38\sqrt{7}}{49} n + \frac{240\sqrt{7}}{49} n^2 \right) = \frac{1}{\pi^2}.$$

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