A Conjecture on Poincaré-Betti Series of Modules of Differential Operators on a Generic Hyperplane Arrangement

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CONTENTS

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Holm [Holm 04, Holm 02] studied modules of higher-order differential operators (generalizing derivations) on generic (central) hyperplane arrangements. We use his results to determine the Hilbert series of these modules. We also give a conjecture about the Poincaré-Betti series; these are known for the module of derivations through the work of Yuzvinsky [Yuzvinsky 91] and Rose and Terao [Rose and Terao 91]

1. INTRODUCTION

The module of derivations $\mathcal{D}^{(1)}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A} \in \mathbb{C}^n$ (henceforth called an n-arrangement) is an interesting and much studied object [Orlik and Terao 92, Kung 98, Józefiak and Sagan 93]. In particular, the question whether this module is free, for various classes of arrangements, has received great attention.

On the other hand, the module of higher differential operators $\mathcal{D}^{(m)}(\mathcal{A})$ received their first incisive treatment in the PhD thesis of Pär Holm [Holm 02]. The deepest result in that work concerns so-called *generic arrangements*, which are arrangements where every intersection of $s \leq n$ hyperplanes in $\mathcal{A}$ has the expected codimension s. Holm gave a concrete generating set for $\mathcal{D}^{(m)}(\mathcal{A})$, proved an extension of Saito's determinental criterion for freeness of derivations, and used these results to tackle the question of *higher-order freeness* for generic arrangements, i.e., the question when is $\mathcal{D}^{(m)}(\mathcal{A})$ a free module (in which case we say that $\mathcal{A}$ is *m-free*). In brief, he showed that

(i) all 2-arrangements are m-free for all m;

(ii) all n-arrangements with $|\mathcal{A}| \leq n$ are m-free for all m;

(iii) if $n \geq 3$, $r > n$, and $m < r - n + 1$, then $\mathcal{A}$ is not m-free;

(iv) if $n \geq 3$, $r > n$, and $m = r-n+1$, then $\mathcal{A}$ is m-free.

He conjectured that if $n \geq 3$, $r > n$, and $m > r - n + 1$, then $\mathcal{A}$ is m-free.

For $m = 1$, (iii) becomes $r > n \geq 3$. Yuzvinsky [Yuzvinsky 91] and independently Terao and Rose [Rose and Terao 91] showed that the modules of derivations of generic arrangements are nonfree, with a minimal free resolution of length $n - 2$. More precisely, they showed that the graded Betti numbers are given by

$$\beta_{k,u} = \begin{cases} 1, & \text{if } k = 0 \text{ and } u = 1, \\ \binom{r}{n-k}\binom{r-n+k-2}{k-1}, & \text{if } u + n - r - 1 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

so that the Poincaré-Betti series can be expressed as

$$b + \left[t^{n-r-1}(1+bt)^{r-1}\right] \odot (1-t)^{n-r},$$

where t enumerates homological degree, b enumerates ring degree, and $\odot$ denotes Hadamard product of power series.

This article contains a short exposé of various ways of calculating modules of differential operators (on a computer), a brief review of the work of Holm, and finally some conjectures supported by extensive computer experiments. The most important one is the conjectured formula for the Poincaré-Betti series of $\mathcal{D}^{(m)}(\mathcal{A})$ when $\mathcal{A}$ is a generic n-arrangement with $|\mathcal{A}| = r$, $3 \leq n$, $r \geq m + n$:

$$\mathcal{P}\left(\mathcal{D}(\mathcal{A}_{n,r})^{(m)}\right) = b^m + \Big\{t^{-r+n-1}\big((1+bt)^m - (bt)^m\big)\big(1+bt\big)^{r-m}\Big\} \odot (1-t)^{m-r+n-1}.$$

2. TERMINOLOGY AND NOTATION

For basic terminology regarding hyperplane arrangements, we refer to Orlik and Terao's treatise [Orlik and Terao 92]. For more details on the Grothendieck ring of differential operators, see, for instance, the PhD thesis by Holm [Holm 02], the articles [Holm 04, Tripp 97], or the textbooks [Björk 79, Coutinho 95].

2.1 Notation

Let $\mathcal{A}$ be an affine central hyperplane arrangement in $\mathbb{C}^n$ with $|\mathcal{A}| = r$ and with defining polynomial

$$p = \prod_{i=1}^{r} p_i \in S = \mathbb{C}[x_1, \ldots, x_n].$$

We let $\mathcal{D}(S)$ denote the Weyl algebra of differential operators on S. This is the set of all finite S-linear combinations

$$\delta = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \partial^\alpha, \qquad c_\alpha \in S. \tag{2–1}$$

An element of $\mathcal{D}(S)$ can be regarded as a partial differential operator with polynomial coefficients, and, thus, it induces an S-algebra endomorphism. We use the notation $Q * v$ to denote the action of $Q \in \mathcal{D}(S)$ on $v \in S$.

$\mathcal{D}(S)$ is an S-module in a natural way; the action is given by

$$q \sum_{\alpha \in \mathbb{N}^n} c_\alpha \partial^\alpha = \sum_{\alpha \in \mathbb{N}^n} q c_\alpha \partial^\alpha.$$

If in (2–1) all α with $c_\alpha \neq 0$ have total degree m, we say that δ is a homogeneous mth-order operator and write $|\delta| = m$ or $\delta \in \mathcal{D}^{(m)}(S)$. If, in addition, all c_α occurring in (2–1) are homogeneous polynomials of total degree v, we say that δ is homogeneous of polynomial degree v. Thus, $\mathcal{D}(S)$ is a bigraded S-module, where we use the convention that $x^\alpha \partial^\beta$ has bigrade $(k, m) = (|\alpha|, |\beta|)$.

Let m be a positive integer. We set

$$\mathcal{D}(\mathcal{A}) = \left\{ \delta = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \partial^\alpha \,\middle|\, c_\alpha \in S,\ \delta * \langle p \rangle \subseteq \langle p \rangle \right\},$$
$$\mathcal{D}^{(m)}(\mathcal{A}) = \left\{ \delta = \sum_{|\alpha| = m} c_\alpha \partial^\alpha \,\middle|\, c_\alpha \in S,\ \delta * \langle p \rangle \subseteq \langle p \rangle \right\}. \tag{2–2}$$

In particular, $\mathcal{D}^{(1)}(\mathcal{A})$ is the much studied module of derivations of $\mathcal{A}$.

It is a fact that $\mathcal{D}^{(m)}(\mathcal{A})$ is a graded S-module, where the $\mathbb{N}$-grading is given by polynomial degree. Holm [Holm 04] showed that

$$\mathcal{D}(\mathcal{A}) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(\mathcal{A}).$$

Hence, the S-module $\mathcal{D}(\mathcal{A})$ is bigraded, and, hence, so are all Tor modules. The S-module $\mathcal{D}(\mathcal{A})$ is not necessarily finitely generated, but every $\mathcal{D}^{(m)}(\mathcal{A})$ is. Consequently, we can calculate the graded minimal free resolution

$$0 \leftarrow \mathcal{D}^{(m)}(\mathcal{A}) \leftarrow \bigoplus_i \beta_{1,i}^{(m)} S(-i) \leftarrow \cdots \leftarrow \bigoplus_i \beta_{\ell,i}^{(m)} S(-i) \leftarrow 0, \tag{2–3}$$

where, by the Hilbert syzygy theorem, $\ell \leq n$. We define the Poincaré-Betti series and Hilbert series of $\mathcal{D}^{(m)}(\mathcal{A})$

and of $\mathcal{D}(\mathcal{A})$ by

$$\begin{aligned} \mathcal{P}(\mathcal{D}^{(m)}(\mathcal{A}))(b,t) &= \sum_{j,i} \beta^{(m)}_{j,i} b^i t^j, \\ \mathcal{P}(\mathcal{D}(\mathcal{A}))(a,b,t) &= \sum_m a^m \mathcal{P}(\mathcal{D}^{(m)}(\mathcal{A}))(b,t), \\ \mathcal{H}(\mathcal{D}(\mathcal{A}))(a,b) &= (1-b)^{-n}\mathcal{P}(\mathcal{D}(\mathcal{A}))(a,b,-1), \\ \mathcal{H}(\mathcal{D}^{(m)}(\mathcal{A}))(b) &= (1-b)^{-n}\mathcal{P}(\mathcal{D}^{(m)}(\mathcal{A}))(b,-1). \end{aligned} \tag{2–4}$$

2.2 Additional Notation

We define $\left(\!\binom{a}{b}\!\right) = \binom{a+b-1}{b}$, i.e., $\left(\!\binom{a}{b}\!\right)$ is the number of multisets of weight b on an a-set.

If $f(t) = \sum_{i=0}^{\infty} c_i t^i$ is a formal power series in t, with coefficients in some commutative ring, we set

$$[t^\ell]f(t) = c_\ell.$$

Definition 2.1. Let R be a commutative ring and let $f, g \in R[[t^{-1}, t]]$ be two formal Laurent series, i.e.,

$$f = \left(\sum_{i=-\infty}^{\infty} a_i t^i\right), \qquad g = \left(\sum_{i=-\infty}^{\infty} b_i t^i\right).$$

We define the *Hadamard product* of f and g by

$$f \odot g = \sum_{i=-\infty}^{\infty} a_i b_i t^i.$$

2.3 Reminder

n is the dimension of the ambient space of $\mathcal{A}$;

r denotes the number of hyperplanes in $\mathcal{A}$;

m indicates the order of differential operators.

3. PREVIOUS WORK ON MODULES OF DIFFERENTIAL OPERATORS

Derivations on Stanley-Reisner rings were studied by Brumatti and Simis [Brumatti and Simis 95]. This was generalized to higher-order differential operators by Eriksson [Eriksson 98] and independently by Traves [Traves 99] and Tripp [Tripp 97]. A major result is the following: if $J \subset S$ is generated by square-free monomials, then $\mathcal{D}(J)$, the S-module of differential operators preserving J, is generated by

$$\left\{ \mathbf{x}^{\mathbf{b}}\partial^{\mathbf{a}} \middle| \mathbf{x}^{\mathbf{b}} \in (J : (J : \mathbf{x}^{\mathbf{a}})) \right\}. \tag{3–1}$$

Modules of derivations on hyperplane arrangement are described in the classic textbook by Orlik and Terao [Orlik and Terao 92]. An important determinental criterion by Saito characterizes those arrangements, called *free*, for which the module of derivations is a free S-module. When an arrangement is nonfree, it is of interest to calculate a minimal free resolution of its module of derivations. For *generic arrangements*, this has been done by Yuzvinsky [Yuzvinsky 91]. In particular, the numerical character of the resolution, encoded in the Poincaré-Betti series, is known.

This article gives a conjecture for the Poincaré-Betti series of higher modules of differential operators of a generic arrangement (unfortunately, we have no guess for the *maps* in the minimal free resolution). It uses results from [Holm 02], which consists of three parts, with the following content:

- The first part, "Differential Operators on Hyperplane Arrangements," published in [Holm 04], describes generators of $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}^{(m)}(\mathcal{A})$ for *generic arrangements*. It contains a formula similar to (3–1). This is proved by a short exact sequence arising from extraction-deletion. Holm also shows that $D(\mathcal{A})$ is finitely generated as a $\mathbb{C}$-algebra. We will use several results from [Holm 04] in what follows.

- In the second part, "Gelfand-Kirillov Dimension of a Class of Differential Operator Rings," Holm calculates the Gelfand-Kirillov dimension of the noncommutative algebra of differential operators on a certain kind of Noetherian C-algebra, including coordinate rings of central hyperplane arrangements.

- In the third part, "Higher-Order Freeness of Hyperplane Arrangements," Holm generalizes Saito's criterion to higher-order modules of differential operators. He also shows that when r, the number of hyperplanes in the arrangement, is no greater than n, the dimension of the ambient space, then if the arrangement is generic, its module of mth-order differential operators is free (the arrangement is then said to be *m-free*. The same is true for any *2-arrangement*. On the other hand, Holm shows that when $r > n \geq 3$ and $m < r - n + 1$, the module of mth-order differential operators on a generic arrangement is *not free*. He shows that it is free for $m = r - n + 1$ and conjectures that it continues to be free for $m > r - n + 1$.

4. CALCULATING MODULES OF DIFFERENTIAL OPERATORS ON HYPERPLANE ARRANGEMENTS

We shall review some methods of calculating generators of the S-module $\mathcal{D}^{(m)}(\mathcal{A})$.

4.1 The "Jacobian" Method

First, we describe the most straightforward method, a slight variant of which is used in the Macaulay 2 [Grayson and Stillman 02] package `D-modules.m2` [Tsai and Leykin 02] by Harry Tsai and Anton Leykin.

Lemma 4.1. *Let $\mathcal{D}^{(m)}(S) \ni \delta = \sum_{|\alpha|=m} c_\alpha \partial^\alpha$ be homogeneous of polynomial degree v. Then $\delta \in \mathcal{D}^{(m)}(\mathcal{A})$ iff $\delta * (x^\beta p) \in \langle p \rangle$ for all $|\beta| < v$.*

Corollary 4.2. *Let m be a positive integer. Let G be a row matrix whose entries are*

$$\{\, \partial^\alpha \,|\, |\alpha| = m \,\},$$

let H be a column matrix whose entries are

$$\{\, x^\beta \,|\, |\beta| < m \,\},$$

and let A be the matrix indexed by G and H where the $(\partial^\alpha, x^\beta)$ entry is

$$\partial^\alpha * (x^\beta p).$$

Let $B = [A|pI]$, where I is the identity matrix of appropriate dimension. Then, the syzygy module of the columns of B correspond to $\mathcal{D}^{(m)}(\mathcal{A})$. More precisely, if

$$[A|pI]\begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix} = \mathbf{0},$$

then

$$\sum_\alpha u_\alpha \partial_\alpha \in \mathcal{D}^{(m)}(\mathcal{A}),$$

and this is an isomorphism of S-modules.

Example 4.3. Suppose that $S = \mathbb{C}[x_1, x_2]$ and that $p = x_1$. For $m = 2$, the matrix A is

	∂_1^2	$\partial_1\partial_2$	∂_2^2
1	0	0	0
x_1	2	0	0
x_2	0	1	0

so in order to calculate $\mathcal{D}^{(2)}(\mathcal{A})$, we should calculate the syzygies of the columns of

$$B = \begin{pmatrix} 0 & 0 & 0 & x_1 & 0 & 0 \\ 2 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & x_1 \end{pmatrix}.$$

A generating set is

$$\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-1/2x_1\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-x_1\\0\\0\\0\\1\end{bmatrix},$$

so $\mathcal{D}^{(2)}(\mathcal{A})$ is generated by ∂_2^2, $-1/2x_1\partial_1^2$, and $-x_1\partial_1\partial_2$.

4.2 The Method of Intersecting Modules

The following results by Holm are proved in [Holm 04].

Theorem 4.4. [Holm 04] $\mathcal{D}^{(m)}(p) = \cap_{i=1}^r \mathcal{D}^{(m)}(p_i)$.

Holm also showed how to calculate $\mathcal{D}^m(p_i)$. Let $q \in S_1$ be a linear form and let $H \subset \mathbb{C}^n$ be its associated linear variety (a hyperplane through the origin).

Definition 4.5. Let V be the $\mathbb{C}$-vector space $V = \sum_{i=1}^n \mathbb{C}\partial^i$ and define

$$V_H = \left\{ \sum_{i=1} b_i \partial_i \middle| (b_1, \dots, b_n) \in H \right\}.$$

Then, V_H is a codimension-one subspace of V.

Lemma 4.6. [Holm 04] *Let $\mathcal{N}$ be the module of derivations annihilating q. Then, $\mathcal{N} = SV_H$, and if δ is any derivation such that $\delta * q = aq$, $a \in \mathbb{C}^*$, then $\mathcal{D}^{(1)}(q) = \mathcal{N} + S\delta$.*

Proposition 4.7. [Holm 04] *Let $M = \{\delta_1, \dots, \delta_n\}$ be a basis for V such that $\{\delta_1, \dots, \delta_{n-1}\}$ is a basis for V_h. Then, $N = \{\delta_1, \dots, \delta_{n-1}, q\delta_n\}$ generates $\mathcal{D}^{(1)}(q)$, and*

$$\mathcal{D}^{(m)}(q) = \sum_{\substack{|\alpha|=m \\ \alpha_n = 0}} S\delta^\alpha + \sum_{\substack{|\alpha|=m \\ \alpha_n > 0}} Sq\delta^\alpha. \tag{4–1}$$

The above results can be succinctly summarized as follows: let $A(p)$ be the $n \times n$ coordinate matrix of N, i.e., the matrix formed by the coordinate vectors of elements of N, and let $S^m A(p)$ denote the mth symmetric power. Let $B^m(p)$ be the result of replacing any occurrence of q^i with $i > 1$, by q. In other words, if $A(p)$ is regarded as the matrix of an endomorphism

$$\phi : S^n \to S^n,$$

then $S^m A(p)$ is the matrix of the endomorphism

$$S^m\phi : S^m S^n \to S^m S^n,$$

and $B^m(p)$ is the matrix of the associated endomorphism on $S^m T$, where

$$T = S/(q - q^2).$$

Example 4.8. (Example 4.3 continued.) If $S = \mathbb{C}[x_1, x_2]$ and $p = x_1$, then V_H is spanned by ∂_2. Hence, we can take $M = \{\partial_2, \partial_1\}$ and $N = \{\partial_2, x_1\partial_1\}$, so if we order the monomials of degree two as x_1^2, x_1x_2, x_2, then

$$A = \begin{bmatrix} 0 & x_1 \\ 1 & 0 \end{bmatrix}, S^2A = \begin{bmatrix} 0 & 0 & x_1^2 \\ 0 & x_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & x_1 \\ 0 & x_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus, we recover the result that $\mathcal{D}^{(2)}(x_1)$ is generated by

$$[\partial^2, \partial_1\partial_2, \partial_2^2]B = [\partial_2^2, x_1\partial_1\partial_2, x_1\partial_1^2].$$

Now recall (see for instance [Adams and Loustaunau 94, Theorem 3.8.3]) the following method of computing the intersection of submodules of free modules. Suppose that $M_1, \ldots, M_\ell$ are submodules of the free module S^s, that $\underline{\mathbf{e}} = [\mathbf{e}_1, \ldots, \mathbf{e}_s]$ is a basis of S^s, and that the matrix A_i consists of the coordinate vectors (as column vectors) for a generating set of M_i. Then, the truncations of the syzygies of the matrix

$$\begin{bmatrix} I_s & A_1 & 0 & 0 & \cdots & 0 \\ I_s & 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ I_s & 0 & 0 & 0 & \cdots & A_s \end{bmatrix}$$

correspond to elements in $\cap_{i=1}^s M_i$.

Example 4.9. We have that

$$\mathcal{D}^{(2)}(x_1x_2) = \mathcal{D}^{(2)}(x_1) \cap \mathcal{D}^{(2)}(x_2).$$

With respect to the basis $[\partial_1, \partial_1\partial_2, \partial_2^2]$ for $\mathcal{D}^{(2)}(S)$, the matrices of $\mathcal{D}^{(2)}(x_1)$ and $\mathcal{D}^{(2)}(x_2)$ can be taken to be

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix}.$$

The syzygies of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & x_2 \end{bmatrix}$$

are generated by

$$\begin{bmatrix} 0 \\ 0 \\ -x_2 \\ 0 \\ 0 \\ x_2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -x_1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ x_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_1x_2 \\ 0 \\ 0 \\ -x_2 \\ 0 \\ 0 \\ -x_1 \\ 0 \end{bmatrix},$$

so the intersection of the two modules is generated by

$$\begin{bmatrix} 0 \\ 0 \\ -x_2 \end{bmatrix}, \begin{bmatrix} -x_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_1x_2 \\ 0 \end{bmatrix},$$

and $\mathcal{D}^{(2)}(x_1x_2)$ is generated by $-x_2\partial_2, -x_1\partial_1, x_1x_2\partial_1\partial_2$.

4.3 Calculating Modules of Differential Operators on Generic Arrangements

Holm [Holm 02, Paper III, Theorem 5.8] generalizes Saito's criterion for freeness of arrangement as follows:

Theorem 4.10. (Holm-Saito criterion.) *Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^n$ consisting of r hyperplanes and having defining polynomial p. Let $m \geq 1$, $s_m = \left(\!\binom{n}{m}\!\right)$, and $t_m = s_{m-1}$. Then, if $\mathcal{D}^{(m)}(\mathcal{A})$ is free, it has a basis consisting of exactly s_m differential operators.*

Given operators $\theta_1, \ldots, \theta_{s_m} \in \mathcal{D}^{(m)}(\mathcal{A})$, we order the n-multi-indices of weight m lexicographically as $\alpha^1, \ldots, \alpha^{s_m}$ and define the $s_m \times s_m$ coefficient matrix of $\theta_1, \ldots, \theta_{s_m}$ as

$$M_m(\theta_1, \ldots, \theta_{s_m}) =$$

$$\begin{pmatrix} \frac{1}{\alpha^1}!\theta_1 * x^{\alpha^1} & \cdot & \cdot & \cdot & \frac{1}{\alpha^1}!\theta_{s_m} * x^{\alpha^1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\alpha^{s_m}}!\theta_1 * x^{\alpha^{s_m}} & \cdot & \cdot & \cdot & \frac{1}{\alpha^{s_m}}!\theta_{s_m} * x^{\alpha^{s_m}} \end{pmatrix}.$$

Then, $\det M_m(\theta_1, \ldots, \theta_{s_m})$ *lies in the principal ideal generated by p^{t_m}. Furthermore, the following conditions are equivalent:*

(i) $\det M_m(\theta_1, \ldots, \theta_{s_m}) = cp^{t_m}$ *for some $c \in \mathbb{C} \setminus \{0\}$;*

(ii) $\theta_1, \ldots, \theta_{s_m}$ is a basis for $\mathcal{D}^{(m)}(\mathcal{A})$.

In [Holm 04], Holm gives a method for constructing a (not necessarily minimal) generating set of $\mathcal{D}^{(m)}(\mathcal{A})$, when $\mathcal{A}$ is a generic n-arrangement.

4.3.1 The case $r > n$. Holm showed that for any positive m and for any central arrangement $\mathcal{A}$, the modified Euler derivation

$$\varepsilon_m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha \partial^\alpha$$

belongs to $\mathcal{D}^{(m)}(\mathcal{A})$. He then proceeded to find other generators as follows.

Recall the definition of V and of V_H from Definition 4.5. We let H_i be the hyperplane associated with the linear form p_i and set $V_i = V_{H_i}$. Choose a basis element for each intersection of $n-1$ of the V_i (by genericity, this intersection is one-dimensional) and let $M = \{\delta_1, \ldots, \delta_t\}$, with $t = \binom{r}{n-1}$, be the set of all of these. We define a subset D of the derivations on S by

$$D = \{P_1\delta_1, \ldots, P_t\delta_t\},$$

where each P_i is the product of those p_j that are not annihilated by δ_i. If all the p_j are annihilated by δ_i, we set $P_i = 1$.

Holm [Holm 04] showed that $D \subset \mathcal{D}^{(1)}(\mathcal{A})$, and, furthermore, that:

Theorem 4.11. [Holm 04] *Suppose $P = p_1 \cdots p_r$ is the defining polynomial of a generic arrangement $\mathcal{A}$. Let M and D be as above and let I be the principal ideal on P. Then,*

$$\mathcal{D}^{(m)}(\mathcal{A}) = \bigoplus_{m\geq 0}\left(\sum_{|\alpha|=m}(I:(I:P^{\alpha}))\delta^{\alpha} + S\varepsilon_m\right)$$

as a bigraded S-module.

Here, the double colon ideal $(I:(I:P^{\alpha}))$ is a principal ideal in S, with the generator given by $p_{i_1}\cdots p_{i_\ell}$, the product of those p_j such that some δ_i with $\alpha_i \neq 0$ does not annihilate p_j.

Example 4.12. Let $p = xy(y-x) \in \mathbb{C}[x,y] = S$. Then, the associated arrangement is generic. We get that

$$P_1 = y(y-x), \quad P_2 = x(y-x), \quad P_3 = xy,$$

and

$$\delta_1 = \partial_y, \quad \delta_2 = \partial_x, \quad \delta_3 = \partial_x + \partial_y.$$

Hence, $\mathcal{D}^{(0)}(I) = S$, and $\mathcal{D}^{(1)}(I)$ is generated by

$$y(y-x)\partial_y, \quad x(y-x)\partial_x, \quad xy(\partial_x+\partial_y), \quad x\partial_x + y\partial_y.$$

Since any product of distinct δ_i is equal to $P = xy(x-y)$, we have that for $m \geq 2$

$$\begin{aligned}\mathcal{D}(I)^{(m)} &= Sy(y-x)\partial_y^m + Sx(y-x)\partial_x^m \\ &\quad + Sxy(\partial_x+\partial_y)^m + S\varepsilon_m + I\mathcal{D}^{(m)}(S).\end{aligned}$$

4.3.2 The case $r \leq n$. For the generic arrangement $\mathcal{A}_{n,r}$ with $r \leq n$, we can perform a linear change of variables, so that $p_i = x_i$. Holm [Holm 02, Paper III, Proposition 6.2] showed the following:

Proposition 4.13. [Holm 02] *$\mathcal{D}^{(m)}(x_1\cdots x_r)$ is free with basis*

$$\{\, x_\alpha \partial^\alpha | \alpha \in \mathbb{N}^n,\, |\alpha| = m \,\},$$

where x_α is the monic generator of $\sqrt{\langle x^{\tilde{\alpha}}\rangle}$, the radical of $\langle x^{\tilde{\alpha}}\rangle$, and where

$$\tilde{\alpha} = (\alpha_1, \ldots, \alpha_r, 0, \ldots, 0) \in \mathbb{N}^n.$$

4.3.3 The case $n = 2$. Holm [Holm 02, Paper III, Proposition 6.7] notes that an arrangement $\mathcal{A}$ in $\mathbb{C}^2$ is generic. Furthermore, he proves the following:

Proposition 4.14. [Holm 02] *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^2$, with defining polynomial $p = p_1\cdots p_r \in \mathbb{C}[x_1,x_2]$, and let m be a positive integer. Let $P_i = p/p_i$ for $1 \leq i \leq r$ and define*

$$\delta_i = \begin{cases}\partial_2, & \text{if } p_i = ax_1, a \in \mathbb{C}^*,\\ \partial_1 + a_i\partial_2, & \text{if } p_i = a(x_2 - ax_1), a \in C^*.\end{cases}$$

Let

$$\left\{q_1, \ldots, q_{\left(\binom{n}{m}\right)}\right\} = \{\, \partial^\alpha \,||\alpha| = m \,\}.$$

Then, $\mathcal{D}^{(m)}(\mathcal{A})$ is free, minimally generated by

$$\begin{cases}\{\varepsilon_m, P_1\delta_1^m, \ldots, P_r\delta_r^m\}, & \text{if } 1 \leq m \leq r-2,\\ \{P_1\delta_1^{r-1}, \ldots, P_r\delta_r^{r-1}\}, & \text{if } m = r-1,\\ \{P_1\delta_1^m, \ldots, P_r\delta_r^m, pq_r, \ldots, pq_m\}, & \text{if } m \geq r.\end{cases} \tag{4–2}$$

5. AN EXACT SEQUENCE

If $\mathcal{A}$ is an n-arrangement consisting of the hyperplanes $H_1, \ldots, H_r$, recall that the *deleted* arrangement $\mathcal{A}'$ is the arrangement in $\mathbb{C}^n$ consisting of the hyperplanes $H_2, \ldots, H_r$. The *restricted* arrangement $\mathcal{A}''$ is the arrangement

$$H_2 \cap H_1, \ldots, H_r \cap H_1 \subset H_1 \simeq \mathbb{C}^{n-1}.$$

Clearly, if $\mathcal{A}$ is generic, then so is $\mathcal{A}'$ and $\mathcal{A}''$, and we can write

$$\mathcal{A}'_{n,r} = \mathcal{A}_{n,r-1}, \qquad \mathcal{A}''_{n,r} = \mathcal{A}_{n-1,r-1}.$$

We can perform a change of coordinates so that the defining polynomial of H_1 is x_n. Then, multiplication with x_n gives an S-module homomorphism of degree 1

$$\mathcal{D}^{(m)}(\mathcal{A}_{n,r-1}) \xrightarrow{\cdot x_n} \mathcal{D}^{(m)}(\mathcal{A}_{n,r}),$$

and the natural projection

$$\pi : S = \mathbb{C}[x_1, \ldots, x_n] \to S' = \frac{S}{x_n} \simeq \mathbb{C}[x_1, \ldots, x_{n-1}]$$

induces a degree 0 map of graded vector spaces,

$$\mathcal{D}^{(m)}(\mathcal{A}_{n,r}) \to \mathcal{D}^{(m)}(\mathcal{A}_{n-1,r-1}), \tag{5–1}$$

by replacing every occurrence of x_n by zero. The projection π can be used to give any S'-module the structure of an S-module (via extension of scalars), and so the map (5–1) becomes an S-module homomorphism.

Theorem 5.1. [Holm 04] *If m, r, n are positive integers with $n > 2$ and $\mathcal{A}_{n,r}$ is a generic arrangement, then there is a short exact sequence of graded S-modules*

$$0 \to \mathcal{D}^{(m)}(\mathcal{A}_{n,r-1})(-1) \to \mathcal{D}^{(m)}(\mathcal{A}_{n,r}) \to \mathcal{D}^{(m)}(\mathcal{A}_{n-1,r-1}) \to 0. \tag{5–2}$$

Corollary 5.2. *If m, r, n are positive integers with $n > 2$, then*

$$\mathcal{H}(\mathcal{D}^{(m)}(\mathcal{A}_{n,r}))(b) = b\mathcal{H}(\mathcal{D}^{(m)}(\mathcal{A}_{n,r-1}))(b) + \mathcal{H}(\mathcal{D}^{(m)}(\mathcal{A}_{n-1,r-1}))(b). \tag{5–3}$$

6. THE HILBERT AND POINCARÉ-BETTI SERIES OF $\mathcal{D}^{(m)}(\mathcal{A})$ FOR GENERIC $\mathcal{A}$

If $\mathcal{A}$ and $\mathcal{A}'$ are two generic n-arrangements with $|\mathcal{A}| = |\mathcal{A}'| = r$, then their Hilbert series and Poincaré-Betti series coincide. We let $\mathcal{A}_{n,r}$ denote any generic n-arrangement consisting of r hyperplanes.

6.1 Derivations, the Case $m = 1$

Yuzvinsky [Yuzvinsky 91] has given a minimal free resolution of $\mathcal{D}^{(1)}(\mathcal{A}_{n,r})$.

Theorem 6.1. [Yuzvinsky 91] *Let $r > n \geq 3$ and let $\mathcal{A}_{n,r}$ be a generic n-arrangement with defining polynomial $p = p_1 \cdots p_r \in S = \mathbb{C}[x_1, \ldots, x_n]$. Define*

$$\mathcal{D}_0 = \left\{ \theta \in \mathcal{D}^{(1)}(\mathcal{A}_{n,r}) \middle| \theta(p_r) = 0 \right\}.$$

Then, $\mathcal{D}^{(1)}(\mathcal{A}_{n,r}) = S\varepsilon_1 \oplus \mathcal{D}_0$ as an S-module, and the minimal free resolution of $\mathcal{D}_0$ has length $r-1$ and is $(r-n+1)$-linear. More precisely, the graded Betti numbers of $\mathcal{D}_0$ are given by

$$\beta_{k,u} = \begin{cases} \binom{r}{n-k}\binom{r-n+k-2}{k-1}, & \text{if } u+n-r-1 = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{6–1}$$

Example 6.2. If $n = 3$ and $r = 5$, then the Poincaré-Betti series of $\mathcal{D}_0$ is $4b^3 + 2b^4t$, so the Poincaré-Betti series of $\mathcal{D}^{(1)}(\mathcal{A}_{3,5})$ is $b + 4b^3 + 2b^4t$.

We will give a compact formula for the Poincaré-Betti series of derivations, a formula that will give a hint as to what the Poincaré-Betti series of higher-order differential operators might look like.

Lemma 6.3. *Let $r > n \geq 3$. Then,*

$$\mathcal{P}\left(\mathcal{D}^{(1)}(\mathcal{A}_{n,r})\right)(b,t) = b + \left[t^{n-r-1}(1+bt)^{r-1}\right] \odot (1-t)^{n-r}. \tag{6–2}$$

Example 6.4. To continue with the previous example, if $n = 3$ and $r = 5$, then (6–2) becomes

$$\begin{aligned} \mathcal{P}\left(\mathcal{D}^{(1)}(\mathcal{A}_{3,5})\right) &= b + \left[t^{-3}(1+bt)^4\right] \odot (1-t)^{-2} \\ &= b + \left[t^{-3} + 4bt^{-2} + 6b^2t^{-1} + 4b^3 + b^4t\right] \\ &\qquad \odot (1 + 2t + 3t^2 + \ldots) \\ &= b + 4b^3 + 2b^4t. \end{aligned}$$

6.2 The Case $n = 2$

Proposition 4.14 shows that, when $n = 2$, the Poincaré-Betti series of $\mathcal{D}^{(m)}(\mathcal{A}_{2,r})$ is given by

$$\mathcal{P}(\mathcal{D}^{(m)}(\mathcal{A}_{2,r}))(b,t) = \begin{cases} rb^{r-1} + (m-r+1)b^r, & \text{if } r \leq 2 \text{ or } m > r-2, \\ b^m + mb^{r-1}, & \text{otherwise.} \end{cases} \tag{6–3}$$

It follows that the Hilbert series is

$$\mathcal{H}(\mathcal{D}^{(m)}(\mathcal{A}_{2,r}))(b,t) = \begin{cases} \frac{rb^{r-1}+(m-r+1)b^r}{(1-b)^2}, & \text{if } r \leq 2 \text{ or } m > r-2, \\ \frac{b^m+mb^{r-1}}{(1-b)^2}, & \text{otherwise.} \end{cases} \tag{6–4}$$

Together with (5–3), the initial values (6–4) determine $\mathcal{H}(\mathcal{D}^{(m)}(\mathcal{A}_{n,r}))(b)$ for all $n \geq 2$.

6.3 The Case $r \leq n$

Proposition 4.13 yields the following:

Lemma 6.5. *If $r \leq n$, then*

$$\beta^{(m)}_{u,k}(\mathcal{A}_{n,r}) = \begin{cases} 0, & \text{if } u > 0, \\ \binom{r}{k}\left(\!\binom{n-r+k}{m-k}\!\right), & \text{if } u = 0, \end{cases}$$

$$
\begin{aligned}
\mathcal{P}\left(\mathcal{D}(\mathcal{A}_{n,r})\right)(a,b,t) &= \frac{(1-a+ab)^r}{(1-a)^n},\\
\mathcal{H}\left(\mathcal{D}(\mathcal{A}_{n,r})\right)(a,b) &= \frac{(1-a+ab)^r}{(1-a)^n(1-b)^{-n}},\\
\mathcal{P}\left(\mathcal{D}^{(m)}(\mathcal{A}_{n,r})\right)(b,t) &= [a^m]\frac{(1-a+ab)^r}{(1-a)^n},\\
\mathcal{H}\left(\mathcal{D}^{(m)}(\mathcal{A}_{n,r})\right)(b) &= [a^m]\frac{(1-a+ab)^r}{(1-a)^n(1-b)^{-n}}.
\end{aligned}
\tag{6–5}
$$

Proof: There are $\binom{r}{k}$ different square-free $\gamma \in \mathbb{N}^n$ of weight k such that

$$\gamma_{r+1} = \cdots = \gamma_n = 0.$$

For a fixed such γ, let $g = g(\gamma)$ denote the number of $\alpha \in \mathbb{N}^n$ with $\bar{\alpha} = \gamma$. Then,

$$\alpha \mapsto \alpha - \gamma$$

gives a bijection between the set of multisets α on $\{1,\dots,n\}$, with weight k and with $\operatorname{supp}(\tilde{\alpha}) = \gamma$, and the set of multisets on

$$\{r+1, r+2, \dots, n\} \cup \operatorname{supp}(\gamma)$$

of weight $m-k$. So, g is independent of γ and is equal to $\left(\!\binom{n-r+k}{m-k}\!\right)$. It follows that the number of minimal generators of $\mathcal{D}^m(\mathcal{A})$ of polynomial degree k is given by $\binom{r}{k}\left(\!\binom{n-r+k}{m-k}\!\right)$.

To prove the second identity, we take advantage of the fact that we may assume that the linear forms defining the arrangement are the monomials $x_1,\dots,x_r$. For this particular arrangement, the modules of differentials will be multigraded, by giving the operator $x^\alpha\partial^\beta$ the multidegree (α,β). We will calculate Poincaré-Betti series with respect to this fine grading, then specialize to get the desired series (which itself can not be given this fine grading).

We start by calculating the generating function

$$G(a_1,\dots,a_n,b_1,\dots,b_n),$$

where the sum is over all minimal generators of $\mathcal{D}(\mathcal{A}_{n,r})$, and where $x^\alpha\partial^\beta$ contributes $a_1^{\beta_1}\cdots a_n^{\beta_n}b_1^{\alpha_1}\cdots b_n^{\alpha_n}$. Then, as before, we are looking for all pairs (α,β) with $\alpha \subset \{0,\dots,r\}$ and $\operatorname{supp}(\tilde{\alpha}) = \beta$. Hence,

$$
\begin{aligned}
&G(a_1,\dots,a_n,b_1,\dots,b_n) =\\
&\sum_{\beta\subseteq\{1,2,\dots,r\}} b^\beta \left(\sum_{\operatorname{supp}(\gamma)\subset\beta} a^\gamma\right)\left(\sum_{\operatorname{supp}(\theta)\subseteq\{r+1,\dots,n\}} a^\theta\right)\\
&= \sum_{\beta\subseteq\{1,2,\dots,r\}} a^\beta b^\beta \prod_{i=r+1}^{n}(1-a_j)^{-1}\\
&= \prod_{i=1}^{r}\frac{1-a_i+a_ib_i}{1-a_i}\prod_{j=r+1}^{n}(1-a_j)^{-1}.
\end{aligned}
\tag{6–6}
$$

Specializing gives

$$
\begin{aligned}
P(\mathcal{A}_{n,r})(a,b,t) &= G(a,\dots,a,b,\dots,b)\\
&= \frac{(1-a+ab)^r}{(1-a)^n}.
\end{aligned}
\tag{6–7}
$$

□

6.4 The Cases $m = r - n + 1$ and $m > r - n + 1$

As we have already noted, it is known that $\mathcal{D}^{(m)}(\mathcal{A}_{n,r})$ is free when $r \le n$ or when $n \le 2$. Holm [Holm 02, Paper III] showed that when $r > n \ge 3$ and $m = r-n+1$, then $\mathcal{D}^{(m)}(\mathcal{A}_{n,r})$ is a free module, minimally generated by $\binom{r}{n-1}$ differential operators of order polynomial degree m. He conjectured that $\mathcal{D}^{(m)}(\mathcal{A}_{n,r})$ is free when $r > n \ge 3$ and $m \ge r-n+1$. More precisely, it is reasonable to conjecture that (6–5) holds also for this range. This is certainly true for the Hilbert series.

Lemma 6.6. *For $r \le m+n-1$,*

$$\mathcal{H}(\mathcal{D}^{(m)}(\mathcal{A}_{n,r}))(b) = [a^m]\frac{(1-a+ab)^r}{(1-a)^n(1-b)^n}. \tag{6–8}$$

Proof: This holds for $n = 2$ by (6–4). Furthermore, if $r \le m+n-1$, then $r-1 \le m+n-1$ and $r-1 \le m+n-1-1$, so the assertion follows by induction, since

$$
\begin{aligned}
[a^m]\frac{(1-a+ab)^r}{(1-a)^n(1-b)^n} &= [a^m]\frac{(1-a+ab)^{r-1}}{(1-a)^n(1-b)^n}\\
&\quad + [a^m]\frac{(1-a+ab)^{r-1}}{(1-a)^{n-1}(1-b)^{n-1}}.
\end{aligned}
$$

□

6.5 The Case $r \ge m + n$

Holm [Holm 02, Paper III] showed that when $r > n \ge 3$ and $r \ge m+n$, then $\mathcal{D}^{(m)}(\mathcal{A}_{n,r})$ is *not* a free module. This is therefore the "interesting range." We will eventually formulate a conjecture regarding the Poincaré-Betti series of $\mathcal{D}^{(m)}(\mathcal{A}_{n,r})$ for m,n,r in this range.

We will simplify the problem slightly by identifying a direct summand of these modules. Recall the notations of Theorem 4.11:

r	$n=3$	$n=4$	$n=5$
5	$2b^4t+7b^3$	-	-
6	$4b^5t+9b^4$	$2b^5t^2+9b^4t+16b^3$	-
7	$6b^6t+11b^5$	$6b^6t^2+22b^5t+25b^4$	$2b^6t^3+11b^5t^2+25b^4t+30b^3$
8	$8b^7t+13b^6$	$12b^7t^2+39b^6t+36b^5$	$8b^7t^3+39b^6t^2+72b^5t+55b^4$
9	$10b^8t+15b^7$	$20b^8t^2+60b^7t+49b^6$	
10	$12b^9t+17b^8$		

TABLE 1.

r	$n=3$	$n=4$	$n=5$
6	$3b^5t+12b^4$	-	-
7	$6b^6t+15b^5$	$3b^6t^2+15b^5t+31b^4$	-
8	$9b^7t+18b^6$	$9b^7t^2+36b^6t+46b^5$	$3b^7t^3+18b^6t^2+46b^5t+65b^4$
9	$12b^8t+21b^7$	$18b^8t^2+63b^7t+64b^6$	$111b^5+128tb^6+63t^2b^7+12t^3b^8$

TABLE 2.

$$D = \{P_1\delta_1, \ldots, P_t\delta_t\}$$

is a certain subset of $\mathcal{D}^{(1)}(\mathcal{A}_{n,r})$ with the property that

$$\left\{ P_\alpha \delta^\alpha \middle| \alpha \in \mathbb{N}^t,\, |\alpha| = m \right\} \cup \{\varepsilon_m\}$$

generates $\mathcal{D}^{(m)}(\mathcal{A}_{n,r})$, where P_α is the product of those p_i such that some δ_j with $\alpha_j \neq 0$ do not annihilate p_i.

Let $\Xi^{(m)}(\mathcal{A}_{n,r})$ be the module generated by

$$\left\{ P_\alpha \delta^\alpha \middle| \alpha \in \mathbb{N}^t,\, |\alpha| = m \right\}.$$

Then, Holm's result can be stated as

$$\mathcal{D}^{(m)}(\mathcal{A}_{n,r}) = \Xi^{(m)}(\mathcal{A}) + S\varepsilon_m. \tag{6–9}$$

Holm showed [Holm 02, Paper I, Lemma 5.27] that for $r \leq n$, $\varepsilon_m \in \Xi^{(m)}(\mathcal{A})$. Furthermore, we have:

Lemma 6.7. *Suppose that $r > n \geq 3$, $r \geq m+n$. Then, $\varepsilon_m \notin \Xi^{(m)}(\mathcal{A}_{n,r})$, so*

$$\mathcal{D}^{(m)}(\mathcal{A}_{n,r}) = \Xi^{(m)}(\mathcal{A}) \oplus S\varepsilon_m. \tag{6–10}$$

Hence,

$$\mathcal{P}\left(\mathcal{D}^{(m)}(\mathcal{A}_{n,r})\right)(b,t) = b^m + \mathcal{P}\left(\Xi^{(m)}(\mathcal{A}_{n,r})\right)(b,t). \tag{6–11}$$

Proof: All P_i have degree $r-n+1$, so all P_α have degree $\geq r-n+1 > m$, hence all $P_\alpha\delta^\alpha$ have polynomial degree $> m$. But ε_m has polynomial degree m, hence can not be expressed as an S-linear combination of the $P_\alpha\delta^\alpha$. This shows that $\varepsilon_m \notin \Xi^{(m)}(\mathcal{A})$, which together with (6–9) yields (6–10). □

We tabulate $\mathcal{P}(\Xi^{(m)}(\mathcal{A}_{n,r}))(b,t)$ for $m = 2,3,4$ (these Poincaré-Betti series were calculated using Macaulay 2 [Grayson and Stillman 02] and the method described in the beginning of this paper).

For $m = 2$, the Poincaré-Betti series are as in Table 1. We conjecture that

$$\mathcal{P}(\Xi^{(2)}(\mathcal{A}_{n,r})(b,t) = \left[t^{n-r-1}(1+2bt)(1+bt)^{r-2}\right] \odot (1-t)^{n-r+1}. \tag{6–12}$$

For $m = 3$, we get the values given in Table 2. We conjecture that

$$\mathcal{P}(\Xi^{(3)}(\mathcal{A}_{n,r}))(b,t) = \left[t^{n-r-1}(1+3bt+3b^2t^2)(1+bt)^{r-3}\right] \odot (1-t)^{n-r+2}. \tag{6–13}$$

For $m = 4$ we get the values given in Table 3. We conjecture that

$$\mathcal{P}(\Xi^{(4)}(\mathcal{A}_{n,r}))(b,t) = \left[t^{n-r-1}(1+4bt+6b^2t^2+4b^3t^3)(1+bt)^{r-4}\right] \odot (1-t)^{n-r+3}. \tag{6–14}$$

Based on these computations, we make the following conjecture, which by Yuzvinsky's result is true for derivations, i.e., when $m = 1$.

r	$n=3$	$n=4$
7	$18b^5+4tb^6$	-
8	$52b^5+22tb^6+4t^2b^7$	$8b^7t+22b^6$

TABLE 3.

Conjecture 6.8. *Suppose that $3 \leq n$ and $r \geq m+n$. Let $\mathcal{A}_{n,r}$ be a generic n-arrangement with $|\mathcal{A}_{n,r}| = r$. Then,*

$$\mathcal{P}\left(\Xi^{(m)}(\mathcal{A}_{n,r})\right) = \left\{t^{-r+n-1}\big((1+bt)^m - (bt)^m\big)\big(1+bt\big)^{r-m}\right\} \odot (1-t)^{m-r+n-1}, \quad (6\text{–}15)$$

and, hence,

$$\mathcal{P}\left(\mathcal{D}^{(m)}(\mathcal{A}_{n,r})\right) = b^m + \left\{t^{-r+n-1}\big((1+bt)^m - (bt)^m\big)\big(1+bt\big)^{r-m}\right\} \odot (1-t)^{m-r+n-1}. \quad (6\text{–}16)$$

Note that this conjecture implies that

(i) the homological dimension of the S-module $\Xi^{(m)}(\mathcal{A}_{n,r})$ (and hence of $\mathcal{D}^{(m)}(\mathcal{A}_{n,r})$) is $n-2$;

(ii) the minimal free resolution of $\Xi^{(m)}(\mathcal{A}_{n,r})$) is $(r-n)$-linear;

(iii) $\Xi^{(m)}(\mathcal{A}_{n,r})$ is minimally generated by

$$\binom{r}{r-n+1} - \binom{r-m}{r-n+1-m} = \binom{r}{n-1} - \binom{r-m}{n-1}$$

differential operators of polynomial degree $r-n+1$; $\mathcal{D}^{(m)}(\mathcal{A}_{n,r})$ is minimally generated by these differential operators and ε_m.

We now indicate a possible way of proving (6–15). The short exact sequence (5–2), together with Lemma 6.7, gives a short exact sequence

$$0 \to \Xi^{(m)}(\mathcal{A}_{n,r-1})(-1) \to \Xi^{(m)}(\mathcal{A}_{n,r}) \to \Xi^{(m)}(\mathcal{A}_{n-1,r-1}) \to 0, \quad (6\text{–}17)$$

which gives rise to a long exact sequence in homology (we have omitted the shifts that are necessary to make the morphisms below homogeneous of degree zero)

$$\cdots \to \mathrm{Tor}_S^1(\mathcal{D}^{(m)}(\mathcal{A}_{n,r-1}),\mathbb{C}) \to \mathrm{Tor}_S^1(\mathcal{D}^{(m)}(\mathcal{A}_{n,r}),\mathbb{C}) \to \mathrm{Tor}_S^1(\mathcal{D}^{(m)}(\mathcal{A}_{n-1,r-1}),\mathbb{C}) \xrightarrow{\delta_1} \Xi^{(m)}(\mathcal{A}_{n,r-1}) \otimes_S \mathbb{C} \to \Xi^{(m)}(\mathcal{A}_{n,r}) \otimes_S \mathbb{C} \to \Xi^{(m)}(\mathcal{A}_{n-1,r-1}) \otimes_S \mathbb{C} \to 0, \quad (6\text{–}18)$$

which controls the "deviation"

$$q(m,n,r) = \mathcal{P}(\Xi^{(m)}(\mathcal{A}_{n,r})) - b\mathcal{P}(\Xi^{(m)}(\mathcal{A}_{n,r-1})) - (1+bt)\mathcal{P}(\Xi^{(m)}(\mathcal{A}_{n-1,r-1})).$$

If all the connecting homomorphisms δ_i are zero, then so is this deviation, and the Poincaré-Betti series can be computed recursively using deletion-restriction. In the "interesting range" $3 \leq n < r$ and $r \geq m+n$, assuming the conjectured Formula (6–15) and using Lemma 6.9, we get that $q(m,n,r) = 0$. This indicates (but does not prove) that the connecting homomorphisms are zero. Conversely, a proof of the vanishing of all connecting homomorphisms would also prove (6–15).

Lemma 6.9. *For $3 \leq n < r$ and $r \geq m+n$, let*

$$A = \left\{t^{-r+n-1}\big((1+bt)^m - (bt)^m\big)\big(1+bt\big)^{r-m}\right\}, \quad (6\text{–}19)$$

$$B = b\left\{t^{-r+n}\big((1+bt)^m - (bt)^m\big)\big(1+bt\big)^{r-1-m}\right\}, \quad (6\text{–}20)$$

$$C = (1+bt)\left\{t^{-r+n-1}\big((1+bt)^m - (bt)^m\big)\big(1+bt\big)^{r-1-m}\right\}. \quad (6\text{–}21)$$

It then holds that

$$A \odot (1-t)^{m-r+n-1} - B \odot (1-t)^{m-r+n} - C \odot (1-t)^{m-r+n-1} = 0. \quad (6\text{–}22)$$

Proof: Let $k = r-n$ and

$$U(r,k,m) = \left[\big((1+bt)^m - (bt)^m\big)\big(1+bt\big)^{r-m}\right] = t^{-k-1}(1+bt)^r - t^{m-k-1}(1+bt)^{r-m}.$$

Then, for $\ell > 0$, we have that

$$[x^\ell] \quad U(r,k,m) = b^{\ell+k+1}\left[\binom{r}{\ell+k+1} - \binom{r-m}{\ell-m+k+1}\right],$$

$$[x^\ell] \quad b\,U(r-1,k-1,m) = b^{\ell+k+1}\left[\binom{r-1}{\ell+k} - \binom{r-1-m}{\ell-m+k}\right],$$

$$[x^\ell] \quad U(r-1,k,m) = b^{\ell+k+1}\left[\binom{r-1}{\ell+k+1} - \binom{r-1-m}{\ell-m+k+1}\right],$$

$$[x^\ell] \quad bt\,U(r-1,k,m) = b^{\ell+k+1}\left[\binom{r-1}{\ell+k} - \binom{r-1-m}{\ell-m+k}\right].$$

Since

$$(1-t)^{m-r+n-1} = \sum_{\ell=0}^{\infty} \left(\!\!\binom{m-r+n-1}{\ell}\!\!\right),$$

Equation (6–22) is equivalent to the identity

$$\begin{aligned}
&\left[\binom{r}{\ell+k+1} - \binom{r-m}{\ell-m+k+1}\right]\left(\!\!\binom{m-k+1}{\ell}\!\!\right) \\
&\quad - \left[\binom{r-1}{\ell+k} - \binom{r-1-m}{\ell-m+k}\right]\left(\!\!\binom{m-k}{\ell}\!\!\right) \\
&\quad - \left[\binom{r-1}{\ell+k+1} - \binom{r-1-m}{\ell-m+k+1}\right]\left(\!\!\binom{m-k+1}{\ell}\!\!\right) \\
&\quad - \left[\binom{r-1}{\ell+k} - \binom{r-1-m}{\ell-m+k}\right]\left(\!\!\binom{m-k+1}{\ell-1}\!\!\right) = 0,
\end{aligned} \tag{6–23}$$

which can be verified algorithmically.[1]

The case $\ell = 0$ is dealt with similarly. □

7. ELECTRONIC SUPPLEMENTS

To help those interested in performing further experiments, some Macaulay 2 [Grayson and Stillman 02] and Maple [Char et al. 91] code is included as an electronic supplement to this article. It is available at http://www.expmath.org/expmath/volumes/14/14.4/snellman/code.zip. An accompanying `README` file describes how the code can be used.

REFERENCES

[Adams and Loustaunau 94] William W. Adams and Philippe Loustaunau. *An Introduction to Gröbner Bases*, Graduate Studies in Mathematics, 3. Providence, RI: American Mathematical Society, 1994.

[Björk 79] J. -E. Björk. *Rings of Differential Operators*, North-Holland Mathematical Library, 21. Amsterdam: North-Holland Publishing Co., 1979.

[Brumatti and Simis 95] Paulo Brumatti and Aron Simis. "The Module of Derivations of a Stanley-Reisner Ring." *Proc. Amer. Math. Soc.* 123:5 (1995), 1309–1318.

[Char et al. 91] Bruce W. Char, Keith O. Geddes, Gaston H. Gonnet, Benton L. Leong, Michael B. Monagan, and Stephen M. Watt. *Maple V Library Reference Manual.* New York: Springer-Verlag, 1991.

[1] We used the `simplify(expr,GAMMA)` command of the computer algebra system Maple [Char et al. 91].

[Coutinho 95] S. C. Coutinho. *A Primer of Algebraic D-Modules*, London Mathematical Society Student Texts, 33. Cambridge, UK: Cambridge University Press, 1995.

[Eriksson 98] Anders Eriksson. "The Ring of Differential Operators of a Stanley-Reisner Ring." *Comm. Algebra* 26:12 (1998), 4007–4013.

[Grayson and Stillman 02] Daniel R. Grayson and Michael E. Stillman. "Macaulay 2." Computer Algebra Program. Available from World Wide Web (http://www.math.uiuc.edu/Macaulay2/), 2002.

[Holm 02] Pär Holm. *Differential Operators on Arrangements of Hyperplanes.* PhD diss., Stockholm University, 2002.

[Holm 04] Pär Holm. "Differential Operators on Hyperplane Arrangements." *Communications in Algebra* 32:6 (2004), 2177–2201.

[Józefiak and Sagan 93] Tadeusz Józefiak and Bruce E. Sagan. "Basic Derivations for Subarrangements of Coxeter Arrangements." *J. Algebraic Combin.* 2:3 (1993), 291–320.

[Kung 98] Joseph P. S. Kung. "A Geometric Condition for a Hyperplane Arrangement to Be Free." *Adv. Math.* 135:2 (1998), 303–329.

[Orlik and Terao 92] Peter Orlik and Hiroaki Terao. *Arrangements of Hyperplanes*, Grundlehren der Mathematischen Wissenschaften, 300. Berlin: Springer-Verlag, 1992.

[Rose and Terao 91] Lauren L. Rose and Hiroaki Terao. "A Free Resolution of the Module of Logarithmic Forms of a Generic Arrangement." *Journal of Algebra* 136 (1991), 376–400.

[Traves 99] William N. Traves. "Differential Operators on Monomial Rings." *J. Pure Appl. Algebra* 136:2 (1999), 183–197.

[Tripp 97] J. R. Tripp. "Differential Operators on Stanley-Reisner Rings." *Trans. Amer. Math. Soc.* 349:6 (1997), 2507–2523.

[Tsai and Leykin 02] Harry Tsai and Anton Leykin. "D-Modules.m2." Macaulay 2 package. Available from World Wide Web (http://www2.math.uic.edu/~leykin/Research/Dmodules/Dmods.html), 2002.

[Yuzvinsky 91] Sergey Yuzvinsky. "A Free Resolution of the Module of Derivations for Generic Arrangements." *Journal of Algebra* 136 (1991), 32–436.

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