On Lower Bounds of the Density of Delone Sets and Holes in Sequences of Sphere Packings

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CONTENTS

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We study lower bounds of the packing density of a system of nonoverlapping equal spheres in $\mathbb{R}^n, n \geq 2$, as a function of the maximal circumradius of its Voronoi cells. Our viewpoint, using Delone sets, allows us to investigate the gap between the upper bounds of Rogers or Kabatjanskii-Levenstein and the Minkowski-Hlawka type lower bounds for the density of lattice-packings, without entering the fundamental problem of constructing Delone sets with Delone constants between $2^{-0.401}$ and 1. As a consequence we provide explicit asymptotic lower bounds of the covering radii (holes) of the Barnes-Wall, Craig, and Mordell-Weil lattices, respectively BW_n, $\mathbb{A}_n^{(r)}$, and MW_n, and of the Delone constants of the BCH packings, when n goes to infinity.

1. INTRODUCTION

The maximal packing density of equal spheres in $\mathbb{R}^n$ has received a lot of attention [Rogers 64, Goodman and O'Rourke 97, Cassels 59, Martinet 96, Conway and Sloane 88, Oesterlé 90, Gruber and Lekkerkerker 87, Zong 99]. Similar problems are encountered in coding theory, data transmission, combinatorial geometry, and cryptology [Hoffstein et al. 01]. We will consider the problem through the context of Delone sets. We will give explicit lower bounds of the density of a Delone set as a function of n and its so-called Delone constant R expressing the maximal size of its holes.

Blichfeldt, Rogers, Levenštein, Sidel'nikov, Kabatjanskiĭ, and Levenštein [Goodman and O'Rourke 97, Gruber and Lekkerkerker 87, Conway and Sloane 88] have given upper bounds of the packing density, while lower bounds of the latice-packing density were given by Minkowski, Davenport-Rogers, Ball [Ball 92], etc. (see Section 2). In between the situation is considered fairly vague. The present paper contributes to our knowledge of the range between both types of bounds although the fundamental problem, far from obvious, of constructing Delone sets of

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very small Delone constant, namely less than 1, is not considered here.

For this we will recall the language of uniformly discrete sets and Delone sets instead of that of systems of spheres. A discrete subset Λ of $\mathbb{R}^n$ is said to be uniformly discrete if there exists a constant $r > 0$ such that $x, y \in \Lambda, x \neq y$ implies $\|x - y\| \geq r$. Thus a uniformly discrete set is either the empty set, a subset $\{x\}$ reduced to one element, or, if it contains at least two points, they satisfy such an inequality. If r is equal to the minimal interpoint distance $\inf\{\|x-y\| \mid x, y \in \Lambda, x \neq y\}$, Λ is said to be a uniformly discrete set of constant r of $\mathbb{R}^n$. Uniformly discrete sets of constant 1 will be called $\mathcal{UD}$-sets and the set of $\mathcal{UD}$-sets will be denoted by $\mathcal{UD}$ (without mentioning the dimension n of the ambient space). There is a one-to-one correspondence between the set SS, of systems of equal spheres of radius 1/2, and the set $\mathcal{UD}$: $\Lambda = (a_i)_{i\in\mathbb{N}} \in \mathcal{UD}$ is the set of sphere centres of $\mathcal{B}(\Lambda) = \{a_i + B \mid i \in \mathbb{N}\} \in SS$ where $B(z, t)$ generically denotes the closed ball centred at $z \in \mathbb{R}^n$ of radius $t > 0$, and $B := B(0, 1/2)$. We will take 1/2 in the sequel for the common radius of spheres to be packed and will consider $\mathcal{UD}$-sets instead of systems of equal spheres of radius 1/2.

Let $\Lambda \in \mathcal{UD}$. The density of the system of spheres $\mathcal{B}(\Lambda)$ is defined by

$$\delta(\mathcal{B}(\Lambda)) := \limsup_{R\to+\infty} \left[\frac{\mathrm{vol}\left(\left(\bigcup_{i\in\mathbb{N}}(a_i + B)\right) \bigcap B(0,R)\right)}{\mathrm{vol}(B(0,R))} \right].$$

Let us denote by $\mathcal{L}$ the space of (n-dimensional) lattices of $\mathbb{R}^n$. We will denote:

$$\delta := \sup_{\Lambda\in\mathcal{UD}} \delta(\mathcal{B}(\Lambda)), \quad \delta_L := \sup_{\Lambda\in\mathcal{L}\cap\mathcal{UD}} \delta(\mathcal{B}(\Lambda))$$

and will call them respectively the *packing density* and the *lattice-packing density*.

A $\mathcal{UD}$-set Λ is said to be a Delone set if there exists a constant $R > 0$ such that, for all $z \in \mathbb{R}^n$, there exists an element $\lambda \in \Lambda$ such that $\|z - \lambda\| \leq R$ (property of *relative denseness* of Besicovitch). If Λ is a Delone set, then $R(\Lambda) := \sup_{z\in\mathbb{R}^n} \inf_{\lambda\in\Lambda} \|z - \lambda\|$ is called the *Delone constant* of Λ. Let $R_c = R_c(n) := \inf\{R(\Lambda) | \Lambda \in \mathcal{UD}\}$. This lower bound is an invariant of the ambient space which is only a function of n and the Euclidean metric on $\mathbb{R}^n$. We will call it the *Delone covering constant*.

In Section 2, we will recall the asymptotic expressions of the classical upper bounds of the packing density and the lower bounds of the lattice-packing density, when n goes to infinity.

In Section 3, we will recall known lower bounds of the minimal hole constant, in the case of lattice packings, and state some results concerning lower bounds of R_c in the general case of arbitrary packings.

The Delone constant of a Delone set $\Lambda \in \mathcal{UD}$ is the maximal circumradius of the Voronoi cells in the Voronoi decomposition of space by Λ (Section 3); if Λ is a lattice, it is the covering radius of the lattice, if Λ is a nonperiodic $\mathcal{UD}$-set, it is the "maximal size of the holes in Λ." In Section 4, we will prove Theorem 1.1.

Theorem 1.1. *Let $n \geq 2$. If Λ is a Delone set of $\mathbb{R}^n$ of Delone constant R, then*

$$(2R)^{-n} \leq \delta(\mathcal{B}(\Lambda)) \leq \delta \text{ for all } R_c \leq R. \tag{1–1}$$

Let us denote $\mu_n(R) := (2R)^{-n}$. The $(2R)^{-n}$ dependence of the expression of $\mu_n(R)$ with n is very important and constitutes a key result. It allows us to study the minimal asymptotic values of the Delone covering constant $R_c(n)$ when n tends to infinity. Namely, we will prove Theorem 1.2.

Theorem 1.2. *For all $\epsilon > 0$ there exists $n(\epsilon)$ such that for $n > n(\epsilon)$, $R_c(n) \geq 2^{-0.401} - \epsilon$.*

Remark 1.3. Theorem 1.2 asserts the existence of an infinite collection of *middle-sized* Voronoi cells in any densest or saturated packing of equal spheres of $\mathbb{R}^n$ of radius 1/2 of circumradii greater than

$$2^{-0.401} + o(1) = 0.757333... + o(1).$$

The small values of R between the bound

$$\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}$$

and 1 are discussed in Section 3

In Section 5, as an application of Theorem 1.1, we will obtain explicit lower bounds as a function of n of the covering radii (holes) of known lattices, namely Barnes-Wall BW_n, Craig $\mathbb{A}_n^{(r)}$, Mordell-Weil MW_n, and of the Delone constants of BCH packings.

In Section 6, we will show the pertinency of the lower bound $\mu_n(R)$, and "its continuity with R" by comparing it to known classical asymptotic bounds. The construction of Delone sets with very small Delone constants is a difficult problem which is not considered here. Concerning lattice packings, our results give credit to the conjecture stating that (recall that the space $\mathcal{UD}$ depends

$cn\,2^{-n/2}$ (c a const.)	Blichfeldt [Blichfeldt 29]
$\frac{n}{e}\,2^{-n/2}$	Rogers [Rogers 58]
$2^{(-0.5096+o(1))n}$	Sidel'nikov [Sidel'nikov 73]
$2^{(-0.5237+o(1))n}$	Levenštein [Levenštein 79]
$2^{(-0.5990+o(1))n}$	Kabatjanskiĭ and Levenštein [Kabatjanskiĭ and Levenštein 78]

TABLE 1. Upper bounds of δ as a function of n.

upon n): for all $\epsilon > 0$, there exists $n_L(\epsilon)$ such that for $n > n_L(\epsilon)$ and for all (n-dimensional) lattices $L \in \mathcal{UD}$, $R(L) \geq 1 - \epsilon$.

2. ASYMPTOTIC BEHAVIOUR OF THE UPPER BOUNDS OF δ AND OF THE LOWER BOUNDS OF δ_L

The upper bounds of δ, as a function of n, are recalled in Table 1, the best one being the one of Kabatjanskiĭ and Levenštein ([Rogers 64], [Gruber and Lekkerkerker 87, Section 19 and Section 38, pages 390–391], [Conway and Sloane 88, Chapters 1 and 9], [Zong 99, Chapter 3]).

Their asymptotic expressions, when n goes to infinity, all exhibit a dominant exponential term of the type $2^{-\alpha n}$ where α is close to 1/2. As for lower bounds, non-trivial lower bounds of the packing constant δ do not seem to exist yet (see Section 6); [Elkies 00a]). The basic result is concerned with lattice packings: the Conjecture of Minkowski (1905) proved by Hlawka [Cassels 59, Gruber and Lekkerkerker 87] states

$$\frac{\zeta(n)}{2^{n-1}} \leq \delta_L \qquad (2–1)$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ denotes the Riemann ζ-function. Proofs of this lower bound do not provide explicit constructions of very dense lattices. This lower bound was improved by Davenport and Rogers [Davenport and Rogers 47] who gave: $(\ln\sqrt{2} + o(1))\,n\,2^{-n}$, for n sufficiently large, and by Ball [Ball 92] who recently obtained better: $2(n-1)\zeta(n)2^{-n}$. For details, see [Goodman and O'Rourke 97], Chapter VI in [Cassels 59], Chapter 9 in [Conway and Sloane 88], [Gruber and Lekkerkerker 87], or [Zong 99]. One can remark that these asymptotic expressions all exhibit a dominant exponential term in $2^{-\alpha' n}$ with $\alpha' = 1$, and that there exists a close similarity between the asymptotic expressions of the lower and upper bounds and Theorem 1.1. Theorem 1.1 will allow to "go continuously" in some sense from the first type ("$\alpha \simeq 1/2$" case) to the second type ("$\alpha' = 1$" case) of bounds; see Section 6

3. LOWER BOUNDS OF THE MINIMAL HOLE CONSTANT $R_L(n)$ AND OF THE DELONE COVERING CONSTANT $R_C(n)$

Bounds for the (lattice-)packing density are obviously linked to holes. Let us recall some definitions. If a lattice $\Lambda \in \mathcal{UD}$ of $\mathbb{R}^n$ is a Delone set of Delone constant R, then classically the quantity R is called the *covering radius* of Λ. Given a $\mathcal{UD}$-set $\Lambda := \{\lambda_i\}$, to each element $\lambda_i \in \Lambda$ is associated its local cell $C(\lambda_i, \Lambda)$, also denoted by $C(\lambda_i, \mathcal{B}(\Lambda))$, defined by the closed subset (not necessarily bounded), called Voronoi cell at λ_i,

$$C(\lambda_i, \Lambda) := \left\{ \begin{array}{l} x \in \mathbb{R}^n | \|x - \lambda_i\| \leq \|x - \lambda_j\| \\ \text{for all } j \neq i \end{array} \right\}.$$

As soon as Λ is a Delone set of Delone constant $R > 0 (R < +\infty)$, all the Voronoi cells at its points are bounded closed convex polyhedra. In this case, for all $\lambda_i \in \Lambda$, we have

$$C(\lambda_i, \Lambda) := \left\{ \begin{array}{l} x \in \mathbb{R}^n | \|x - \lambda_i\| \leq \|x - \lambda_j\| \\ \text{for all } j \neq i \text{ with } \|\lambda_j - \lambda_i\| < 2R \end{array} \right\}.$$

By definition the circumradius of the Voronoi cell at λ_i is $\rho_i := \max_v \|\lambda_i - v\|$ where the supremum (reached) is taken over all the vertices v of the Voronoi cell $C(\lambda_i, \Lambda)$ at λ_i and the Delone constant R of Λ is equal to $\max_i \rho_i$. The elements $z \in \mathbb{R}^n$ lying at a distance $R(\Lambda)$ of Λ will be called (spherical) deep holes (or deepest holes) of Λ. The other vertices of Voronoi cells will be called holes.

In the particular case of a lattice L the covering radius $R(L)$ is the circumradius of the Voronoi cell of the lattice L at the origin. Any vertex of this Voronoi cell at a distance of L less than $R(L)$ from L is called shallow hole [Conway and Sloane 88]. All the vertices of the Voronoi cell of a lattice at the origin may be simultaneously deepest holes when this Voronoi cell is highly symmetrical [Verger-Gaugry 97].

Let us define *the minimal hole constant* by

$$R_L = R_L(n) := \min_{L \in \mathcal{UD} \cap \mathcal{L}} R(L)$$

over all lattices L of $\mathbb{R}^n$ which are $\mathcal{UD}$-sets. Its determination is an important problem, already mentioned by

$n = 3$	Böröczky [Böröczky 86]	$= \sqrt{5}/(2\sqrt{3}) \simeq 0.645497...$
$n = 4$	Horvath [Horvath 82]	$= (\sqrt{3}-1)3^{1/4}/\sqrt{2} \simeq 0.68125...$
$n = 5$	Horvath [Horvath 82]	$= \sqrt{9+\sqrt{13}}/(2\sqrt{6}) \simeq 0.72473...$
$n \geq 2$	Rogers [Rogers 50]	< 1.5
$n \geq 2$	Henk [Henk 95]	$\leq \sqrt{21}/4 \simeq 1.1456...$
$n >> 1$	Butler [Butler 72]	$\leq n^{(\log_2 \ln n + c)/n} = 1 + o(1)$ (c is a constant)

TABLE 2. Minimal hole constant $R_L(n)$ for lattice-packings of spheres of radius 1/2 in $\mathbb{R}^n$.

Fejes-Toth [Fejes-Toth 79]. It corresponds to the smallest possible holes in lattice packings $L + B$. Our knowledge about it is comparatively limited and the lattices for which the covering radius is equal to the minimal hole constant are unknown as soon as n is large enough. In Table 2 we summarize some values and known upper bounds of $R_L(n)$.

The following theorem is fundamental but nonconstructive.

Theorem 3.1. (Butler.) [Butler 72]

$R_L(n) \leq 1 + o(1)$ *when n is sufficiently large.*

This leads to the following question:

Question 3.2. For all $\epsilon > 0$, does there exist $n_0(\epsilon)$ such that the inequality $R_L(n) \geq 1-\epsilon$ holds for all $n \geq n_0(\epsilon)$?

If the answer to this fundamental question is yes, then Butler's Theorem [Butler 72] would imply that $R_L(n) = 1+o(1)$. Then this result would be a very important step towards a proof of the conjecture stating that the strict‘ inequality "$\delta > \delta_L$" holds for n large enough. The affirmative answer to Question 3.2 is a conjecture [Conway and Sloane 88]. Consequently, the search for lower bounds of $R_L(n)$ is crucial.

The lower bound $\sqrt{2}/2 + o(1)$ for $R_L(n)$ when n is large enough was given by Blichfeldt (see [Butler 72, page 722]). Let us note that the normalized (see Section 5) Leech lattice $\Lambda_{24}/\sqrt{N(\Lambda_{24})}$ [Elkies 00b] has a small value of its covering radius by the theorem of Conway, Parker, and Sloane (in [Conway and Sloane 88, Chapter 23]): $R(\Lambda_{24}/\sqrt{N(\Lambda_{24})}) = \sqrt{2}/2$. In low dimension, this value is rarely reached [Conway and Sloane 88]. In general, for lattices, the information about its holes is limited (see Chapter 22 by Norton in [Conway and Sloane 88]) because of the difficulty of computing explicitely the Voronoi cells of a lattice from the lattice itself when n is large.

Let us now turn to the notion of *saturation*, linked to the possible filling of holes. We will say that a $\mathcal{UD}$-set Λ is *saturated*, or *maximal*, if it is impossible to add a sphere to $\mathcal{B}(\Lambda)$ without destroying the fact that it is a packing of spheres, i.e., without creating an overlap of spheres. The set SS of systems of spheres of radius 1/2, is partially ordered by the relation $\prec$, defined by

$$\Lambda_1, \Lambda_2 \in \mathcal{UD}, \quad \mathcal{B}(\Lambda_1) \prec \mathcal{B}(\Lambda_2) \iff \Lambda_1 \subset \Lambda_2.$$

By Zorn's Lemma, maximal sphere packings exist. The saturation operation of a sphere packing consists of adding spheres to obtain a maximal sphere packing. It is fairly arbitrary and may be finite or infinite. Note that it is not because a sphere packing is maximal (saturated) that its density is equal to δ.

Let $X_R \subset \mathcal{UD}$ be the subset of Delone sets of Delone constant $R > 0$ of $\mathbb{R}^n$. By saturating a Delone set of Delone constant $R > 0$ we will always obtain a Delone set of constant less than 1, but not a Delone set of Delone constant $= R_c$ in general. Let $R^{(s)} := \sup\{R(\Lambda) \mid \Lambda \text{ saturated}\}$. It is obvious that $1/2 \leq R_c < R^{(s)} \leq 1$, $R_c(n) \leq R_L(n)$ and that the subset of saturated Delone sets of $\mathbb{R}^n$ is included in $\bigcup_{R_c \leq R \leq R^{(s)}} X_R$. More precisely we have the following facts.

Lemma 3.3.

(i) $R^{(s)} = 1$;

(ii) $R_c(n) \geq \frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}} = \frac{\sqrt{2}}{2}(1 + O(1/n))$ *for n large.*

Proof:

(i) Let us assume $R^{(s)} < 1$ and that $R^{(s)}$ is the Delone constant of a saturated Delone set Λ. We will obtain a contradiction. Then there exists $z \in \mathbb{R}^n$ such that $\inf_{\lambda \in \Lambda} \|z - \lambda\| = R^{(s)}$. Up to a translation, we may assume $z = 0$. Let $\epsilon > 0$ be small enough such that $(1+\epsilon)R^{(s)} < 1$. Let $\eta \in (3, 4)$ such that the system of spheres

$$\mathcal{B}\left(\Lambda \cap B(0, \eta R^{(s)})\right) := \{B(c_1, 1/2), \ldots, B(c_m, 1/2)\}$$

(with $m \geq 1$) is such that $\|hc_j\| < \eta R^{(s)}$ for all $j = 1, 2, \ldots, m$ and all $h \in [1, 1+\epsilon)$.

Now let $h \in (1, 1+\epsilon)$ and let us create the new Delone set Λ_h from Λ as follows: first, $\Lambda_h \cap B(0, \eta R^{(s)})$ is exactly equal to the set $\{hc_1, hc_2, \ldots, hc_m\}$ so that, "inside" the ball $B(0, \eta R^{(s)})$, $\mathcal{B}\left(\Lambda_h \cap B(0, \eta R^{(s)})\right) = \{B(hc_1, 1/2), \ldots, B(hc_m, 1/2)\}$. For constructing $\mathcal{B}\left(\Lambda_h \cap \left(\mathbb{R}^n \setminus B(0, \eta R^{(s)})\right)\right)$, we take any infinite packing $\mathcal{B}_1$ of balls of radius 1/2 centred at points which lie in $\mathbb{R}^n \setminus B(0, \eta R^{(s)})$ so that: (1)$\mathcal{B}_1 \cup \mathcal{B}\left(\Lambda_h \cap B(0, \eta R^{(s)})\right)$ is a packing of balls of $\mathbb{R}^n$, and (2) $\mathcal{B}_1$ is saturated. We obtain the Delone set $\Lambda_h \in \mathcal{UD}$ defined by $\mathcal{B}(\Lambda_h) := \mathcal{B}_1 \cup \mathcal{B}\left(\Lambda_h \cap B(0, \eta R^{(s)})\right)$.

We will take ϵ small enough such that all the Voronoi cells at the points hc_j, with $j = 1, 2, \ldots, m$ and $h \in [1, 1+\epsilon)$, have a circumradius always strictly less than 1 (this is always possible because of the continuity of the maps defining the vertices of Voronoi cells as functions of the centres of balls). Since the restriction of the system of balls $\mathcal{B}(\Lambda_h)$ to the portion of space outside the cluster $\{B(hc_1, 1/2), \ldots, B(hc_m, 1/2)\} \cup B(0, \eta R^{(s)})$, is saturated, all the Voronoi cells at the centres of the balls of $\mathcal{B}_1$ have a circumradius $\leq R^{(s)} < 1$.

Then, on one hand, since the distance between 0 and Λ_h is $hR^{(s)} > R^{(s)}$, for $h > 1$, the Delone set Λ_h has a Delone constant strictly greater than $R^{(s)}$. Hence it is not saturated, by definition of $R^{(s)}$. On the other hand, since all the Voronoi cells of Λ_h, at the centres of balls located "outside" and "inside" $B(0, \eta R^{(s)})$, have a circumradius strictly less than 1, it is impossible to add a sphere at any of their vertices to saturate Λ_h, and therefore there is no place in $\mathbb{R}^n$ to add a ball of radius 1/2 to saturate Λ_h. Contradiction.

In the case where the supremum $R^{(s)} = \sup\{R(\Lambda) \mid \Lambda \text{ saturated}\}$ is not reached, let us still assume that $R^{(s)} < 1$ and let us show the contradiction. Then, necessarily [Verger-Gaugry 01], there exist a sequence of points $(z_i)_{i \geq 1}$ and a sequence of Delone sets $(\Lambda_i)_{i \geq 1}$ such that: $\|z_i\|$ tends to $+\infty$ when i goes to infinity with the property that, for all $\epsilon > 0$ there exists $i_0(\epsilon)$ such that $i \geq i_0(\epsilon)$ implies $R^{(s)} - \epsilon \leq \inf_{\lambda \in \Lambda_i} \|z_i - \lambda\| \leq R^{(s)}$. Let R_i be the Delone constant of Λ_i. We now take ϵ small enough in order to have $1/R^{(s)} > 1/(1 - \epsilon/R^{(s)})$. It corresponds to values of i large enough. Then, as above, we will consider a new Delone set $\Lambda_{h,i}$ created from Λ_i by a local dilation of scalar factor h about the point z_i. When $1/R^{(s)} > h > 1/(1 - \epsilon/R^{(s)})$ then $hR_i \leq hR^{(s)} < 1$ and $hR_i \geq h(R^{(s)} - \epsilon) > (R^{(s)} - \epsilon)/(1 - \epsilon/R^{(s)}) = R^{(s)}$. As above, we obtain a Delone set $\Lambda_{h,i}$ which is such that its Delone constant is strictly greater than $R^{(s)}$ and strictly smaller than 1, thus not saturated and impossible to saturate. Contradiction.

(ii) Let us show that, if Λ is a Delone set of $\mathbb{R}^n$ of constant R, $n \geq 1$, then $\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}} \leq R$. This inequality comes from an inequality of Blichfeldt (Lemma 1 in [Rogers 64, page 79]; or [Blichfeldt 29]) since the distance from the centre of a Voronoi cell to any point of its $(n-i)$-dimensional plane, in the Voronoi decomposition of space by Λ, is at least $\frac{1}{2}\sqrt{\frac{2i}{i+1}}$ for all $1 \leq i \leq n$. Taking $i = n$ in the above inequality gives the result. Note that in the constructions of Rogers, packings of equal ball of radius 1, and not 1/2, are considered; this justifies the factor 1/2 in front of the expression.

□

We will call $\sqrt{2}/2$ the Blichfeldt bound.

If $n = 1$, $X_{R_c} = X_{1/2}$ is not empty since it contains $\mathbb{Z}$. If $n = 2$, the set $X_{R_c} = X_{\frac{1}{\sqrt{3}}}$ is not empty since it contains the lattice generated by the points with coordinates $(1, 0)$ and $(1/2, \sqrt{3}/2)$ in the plane (extreme lattice) in an orthonormal basis [Kerschner 39]. What happens for $n \geq 3$? The set $X_{\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}}$ is certainly empty since, as soon as $n \geq 3$, the minimal Voronoi cell is not tiling the ambient space periodically [Rogers 64]. (McLaughlin's Theorem is cited in [Hales 00, Oesterlé 99, Verger-Gaugry 01, Hales 97a], and for $n = 3$ [Hales 97b].)

Question 3.4. For which values of n and R is X_R not empty?

This fairly old question (see [Ryshkov 75]) is partially answered by Theorem 1.2.

4. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1: Let $R_c \leq R$ and $T > R$ be a real number. If Λ is a Delone set of constant R of $\mathbb{R}^n$, then $(B(0, R) + \Lambda) \cap B(0, T)$ covers the ball $B(0, T - R)$.

Hence, the number of elements of $\Lambda \cap B(0,T)$ is at least $((T-R)/R)^n$. On the other hand, since all the balls of radius 1/2 centred at the elements of $\Lambda \cap B(0,T)$ lie within $B(0,T+1/2)$, the proportion of space they occupy in $B(0,T+1/2)$ is at least

$$\left(\frac{T-R}{R}\right)^n \frac{\mathrm{vol}(B(0,1/2))}{\mathrm{vol}(B(0,T+1/2))} = \left(\frac{T-R}{2R(T+1/2)}\right)^n.$$

When T tends to infinity the above quantity tends to $(2R)^{-n}$ which is a lower bound of the density $\delta(\mathcal{B}(\Lambda))$.

□

Proof of Theorem 1.2: Let $\sigma_{KL}(n) = 2^{-0.599n}$ be the upper bound of Kabatjanskiĭ-Levenštein of the packing density δ. By Theorem 1.1 we deduce that, with $R_c \leq R \leq 1$,

$$\mu_n(R) \leq \delta \leq 2^{-0.599n}.$$

Raising this equation to the power $1/n$ gives readily $2R \geq 2^{0.599} + o(1)$ that is $R \geq 2^{-0.401} + o(1)$. □

5. ASYMPTOTIC BEHAVIOUR OF HOLES IN SEQUENCES OF LATTICES AND PACKINGS

The expression of the bound $\mu_n(R)$ will be used to compute a lower bound of the Delone constant of a Delone set, or a lower bound of the covering radius of a given lattice $L \in \mathcal{UD} \cap \mathcal{L}$, when its density and its minimal interpoint distance are known.

In the case of a lattice L, the minimal interpoint distance of L is the square root of the norm $N(L)$ of the lattice [Martinet 96]. We will consider the normalized lattice

$$\frac{1}{\sqrt{N(L)}} L$$

instead of the lattice L to apply the preceding considerations with packings of spheres of common radius 1/2. The situation is similar for a Delone set which will be normalized by its minimal interpoint distance. We will denote by $\mathrm{dens}(L) := \delta(\mathcal{B}(L/\sqrt{N(L)}))$ (Theorem 1.7 in [Rogers 64]) the density of the system of spheres $L + B(0, \sqrt{N(L)}/2)$ if L is a lattice and by $\mathrm{dens}(\Lambda) := \delta(\mathcal{B}(\Lambda/n(\Lambda)))$ (Theorem 1.7 in [Rogers 64]) the density of the system of spheres $\Lambda + B(0, n(\Lambda)/2)$ if Λ is a Delone set of minimal interpoint distance $n(\Lambda)$.

Let us observe that, for all Delone sets Λ and all nonnegative scalar factors λ such that $\Lambda \in \mathcal{UD}$ and $\lambda\Lambda \in \mathcal{UD}$, the equality $R(\lambda\Lambda) = \lambda R(\Lambda)$ holds. Then, from Theorem 1.1, we readily obtain the following inequalities:

(i) $\frac{n(\Lambda)}{2}\,\mathrm{dens}(\Lambda)^{-1/n} \leq R(\Lambda)$, for all Delone sets $\Lambda \in \mathcal{UD}$ of minimal interpoint distance $n(\Lambda)$, and

(ii) $\frac{\sqrt{N(L)}}{2}\,\mathrm{dens}(L)^{-1/n} \leq R(L)$, for all lattice $L \in \mathcal{UD} \cap \mathcal{L}$ of norm $N(L)$.

In the sequel the following notations will be used: $t_L := \sqrt{N(L)}\,\tilde{t_L}$ with $\tilde{t_L} := \frac{1}{2}\,\mathrm{dens}(L)^{-1/n}$; and $t_\Lambda := n(\Lambda)\,\tilde{t_\Lambda}$ with $\tilde{t_\Lambda} := \frac{1}{2}\,\mathrm{dens}(\Lambda)^{-1/n}$ for L and Λ as above.

Let us now apply these inequalities to some known sequences of lattices and packings, as given by [Conway and Sloane 88, Chapters 5 and 8] and [Martinet 96, Chapter V], to obtain an estimation of the size of the deep holes.

5.1 Leech Lattice

For the Leech lattice Λ_{24} in $\mathbb{R}^{24}$ the density $\delta(\Lambda_{24}) = \pi^{12}/479001600 = 0.001930...$ and the covering radius $R(\Lambda_{24}/\sqrt{N(\Lambda_{24})}) = \sqrt{2}/2$ are both known [Conway and Sloane 88, Elkies 00a]. We obtain $t_{\Lambda_{24}} = 0.6487....$ This numerical value is within 10% of the true value 0.707.... This estimation of the size of the deep hole in Λ_{24} is fairly realistic.

5.2 Barnes-Wall Lattices

The density of the Barnes-Wall lattice BW_n ([Leech 64], [Conway and Sloane 88, page 234 or page 151]), in $\mathbb{R}^n$, $n = 2^m$, $m \geq 2$, is equal to $2^{-5n/4} n^{n/4} \pi^{n/2}/\Gamma(1+n/2)$. The norm $N(BW_n)$ is equal to n ([Leech 64, page 678]).

Proposition 5.1. *Let $n = 2^m$ with $m \geq 2$. The covering radius $R(BW_n) \geq t_{BW_n}$ of the Barnes-Wall lattice BW_n is such that the size of its (deepest) hole tends to infinity as (and better than)*

$$t_{BW_n} := \frac{2^{-1/4}}{\sqrt{\pi e}} n^{3/4}\,(1+o(1))$$

when n goes to infinity.

Proof: Raising the equation

$$2^{-5n/4} n^{n/4} \pi^{n/2}/\Gamma(1+n/2) = \delta(\mathcal{B}(BW_n/\sqrt{n})) = \mu_n(t)$$

to the power $1/n$ and allowing n to tend to infinity leads easily to the claimed asymptotic expression of $t_{BW_n/\sqrt{n}}$ as a function of n. The multiplication of $t_{BW_n/\sqrt{n}} = \tilde{t_{BW_n}}$ by the minimal interpoint distance $\sqrt{n}$ gives the claimed lower bound t_{BW_n} of the covering radius $R(BW_n)$ of BW_n. □

5.3 BCH Packings

In this section, the reference will be [Conway and Sloane 88, page 155]. Let $n = 2^m, m \geq 4$. The packings of equal spheres considered below are obtained using extended BCH codes in construction C of length n. They are not lattices. There are two packings (a and b) which use two different codes of the Hamming distances. Let us denote the second one by P_{nb}. Its density $\text{dens}(P_{nb})$ satisfies

$$\log_2 \text{dens}(P_{nb}) \simeq -\frac{1}{2} n \log_2 \log_2 n, \quad \text{as } n \to +\infty$$

and its minimal interpoint distance is [Conway and Sloane 88, page 150] $n(P_{nb}) = \sqrt{\gamma}\, 2^a$ with $\gamma = 2$ and $a = [(m-1)/2]$. We deduce the following proposition.

Proposition 5.2. *Let $n = 2^m$ with $m \geq 4$. The Delone constant $R(P_{nb}) \geq t_{P_{nb}}$ of the BCH packing P_{nb} tends to infinity as (and better than)*

$$t_{P_{nb}} = 2^{-\frac{1}{2}+[(-1+\log_2 n)/2]} \sqrt{\log_2 n}\,(1+o(1))$$
$$\simeq \frac{1}{\sqrt{2}} \log_2 n\,(1+o(1))$$

when n goes to infinity.

The proof can be made with the same arguments as in the proof of Proposition 5.1.

5.4 Craig Lattices

These lattices are known to be among the densest ones (see [Martinet 96, pages 163–171], [Conway and Sloane 88, pages 222–224]). The density $\text{dens}(\mathbb{A}_n^{(r)})$ of the Craig lattice $\mathbb{A}_n^{(r)}, n \geq 1, r \geq 1$, in $\mathbb{R}^n$ is at least

$$\frac{(r/2)^{n/2}}{(n+1)^{r-1/2}} \frac{\pi^{n/2}}{\Gamma(1+n/2)},$$

with equality if the norm of the lattice is $2r$. The norm of Craig lattices is not known in general and lower bounds of $N(\mathbb{A}_n^{(r)})$ were obtained by Craig (see [Martinet 96, Bachoc and Batut 92, Craig 78]). The determination of $N(\mathbb{A}_n^{(r)})$ is equivalent to the so-called Tarry-Escott problem in combinatorics and does not seem to be solved yet. However, for some values of n and r this norm is known.

Theorem 5.3. *Let $n \geq 2$.*

(i) [Craig 78] *If $n+1$ is a prime number p and $r < n/2$, then $N(\mathbb{A}_n^{(r)}) \geq 2r$.*

(ii) [Bachoc and Batut 92] *If $n+1$ is a prime number p with r a strict divisor of $n = p-1$, then $N(\mathbb{A}_n^{(r)}) = 2r$.*

Bachoc and Batut [Bachoc and Batut 92] made an exhaustive investigation of Craig lattices for the prime numbers $p \leq 23$. The equality $N(\mathbb{A}_{p-1}^{(r)}) = 2r$ holds for $r = 1$, $r = 2$, $r = 3$ and also for $r = (p+1)/4$ with $p \equiv 3 \bmod 4$. This last case was proved by Elkies (cited in [Gross 90]), from the general theory of Mordell-Weil lattices developed by Elkies and Shioda concerning the groups of rational points of elliptic curves over function fields [Shioda 92]. The equality $N(\mathbb{A}_{p-1}^{(r)}) = 2r$ was also proved to be true for $p \leq 37$ and $r \in [1, \frac{p+1}{4}]$ [Martinet 96, page 169], but wrong for higher values of p.

Using the assertion (ii) in Theorem 5.3 we obtain the following proposition.

Proposition 5.4. *Let $n \geq 2$ such that $n+1$ is a prime number and r a strict divisor of n. Then, the covering radius $R(\mathbb{A}_n^{(r)}) \geq t_{\mathbb{A}_n^{(r)}}$ of the Craig lattice $\mathbb{A}_n^{(r)}$ is such that the size of its (deepest) hole tends to infinity as (and better than)*

$$t_{\mathbb{A}_n^{(r)}} := \frac{1}{\sqrt{2\pi e}} \sqrt{n}(1+o(1))$$

when n goes to infinity.

Let us remark that $t_{\mathbb{A}_n^{(r)}}$ is independent of r when n is large enough.

As shown by Propositions 5.1 and 5.4 the deep holes of the Barnes-Wall and Craig lattices, BW_n and $\mathbb{A}_n^{(r)}$, have sizes which goes to infinity with n (r fixed). In order to allow comparison between them and with Butler's Theorem (Theorem 3.1), we have to consider the normalized lattices

$$\frac{1}{\sqrt{n}} BW_n \text{ and } \frac{1}{\sqrt{2r}} \mathbb{A}_n^{(r)},$$

assuming that n is such that $n+1$ is a prime number. In the first case, the covering radius tends to infinity with n leaving no hope to obtain very dense packings of spheres from the lattices BW_n when n is large enough. In the second case, since

$$t_{\mathbb{A}_n^{(r)}/\sqrt{2r}} = \frac{1}{2\sqrt{\pi e}} \sqrt{\frac{n}{r}}$$

we see that $t_{\mathbb{A}_n^{(r)}/\sqrt{2r}} > 1$ if $r < \frac{1}{4\pi e} n$. Let us recall, from Theorem 3.1, that the existence of very dense lattices (of minimal interpoint distance one) of covering radius as close as 1 is expected. Therefore we can expect to find very dense Craig lattices satisfying this condition when $r = r(n)$ is a suitable function of n and large enough, namely: $r(n) > \frac{1}{4\pi e} n$ for which the lower bound $t_{\mathbb{A}_n^{(r)}/\sqrt{2r}}$ of $R(\mathbb{A}_n^{(r)}/\sqrt{2r})$ is then less than unity. On the other hand, the density $\text{dens}(\mathbb{A}_n^{(r)})$ reaches its maximum when r is the integer the closest to $\frac{n}{2\ln(n+1)}$ (obtained by cancelling the derivative of $\text{dens}(\mathbb{A}_n^{(r)})$ with respect to r, with n fixed, assuming that the norm of the lattice $\mathbb{A}_n^{(r)}$ is exactly $2r$).

Since $\frac{n}{2\ln(n+1)} \leq \frac{1}{4\pi e}\, n$, as soon as n is large enough (for $n \geq e^{2\pi e} - 1$), a good compromise for the value of r, assuming that the norm of the lattice $\mathbb{A}_n^{(r)}$ is exactly $2r$, would be $r :=$ the smallest integer $> \frac{1}{4\pi e}\, n$.

Question 5.5. Do there exist normalized Craig lattices

$$\mathbb{A}_n^{(r)} / \sqrt{N(\mathbb{A}_n^{(r)})}$$

(for general n and r) which exhibit a Delone constant (covering radius) smaller than 1?

5.5 Mordell-Weil Lattices

We will refer here to the class of Mordell-Weil lattices given by the following theorem of Shioda [Shioda 91, Theorem 1.1].

Theorem 5.6. [Shioda 91] *Let p be a prime number such that $p + 1 \equiv 0 (\text{mod } 6)$ and k any field containing $\mathbb{F}_{p^2}$. The Mordell-Weil lattice $E(K)$ of the elliptic curve E*

$$y^2 = x^3 + 1 + u^{p+1} \tag{5–1}$$

defined over the rational function field K, where $K = k(u)$, is a positive-definite even integral lattice with the following invariants:

$$\begin{aligned} \text{rank} &= 2p - 2 \\ \det &= p^{\frac{p-5}{3}} \\ N(E(K)) &= \tfrac{p+1}{3} \\ \text{centre density } \Delta &= \frac{\left(\frac{p+1}{12}\right)^{p-1}}{p^{(p-5)/6}} \\ \text{kissing number} &\geq 6p(p-1). \end{aligned}$$

Recall that the centre density Δ is the quotient of the density of the lattice divided by the volume $\pi^{n/2}/\Gamma(1 + n/2)$ of the unit ball of $\mathbb{R}^n$. Such a lattice in $\mathbb{R}^{2p-2}$, denoted by MW_n with $n = 2p - 2$, has a minimal interpoint distance equal to $\sqrt{(p + 1)/3}$ and a density equal to $\text{dens}(MW_n) = \Delta \frac{\pi^{p-1}}{\Gamma(p)}$. We deduce that

$$\tilde{t}_{MW_n} \simeq \frac{1}{2} \frac{\sqrt{\pi}}{(\Gamma(p))^{1/(2p-2)}} \frac{\left(\frac{p+1}{12}\right)^{1/2}}{p^{(p-5)/(12(p-1))}}$$

$$\simeq \frac{\sqrt{\pi\, e}}{4\sqrt{3}}\, p^{-1/12} \simeq 2^{-2+1/12} \frac{\sqrt{\pi\, e}}{\sqrt{3}}\, n^{-1/12}.$$

This value goes to zero while

$$t_{MW_n} \simeq 2^{1/12} \frac{\sqrt{\pi\, e}}{12\sqrt{2}}\, n^{5/12}$$

goes to infinity when p (or n) tends to infinity. This result indicates that the deep holes of the normalized Mordell-Weil lattice $MW_n/\sqrt{N(MW_n)}$ are in fact very shallow, and probably may be bounded above independently of n. This leads to the following question.

Question 5.7. Do there exist normalized Mordell-Weil lattices $MW_n/\sqrt{N(MW_n)}$ which exhibit a Delone constant (covering radius) smaller than 1?

6. COMMENTS AND CONJECTURE

The lower bound $\mu_n(R)$ of δ is particularly interesting for saturated Delone sets of Delone constant R of $\mathbb{R}^n$, that is for $R \leq R^{(s)}$. Since $R^{(s)} = 1$ by Lemma 3.3, we readily obtain a lower bound for δ which is 2^{-n} [Elkies 00a, Elkies 00b]. More generally, the lower bound $\mu_n(R)$ exhibits a dependence with n which is in

$$(2R)^{-n} = 2^{-n(1+\log_2 R)}.$$

Taking $R = R^{(s)} = 1$, gives a 2^{-n} dependence typical of the Minkowski-Hlawka type lower bounds of δ_L, while taking $R = \sqrt{2}/2$ (the Blichfeldt bound, Lemma 3.3) provides a $2^{-n/2}$ dependence typical of the Rogers bound σ_n. In between, all values of R are formally possible but the range is limited (Theorem 1.2).

Here the viewpoint does not include explicit constructions. Working with packings of spheres arising from Delone sets for which we only control the constant R would seem, a priori, to give more freedom to the constructions. Very dense packings are likely to occur with 'almost-touching' spheres everywhere, that is from Delone sets of Delone constants R, as small as possible, close to $R_c(n)$. The corresponding kissing numbers deduced from all the local clusters of spheres would lie between the Coxeter-Böröczky/ Kabatjanskiĭ-Levenštein upper bounds [Böröczky 78, Kabatjanskiĭ and Levenštein 78] and the lower bound of Wyner [Wyner 65], probably closer to the upper bounds. Local arrangements of spheres in a densest sphere packing can be extremely diversified (see [Hales 00, Hales 97a, Hales 97b] on Hales-Ferguson Theorem, for $n = 3$).

In this sense, Theorem 1.1 gives a partial answer to old expectations when R lies between $R_c(n)$ and 1. Indeed, recall [Gruber and Lekkerkerker 87, page 391]: "the best known upper and lower bounds for δ differ by

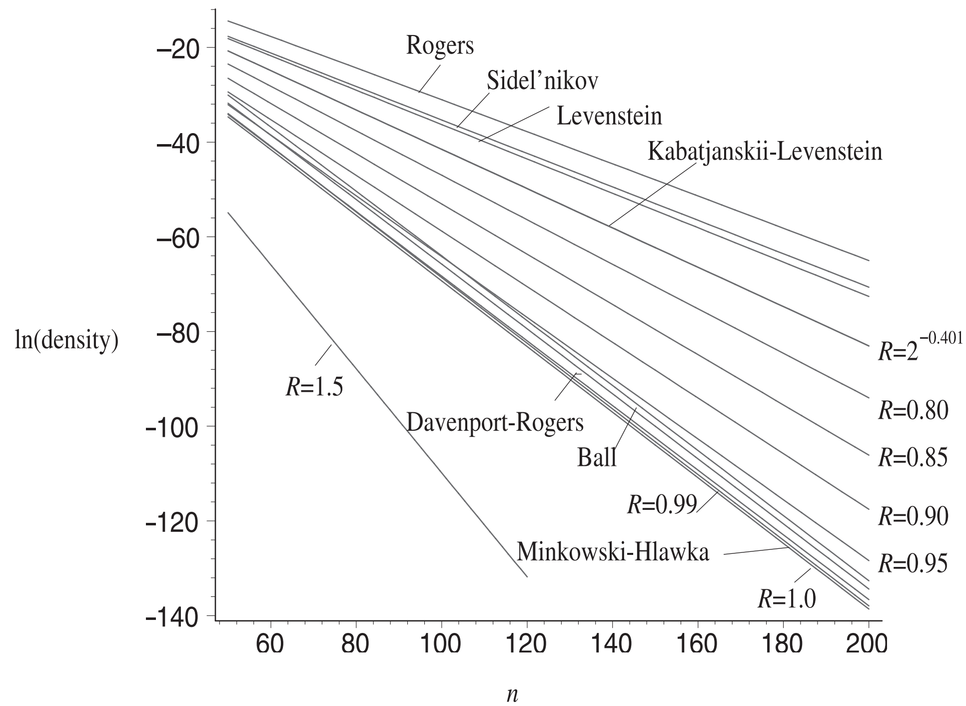

FIGURE 1. Upper bounds of the packing density δ and lower bounds of the lattice-packing density δ_L. The R-dependent lower bounds $\mu_n(R)$ are plotted for $R = 2^{-0.401}, 0.8, 0.85, 0.90, 0.95, 0.99, 1.5$ as a function of the dimension n.

Type	Name	$\log_2 \Delta$
constructions	Barnes-Wall BW_{65536}	180224
	B_{65536}	290998
	$\eta(\overline{\Lambda_{32}})$	295120
	Craig $\mathbb{A}_{65536}^{(2954)}$	297740
(existence) lower bounds of δ_L	Minkowski-Hlawka	324603
	Davenport-Rogers	324616
	Ball	324620
$\mu_{65536}(R)$	$R = 1.5$	286266
	$R = 1.0$	324602
lower bounds	$R = 0.99$	325553
from Theorem 1.1	$R = 0.95$	329452
	$R = 0.90$	334564
	$R = 0.85$	339968
	$R = 0.80$	345700
	$R = 2^{-0.401}$	350882
upper bounds	Kabatjanskiĭ-Levenštein	350882
	Levenštein	355818
of δ	Sidel'nikov	356742
	Rogers	357385

TABLE 3. Table 1.4 of [Conway and Sloane 88, Chapter 1] to which we have added the lower bounds $\mu_{65536}(R)$ for different values of R (the values of the centre density $\log_2 \Delta$ are recomputed from the original references).

a factor which is approximately $2^{n/2}$. This means that the problem of closest packing of spheres is still far from its solution (except for low values of n)." Also recall [Rogers 64, page 9]: we were still, up till now, in the situation where "There remains a wide gap between the results of the Minkowski-Hlawka type, ..., and the results of Blichfeldt type, ..."

In Figure 1 we plot the R-dependent bound $\mu_n(R)$ for several values of R, the upper bounds of Rogers, Sidel'nikov, Levenštein, Kabatjanskiĭ-Levenštein; the lower bounds of Davenport-Rogers, Ball, and of Minkowski-Hlawka, as a function of the dimension n. All values between these two types of bounds can be reached by $\mu_n(R)$ when R is suitably chosen below 1.

The curve $n \to \mu_n(R)$ for $R = 1$ is slightly below the Minkowski-Hlawka bound. When R is greater than 1, the curves $n \to \mu_n(R)$ are entirely below the Minkowski-Hlawka bound. On the contrary, when $R < 1$ is close to unity, the curve $\mu_n(R)$ lies below the Minkowski-Hlawka bound up till a certain value of n and then, as expected, dominates it asymptotically. When $2^{-0.401} < R < 1$ lies far enough from 1 the entire curve $n \to \mu_n(R)$ lies strictly between the two types of bounds (Kabatjanskiĭ-Levenštein and Minkowki-Hlawka).

Theorem 1.2 does not say anything about the frequency and the density of such middle-sized Voronoi cells of circumradius R approximately equal to $2^{-0.401}$ in a general saturated Delone set of $\mathbb{R}^n$ of constant R when n is sufficiently large, in particular in the densest ones.

To allow comparison with known results in literature and to follow Conway and Sloane [Conway and Sloane 88] we have taken n fairly large, namely $n = 65536$. To appreciate the pertinency of the formula given by Theorem 1.1 we have reproduced in Table 3 the Table 1.4 of [Conway and Sloane 88, Chapter 1] and added therein the values of the centre density Δ deduced from $\mu_{65536}(R)$ for $R = 2^{-0.401}, 0.8, 0.85, 0.90, 0.95, 0.99, 1.0, 1.5$. The value of (the logarithm in base 2 of) the centre density Δ computed from $\mu_{65536}(R)$ now sticks to the Kabatjanskiĭ-Levenštein's bound when R is at its asymptotic maximum $R = 2^{-0.401}$. Is this value reached by the Delone constant of a Delone set?

When n is large enough, the sensitivity of $\mu_n(R)$ to the Delone constant R can be perceived by the following comparison (see Table 3): the centre density 324602 relative to the bound $\mu_{65536}(1)$ is slightly below the lower bound 324603 of Minkowski-Hlawka, as expected, whereas the centre density 325553 relative to $\mu_{65536}(0.99)$ is slightly above the best lower bound 324620 of Ball. This gives credit to the conjecture (see Question 3.2 for a precise formulation) that lattices do not exhibit a covering radius less than 1 when n is sufficiently large.

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