Special Values of the Standard Zeta Functions for Elliptic Modular Forms

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CONTENTS

We give an algorithm for computing the special values of twisted standard zeta functions of elliptic modular forms by using the pullback formula for the Siegel-Eisenstein series of degree 2.

1. INTRODUCTION

Let M and k be positive integers and ϕ a Dirichlet character modulo M. For a normalized cuspidal Hecke eigenform f of weight k and Nebentypus ϕ with respect to $\Gamma_0(M)$, and a Dirichlet character χ modulo N, let $L(f, s, \chi)$ be the standard zeta function of f twisted by χ. (For the precise definition of the standard zeta function, see the paragraph immediately preceding Theorem 3.3.) The twisted standard zeta function of an elliptic modular form is sometimes called a twisted symmetric-square L function, an important subject in number theory, and is related to many other areas, especially to Galois representations. For examples, see [Doi et al. 98] and [Dummigan 01]. The special values of the standard zeta function are particularly important. To be more precise, assume that k is even, and set

$$L^*(f, m, \chi) = \frac{L(f, m, \chi)}{\pi^{k+2m} \langle f, f \rangle}$$

for a positive integer $m \leq k - 1$ such that $(-1)^{m-1} = \chi(-1)$, where $\langle -,- \rangle$ is the normalized Petersson product.

As is well known, these values are algebraic numbers and their qualitative natures have been fully investigated by many people (see [Sturm 80, Shimura 00, Böcherer and Schmidt 00]). To investigate various problems related to these values, it is important to compute these values exactly. Several people have considered algorithms for computing these values and have carried out the computations. Sturm [Sturm 80] gave an algorithm for computing these values for a general χ. However, it seems difficult to give exact values by direct use of his method. Zagier [Zagier 77] gave an explicit formula expressing $L^*(f, m, \chi)$

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in the case where M is a square-free positive integer congruent to 1 modulo 4, ϕ is the Kronecker character $(\frac{M}{*})$ corresponding to the extension $\mathbf{Q}(\sqrt{M})/\mathbf{Q}$, and χ is trivial. Stopple [Stopple 96] gave an explicit formula expressing $L^*(f, m, \chi)$ in the case $M = 1$ and χ is a quadratic character of prime conductor $q \equiv 1 \bmod 4$.

In [Katsurada 03], we announced some formulas which seem useful for the computation of $L^*(f, m, \chi)$ in the case where $M = 1$ or a prime number congruent to 1 modulo 4, $\phi = (\frac{M}{*})$, and χ is not necessarily a quadratic character of prime conductor p such that $\chi(-1) = 1$. In this paper, we give a complete proof of these formulas under more general settings. The main tool is the pullback formula of the Siegel-Eisenstein series of degree 2 due to Böcherer and Schmidt [Böcherer and Schmidt 00] and Shimura [Shimura 00]. Such a formula has been used to study the qualitative nature of the special values of the standard zeta function. However, as far as the author knows, no one has used the formula to give its exact values. In this paper, we carry out such a computation.

To explain our method briefly, for simplicity $M \neq p$. Let k and l be even positive integers such that $l \leq k$. Then we define a certain Siegel-Eisenstein series $E^*_{2,l}(Z, Mp^2, \phi\bar{\chi}, s)$ in Section 2. We write $\mathbf{e}(u) = \exp(2\pi\sqrt{-1}u)$ for a complex number u. Then, as is well known, if $l \geq 4$, $E^*_{2,l}(Z, Mp^2, \phi\bar{\chi}, 0)$ becomes a holomorphic modular form of weight l and of Nebentypus $\phi\bar{\chi}$; and has a Fourier expansion of the following form:

$$E^*_{2,l}(Z; Mp^2, \phi\bar{\chi}, 0) = \sum_A c_{n,l}(A, Mp^2, \phi\bar{\chi}, 0)\mathbf{e}(\mathrm{tr}(AZ)),$$

where A runs over all positive definite half-integral matrices of degree 2, and tr(–) denotes the trace of a matrix. Set

$$\begin{aligned}\tilde{c}_{2,l}(A, 0) &= \tilde{c}_{2,l}(A, Mp^2, \phi\bar{\chi}, 0)\\ &= A(l, 0)^{-1}c_{2,l}(A, Mp^2, \phi\bar{\chi}, 0)\end{aligned}$$

with a suitable normalizing factor $A(l, 0)^{-1}$ (see Theorem 2.1). For two positive integers m_1, m_2 set

$$\epsilon(m_1, m_2; l, 0) = \sum_{r^2 \leq 4m_1m_2} \tilde{c}_{2,l}\left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, 0\right) G_l^{k-l}(m_1m_2, r)\chi(r)\tau(\bar{\chi}),$$

where $G_l^{k-l}(u, v)$ is the polynomial introduced by Zagier [Zagier 77], and $\tau(\bar{\chi})$ is the Gauss sum (see Section 3).

Furthermore, set

$$t(m; l, 0) = \epsilon(p, p^2m; l, 0) - \phi(p)p^{k-2}\epsilon(p, m; l, 0),$$

and

$$\mathcal{F}_{p,p}(z) = \sum_{m=1}^{\infty} t(m; l, 0)\mathbf{e}(mz).$$

Then by the holomorphy of the Eisenstein series and the theory of differential operators on modular forms, due to Ibukiyama [Ibukiyama 99], $\mathcal{F}_{p,p}(z)$ belongs to $S_k(\Gamma_0(Mp), \phi)$ (see Sections 3 and 4). Now, take a basis $\{f_i\}_{i=1}^{d_1}$ of $S_k(\Gamma_0(M), \phi)$ consisting of primitive forms, and write

$$f_i(z) = \sum_{m=1}^{\infty} a_i(m)\mathbf{e}(mz)$$

with $a_i(1) = 1$. Then, by the pullback formula due to Böcherer and Schmidt [Böcherer and Schmidt 00], we have

$$\mathcal{F}_{p,p}(z) = \gamma_{k,l,p,M} \sum_{i=1}^{d_1} L^*(f_i, l-1, \chi)\bar{c_i}^2 \tilde{f}_i(z),$$

where $\gamma_{k,l,p,M}$ is a rational number explicitly determined by k, l, p, M; and c_i is a certain algebraic number with absolute norm 1; and

$$\tilde{f}_i(z) = \sum_{m=1}^{\infty} a_i(pm)\mathbf{e}(mz)$$

(see (2) of Theorem 4.2). We restate an explicit form of $\tilde{c}_{2,l}(A, 0)$ (see Theorem 2.1).

Thus, by the above formula combined with the trace formula of Hecke operators, we can compute the norm $N_{K_{f,\chi}}(L^*(f, m, \chi))$ for a primitive form $f \in S_k(\Gamma_0(M), \phi)$ and for an odd integer m such that $3 \leq m \leq k-1$. Here $K_{f,\chi}$ is the field over $\mathbf{Q}$ generated by all the eigenvalues of Hecke operators relative to f and all the values of χ (see Theorem 4.6). If χ^2 is not trivial, $E^*_{2,2}(Z; Mp^2, \phi\bar{\chi}, 0)$ becomes holomorphic, and by the same procedure, we obtain an exact value for $N_{K_{f,\chi}}(L^*(f, 1, \chi))$. On the other hand, if χ^2 is trivial, $E^*_{2,2}(Z; Mp^2, \phi\bar{\chi}, 0)$ is not holomorphic. However, $E^*_{2,2}(Z; Mp^2, \phi\bar{\chi}, -1/2)$ is holomorphic, and by the same procedure, we obtain an exact value of $N_{K_{f,\chi}}(L^*(f, 0, \chi))$; and by the functional equation due to Li [Li 79], we can also compute $N_{K_{f,\chi}}(L^*(f, 1, \chi))$ (see (2) of Proposition 4.7). In the case $M = p$ we obtain similar results (see (1) of Theorem 4.2 and (1) of Theorem 4.6). In Section 5, we give some numerical examples, and discuss some related topics.

As an application of Theorem 4.2, we show that a prime factor of the denominator of $L^*(f, m, \chi)$ gives a congruence between f and another primitive form (see Theorem 4.10).

By using the method in this paper, we expect more fruitful results about the special values of standard zeta functions of other modular forms, for example, of Siegel modular forms and of Hilbert modular forms. We will discuss these topics in subsequent papers.

2. FOURIER COEFFICIENTS OF SIEGEL-EISENSTEIN SERIES

Let $GSp_n^+(\mathbf{R})$ be the group of proper symplectic similitudes of degree n, and $\mathbf{H}_n$ Siegel's upper half space of degree n. As is usual, we write $\gamma(Z) = (AZ+B)(CZ+D)^{-1}$ and $j(\gamma, Z) = \det(CZ + D)$ for

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n^+(\mathbf{R}).$$

We write $f|_k\gamma(z) = (\det\gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma(z))$ for $\gamma \in GSp_n^+(\mathbf{R})$ and a C^∞-function f on $\mathbf{H}_n$. We simply write $f|\gamma$ for $f|_k\gamma$, if there is no confusion. Let $Sp_n(\mathbf{Z})$ be the Siegel modular group of degree n. For a positive integer M, we denote by $\Gamma_0^{(n)}(M)$ (respectively ${\Gamma'_0}^{(n)}(M)$) the subgroup of $Sp_n(\mathbf{Z})$ consisting of matrices whose lower left $n \times n$ block (respectively upper right $n \times n$ block) is congruent to O modulo M.

For a Dirichlet character ϕ modulo M, we denote by $\tilde{\phi}$ (respectively $\tilde{\phi}'$) the character of $\Gamma_0^{(n)}(M)$ (respectively ${\Gamma'_0}^{(n)}(M)$) defined by $\tilde{\phi}(\gamma) = \phi(\det D)$ (respectively $\tilde{\phi}'(\gamma) = \phi(\det A)$) for

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We denote by $\mathbf{1}_M$ the trivial character modulo M and, in particular, set $\mathbf{1} = \mathbf{1}_1$. For a Dirichlet character ϕ modulo M, we denote by $M_k(\Gamma_0^{(n)}(M), \phi)$ (respectively $M_k^\infty(\Gamma_0^{(n)}(M), \phi)$) the space of holomorphic (respectively C^∞-) modular forms of weight k and Nebentypus ϕ with respect to $\Gamma_0^{(n)}(M)$, and by $S_k(\Gamma_0^{(n)}(M), \phi)$ the subspace of $M_k(\Gamma_0^{(n)}(M), \phi)$ consisting of cusp forms. In particular, if $\phi = \mathbf{1}_M$, we write $S_k(\Gamma_0^{(n)}(M))$ for $S_k(\Gamma_0^{(n)}(M), \phi)$. Furthermore, for a subgroup Γ of $Sp_n(\mathbf{Z})$ we denote by Γ_∞ the subgroup of Γ consisting of matrices whose lower left $n \times n$ block is O.

For a function f on $\mathbf{H}_n$ we write $f^c(Z) = \overline{f(-\bar{Z})}$. Let dv denote the invariant volume element on $\mathbf{H}_n$ given by $dv = \det(\mathrm{Im}(Z))^{-n-1} \wedge_{1\le j\le l\le n} (dx_{jl} \wedge dy_{jl})$. Here, for $Z \in \mathbf{H}_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices (x_{jl}) and (y_{jl}). For two C^∞-modular forms f and g of weight k and Nebentypus ϕ with respect to $\Gamma_0^{(n)}(M)$, we define the Petersson scalar product $\langle f, g\rangle_{\Gamma_0^{(n)}(M)}$ of f and g by

$$\langle f, g\rangle_{\Gamma_0^{(n)}(M)} = \int_{\Gamma_0^{(n)}(M)\backslash \mathbf{H}_n} f(Z)\overline{g(Z)} \det(\mathrm{Im}(Z))^k dv,$$

provided the integral converges.

Let ϕ be a Dirichlet character modulo L. For a positive integer M such that $L|M$, we denote by ϕ_M the Dirichlet character modulo M induced by ϕ. Let f and g be elements of $M_k^\infty(\Gamma_0^{(n)}(M_1), \phi_{M_1})$ and $M_k^\infty(\Gamma_0^{(n)}(M_2), \phi_{M_2})$, respectively. Let N be any common multiple of M_1 and M_2. Then, f and g belong to $M_k^\infty(\Gamma_0^{(n)}(N), \phi_N)$, and the value $m(\Phi_{\Gamma_0^{(n)}(N)})^{-1}\langle f, g\rangle_{\Gamma_0^{(n)}(N)}$ does not depend on the choice of N, where $\Phi_{\Gamma_0^{(n)}(N)}$ is the fundamental domain for

$$\mathbf{H}_n \text{ modulo } \Gamma_0^{(n)}(N),$$

and $m(\Phi_{\Gamma_0^{(n)}(N)}) = \int_{\Gamma_0^{(n)}(N)\backslash \mathbf{H}_n} dv$. We denote this value by $\langle f, g\rangle$ and call it the normalized Petersson product of f and g.

For a Dirichlet character ψ, we denote by $L(s, \psi)$ the Dirichlet L-function associated with ψ. Let n, l, and M be positive integers. For a Dirichlet character ϕ modulo M such that $\phi(-1) = (-1)^l$, we define the Eisenstein series $E'_{n,l}(Z; M, \phi, s)$ by

$$E'_{n,l}(Z; M, \phi, s) = \det \mathrm{Im}(Z)^s L(l + 2s, \phi) \prod_{i=1}^{[n/2]} L(2l + 4s - 2i, \phi^2) \times \sum_{\gamma \in {\Gamma'_0}^{(n)}(M)_\infty \backslash {\Gamma'_0}^{(n)}(M)} \tilde{\phi}'(\gamma) j(\gamma, Z)^{-l} |j(\gamma, Z)|^{-2s}.$$

We then define $E^*_{n,l}(Z; M, \phi, s)$ by

$$E^*_{n,l}(Z; M, \phi, s) = j(\iota, Z)^{-l} E'_{n,l}(\iota(Z); M, \phi, s),$$

where

$$\iota = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}.$$

Let $\mathcal{H}_n(\mathbf{Z})$ denote the set of half-integral matrices of degree n over $\mathbf{Z}$, and denote by $\mathcal{H}_n(\mathbf{Z})_{>0}$ (respectively $\mathcal{H}_n(\mathbf{Z})_{\ge 0}$) the subset of $\mathcal{H}_n(\mathbf{Z})$ consisting of positive definite (respectively semipositive definite) matrices. Then, it is well known that $E^*_{n,l}(Z; M, \phi, s)$ belongs to $M_l^\infty(\Gamma_0^{(n)}(M), \phi)$ and has a Fourier expansion of the following form:

$$E^*_{n,l}(X + \sqrt{-1}Y; M, \phi, s) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})} c_{n,l}(A, Y, M, \phi, s)\mathbf{e}(\mathrm{tr}(AX)).$$

In particular, if $E^*_{n,l}(Z; M, \phi, s)$ belongs to $M_l(\Gamma_0^{(n)}(M), \phi)$, it has the following Fourier expansion:

$$E^*_{n,l}(Z; M, \phi, s) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})_{\ge 0}} c_{n,l}(A, M, \phi, s)\mathbf{e}(\mathrm{tr}(AZ)).$$

Throughout the rest of this paper, we exclusively consider the case $n = 2$.

Let l be an even positive integer. Let $M > 1$ be an integer, and let ϕ be a Dirichlet character modulo M such that $\phi(-1) = 1$. Then, $E^*_{2,l}(Z; M, \phi, 0)$ belongs to $M_l(\Gamma_0^{(2)}(M), \phi)$ in the case $l \geq 4$. Furthermore, $E^*_{2,2}(Z; M, \phi, 0)$ belongs to $M_2(\Gamma_0^{(2)}(M), \phi)$ if $\phi^2 \neq \mathbf{1}_M$. We remark that $E^*_{2,2}(Z; M, \phi, 0)$ is neither holomorphic nor nearly holomorphic in the sense of [Shimura 00] if $\phi^2 = \mathbf{1}_M$. However, $E^*_{2,l}(Z; M, \phi, -1/2)$ belongs to $M_2(\Gamma_0^{(2)}(M), \phi)$ in this case.

Now, to see the Fourier coefficient of the Eisenstein series, for an element

$$A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z}),$$

set $e = e_A = \mathrm{GCD}(a_{11}, a_{12}, a_{22})$. For an element $A \in \mathcal{H}_2(\mathbf{Z})$ such that rank $A = 1$ and for each prime number p, define a polynomial $F_p(A, X)$ as

$$F_p(A, X) = \sum_{i=0}^{\mathrm{ord}_p(e_A)} (pX)^i,$$

where ord_p denotes the normalized additive valuation on the field of p-adic numbers. For an element

$$A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z})_{>0},$$

write $-4 \det A = \mathfrak{d}_A \mathfrak{f}_A^2$ with $\mathfrak{d}_A$ the fundamental discriminant of $\mathbf{Q}(\sqrt{-\det A})$ and $\mathfrak{f}_A$ a positive integer. Furthermore, let $\chi_A = (\frac{\mathfrak{d}_A}{*})$ be the Kronecker character corresponding to $\mathbf{Q}(\sqrt{-\det A})/\mathbf{Q}$. For a prime number p, define a polynomial $F_p(A, X)$ as

$$F_p(A, X) = \sum_{i=0}^{\mathrm{ord}_p(e_A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_A)-i} (p^3X^2)^j - \chi_A(p)pX \times \sum_{i=0}^{\mathrm{ord}_p(e_A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_A)-i-1} (p^3X^2)^j.$$

For a Dirichlet character ψ modulo L, let m_ψ denote its conductor, and $\psi^{(0)}$ the associated primitive character. Furthermore, let $B_{m,\psi}$ be the mth generalized Bernoulli number associated with ψ, and let $\tau(\psi)$ be the Gauss sum defined by

$$\tau(\psi) = \sum_{X mod L} \psi(X)\mathbf{e}(X/L).$$

Let l be an even positive integer, and $s = 0$ or $-1/2$. Let ϕ be a Dirichlet character such that $\phi(-1) = 1$. Now assume that the triple (l, s, ϕ) satisfies one of the following conditions:

(h–1) $l \geq 4$ and $s = 0$;

(h–2) $l = 2, s = 0$ and ϕ^2 is not trivial;

(h–3) $l = 2$ and $s = -1/2$.

Theorem 2.1. [Katsurada 99, Shimura 00]. *Let $M > 1$ be an integer, and ϕ a Dirichlet character modulo M such that $\phi(-1) = 1$. Let l be an even positive integer, and $s = 0$ or $-1/2$. Assume that the triple (l, s, ϕ) satisfies one of the Conditions (h–1), (h–2), or (h–3). First, assume that (l, s, ϕ) satisfies either Condition (h–1) or (h–2). Then for $A \in \mathcal{H}_2(\mathbf{Z})_{\geq 0}$ set*

$$\tilde{c}_{2,l}(A, 0) = \tilde{c}_{2,l}(A; M, \phi, 0) = \begin{cases} (4 \det A)^{l-3/2} \prod_{p \mid \mathfrak{f}_A} F_p(A, \phi(p)p^{-l}) B_{l-1,\overline{(\phi\chi_A)^{(0)}}} \\ \quad \times \tau((\phi\chi_A)^{(0)})(-\sqrt{-1}) m_{(\phi\chi_A)^{(0)}}^{1-l} \\ \quad \times \prod_{p \mid M} (1 - (\phi\chi_A)^{(0)}(p)p^{1-l}) & A > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Next assume that (l, s, ϕ) satisfies Condition (h–3). Then for $A \in \mathcal{H}_2(\mathbf{Z})_{\geq 0}$ set

$$\tilde{c}_{2,2}(A, -1/2) = \tilde{c}_{2,2}(A; M, \phi, -1/2) = \begin{cases} -\prod_{p \mid \mathfrak{f}_A} F_p(A, \phi(p)p^{-1}) B_{1,(\phi\chi_A)^{(0)}} \\ \quad \times \prod_{p \mid M} (1 - (\phi\chi_A)^{(0)}(p)) & A > 0 \\ -1/2 \prod_{p \mid e_A} F_p(A, \phi(p)p^{-1}) \\ \quad \times \prod_{p \mid M} (1 - (\phi^2)^{(0)}(p)p) B_{2,(\phi^2)^{(0)}} & \text{rank } A = 1 \\ 1/8 \prod_{p \mid M} \{(1 - (\phi^2)^{(0)}(p)p)(1 - (\phi)^{(0)}(p)p)\} \\ \quad \times B_{2,(\phi^2)^{(0)}} B_{2,(\phi)^{(0)}} & A = O \end{cases}$$

Let

$$A(l, s) = \frac{(-1)^{l/2} 2^l \pi^{3l-3/2}}{\Gamma(l)^2 \Gamma(l - 1/2)}$$

for $l \geq 2$ and $s = 0$, and let

$$A(l, s) = \frac{8\pi^{5/2}}{\Gamma(3/2)}$$

for $l = 2$ and $s = -1/2$, where Γ (–) is the Gamma function. Then we have

$$c_{2,l}(A; M, \phi, s) = A(l, s)\tilde{c}_{2,l}(A; M, \phi, s).$$

Remark 2.2. Assume that (l, s, ϕ) satisfies Condition (h–1) or (h–2). Let m be the conductor of ϕ, and write $\mathfrak{f}_A = \mathfrak{f}'_A \tilde{\mathfrak{f}}_A$ with $\mathfrak{f}'_A = \prod_{p \mid m} p^{\mathrm{ord}_p(\mathfrak{f}_A)}$ and $(\tilde{\mathfrak{f}}_A, m) = 1$.

Then, by the functional equation of $F_p(A,X)$ (see [Katsurada 99]), we can rewrite $\tilde{c}_{2,l}(A,0)$ as

$$\begin{aligned}\tilde{c}_{2,l}(A,0) = (|\mathfrak{d}_A|\mathfrak{f}_A'^{\,2})^{l-3/2}\phi(\tilde{\mathfrak{f}}_A)^2 \prod_{p|\tilde{\mathfrak{f}}_A} F_p(A,\bar{\phi}(p)p^{l-3}) \\ \times B_{l-1,\overline{(\phi\chi_A)^{(0)}}}(-\sqrt{-1})\tau((\phi\chi_A)^{(0)})m_{(\phi\chi_A)^{(0)}}^{1-l} \\ \times \prod_{p|M}(1-(\phi\chi_A)^{(0)}(p)p^{1-l}).\end{aligned} \quad (2\text{–}1)$$

Let ϕ be a Dirichlet character modulo M with conductor m such that $\phi(-1)=1$. Set $m'=M/m$. If $|\mathfrak{d}_A|$ is prime to m, we have

$$\tau((\phi\chi_A)^{(0)}) = \sqrt{-1}\phi(\mathfrak{d}_A)\chi_A(m)\tau(\phi)|\mathfrak{d}_A|^{1/2},$$

$$m_{(\phi\chi_A)^{(0)}} = m|\mathfrak{d}_A|,$$

and

$$\prod_{p|M}(1-(\phi\chi_A)^{(0)}(p)p^{1-l}) = \prod_{p|m'}(1-\phi^{(0)}\chi_A(p)p^{1-l}).$$

Thus if $4\det A$ is prime to M, we have

$$\begin{aligned}\tilde{c}_{2,l}(A,0) = \phi(-4\det A)\prod_{p|\mathfrak{f}_A} F_p(A,\bar{\phi}(p)p^{l-3}) \\ \times B_{l-1,\overline{(\phi\chi_A)^{(0)}}}\chi_A(m)\tau(\phi)m^{1-l} \\ \times \prod_{p|m'}(1-(\phi\chi_A)^{(0)}(p)p^{1-l}).\end{aligned} \quad (2\text{–}2)$$

In particular, if M is a square-free odd positive integer dividing m_1m_2 and r is an integer prime to m_1m_2, we have

$$\begin{aligned}\tilde{c}_{2,l}(A,0) = \phi(r)^2\prod_{p|\mathfrak{f}_A} F_p(A,\bar{\phi}(p)p^{l-3}) \\ \times B_{l-1,\overline{(\phi\chi_A)^{(0)}}} \\ \times \tau(\phi)m^{1-l}\prod_{p|m'}(1-(\phi\chi_A)^{(0)}(p)p^{1-l})\end{aligned} \quad (2\text{–}3)$$

for

$$A = \begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}.$$

On the other hand, we have

$$\begin{aligned}\tilde{c}_{2,l}(A,0) = \prod_{p|\mathfrak{f}_A} F_p(A,\phi(p)p^{l-3})(|\mathfrak{d}_A|\mathfrak{f}_A'^{\,2})^{l-3/2} \\ \times B_{l-1,\overline{(\phi\chi_A)^{(0)}}}\, m_{(\phi\chi_A)^{(0)}}^{3/2-l} \\ \times \prod_{p|M}(1-(\phi\chi_A)^{(0)}(p)p^{1-l}),\end{aligned} \quad (2\text{–}4)$$

if $\phi^2 = \mathbf{1}_M$. Thus if $\phi^2=\mathbf{1}_M$ and $|\mathfrak{d}_A|$ is prime to m, we have

$$\begin{aligned}\tilde{c}_{2,l}(A,0) = \prod_{p|\mathfrak{f}_A} F_p(A,\phi(p)p^{l-3})B_{l-1,\overline{(\phi\chi_A)^{(0)}}}m^{3/2-l} \\ \times \prod_{p|m'}(1-(\phi\chi_A)^{(0)}(p)p^{1-l}).\end{aligned} \quad (2\text{–}5)$$

Assume that M is a square-free odd positive integer and that $|\mathfrak{d}_A|$ is prime to m. If ϕ is primitive,

$$\prod_{p|M}(1-(\phi\chi_A)^{(0)}(p)p^{1-l}) = 1.$$

On the other hand, let $\phi = \mathbf{1}_M$. Let

$$A = \begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z})_{>0}$$

with m_1m_2 divided by M and r prime to m_1m_2. Then,

$$\tilde{c}_{2,2}(A;M,\mathbf{1}_M,-1/2) = 0. \quad (2\text{–}6)$$

Furthermore, for any positive integer $l \geq 2$, we have

$$\tilde{c}_{2,l}(A;M,\mathbf{1}_M,0) = -\prod_{p|\mathfrak{f}_A} F_p(A,p^{l-3})B_{1,\chi_A^{(0)}}\prod_{p|M}(1-p^{1-l}), \quad (2\text{–}7)$$

provided (l,s,ϕ) satisfies Condition (h–1) or (h–2). On the other hand, if $\phi^2 = \mathbf{1}_M$ but $\phi \neq \mathbf{1}_M$, we have

$$\tilde{c}_{2,2}(A;M,\phi,-1/2) = \begin{cases} -\frac{1}{12}\prod_{p|e_A} F_p(A,\phi(p)p^{-1})\prod_{p|M}(1-p) & \text{rank } A = 1 \\ \frac{1}{48}\prod_{p|M}\{(1-p)(1-(\phi)^{(0)}(p)p)\}B_{2,(\phi)^{(0)}} & A = O. \end{cases} \quad (2\text{–}8)$$

3. PULLBACK FORMULA

Now we define Böcherer's differential operator. For details, see [Böcherer and Schmidt 00]. First we define the differential operator $\mathcal{D}_\alpha$ on the module $C^\infty(\mathbf{H}_2)$ of C^∞-functions on $\mathbf{H}_2$ by

$$\begin{aligned}\mathcal{D}_\alpha(f) = &-(\alpha-1/2)\partial f/\partial z_{12} \\ &+ z_{12}\left(\partial^2 f/\partial z_{11}\partial z_{22} - \frac{1}{4}\partial^2 f/\partial z_{12}^2\right)\end{aligned}$$

for $f \in C^\infty(\mathbf{H}_2)$ and

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbf{H}_2.$$

For a nonnegative integer ν define the differential operators $\mathcal{D}_\alpha^\nu$ and $\tilde{\mathcal{D}}_\alpha^\nu$ by

$$\mathcal{D}_\alpha^\nu = \mathcal{D}_{\alpha+\nu-1} \ldots \mathcal{D}_\alpha,$$

and

$$\tilde{\mathcal{D}}_\alpha^\nu = \mathcal{D}_\alpha^\nu|_{z_{12}=0}.$$

Furthermore, for $s \in \mathbf{C}$ and $f \in C^\infty(\mathbf{H}_2)$, we define $\tilde{\mathcal{D}}_{\alpha,s}^\nu$ by

$$\tilde{\mathcal{D}}_{\alpha,s}^\nu(f)(z_{11}, z_{22}) = (y_{11}y_{22})^s \tilde{\mathcal{D}}_{\alpha+s}^\nu(\det Y^{-s} f(Z)),$$

where $Z = X + \sqrt{-1}Y \in \mathbf{H}_2$ and

$$Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix}.$$

Let ϕ be a Dirichlet character modulo M. Then, it is well known that $\tilde{\mathcal{D}}_l^\nu$ and $\tilde{\mathcal{D}}_{l,s}^\nu$ map $M_l^\infty(\Gamma_0^{(2)}(M), \phi)$ into $M_{l+\nu}^\infty(\Gamma_0^{(1)}(M), \phi) \otimes M_{l+\nu}^\infty(\Gamma_0^{(1)}(M), \phi)$. Furthermore, $\tilde{\mathcal{D}}_l^\nu$ maps $M_l(\Gamma_0^{(2)}(M), \phi)$ into $M_{l+\nu}(\Gamma_0^{(1)}(M), \phi) \otimes M_{l+\nu}(\Gamma_0^{(1)}(M), \phi)$, and, in particular, if $\nu > 0$, its image is contained in $S_{l+\nu}(\Gamma_0^{(1)}(M), \phi) \otimes S_{l+\nu}(\Gamma_0^{(1)}(M), \phi)$. Clearly these two operators, $\tilde{\mathcal{D}}_l^\nu$ and $\tilde{\mathcal{D}}_{l,s}^\nu$, coincide with each other if $s = 0$. Furthermore, for $F(Z) \in M_l^\infty(\Gamma_0^{(2)}(M), \phi)$ and $g(z_1) \in S_{l+\nu}(\Gamma_0^{(1)}(M), \phi)$ we have the following identity as a function of z_2:

$$\langle \tilde{\mathcal{D}}_l^\nu(F)(-, z_2), g \rangle = d_{l,\nu,s} \langle \tilde{\mathcal{D}}_{l,s}^\nu(F)(-, z_2), g \rangle \qquad (3\text{–}1)$$

if both sides are cusp forms (see [Böcherer and Schmidt 00, (1.30)]).

Here, we take the inner product as a function of z_1 and

$$d_{l,\nu,s} = \prod_{\mu=1}^{\nu} \frac{l-1+\nu-\mu/2}{l+s-1+\nu-\mu/2}.$$

In addition to the above notation, let $N \geq 1$ be a positive integer, and χ a Dirichlet character modulo N. Assume that N^2 divides M. For positive even integers l, k such that $l \leq k$ we define a function $\mathfrak{E}(z_1, z_2) = \mathfrak{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ on $\mathbf{H}_1 \times \mathbf{H}_1$:

$$\mathfrak{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) = \tilde{\mathcal{D}}_{l,s}^{k-l}\left(\sum_{x \in \mathbf{Z}/N\mathbf{Z}} \bar{\chi}(x) \times E_{2,l}^*(-; M, \phi\bar{\chi}, s)|_k R(x/N)\right)\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix},$$

where

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then (see [Böcherer and Schmidt 00]), $\mathfrak{E}(z_1, z_2)$ belongs to $M_k^\infty(\Gamma_0^{(1)}(M), \phi) \otimes M_k^\infty(\Gamma_0^{(1)}(M), \phi)$.

Now to see an explicit form of $\tilde{\mathcal{D}}_l^\nu$, for an even positive integer l and nonnegative integer ν we define a polynomial $G_l^{2\nu}(u, v)$ in u, v by

$$G_l^{2\nu}(u, v) = \sum_{\mu=0}^{\nu} (-1)^\mu \frac{(l+2\nu-\mu-2)!}{(2\nu-2\mu)!\mu!} u^\mu v^{2\nu-2\mu}.$$

This polynomial was introduced by Zagier [Zagier 77].

We define Ibukiyama's differential operator $\mathcal{G}_l^{2\nu}$ on $C^\infty(\mathbf{H}_2)$ by

$$\mathcal{G}_l^{2\nu} = G_l^{2\nu}(\partial^2/\partial z_{11}\partial z_{22}, \partial/\partial z_{12})|_{z_{12}=0}.$$

We note that

$$\mathcal{G}_l^{2\nu}(\mathbf{e}(\mathrm{tr}(AZ)) = G_l^{2\nu}(a_{11}a_{22}, a_{12})(2\pi\sqrt{-1})^{2\nu}\mathbf{e}(a_{11}z_{11} + a_{22}z_{22}) \qquad (3\text{–}2)$$

for

$$A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix}$$

and

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}.$$

It is well known that $\mathcal{G}_l^{2\nu}$ is a constant multiple of $\tilde{\mathcal{D}}_l^{2\nu}$ (see [Ibukiyama 99]). More precisely, by calculating $\mathcal{G}_l^{2\nu}(z_{12}^{2\nu})$ and $\tilde{\mathcal{D}}_l^{2\nu}(z_{12}^{2\nu})$ for

$$Z = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \in \mathbf{H}_2,$$

we have

$$\mathcal{G}_l^{2\nu} = \frac{(l+2\nu-2)!}{\prod_{\mu=1}^{2\nu}(\mu/2)(l-1+2\nu-\mu/2)} \tilde{\mathcal{D}}_l^{2\nu}. \qquad (3\text{–}3)$$

By (3–3), for $F(Z) \in M_l^\infty(\Gamma_0^{(2)}(M), \phi)$ and $g(z_1) \in S_{l+\nu}(\Gamma_0^{(1)}(M), \phi)$, we have

$$\langle \mathcal{G}_l^{2\nu}(F)(-, z_2), g \rangle = e_{l,2\nu,s} \langle \tilde{\mathcal{D}}_{l,s}^\nu(F)(-, z_2), g \rangle \qquad (3\text{–}4)$$

if both sides are cusp forms, where

$$e_{l,2\nu,s} = \frac{(l+2\nu-2)!}{\prod_{\mu=1}^{2\nu}(\mu/2)(l-1+2\nu-s-\mu/2)}.$$

Now for even positive integers l, k such that $l \leq k$, set

$$\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) = (2\pi\sqrt{-1})^{l-k}\mathcal{G}_l^{k-l}\left(\sum_{x \in \mathbf{Z}/N\mathbf{Z}} \bar{\chi}(x) \times E_{2,l}^*(-; M, \phi\bar{\chi}, s)|_k R(x/N)\right)\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$

Then, as is easily seen, $\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ is a constant multiple of $\mathfrak{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ as a function of z_1 and z_2, and therefore, it belongs to $M_k^{\infty}(\Gamma_0(M), \phi) \otimes M_k^{\infty}(\Gamma_0(M), \phi)$. Furthermore, regarding the holomorphy and the cuspidality of $\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$, by a careful examination of the behavior at cusps, we have Proposition 3.1.

Proposition 3.1. *Let k and l be positive even integers such that $l \leq k$, and $s = 0$ or $-1/2$. Let ϕ and χ be Dirichlet characters modulo M and N, respectively, that satisfy the above conditions. Assume that the triple $(l, s, \phi\bar{\chi})$ satisfies one of the Conditions (h–1), (h–2), or (h–3). Then $\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ belongs to $M_k(\Gamma_0(M), \phi) \otimes M_k(\Gamma_0(M), \phi)$. Furthermore, assume $l < k$, or $k \geq 4$ and $N > 1$. Then $\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ belongs to $S_k(\Gamma_0(M), \phi) \otimes S_k(\Gamma_0(M), \phi)$.*

Remark 3.2. We remark that in the case $k \geq 4$ and $N > 1$, $\mathcal{E}_{2,k}(z_1, z_2; k, M, \phi, \chi, s)$ belongs to $S_k(\Gamma_0(M), \phi) \otimes S_k(\Gamma_0(M), \phi)$ even if $\chi = \mathbf{1}_N$. On the contrary, in the case $k = 2$ and $\chi = \mathbf{1}_N$, we easily see that it does not belong to $S_k(\Gamma_0(M), \phi) \otimes S_k(\Gamma_0(M), \phi)$ by observing Fourier coefficients of $E^*_{2,2}(Z; M, \phi, -1/2)$ in Theorem 2.1. At present, we don't know about the cuspidality of $\mathcal{E}_{2,2}(z_1, z_2; l, M, \phi, \chi, s)$ for a general χ that satisfies (h–2) and (h–3).

Now by (3–4), for any $f \in S_k(\Gamma_0(M), \phi)$, we have

$$\langle f, \mathcal{E}_{2,k}(z_1, -\bar{z}_2; l, M, \phi, \chi, s)\rangle = (2\pi\sqrt{-1})^{l-k} e_{l,k-l,s} \langle f, \mathfrak{E}_{2,k}(z_1, -\bar{z}_2; l, M, \phi, \chi, s)\rangle \quad (3\text{–}5)$$

if both sides are cusp forms. Furthermore, by (3–2) and [Böcherer and Schmidt 00, (6.11)], we have

$$\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} c_{2,l}\left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, M, \phi\bar{\chi}, s\right) \times G_l^{k-l}(m_1 m_2, r) \times T(r, \bar{\chi}) \mathbf{e}(m_1 x_1) \mathbf{e}(m_2 x_2)$$

where we write $z_1 = x_1 + \sqrt{-1} y_1$, $z_2 = x_1 + \sqrt{-1} y_2$, and

$$T(r, \phi) = \sum_{x \bmod N} \phi(x) \mathbf{e}(rx/N)$$

for a Dirichlet character ϕ modulo N.

From now on, let $\Gamma_0(N) = \Gamma_0^{(1)}(N)$. Let M and k be positive integers and ϕ a Dirichlet character modulo M such that $\phi(-1) = (-1)^k$. Let

$$f(z) = \sum_{m=1}^{\infty} a(m) \mathbf{e}(mz)$$

be a normalized cuspidal Hecke eigenform of weight k and Nebentypus ϕ with respect to $\Gamma_0(M)$. Then, for a Dirichlet character χ modulo N, we define the standard zeta function $L(f, s, \chi)$ twisted by χ as

$$L(f, s, \chi) = \prod_p \{(1 - \chi(p)\alpha_p\beta_p p^{-s-k+1}) \times (1 - \chi(p)\alpha_p^2 p^{-s-k+1}) \times (1 - \chi(p)\beta_p^2 p^{-s-k+1})\}^{-1},$$

where α_p, β_p are complex numbers such that

$$\alpha_p + \beta_p = a(p), \alpha_p\beta_p = \phi(p)p^{k-1} \quad (3\text{–}6)$$

for each prime number p. Then by [Böcherer and Schmidt 00, Theorem 3.1] and (3–5), we have the following theorem:

Theorem 3.3. *In addition to the notation and the assumptions as above, assume that $M > 1$, N^2 divides M, $\phi^2 = \mathbf{1}_M$, and $\chi(-1) = 1$. Let $f \in S_k(\Gamma_0(M), \phi)$ be a common eigenfunction of all Hecke operators. Furthermore, assume one of the following conditions: (a) $k = l$, (b) $s = 0$, or (c) $E^*_{2,l}(Z; M, \phi\bar{\chi}, s)$ belongs to $M_l(\Gamma_0^{(2)}(M), \phi\bar{\chi})$. Then we have*

$$\langle f, \mathcal{E}_{2,k}(-, -\bar{z}; l, M, \phi, \chi, \bar{s}))\rangle_{\Gamma_0(M)} = \kappa_{l,k}(s) N^{k+l+2s-2} \times M^{1-k/2} L(f|W_M, l + 2s - 1, \chi) f|W_M|T(M/N^2)(z),$$

where

$$\kappa_{l,k}(s) = \frac{(-1)^{l/2}}{2^{-3+2k-l+2s}\pi^{k-l-1}} \times \frac{\Gamma(k + s - 1/2)\Gamma(k + s - 1)}{\Gamma(l + s)\Gamma(l + s - 1/2)} \times \frac{\Gamma(k - 1)}{\prod_{\mu=1}^{k-l}(\mu/2)(k - 1 - s - \mu/2)},$$

$T(M/N^2)$ is the Hecke operator, and

$$W_M = \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}.$$

Remark 3.4. We slightly change the notation in [Böcherer and Schmidt 00]. It is not certain that the assertion of [Katsurada 03, Theorem 3.1] holds in general. Thus we impose some conditions here. This does not affect our main results. There is a minor misprint in [Böcherer and Schmidt 00, Theorem 3.1]. On page 1,339, line 9, "$2^{1+n(n+1)/2-2ns}$" should read "$2^{1-nl+n(n+3)/2-2ns}$," and this correction has been done in [Katsurada 03, Theorem 3.1].

Now, let ϕ be as in Theorem 3.3, and assume that the triple $(l, s, \phi\bar{\chi})$ satisfies one of the Conditions (h–1), (h–2), or (h–3). Then we define a function $\tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ on $\mathbf{H}_1 \times \mathbf{H}_1$ so that

$$\tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{r^2 \leq 4m_1m_2} \tilde{c}_{2,l}\left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, M, \phi\bar{\chi}, s\right) \times G_l^{k-l}(m_1m_2, r)T(r, \bar{\chi})\mathbf{e}(m_1z_1)\mathbf{e}(m_2z_2),$$

where

$$\tilde{c}_{2,l}\left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, M, \phi\bar{\chi}, s\right)$$

is as defined in Theorem 2.1. Then, by Theorem 2.1 we have

$$\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) = A(l, s)\tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \phi, \chi, s).$$

From now on, for a Dirichlet character ψ modulo M_0 we use the same symbol ψ to denote the character modulo M induced from ψ if M_0 divides M. For a positive integer r let

$$\delta_r = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix},$$

and let $S_k(\Gamma_0(M), \phi)^{(r)} = \{f|\delta_r; f \in S_k(\Gamma_0(M), \phi)\}$, and let $S_k(\Gamma_0(M), \phi)^{new}$ be the space of new forms in $S_k(\Gamma_0(M), \phi)$. We note that $S_k(\Gamma_0(M), \phi)^{new} = S_k(\Gamma_0(M), \phi)$ if ϕ is a primitive character of conductor M. Furthermore for a primitive form f in $S_k(\Gamma_0(M), \phi)^{new}$ let c_f be the complex number such that $f|W_M = c_f f^c$. Let $\lambda_f(m)$ be the eigenvalue of the Hecke operator $T(m)$ for a positive integer m. For an odd positive integer $m \leq k-1$, let

$$\Lambda(f, m, \chi) = \frac{\Gamma(k-1)\Gamma(k+m-1)\Gamma(m+1)}{\Gamma(k-m)} \times \frac{L(f, m, \chi)}{2^{2k+2m-4}\pi^{k+2m}\langle f, f\rangle},$$

and

$$\Lambda(f, 0, \chi) = \Gamma(k-1)\frac{L(f, 0, \chi)}{2^{2k-3}\pi^k\langle f, f\rangle}.$$

We note that $m(\Phi_{\Gamma_0(N)})) = \frac{\pi}{3}[\Gamma : \Gamma_0(N)]$. Thus by Theorem 3.3 we obtain the following two theorems:

Theorem 3.5. *Let the notation and the assumptions be as before. Let p be a prime number such that $p \equiv 1 \bmod 4$, $\phi = (\frac{p}{*})$, and χ a Dirichlet character modulo p such that $\chi(-1) = 1$.*

(1) Let f be a primitive form in $S_k(\Gamma_0(p^2), \phi)^{new}$. Then,

$$\langle f, \tilde{\mathcal{E}}_{2,k}(-, -\bar{z}; l, p^2, \phi, \chi, s)\rangle = 3[\Gamma : \Gamma_0(p^2)]^{-1}p^{l+2s}\Lambda(f^c, l+2s-1, \chi)\langle f, f\rangle c_f f^c(z).$$

(2) Let f be a primitive form in $S_k(\Gamma_0(p), \phi)$. Then, we have

$$\langle f, \tilde{\mathcal{E}}_{2,k}(-, -\bar{z}; l, p^2, \phi, \chi, s)\rangle = 3[\Gamma : \Gamma_0(p^2)]^{-1}p^{l+2s}\Lambda(f^c|\delta_p, l+2s-1, \chi) \times \langle f|\delta_p, f|\delta_p\rangle c_f f^c|\delta_p(z),$$

and

$$\langle f|\delta_p, \tilde{\mathcal{E}}_{2,k}(-, -\bar{z}; l, p^2, \phi, \chi, s)\rangle = 3[\Gamma : \Gamma_0(p^2)]^{-1}p^{l+2s}\Lambda(f^c, l+2s-1, \chi)\langle f, f\rangle c_f f^c(z).$$

Theorem 3.6. *Let the notation and the assumptions be as before. Let $p_0 = 1$ or a prime number such that $p_0 \equiv 1 \bmod 4$, and $\phi = (\frac{p_0}{*})$. Furthermore, let p be a prime number different from p_0, and χ a Dirichlet character modulo p such that $\chi(-1) = 1$.*

(1) Let f be a primitive form in $S_k(\Gamma_0(p_0p^2), \phi)^{new}$. Then, we have

$$\langle f, \tilde{\mathcal{E}}_{2,k}(-, -\bar{z}; l, p_0p^2, \phi, \chi, s)\rangle = 3[\Gamma : \Gamma_0(p_0p^2)]^{-1}p_0^{1-k/2}p^{l+2s}\Lambda(f^c, l+2s-1, \chi) \times \langle f, f\rangle c_f\overline{\lambda_f(p_0)}f^c(z).$$

(2) Let f be a primitive form in $S_k(\Gamma_0(p_0p), \phi)^{new}$. Then, we have

$$\langle f, \tilde{\mathcal{E}}_{2,k}(-, -\bar{z}; l, p_0p^2, \phi, \chi, s)\rangle = 3[\Gamma : \Gamma_0(p_0p^2)]^{-1}p_0^{1-k/2}p^{l+2s}\Lambda(f^c|\delta_p, l+2s-1, \chi) \times \langle f|\delta_p, f|\delta_p\rangle c_f\overline{\lambda_f(p_0)}f^c|\delta_p(z),$$

and

$$\langle f|\delta_p, \tilde{\mathcal{E}}_{2,k}(-, -\bar{z}; l, p_0p^2, \phi, \chi, s)\rangle = 3[\Gamma : \Gamma_0(p_0p^2)]^{-1}p_0^{1-k/2}p^{l+2s}\Lambda(f^c, l+2s-1, \chi) \times \langle f, f\rangle c_f\overline{\lambda_f(p_0)}f^c(z).$$

(3) Let f be a primitive form in $S_k(\Gamma_0(p_0),\phi)$. Then, we have

$$\langle f, \tilde{\mathcal{E}}_{2,k}(-,-\bar{z};l,p_0p^2,\phi,\chi,s)\rangle = 3[\Gamma:\Gamma_0(p_0p^2)]^{-1}p_0^{1-k/2}p^{l+2s}\Lambda(f^c|\delta_{p^2},l+2s-1,\chi) \times \langle f|\delta_{p^2}, f|\delta_{p^2}\rangle c_f\overline{\lambda_f(p_0)}f^c|\delta_{p^2}(z),$$

$$\langle f|\delta_p, \tilde{\mathcal{E}}_{2,k}(-,-\bar{z};l,p_0p^2,\phi,\chi,s)\rangle = 3[\Gamma:\Gamma_0(p_0p^2)]^{-1}p_0^{1-k/2}p^{l+2s}\Lambda(f^c|\delta_p,l+2s-1,\chi) \times \langle f|\delta_p, f|\delta_p\rangle c_f\overline{\lambda_f(p_0)}f^c|\delta_p(z),$$

and

$$\langle f|\delta_{p^2}, \tilde{\mathcal{E}}_{2,k}(-,-\bar{z};l,p_0p^2,\phi,\chi,s)\rangle = 3[\Gamma:\Gamma_0(p_0p^2)]^{-1}p_0^{1-k/2}p^{l+2s}\Lambda(f^c,l+2s-1,\chi) \times \langle f, f\rangle c_f\overline{\lambda_f(p_0)}f^c(z).$$

Remark 3.7. In Theorems 3.5 and 3.6, we imposed the same restrictions on M, N, and χ as in Theorem 3.3. In the general case, the formula becomes more complicated. However, we can give a similar formula in the case where M and N are square-free, which we will discuss in a subsequent paper.

4. COMPUTATION OF $L(f,l,\chi)$

In this section, we give some formulas to compute $L(f,m,\chi)$ for a primitive form $f \in S_k(\Gamma_0(N),\psi)$ in the following two cases:

(c–1) N is a prime number p such that $p \equiv 1 \bmod 4$, $\psi = (\frac{p}{*})$, and χ is a Dirichlet character modulo p such that $\chi(-1)=1$.

(c–2) N is 1 or a prime number p_0 such that $p_0 \equiv 1 \bmod 4$, $\psi = (\frac{p_0}{*})$, and χ is a Dirichlet character modulo p such that $\chi(-1)=1$, where p is a prime number different from p_0.

In either case, χ is a primitive character modulo p, or $\mathbf{1}_p$. Furthermore, we note that $S_k(\Gamma_0(N),\psi)^{new} = S_k(\Gamma_0(N),\psi)$ in both cases. Let $M=p^2$ or p_0p^2 for Case (c–1) or (c–2), respectively. From now on, let k be an even integer not smaller than 4, and l an even integer such that $2 \le l \le k$. Assume that the triple $(l,s,\psi\bar{\chi})$ satisfies one of the Conditions (h–1), (h–2), or (h-3). For two positive integers m_1, m_2 set

$$\epsilon(m_1,m_2;l,s) = \epsilon(m_1,m_2;l,M,\psi,\chi,s) = \sum_{r^2\le 4m_1m_2} \tilde{c}_{2,l}\left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2\end{pmatrix}, M, \psi\bar{\chi}, s\right) \times G_l^{k-l}(m_1m_2,r)T(r,\bar{\chi}).$$

We note that $T(r,\bar{\chi}) = p-1$ or $\chi(r)\tau(\bar{\chi})$ if respectively, $\chi = \mathbf{1}_p$ and $r \equiv 0 \bmod p$, or not. Furthermore, for each positive integer m_1 set

$$\mathcal{F}_{m_1}(z_2) = \sum_{m_2=1}^{\infty} \epsilon(m_1,m_2;l,s)\mathbf{e}(m_2z_2),$$

and for a prime number p let

$$\mathcal{F}_{m_1,p}(z_2) = \sum_{m_2=1}^{\infty}(\epsilon(m_1,p^2m_2;l,s)-\psi(p)p^{k-2}\epsilon(m_1,m_2;l,s))\mathbf{e}(m_2z_2).$$

We note that

$$\tilde{\mathcal{E}}_{2,k}(z_1,z_2;l,M,\psi,\chi,s) = \sum_{m_1=1}^{\infty}\mathcal{F}_{m_1}(z_2)\mathbf{e}(m_1z_1). \quad (4–1)$$

Take a basis $\{f_i\}_{i=1}^{d_1}$ of $S_k(\Gamma_0(N),\psi)$ consisting of primitive forms. We note that, in this case, $S_k(\Gamma_0(N),\psi) = S_k(\Gamma_0(N),\psi)^{new}$. Let $f_i|W_N = c_if_i^c$ with constant c_i, and write

$$f_i(z) = \sum_{m=1}^{\infty} a_i(m)\mathbf{e}(mz)$$

with $a_i(1)=1$.

First, we have the following lemma:

Lemma 4.1. *Let N be a positive integer and ψ a Dirichlet character modulo N.*

(1) Let f and g be Hecke eigenforms in $S_k(\Gamma_0(N),\psi)$, and let p be a prime number. Then we have

$$\langle f|\delta_p, g\rangle = p^{-k/2}\overline{\lambda_g(p)}\langle f,g\rangle - \alpha(N,p)\bar{\psi}(p)p^{-1}\overline{\langle g|\delta_p, f\rangle},$$

where $\lambda_g(p)$ denotes the eigenvalue of the Hecke operator $T(p)$ with respect to g, and $\alpha(N,p)$ is 0 or 1 according to whether p divides N, or not.

(2) Let $g \in S_k(\Gamma_0(N),\psi)$ be a Hecke eigenform. Let p be a prime number dividing N. Then we have

$$\langle g|\delta_p, g\rangle = p^{-k/2}\overline{\lambda_g(p)}\langle g,g\rangle.$$

(3) Let $f \in S_k(\Gamma_0(N), \psi)$ be a primitive form. Let p be a prime number that does not divide N. Then we have

$$\langle f|\delta_p, f\rangle = \frac{p^{-k/2}\bar{\psi}(p)\lambda_f(p)}{1+p^{-1}}\langle f, f\rangle,$$

and

$$\langle f|\delta_{p^2}, f\rangle = \frac{p^{-k}\bar{\psi}(p)^2\lambda_f(p)^2 - \bar{\psi}(p)p^{-1}(1+p^{-1})}{1+p^{-1}}\langle f, f\rangle.$$

Proof: The assertions follow immediately from [Shimura 76, (2.5)] and [Shimura 76, Lemma 1]. □

Now we compute the value $\Lambda(f, l, \chi)$.

Theorem 4.2. *Let the notation and the assumptions be as before.*

(1) In Case (c–1), for any even positive integer $l \le k$, we have

$$\mathcal{F}_p(z_2) = 3p^{k/2+l+2s-1}(p+1)^{-1} \times \sum_{i=1}^{d_1} \Lambda(f_i, l+2s-1, \chi)\overline{c_i} f_i(z_2).$$

(2) Let $t_{p_0} = p_0 + 1$ or 1 according to whether p_0 is a prime number or 1. Then, in Case (c–2), for any even positive integer $l \le k$, we have

$$\mathcal{F}_{p,p}(z_2) = 3t_{p_0}^{-1} p_0^{1/2} p^{k+l+2s-2} \times \sum_{i=1}^{d_1} \Lambda(f_i, l+2s-1, \chi)\overline{c_i}^2 \tilde{f}_i(z_2),$$

where we write $\tilde{f}(z) = \sum_{m=1}^{\infty} a(pm)\mathbf{e}(mz)$ for a modular form $f(z) = \sum_{m=1}^{\infty} a(m)\mathbf{e}(mz)$.

Proof:

(1) Set $\tilde{\mathcal{E}}(z_1, z_2) = \tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, p^2, \psi, \chi, s)$. Then by Proposition 3.1, $\tilde{\mathcal{E}}(z_1, z_2)$ belongs to $S_k(\Gamma_0(p^2), \psi) \otimes S_k(\Gamma_0(p^2), \psi)$. As is well known,

$$S_k(\Gamma_0(p^2), \psi) = S_k(\Gamma_0(p), \psi) \oplus S_k(\Gamma_0(p), \psi)^{(p)} \perp S_k(\Gamma_0(p^2), \psi)^{new}.$$

Let

$$d_1 = \dim S_k(\Gamma_0(p), \psi)$$

and

$$d_2 = \dim S_k(\Gamma_0(p^2), \psi)^{new}.$$

Take a basis $\{g_i\}_{i=1}^{d_2}$ of $S_k(\Gamma_0(p^2), \psi)^{new}$ consisting of common eigenfunctions of Hecke operators. Then

$$\{f_i \ (i = 1, 2, ..., d_1),\ f_i|\delta_p \ (i = 1, 2, ..., d_1),\ g_i \ (i = 1, 2, ..., d_2)\}$$

forms a basis of $S_k(\Gamma_0(p^2), \psi)$. Let c_i be as above. Then, we have $f_i|W_p = c_i f_i^c$, $f_i|W_{p^2} = c_i f_i^c|\delta_p$. Furthermore, we have $g_i|W_{p^2} = c_i' g_i^c$ with constant c_i'. From this we have $f_i|\delta_p|W_{p^2} = c_i f_i^c$. We note that $\langle g_i, g_j\rangle = 0$ for any $1 \le i \ne j \le d_2$, and $\langle f_i, g_j\rangle = \langle g_j, f_i\rangle = 0$ for any $1 \le i \le d_1$ and $1 \le j \le d_2$. Thus, by Proposition 3.1, we have

$$\begin{aligned}\tilde{\mathcal{E}}(z_1, z_2) = &\sum_{i,j=1}^{d_2} b_{ij} g_i(z_1) g_j(z_2) \\ &+ \sum_{i,j=1}^{d_1} a_{ij}^{(0,0)} f_i(z_1) f_j(z_2) \\ &+ \sum_{i,j=1}^{d_1} a_{ij}^{(0,1)} f_i(z_1) f_j|\delta_p(z_2) \\ &+ \sum_{i,j=1}^{d_1} a_{ij}^{(1,0)} f_i|\delta_p(z_1) f_j(z_2) \\ &+ \sum_{i,j=1}^{d_1} a_{ij}^{(1,1)} f_i|\delta_p(z_1) f_j|\delta_p(z_2).\end{aligned}$$

Now let

$$g_i(z) = \sum_{m=1}^{\infty} b_i(m)\mathbf{e}(mz) \quad (i = 1, 2, ..., d_2)$$

with $b_i(1) = 1$. Then by (1) of Theorem 3.5, we have

$$\begin{aligned}\langle g_i, \tilde{\mathcal{E}}(-, -\overline{z_2})\rangle &= 3p^{-1}(p+1)^{-1}p^{l+2s}\Lambda(g_i^c, l+2s-1, \chi) \times \langle g_i, g_i\rangle c_i' g_i^c(z_2) \\ &= \sum_{j=1}^{d_2} \overline{b_{ij}}\langle g_i, g_i\rangle g_j^c(z_2).\end{aligned}$$

Since $g_1, ..., g_{d_2}$ are orthogonal with each other, we have $b_{ij} = 3p^{-1}(p+1)^{-1}p^{l+2s}\Lambda(g_i, l+2s-1, \chi)\overline{c_i'}$ or 0 according to whether $i = j$, or not. Furthermore, by (2) of Theorem 3.5, we have

$$\begin{aligned}\langle f_i, \tilde{\mathcal{E}}(-,-\overline{z_2})\rangle &= 3p^{-1}(p+1)^{-1} \\ &\qquad \times p^{l+2s}\Lambda(f_i^c|\delta_p, l+2s-1,\chi) \\ &\qquad \times \langle f_i|\delta_p, f_i|\delta_p\rangle c_i f_i^c|\delta_p(z_2) \\ &= \sum_{j=1}^{d_1} \overline{a_{ij}^{(0,0)}}\langle f_i, f_i\rangle f_j^c(z_2) \\ &\quad + \sum_{j=1}^{d_1} \overline{a_{ij}^{(0,1)}}\langle f_i, f_i\rangle f_j^c|\delta_p(z_2) \\ &\quad + \sum_{j=1}^{d_1} \overline{a_{ij}^{(1,0)}}\langle f_i, f_i|\delta_p\rangle f_j^c(z_2) \\ &\quad + \sum_{j=1}^{d_1} \overline{a_{ij}^{(1,1)}}\langle f_i, f_i|\delta_p\rangle f_j^c|\delta_p(z_2).\end{aligned}$$

We note that

$$\Lambda(f_i^c|\delta_p, l+2s-1,\chi) = \overline{\Lambda(f_i, l+2s-1,\chi)},$$

and

$$\begin{aligned}\langle f_i, f_i|\delta_p\rangle &= \lambda_i(p)p^{-k/2}\langle f_i, f_i\rangle, \langle f_i|\delta_p, f_i|\delta_p\rangle \\ &= \langle f_i, f_i\rangle.\end{aligned}$$

Thus for any $1 \le i \le d_1$ we have

$$\begin{aligned}&a_{ij}^{(0,1)} + \overline{\lambda_i(p)}p^{-k/2}a_{ij}^{(1,1)} = \\ &\quad 3p^{-1}(p+1)^{-1}p^{l+2s}\Lambda(f_i, l+2s-1,\chi)\overline{c_i} \text{ or } 0\end{aligned}$$

according to whether $i = j$, or not. Similarly, we have

$$a_{ij}^{(0,0)} + \overline{\lambda_i(p)}p^{-k/2}a_{ij}^{(1,0)} = 0$$

for any $1 \le i,\ j \le d_1$. Similarly, by taking the inner product of $f_i|\delta_p(z_1)$ against $\tilde{\mathcal{E}}(z_1, -\overline{z_2})$, we have

$$\begin{aligned}&\lambda_i(p)p^{-k/2}a_{ij}^{(0,0)} + a_{ij}^{(1,0)} = \\ &\quad 3p^{-1}(p+1)^{-1}p^{l+2s}\Lambda(f_i, l+2s-1,\chi)\overline{c_i} \text{ or } 0\end{aligned}$$

according to whether $i = j$, or not, and

$$\lambda_i(p)p^{-k/2}a_{ij}^{(0,1)} + a_{ij}^{(1,1)} = 0$$

for any $1 \le i, j \le d_1$. Thus we have

$$\begin{aligned}a_{ii}^{(0,0)} &= \frac{-3p^{-1}(p+1)^{-1}p^{l+2s-k/2}\overline{\lambda_i(p)}}{1-p^{-k}|\lambda_i(p)|^2} \\ &\qquad \times \Lambda(f_i, l+2s-1,\chi)\overline{c_i}\end{aligned}$$

and

$$\begin{aligned}a_{ii}^{(1,0)} &= a_{ii}^{(0,1)} = -p^{1-k/2}\lambda_i(p)a_{ii}^{(0,0)}, \\ a_{ii}^{(1,1)} &= p^{1-k}\lambda_i(p)^2 a_{ii}^{(0,0)}\end{aligned}$$

for any $i = 1, ..., d_1$; and

$$a_{ij}^{(0,0)} = a_{ij}^{(1,0)} = a_{ij}^{(0,1)} = a_{ij}^{(1,1)} = 0$$

for any $1 \le i \ne j \le d_1$. Thus we have

$$\begin{aligned}\tilde{\mathcal{E}}(z_1, z_2) &= \sum_{i=1}^{d_2} b_{ii}g_i(z_1)g_i(z_2) \\ &\quad + \sum_{i=1}^{d_1} a_{ii}^{(0,0)}\Big\{f_i(z_1)f_i(z_2) \\ &\qquad - p^{1-k/2}\lambda_i(p)f_i(z_1)f_i|\delta_p(z_2) \\ &\qquad - p^{1-k/2}\lambda_i(p)f_i|\delta_p(z_1)f_i(z_2) \\ &\qquad + p^{1-k}\lambda_i(p)^2 f_i|\delta_p(z_1)f_i|\delta_p(z_2)\Big\}.\end{aligned}$$

We note that $b_i(pm) = 0$ and $a_i(pm) = \lambda_i(p)a_i(m)$. Thus, comparing both sides of (4–1), we have

$$\begin{aligned}\mathcal{F}_p(z_2) &= \sum_{i=1}^{d_1} a_{ii}^{(0,0)}\Big\{\lambda_i(p)f_i(z_2) \\ &\qquad - p^{1-k/2}\lambda_i(p)^2 f_i|\delta_p(z_2) \\ &\qquad - p^{1-k/2}\lambda_i(p)p^{k/2}f_i(z_2) \\ &\qquad + p^{1-k}\lambda_i(p)^2 p^{k/2}f_i|\delta_p(z_2)\Big\} \\ &= \sum_{i=1}^{d_1}(1-p)a_{ii}^{(0,0)}\lambda_i(p)f_i(z_2).\end{aligned}$$

We note that $|\lambda_i(p)|^2 = p^{k-1}$. This proves assertion (1).

(2) Set $\tilde{\mathcal{E}}(z_1, z_2) = \tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, p_0p^2, \psi, \chi, s)$. As is well known,

$$\begin{aligned}&S_k(\Gamma_0(p_0p^2), \psi) = \\ &S_k(\Gamma_0(p_0), \psi) \oplus S_k(\Gamma_0(p_0), \psi)^{(p)} \oplus S_k(\Gamma_0(p_0), \psi)^{(p^2)} \\ &\quad \perp S_k(\Gamma_0(p_0p), \psi)^{new} \oplus S_k(\Gamma_0(p_0p), \psi)^{new\,(p)} \\ &\quad \perp S_k(\Gamma_0(p_0p^2), \psi)^{new}.\end{aligned}$$

Take the bases $\{g_i\}_{i=1}^{d_2}$ of $S_k(\Gamma_0(p_0p), \psi)^{new}$ and $\{h_i\}_{i=1}^{d_3}$ of $S_k(\Gamma_0(p_0p^2), \psi)^{new}$ consisting of primitive forms. Then

$$\begin{aligned}&\{f_i\ (i = 1, 2, ..., d_1),\ f_i|\delta_p\ (i = 1, 2, ..., d_1), \\ &\quad f_i|\delta_{p^2}\ (i = 1, 2, ..., d_1),\ g_i\ (i = 1, 2, ..., d_2), \\ &\quad g_i|\delta_p\ (i = 1, 2, ..., d_2),\ h_i\ (i = 1, 2, ..., d_3)\}\end{aligned}$$

forms a basis of $S_k(\Gamma_0(p_0p^2),\psi)$. Thus, similarly to the proof of (1), by using Proposition 3.1, we have

$$\begin{aligned}\tilde{\mathcal{E}}(z_1,z_2) &= \sum_{i,j=1}^{d_3} c_{ij}^{(0,0)} h_i(z_1)h_j(z_2)\\ &+ \sum_{\alpha,\beta=0}^{1}\sum_{i,j=1}^{d_2} b_{ij}^{(\alpha,\beta)} g_i|\delta_{p^\alpha}(z_1)g_j|\delta_{p^\beta}(z_2)\\ &+ \sum_{\alpha,\beta=0}^{2}\sum_{i,j=1}^{d_1} a_{ij}^{(\alpha,\beta)} f_i|\delta_{p^\alpha}(z_1)f_j|\delta_{p^\beta}(z_2)\end{aligned}$$

with $c_{ij}^{(0,0)}, b_{ij}^{(\alpha,\beta)}, c_{ij}^{(\alpha,\beta)} \in \mathbf{C}$. Now let $f_i|W_{p_0} = c_i f_i^c$, $g_i|W_{p_0p} = c_i' g_i^c$, and $h_i|W_{p_0p^2} = c_i'' h_i^c$ with constant c_i''. Then, we have $f_i|W_{p_0p^2} = c_i f_i^c|\delta_{p^2}$, $f_i|\delta_p|W_{p_0p^2} = c_i f_i^c|\delta_p$, and $g_i|\delta_p|W_{p_0p^2} = c_i' g_i^c$. For a positive integer write $\lambda_i(m) = \lambda_{f_i}(m)$, $\lambda_i(m)' = \lambda_{g_i}(m)$, and $\lambda_i(m)'' = \lambda_{h_i}(m)$. We have $\overline{\lambda_i(p)} = \psi(p)\lambda_i(p)$. Then by direct computation combined with Theorem 3.6 and Lemma 4.1 we have

$$\begin{aligned}\tilde{\mathcal{E}}(z_1,z_2) &= \sum_{i}^{d_3} c_{ii}h_i(z_1)h_i(z_2)\\ &+ \sum_{i=1}^{d_2} b_{ii}\Big\{-p^{-k/2}\lambda_i(p)'g_i(z_1)g_i(z_2)\\ &\quad + g_i(z_1)g_i|\delta_p(z_2) + g_i|\delta_p(z_1)g_i(z_2)\\ &\quad - p^{-k/2}\lambda_i(p)'g_i|\delta_p(z_1)g_i|\delta_p(z_2)\Big\}\\ &+ \sum_{i=1}^{d_1} a_{ii}\Big\{p^{-1}f_i|\delta_{p^2}(z_1)f_i|\delta_{p^2}(z_2)\\ &\quad - p^{-k/2}\lambda_i(p)f_i|\delta_p(z_1)f_i|\delta_{p^2}(z_2)\\ &\quad + f_i(z_1)f_i|\delta_{p^2}(z_2)\\ &\quad - p^{-k/2}\lambda_i(p)f_i|\delta_{p^2}(z_1)f_i|\delta_p(z_2)\\ &\quad + (1+\psi(p)\lambda_i(p)^2p^{-k} - p^{-2})f_i|\delta_p(z_1)\\ &\qquad \times f_i|\delta_p(z_2)\\ &\quad - \psi(p)\lambda_i(p)p^{-k/2}f_i(z_1)f_i|\delta_p(z_2)\\ &\quad + f_i|\delta_{p^2}(z_1)f_i(z_2)\\ &\quad - \psi(p)p^{-k/2}\lambda_i(p)f_i|\delta_p(z_1)f_i(z_2)\\ &\quad + \psi(p)p^{-1}f_i(z_1)f_i(z_2)\Big\},\end{aligned}$$

where

$$\begin{aligned}c_{ii} &= 3p^{-1}(p+1)^{-1}t_{p_0}^{-1}p^{l+2s}p_0^{1-k/2}\\ &\quad \times \Lambda(h_i, l+2s-1,\chi)\overline{c_i''}\lambda_i(p_0)'',\end{aligned}$$

$$\begin{aligned}b_{ii} &= 3p^{-1}(p+1)^{-1}t_{p_0}^{-1}\\ &\quad \times \frac{p^{l+2s}p_0^{1-k/2}\Lambda(g_i,l+2s-1,\chi)\overline{c_i'}\lambda_i(p_0)'}{1-p^{-2}},\end{aligned}$$

and

$$\begin{aligned}a_{ii} &= 3p^{-1}(p+1)^{-1}t_{p_0}^{-1}\\ &\quad \times \frac{(1+p^{-1})p^{l+2s}p_0^{1-k/2}\overline{c_i}\lambda_i(p_0)}{(1-p^{-1})((1+p^{-1})^2-\psi(p)p^{-k}\lambda_i(p)^2)}\\ &\quad \times \Lambda(f_i,l+2s-1,\chi).\end{aligned}$$

Now let

$$g_i(z) = \sum_{m=1}^{\infty} b_i(m)\mathbf{e}(mz)\ (i=1,2,...,d_2)$$

and

$$h_i(z) = \sum_{m=1}^{\infty} c_i(m)\mathbf{e}(mz)\ (i=1,2,...,d_3)$$

with $b_i(1) = c_i(1) = 1$. We note that $c_i(pm) = 0$ and $b_i(pm) = \lambda_i(p)'b_i(m)$. Thus we have

$$\begin{aligned}\mathcal{F}_p(z_2) &= \sum_{i=1}^{d_1} a_{ii}\Big\{p^{k/2}(1-p^{-2})f_i|\delta_p(z_2)\\ &\qquad -\psi(p)(1-p^{-1})\lambda_i(p)f_i(z_2)\Big\}\\ &\quad + \sum_{i=1}^{d_2} p^{k/2}(1-p^{-2})b_{ii}g_i(z_2).\end{aligned}$$

We note that $b_i(p^2m) = \psi(p)p^{k-2}b_i(m)$ (see [Miyake 89, Theorem 4.6.17]). Thus we have

$$\begin{aligned}&\epsilon(p,p^2m;l,s) - \psi(p)p^{k-2}\epsilon(p,m;l,s)\\ &= \sum_{i=1}^{d_1} a_{ii}\,\{p^k(1-p^{-2})a_i(pm)\\ &\qquad - \psi(p)(1-p^{-1})\lambda_i(p)a_i(p^2m)\\ &\qquad - \psi(p)p^{k-2}(p^k(1-p^{-2})a_i(p^{-1}m)\\ &\qquad -\psi(p)(1-p^{-1})\lambda_i(p)a_i(m))\}\\ &= \sum_{i=1}^{d_1} a_{ii}p^k(1-p^{-1})((1+p^{-1})^2\\ &\qquad - \psi(p)\lambda_i(p)^2p^{-k})a_i(pm)\\ &= 3t_{p_0}^{-1}p^{k+l+2s-2}p_0^{1-k/2}\\ &\qquad \times \sum_{i=1}^{d_1}\Lambda(f_i,l+2s-1,\chi)\overline{c_i}\lambda_i(p_0)a_i(pm)\end{aligned}$$

for any positive integer m. This proves assertion (2). □

Corollary 4.3. *Let the notation and assumptions be as before. Furthermore, set*

$$t(m;l,s)=\begin{cases} 3^{-1}(p+1)p^{-k/2-l-2s+1}\epsilon(p,m;l,s) & \text{Case (c–1)} \\ 3^{-1}t_{p_0}p_0^{-1/2}p^{-k-l-2s+2} \\ \quad \times(\epsilon(p,p^2m;l,s)-\psi(p)p^{k-2}\epsilon(p,m;l,s)) & \text{Case (c–2).} \end{cases}$$

(1) In Case (c–1), for any positive integer m we have

$$t(m;l,s)=\sum_{i=1}^{d_1}\Lambda(f_i,l+2s-1,\psi)\overline{c_i}a_i(m).$$

(2) In Case (c–2), for any positive integer m we have

$$t(m;l,s)=\sum_{i=1}^{d_1}\Lambda(f_i,l+2s-1,\chi)\overline{c_i}^2a_i(pm).$$

The above corollary is a certain generalization of [Katsurada 03, Theorem 4.1]. Namely, in that theorem, we restricted ourselves to the case where $l\le k-2$. Furthermore, in (3) of that theorem, we restricted ourselves to the case where $(\frac{p_0}{p})=1$ and m is prime to p_0p, and in the above corollary such conditions have been removed. We also note that, $\Lambda(f,m,\mathbf{1}_p)=\Lambda(f,m,\mathbf{1})(1-p^{-m-k+1}a(p)^2)$ for a primitive form f in $S_k(\Gamma_0(p),(\frac{p}{*}))$, where $a(p)$ denotes the pth Fourier coefficient of f. Thus (1) of that theorem is essentially included in (1) of the above corollary as a special case. However, for practical computation, we include the following statement, which can be easily proved in a way similar to Theorem 4.2.

Proposition 4.4. *Let f_i $(i=1,\ldots,d_1)$ and c_i be as in (1) of Theorem 4.2. Then for any even positive integer $l\le k-2$, we have*

$$\mathcal{F}_1(z_2)=3(p+1)^{-1}p^{1/2}\sum_{i=1}^{d_1}\Lambda(f_i,l+2s-1,\mathbf{1})\overline{c_i}^2f_i(z_2).$$

Corollary 4.5. *In addition to the previous notation, set*

$$t(m;l,s)=3^{-1}(p+1)p^{-1/2}\epsilon(1,m;l,s).$$

Then, for any positive integer m we have

$$t(m;l,s)=\sum_{i=1}^{d_1}\Lambda(f_i,l+2s-1,1)\overline{c_i}^2a_i(m).$$

In the previous case, $\tilde{\mathcal{E}}_{2,k}(z_1,z_2;k,p,\psi,\mathbf{1},0)$ does not belong to $S_k(\Gamma_0(p),\psi)\otimes S_k(\Gamma_0(p),\psi)$ but belongs to $M_k(\Gamma_0(p),\psi)\otimes M_k(\Gamma_0(p),\psi)$. Thus, by modifying the previous method, we obtain a similar formula for the value of $\Lambda(f,k-1,\mathbf{1})$. Now for a prime number q that does not divide p_0p let

$$\begin{aligned}\beta_{i+1}&=\beta(i+1,q;l,s)\\&=\sum_{r=0}^{[i/2]}({}_iC_r-{}_iC_{r-1})q^{r(k-1)}t(q^{i-2r};l,s)\end{aligned}$$

for $i=0,1,....,d_1-1$, where ${}_iC_r=\frac{i!}{r!(i-r)!}$. We understand that ${}_iC_{-1}=0$. For a Hecke eigenform f let K_f be the field over $\mathbf{Q}$ generated by all the eigenvalues of the Hecke operators. Furthermore, for a character χ, let $K_{f,\chi}$ be the field generated over K_f by all the values of χ. Set $e_f=[K_f:\mathbf{Q}]$, and denote by $N_{K_f}(\alpha)$ the norm of α over $\mathbf{Q}$ for $\alpha\in K_f$. Similarly, we define $e_{f,\chi}$ and $N_{K_{f,\chi}}(\alpha)$ for $\alpha\in K_{f,\chi}$. Let $\{f_i\}_{i=1}^d$ be the basis of $S_k(\Gamma_0(N),\psi)$ as above and write $K_i=K_{f_i}$ and $e_i=e_{f_i}$. Let $\Phi(X)=\Phi_{T(m)}(X)$ be the characteristic polynomial of $T(m)$ on $S_k(\Gamma_0(N),\psi)$. We note that $N_{K_i}(c_i)=1$ in Theorem 4.2 and Proposition 4.4. Thus by [Goto 98, Lemma 2.2], we obtain:

Theorem 4.6. *Let the notation and assumptions be as in Theorem 4.2 and Proposition 4.4. Let f be a primitive form in $S_k(\Gamma_0(N),\psi)$, and $a(q)$ be the qth Fourier coefficient of f. Assume that $\Phi'_{T(q)}(a(q))\neq 0$. Write $\Phi_{T(q)}(X)=\sum_{i=0}^{d_1}b_{d_1-i}X^i$, $K=K_{f,\chi}$, and $e=e_{f,\chi}$.*

(1) In (1) of Theorem 4.2 or Proposition 4.4, we have

$$N_K(\Lambda(f,l+2s-1,\chi))=N_K\left(\frac{\sum_{i=0}^{d_1-1}\sum_{j=i}^{d_1-1}\beta_{d_1-j}b_{j-i}a(q)^i}{\Phi'_{T(q)}(a(q))}\right).$$

(2) In (2) of Theorem 4.2, if $N(a(p))\neq 0$, we have

$$N_K(\Lambda(f,l+2s-1,\chi))=N_K\left(\frac{\sum_{i=0}^{d_1-1}\sum_{j=i}^{d_1-1}\beta_{d_1-j}b_{j-i}a(q)^i}{a(p)\Phi'_{T(q)}(a(q))}\right).$$

Now we give an exact value of $\Lambda(f,1,\chi)$ for a Dirichlet character such that χ^2 is trivial. For a Hecke eigenform $f\in S_k(\Gamma_0(M),\psi)$, set

$$\tilde{L}(f,s,\mathbf{1})=\frac{L(f,s,\mathbf{1})}{\prod_{p|M}(1-\overline{a(p)}^2p^{-s-k+1})},$$

and

$$\tilde{L}(f,s,\psi) = \frac{L(f,s,\psi)}{\prod_{p|M}(1-p^{-s})}.$$

Furthermore, set, for an odd positive integer $l \leq k-1$,

$$\tilde{\Lambda}(f,l,\chi) = \frac{\Gamma(k-1)\Gamma(k+l-1)\Gamma(l+1)}{\Gamma(k-l)} \times \frac{\tilde{L}(f,l,\chi)}{2^{2k+2l-4}\pi^{k+2l}\langle f,f\rangle},$$

and

$$\tilde{\Lambda}(f,0,\chi) = \Gamma(k-1)\frac{\tilde{L}(f,0,\chi)}{2^{2k-3}\pi^k\langle f,f\rangle}$$

for $\chi = \mathbf{1}$ or ψ. If χ^2 is trivial, $E^*_{2,2}(Z;M,\psi\chi,0)$ does not belong to $M_2(\Gamma_0(M),\psi\bar{\chi})$, and thus we cannot give an exact value of $\Lambda(f,1,\chi)$ by direct use of the previous method. However, we can relate the value $\Lambda(f,1,\chi)$ to $\Lambda(f,0,\chi)$ by using the functional equation. We explain this in the following three cases:

(c–3) M is a prime number p and $\chi = \mathbf{1}$;

(c–4) M is a prime number p and $\chi = \psi = (\frac{p}{*})$;

(c–5) $M = 1$ and $\chi = (\frac{p}{*})$ with p a prime number such that $p \equiv 1 \bmod 4$.

First, in Case (c–3), set

$$\tilde{R}(f,s,\mathbf{1}) = p^{(s+k-1)/2}\pi^{-3/2(s+k-1)}\Gamma((s+k-1)/2) \times \Gamma((s+k)/2)\Gamma((s+1)/2)\tilde{L}(f,s,\mathbf{1}).$$

Next, in Case (c–5), for $f \in S_k(\Gamma_0(1))$ and the character χ modulo p, set

$$R(f,s,\chi) = p^{3(s+k-1)/2}\pi^{-3/2(s+k-1)}\Gamma((s+k-1)/2) \times \Gamma((s+k)/2)\Gamma((s+1)/2)L(f,s,\chi).$$

Then by Li [Li 79], we have the following functional equation:

Proposition 4.7.

(1) In Case (c–3),

$$\tilde{R}(f,1-s,\mathbf{1}) = \tilde{R}(f,s,\mathbf{1}).$$

In particular,

$$\tilde{\Lambda}(f,1,\mathbf{1}) = p^{-1/2}\tilde{\Lambda}(f,0,\mathbf{1}).$$

(2) In Case (c–5), under the previous notation and assumptions, we have

$$R(f,1-s,\chi) = R(f,s,\chi).$$

In particular, in Case (c–5), we have

$$\Lambda(f,1,\chi) = p^{-3/2}\Lambda(f,0,\chi).$$

In Case (c–4), the value $\Lambda(f,1,\phi)$ can be given by a different method (see [Zagier 77]).

Proposition 4.8. *In Case (c–4), we have*

$$\Lambda(f,1,\phi) = \frac{2}{3}(1-p^{-2}).$$

Now we discuss congruence among modular forms. Let K be an algebraic field, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K. Let $\mathfrak{q}$ be a prime ideal of $\mathfrak{O}$, and $\mathfrak{O}_\mathfrak{q}$ the localization of $\mathfrak{O}$ at $\mathfrak{q}$. Let $f(z) = \sum_{n=1} a(m)\mathbf{e}(mz)$ and $g(z) = \sum_{n=1} b(m)\mathbf{e}(mz)$ be elements of $S_k(\Gamma_0(M),\phi)$ whose Fourier coefficients belong to $\mathfrak{O}_\mathfrak{q}$. Then, we write $f \equiv g \bmod \mathfrak{q}$ if $a(m) \equiv b(m) \bmod \mathfrak{q}$ for any positive integer m. We give the following lemma which is essentially the same as Lemma 1.4 of [Doi et al. 98].

Lemma 4.9. *Let $f_1,...,f_r$ be a basis of $S_k(\Gamma_0(M),\phi)$ consisting of Hecke eigenforms. Let K be the composite field of all $K_{f_1},...,K_{f_r}$, and $\mathfrak{O}$ the ring of integers in K. Let $\mathfrak{q}$ be a prime ideal of $\mathfrak{O}$. Assume that all the Fourier coefficients and eigenvalues of f_i $(i=1,..,r)$ belong to $\mathfrak{O}_\mathfrak{q}$. Let h be an element of $S_k(\Gamma_0(M),\phi)$ whose Fourier coefficients belong to $\mathfrak{O}_\mathfrak{q}$ such that*

$$h = \sum_{i=1}^{r} l_i f_i$$

with $l_i \in K$. Assume that $f_1 \not\equiv 0 \bmod \mathfrak{q}$ and $\mathrm{ord}_\mathfrak{q}(l_1) < 0$. Then there exists $2 \leq i \leq r$ such that

$$f_i \equiv f_1 \bmod \mathfrak{q}.$$

Now by Theorem 4.2, combined with Lemma 4.9, we have the following theorem.

Theorem 4.10. *Let N, p, ψ, and χ be as in Theorem 4.2. Let f be a primitive form in $S_k(\Gamma_0(N),\psi)$. Let $\mathfrak{O}_{K_f}$ be the ring of integers in K_f, and $\mathfrak{q}$ a prime ideal of $\mathfrak{O}_{K_f}$ dividing the denominator of $N_{K_{f,\chi}/K_f}(\Lambda(f,l+2s-1,\chi))$*

but not dividing Npr, where $r = 6$ or $(2l-1)!$ according to whether (2) of Theorem 4.2 holds, or not. Then, there exists a primitive form g in $S_k(\Gamma_0(N), \psi)$ different from f such that

$$g \equiv f \bmod \mathfrak{q}',$$

where $\mathfrak{q}'$ is a prime ideal of $\mathfrak{O}_{K_f K_g}$ lying above $\mathfrak{q}$.

Proof: We note that the generalized Bernoulli number associated with a Dirichlet character ϕ is an algebraic integer if the conductor of ϕ is not a power of a prime number (see [Carlitz 59, Leopoldt 58]). Thus by Equations (2–4) and (2–5), all the Fourier coefficients of $\mathcal{F}_p(z)$ belong to $\mathfrak{O}_\mathfrak{q}$ in Case (c–1). Thus the assertion in Case (c–1) follows directly from (1) of Theorem 4.2 and Lemma 4.9. In Case (c–2), $\mathcal{F}_{p,p}$ belongs to $S_k(\Gamma_0(pN), \psi)$ and its eigenvalues and Fourier coefficients belong to $\mathfrak{O}_\mathfrak{q}$. In this case, we remark that $\mathfrak{q}$ does not divide both the pth and the p^2th Fourier coefficients of f. Thus we have $\tilde{f} \not\equiv 0 \bmod \mathfrak{q}$. Thus, again by (1) of Theorem 4.2 and Lemma 4.9, we can show that there exists a primitive form g in $S_k(\Gamma_0(N), \psi)$ different from f such that

$$\tilde{g} \equiv \tilde{f} \bmod \mathfrak{q}',$$

where $\tilde{f}$ (respectively $\tilde{g}$) is the modular form in (2) of Theorem 4.2 for f (respectively g). Thus the assertion can be proved by the above remark. □

5. NUMERICAL EXAMPLES AND COMMENTS

Using Theorem 4.6 combined with Proposition 4.7 we can compute the norm of $\Lambda(f, m, \chi)$. A subspace S of $S_k(\Gamma_0(N), \psi)$ is called nonsplitting if it is spanned by all Galois conjugates of a primitive form in S. Take a primitive form f of $S_k(\Gamma_0(1))$. Assume that $S_k(\Gamma_0(1))$ is nonsplitting. Then, $N_{K_f}(\Lambda(f, l, (\frac{q}{*})))$ is independent of f. Thus, in this case, we denote this value by $\mathbf{L}(k; l, q)$. Similarly, in the case $S_k(\Gamma_0(q), (\frac{q}{*}))$ is nonsplitting, we define $\tilde{\mathbf{L}}(k, q; l, 1)$ and $\tilde{\mathbf{L}}(k, q; l, q)$ as $N_{K_f}(\tilde{\Lambda}(f, l, \mathbf{1}))$ and $N_{K_f}(\tilde{\Lambda}(f, l, (\frac{q}{*})))$, respectively, for a primitive form f of $S_k(\Gamma_0(q), (\frac{q}{*}))$. We have computed some values using Mathematica.

Example 5.1. It is conjectured by Maeda that $S_k(\Gamma_0(1))$ is nonsplitting, and so far, this conjecture has been verified at least for $k \leq 2000$ (see [Hida and Maeda 97, Farmer and James 02]).

We show some examples of $\mathbf{L}(k; l, q)$ for various k, l, q. From now on let

$$[a_1, a_2, a_3] = \begin{pmatrix} a_1 & a_3/2 \\ a_3/2 & a_2 \end{pmatrix}.$$

To make computations easy, for an odd positive integer $l \leq k-1$ and a positive integer m prime to q, set $t(m; l) = q^{-3/2} t(m, 2, -1/2)$ or $t(m; l+1, 0)$ according to whether $l = 1$, or not. We note that the Gauss sum $\tau(\chi)$ for χ is $q^{1/2}$. Thus by (2–5) we have

$$\begin{aligned} t(m; l) &= 2/3 q^{-k-2l+2} \\ &\times \Big\{ \sum_{r=1}^{[2q\sqrt{qm}]} \prod_{p \mid \mathfrak{f}_{[q, q^2 m, r]}} F_p([q, q^2 m, r], \chi(p) p^{l-2}) \\ &\qquad \times B_{l, (\chi\chi_{[q,q^2m,r]})^{(0)}} G_{l+1}^{k-l-1}(q^3 m, r) \chi(r) \\ &\quad - q^{k-2} \sum_{r=1}^{[2\sqrt{qm}]} \prod_{p \mid \mathfrak{f}_{[q,m,r]}} F_p([q, m, r], \chi(p) p^{l-2}) \\ &\qquad \times B_{l, (\chi\chi_{[q,m,r]})^{(0)}} G_{l+1}^{k-l-1}(qm, r) \chi(r) \Big\}. \end{aligned}$$

Example 5.1.1. We compute $\mathbf{L}(12; l, q)$ for $1 \leq l \leq k-1$ and some prime numbers q. In this case $\dim S_{12}(\Gamma_0(1)) = 1$. Take a unique primitive form $f(z) = \sum_{m=1}^{\infty} a(m)\mathbf{e}(mz)$ in $S_{12}(\Gamma_0(1))$. Thus, by (2) of Theorem 4.6 and (2) of Proposition 4.7, we have

$$\mathbf{L}(12; l, q) = a(q)^{-1} t(1; l)$$

if $a(q) \neq 0$. Numerical examples are displayed in Tables 1–4.

The values in Tables 1–4 were computed in [Katsurada 03] except the following two cases: (a) $k = 18$; and (b) $k = 12$ and $l = 11$. We note that, the value of $\mathbf{L}(12; l, q)$ has been obtained by Stopple [Stopple 96] in the case where $q = 5$ and $l = 1, 3, 5, 7$, or $q = 13, 17, 29, 37, 41$ and $l = 1$, and that all the values in the tables can also be obtained by his method. We note that relatively large prime numbers appear in the numerator of $\mathbf{L}(k; l, 5)$, contrary to the untwisted case in [Dummigan 01]. We note that the numerator of $\Lambda(f, l, \mathbf{1})$ is conjecturally related to the order of the Shafarevich-Tate group (see [Dummigan 01]). In

l	$\mathbf{L}(12; l, 5)$
1	$2^{14} \cdot 3^5 \cdot 7/5^{10}$
3	$2^{14} \cdot 3^5 \cdot 7 \cdot 2851/5^{13}$
5	$2^{19} \cdot 3^5 \cdot 7 \cdot 1511599/5^{16}$
7	$2^{19} \cdot 3^8 \cdot 7^3 \cdot 521 \cdot 295387/5^{20}$
9	$2^{26} \cdot 3^{10} \cdot 7^2 \cdot 110308273279/5^{24}$
11	$2^{21} \cdot 3^{10} \cdot 7^2 \cdot 11 \cdot 2963 \cdot 5523341/5^{29}$

TABLE 1.

l	$\mathbf{L}(12; l, 13)$
1	$2^{14} \cdot 3^8 \cdot 5^3 \cdot 7 \cdot 563/13^{12}$
3	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 41177 \cdot 1445419/13^{16}$
5	$2^{19} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 299696968678699/13^{20}$
7	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3 \cdot 31^2 \cdot 5479 \cdot 306945156059/13^{23}$
9	$2^{26} \cdot 3^{10} \cdot 5^5 \cdot 7^2 \cdot 547 \cdot 10267 \cdot 1634679978646831/13^{28}$
11	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 17 \cdot 29 \cdot 131 \cdot 3331 \cdot 868032338256361/13^{32}$

TABLE 2.

l	$\mathbf{L}(12; l, 17)$
1	$2^{21} \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 2389/17^{12}$
3	$2^{17} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 477200018623/17^{14}$
5	$2^{20} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 23 \cdot 29 \cdot 997 \cdot 46316422211/17^{20}$
7	$2^{22} \cdot 3^8 \cdot 5^3 \cdot 7^3 \cdot 167 \cdot 11003 \cdot 322079 \cdot 915248119/17^{24}$
9	$2^{30} \cdot 3^{10} \cdot 5^5 \cdot 7^2 \cdot 43 \cdot 892028959 \cdot 1604767911433/17^{27}$
11	$2^{20} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 207904571011548650027085553/17^{21}$

TABLE 3.

l	$\mathbf{L}(18; l, 5)$
1	$2^{13} \cdot 3^4 \cdot 7 \cdot 11 \cdot 13/5^{13}$
3	$2^{13} \cdot 3^4 \cdot 7 \cdot 11 \cdot 13 \cdot 227 \cdot 769/5^{17}$
5	$2^{18} \cdot 3^4 \cdot 7 \cdot 11 \cdot 13 \cdot 397 \cdot 140407/5^{20}$
7	$2^{17} \cdot 3^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 1279 \cdot 2959715807/5^{24}$
9	$2^{20} \cdot 3^8 \cdot 7^2 \cdot 11 \cdot 13 \cdot 673 \cdot 1709 \cdot 43867/5^{28}$
11	$2^{22} \cdot 3^7 \cdot 7^2 \cdot 11^3 \cdot 13 \cdot 23 \cdot 79265243 \cdot 681985859041/5^{31}$
13	$2^{30} \cdot 3^9 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 79 \cdot 63077797 \cdot 1535138971999/5^{35}$
15	$2^{31} \cdot 3^{11} \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 673 \cdot 10420198073 \cdot 99670080988216447/5^{38}$
17	$2^{24} \cdot 3^{10} \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 131 \cdot 263 \cdot 205502593 \cdot 31922855977/5^{45}$

TABLE 4.

our case, as suggested by the referee, arguing as in Theorem 14.2 of [Dummigan 01], one would expect that Eisenstein primes should divide the numerator of $\mathbf{L}(k; k/2, q)$ under certain conditions when $k/2$ is odd. The value $\mathbf{L}(18; 9, 5)$ shows that this observation is true in the case $k = 18$ and $q = 5$, because 43867 is an Eisenstein prime.

Example 5.1.2. The values of the standard zeta functions at $s = 1$ are particularly important. To explain this, let q be a prime number congruent to 1 modulo 4, and let $\mathfrak{O}_{\mathbf{Q}(\sqrt{q})}$ be the ring of integers in $\mathbf{Q}(\sqrt{q})$. Assume that the class number of $\mathbf{Q}(\sqrt{q})$, in the narrow sense, is one. Let $S_{k,k}(SL_2(\mathfrak{O}_{\mathbf{Q}(\sqrt{q})}))$ be the space of cusp forms of weight (k, k) with respect to $SL_2(\mathfrak{O}_{\mathbf{Q}(\sqrt{q})})$. Then $S_{k,k}(SL_2(\mathfrak{O}_{\mathbf{Q}(\sqrt{q})}))$ has the following decomposition:

$$S_{k,k}(SL_2(\mathfrak{O}_{\mathbf{Q}(\sqrt{q})})) = \hat{S}_k(\Gamma_0(1)) \perp \hat{S}_k\left(\Gamma_0(q), \left(\frac{q}{*}\right)\right) \perp S^0_{k,k},$$

where $\hat{S}_k(\Gamma_0(1))$ (respectively $\hat{S}_k(\Gamma_0(q), (\frac{q}{*})))$ is the image of $S_k(\Gamma_0(1))$ (respectively $S_k(\Gamma_0(q), (\frac{q}{*})))$ under the Doi-Naganuma map, and $S^0_{k,k}$ is the orthogonal complement of $\hat{S}_k(\Gamma_0(1)) \perp \hat{S}_k(\Gamma_0(q), (\frac{q}{*}))$ in $S_{k,k}(SL_2(\mathfrak{O}_{\mathbf{Q}(\sqrt{q})}))$ with respect to the Petersson product. Take a primitive form $g \in S^0_{k,k}$ and for an integral ideal $\mathfrak{A}$ in $\mathbf{Q}(\sqrt{q})$ let $c(\mathfrak{A}; g)$ be the $\mathfrak{A}$th Fourier coefficient of g. Let K_g be the field generated over $\mathbf{Q}$ by all $c(\mathfrak{A}; g)$ and K_g^+ the subfield of K_g generated by $c((p); g)$ for all rational primes p. We denote by $D(K_g/K_g^+)$ the relative discriminant of K_g/K_g^+. Assume that $S^0_{k,k}$ is nonsplitting, that is, it is spanned by all Galois conjugates of a primitive form in $S^0_{k,k}$. Then, $D(K_g/K_g^+)$ does not depend on g. Hence we denote this value by $D_{k,q}$. Then, on page 567 of [Doi et al. 98] Doi, Hida, and Ishii, among others, conjectured the following:

> Any odd prime factor of $D_{k,q}$ divides either the numerator of $N_{K_f}(\Lambda(f, 1, (\frac{q}{*})))$ for some primitive form f in $S_k(\Gamma_0(1))$ or the numerator of $N_{K_f}(\tilde{\Lambda}(f, 1, \mathbf{1}))$ for some primitive form f in $S_k(\Gamma_0(q), (\frac{q}{*}))$.

This is actually a counterpart of the conjecture formulated in Goto [Goto 98]. Doi, Hida, and Ishii [Doi et

k	D	$\mathbf{L}(k;1,5)$
24	144169	$2^{29}\cdot 3^9\cdot 7^4\cdot 11^4\cdot 13\cdot 17\cdot 19\cdot 109\cdot 54449/5^{39}\cdot 144169$
28	$131\cdot 139$	$2^{26}\cdot 3^8\cdot 7^3\cdot 11^2\cdot 13^3\cdot 17\cdot 19\cdot 23\cdot 66876860429/5^{44}\cdot 131\cdot 139$
30	51349	$2^{26}\cdot 3^{10}\cdot 7^4\cdot 11^2\cdot 13\cdot 17\cdot 19\cdot 23\cdot 3253\cdot 20017939/5^{43}\cdot 51349$
32	$67\cdot 273067$	$2^{26}\cdot 3^{10}\cdot 7^4\cdot 11\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 37\cdot 157\cdot p_1/5^{54}\cdot 67\cdot 273067$
34	$479\cdot 4919$	$2^{26}\cdot 3^{10}\cdot 7^3\cdot 11^3\cdot 13^2\cdot 19\cdot 23\cdot 29\cdot 191\cdot 3191\cdot p_2/5^{51}\cdot 479\cdot 4919$
38	$181\cdot 349\cdot 1009$	$2^{28}\cdot 3^8\cdot 7^4\cdot 11^2\cdot 13\cdot 17\cdot 23\cdot 29\cdot 31\cdot p_3/5^{60}\cdot 181\cdot 349\cdot 1009$

TABLE 5. $p_1 = 222142617427425679, p_2 = 121120620073, p_3 = 2453963035201979961522108 7$

al. 98] computed an exact value of $\tilde{\mathbf{L}}(k,q;1,1)$ and verified the above conjecture in some cases. Goto [Goto 98] computed the value $\mathbf{L}(20,1,5)$ and Hiraoka [Hiraoka 00] computed the values $\mathbf{L}(22;1,5)$ and $\mathbf{L}(24;1,5)$. Then, combining the results of [Doi et al. 98], they verified the conjecture for $(k,q) = (20,5), (22,5)$, and $(24,5)$. Now let $k = 12$ and $q = 13$. In this case, $S^0_{12,12}$ and $S_k(\Gamma_0(13),(\frac{13}{*}))$ are nonsplitting. Furthermore, according to Table 1 of [Doi et al. 98], the odd prime factors of $D_{12,13}$ are 13, 563, and 6205151, and the numerator of $\tilde{\mathbf{L}}(12,13;1,1)$ is $5\cdot 7\cdot 13^{29}\cdot 6205151$. Thus it is expected that 563 appears in the numerator of $\mathbf{L}(12;1,13)$. Table 2 shows that this is true.

Now we compute $\mathbf{L}(k;1,5)$ for $16 \le k \le 38$. For other numerical examples, see [Stopple 96].

First, let $k = 16, 18, 20, 22, 26$. Then, we have

$$\dim S_k(\Gamma_0(1)) = 1.$$

Take a unique primitive form $f(z) = \sum_{m=1}^{\infty} a(m)\mathbf{e}(mz)$ in $S_k(\Gamma_0(1))$. Then, we have

$$\mathbf{L}(12;l,5) = a(5)^{-1}t(1;1).$$

Thus we have Table 6.

k	$\mathbf{L}(k;1,5)$
16	$2^{13}\cdot 3^4\cdot 7^3\cdot 11/5^{14}$
18	$2^{13}\cdot 3^4\cdot 7\cdot 11\cdot 13/5^{13}$
20	$2^{15}\cdot 3^4\cdot 7\cdot 11\cdot 13\cdot 977/5^{18}$
22	$2^{15}\cdot 3^5\cdot 7^2\cdot 13\cdot 17\cdot 71/5^{18}$
26	$2^{13}\cdot 3^4\cdot 7\cdot 11\cdot 17\cdot 19\cdot 337\cdot 1409/5^{22}$

TABLE 6.

Next let $k = 24, 28, 30, 32, 34, 38$. Then we have $\dim S_k(\Gamma_0(1)) = 2$. Take a basis f_1, f_2, of $S_k(\Gamma_0(1))$, consisting of primitive forms. Then, $K_{f_1} = K_{f_2}$ is a real quadratic field, thus this field can be expressed as $K = \mathbf{Q}(\sqrt{D})$ with D a nonsquare positive integer. Let $\Phi_{T(2)}(X) = X^2 + b_1X + b_2$. Then, by (2) of Theorem 4.6, we have

$$\mathbf{L}(k;1,5) = \frac{t(1;1)t(2;1)b_1 + t(2;1)^2 + t(1;1)^2b_2}{(-b_1^2 + 4b_2)N_K(a(5))},$$

where $a(5)$ is the fifth Fourier coefficient of f_1. The polynomial $\Phi(X)$ and the value $N_K(a(5))$ can easily be computed by the trace formula (see [Miyake 89]). Thus we have Table 5.

Example 5.2. We compute $\tilde{\mathbf{L}}(k,q;l,1)$ and $\tilde{\mathbf{L}}(k,q;l,q)$ when $S_k(\Gamma_0(q),(\frac{q}{*}))$ is nonsplitting. In Proposition 4.4, let $t(m;l) = p^{-1/2}t(m;2,-1/2)$ or $t(m;l+1,0)$ according to whether $l = 1$, or not. Let $\chi = (\frac{q}{*})$. Then for $l \ge 1$ we have

$$\begin{aligned} t(m;l) &= 1/3(q+1)q^{-l} \\ &\times \Big\{2\sum_{1\le r<2\sqrt{m}}\prod_{p\mid \mathfrak{f}_{[1,m,r]}} F_p([1,m,r],\chi(p)p^{l-2}) \\ &\quad\times B_{l,(\chi\chi_{[1,m,r]})^{(0)}}G^{k-l-1}_{l+1}(m,r) \\ &\quad\times q^{(l-3/2)(\mathrm{ord}_q(4\det[1,m,r])+\mathrm{ord}_q(\mathfrak{d}_{[1,m,r]}))} \\ &\quad\times (1-(\chi\chi_{[1,m,r]})^{(0)}(q)q^{-l}) - \delta(l;m,r)(1-q)/12\Big\} \end{aligned}$$

where $\delta(l;m,r) = 1$ or 0 according to whether $l = 1$ and $4m - r^2 = 0$, or not. Furthermore, in (1) of Theorem 4.6, for $l \ge 3$ let $\tilde{t}(m;l) = (1-q^{-l})^{-1}t(m;l+1,0)$. Then, by Equation (2–7), we have

$$\begin{aligned} \tilde{t}(m;l) &= 2/3(q+1)q^{-k/2-l+1/2} \\ &\quad\times \sum_{1\le r\le 2\sqrt{qm}}\prod_{p\mid \mathfrak{f}_{[q,m,r]}} F_p([q,m,r],p^{l-2}) \\ &\quad\times B_{l,\chi^{(0)}_{[q,m,r]}}G^{k-l-1}_{l+1}(qm,r)\chi(r). \end{aligned}$$

Assume that $S_k(\Gamma_0(q),(\frac{q}{*}))$ is nonsplitting. Take a primitive form

$$f(z) = \sum_{m=1}^{\infty} a(m)\mathbf{e}(mz) \in S_k\left(\Gamma_0(q),\left(\frac{q}{*}\right)\right)$$

and set $K = K_f$. Let $\alpha = a(2)$ and assume that $\Phi'_{T(2)}(\alpha) \ne 0$. Write $\Phi_{T(2)}(X) = \sum_{i=0}^{e} b_{e-i}X^i$ and

$$\beta_{i+1} = \sum_{r=0}^{[i/2]}({}_iC_r - {}_iC_{r-1})q^{r(k-1)}t(q^{i-2r};l),$$

as in Section 4. We also define $\tilde{\beta}_{i+1}$ by replacing $t(q^{i-2r};l)$ with $\tilde{t}(q^{i-2r};l)$. Then, we have

$$\tilde{\mathbf{L}}(k,q,l,1) = \frac{N_K(\sum_{i=0}^{e-1}\sum_{j=i}^{e-1}\beta_{e-j}b_{j-i}\alpha^i)}{\Phi_{T(q)}(q^{(k-1+l)/2})\Phi_{T(q)}(-q^{(k-1+l)/2})N_K(\Phi'_{T(2)}(\alpha))}$$

for $l \geq 3$ and

$$\tilde{\mathbf{L}}(k,q;1,1) = \frac{N_K(\sum_{i=0}^{e-1}\sum_{j=i}^{e-1}\beta_{e-j}b_{j-i}\alpha^i)}{\Phi_{T(q)}(q^{(k-1)/2})\Phi_{T(q)}(-q^{(k-1)/2})N_K(\Phi'_{T(2)}(\alpha))}.$$

Furthermore,

$$\tilde{\mathbf{L}}(k,q;l,q) = \frac{N_K(\sum_{i=0}^{e-1}\sum_{j=i}^{e-1}\tilde{\beta}_{e-j}b_{j-i}\alpha^i)}{N_K(\Phi'_{T(2)}(\alpha))}.$$

We note that the value of $\tilde{\mathbf{L}}(k,q;1,1)$ can also be obtained by Zagier's method in [Zagier 77]. On the other hand, the value of $\tilde{\mathbf{L}}(k,q;l,q)$ cannot be obtained by his method.

Here we mention the special values of the standard zeta function of the Doi-Naganuma lift

$$\hat{f} \in S_{k,k}(SL_2(\mathfrak{O}_{\mathbf{Q}(\sqrt{q})}))$$

for a primitive form $f \in S_k(\Gamma_0(q),(\frac{q}{*}))$. For a prime number p, let α_p, β_p be the complex numbers in Equation (3–6). For a prime ideal $\mathfrak{p}$ of $\mathfrak{O}_{\mathbf{Q}(\sqrt{q})}$ let

$$A_{\mathfrak{p}} = \alpha_p^m \text{ and } B_{\mathfrak{p}} = \beta_p^m,$$

if $N(\mathfrak{p}) = p^m$, where $N(\mathfrak{p}) = N_{\mathbf{Q}(\sqrt{q}))/\mathbf{Q}}(\mathfrak{p})$. Then, we define the standard zeta function $\mathbf{L}(\hat{f},s)$ of $\hat{f}$ as

$$L(\hat{f},s) = \prod_{\mathfrak{p}}\Big\{(1-A_{\mathfrak{p}}B_{\mathfrak{p}}N(\mathfrak{p})^{-s-k+1}) \times(1-A_{\mathfrak{p}}^2N(\mathfrak{p})^{-s-k+1})(1-B_{\mathfrak{p}}^2N(\mathfrak{p})^{-s-k+1})\Big\}^{-1}.$$

For the precise definition of the standard zeta function of a general Hilbert modular form, see [Zagier 77]. Set

$$\Lambda(\hat{f},l) = \left(\frac{\Gamma(k-1)\Gamma(k+l-1)\Gamma(l+1)}{2^{2k+2l-5}\pi^{k+2l}\Gamma(k-l)}\right)^2\frac{q^k(q+1)L(\hat{f},l)}{B_{2,(\frac{q}{*})}\langle\hat{f},\hat{f}\rangle},$$

where $\langle\hat{f},\hat{f}\rangle$ denotes the normalized Petersson product of $\hat{f}$ in $S_{k,k}(SL_2(\mathfrak{O}_{\mathbf{Q}(\sqrt{q})}))$. By [Zagier 77, page 158, Equation 97], we have

$$L(\hat{f},s) = \tilde{L}(f,s,\mathbf{1})\tilde{L}\left(f,s,\left(\frac{q}{*}\right)\right).$$

Thus we have

$$\Lambda(\hat{f},l) = \tilde{\Lambda}(f,l,\mathbf{1})\tilde{\Lambda}\left(f,l,\left(\frac{q}{*}\right)\right)\frac{4q^k(q+1)\langle f,f\rangle^2}{B_{2,(\frac{q}{*})}\langle\hat{f},\hat{f}\rangle}.$$

On the other hand, we have

$$\tilde{\Lambda}(f,1,\mathbf{1}) = \frac{4B_{2,(\frac{q}{*})}\langle\hat{f},\hat{f}\rangle}{q^k(q+1)\langle f,f\rangle^2}$$

(see page 158 in [Zagier 77]). Thus we have

$$\Lambda(\hat{f},l) = \frac{\tilde{\Lambda}(f,l,\mathbf{1})\tilde{\Lambda}(f,l,(\frac{q}{*}))}{\tilde{\Lambda}(f,1,\mathbf{1})}. \qquad (5\text{–}1)$$

Let $q = 13$, $k = 8$. Then $S_8(\Gamma_0(13),(\frac{13}{*}))$ is nonsplitting and $\dim S_8(\Gamma_0(13),(\frac{13}{*})) = 6$. Then, by [Doi and Goto 93] we have

$$\begin{aligned}\Phi_{T(13)}(X) &= X^6 + 2\cdot 13\cdot 193X^5 + 7^2\cdot 13^3\cdot 29\cdot 31X^4\\ &\quad + 2^2\cdot 5\cdot 13^6\cdot 47\cdot 179X^3\\ &\quad + 7^2\cdot 13^{10}\cdot 29\cdot 31X^2\\ &\quad + 2\cdot 13^{15}\cdot 193X + 13^{21},\\ \Phi_{T(2)}(X) &= X^6 + 449X^4 + 37224X^2 + 205776,\end{aligned}$$

and

$$N_{K/\mathbf{Q}}(\Phi'_{T(2)}(a(2))) = 2^4\cdot 3^2\cdot 5^4\cdot 41^2\cdot 1429^2\cdot 25104281^2.$$

Thus we have

$$\tilde{\mathbf{L}}(8,13;1,1) = \frac{2^2\cdot 7^6\cdot 13^{14}\cdot 4357^2}{3^{12}\cdot 41^2\cdot 1429^2\cdot 25104281^2}$$

(see Table 1 [Doi et al. 98]). Furthermore,

$$\tilde{\mathbf{L}}(8,13;3,1) = \frac{2^{30}\cdot 3^4\cdot 7^6\cdot 13^4\cdot 5^2}{41^2\cdot 25104281^2},$$

$$\tilde{\mathbf{L}}(8,13;5,1) = \frac{2^{22}\cdot 5^{18}\cdot 7^6\cdot 313^2}{41^2\cdot 25104281^2},$$

$$\tilde{\mathbf{L}}(8,13;3,13) = \frac{2^{30}\cdot 3^{10}\cdot 5^8\cdot 7^6\cdot 4583^2\cdot 10079^2}{13^{16}\cdot 41^2\cdot 25104281^2},$$

and

$$\tilde{\mathbf{L}}(8,13;5,13) = \frac{2^{38}\cdot 3^{10}\cdot 5^{18}\cdot 7^6\cdot 4583^2\cdot 10079^2\cdot 20447687895923^2}{13^{28}\cdot 41^2\cdot 25104281^2}.$$

Set

$$\hat{\mathbf{L}}(8,13;l) = N_{K/\mathbf{Q}}(\Lambda(\hat{f},l)).$$

Then by (5–1), we see that the prime number 4357, a prime factor of the numerator of $\tilde{\mathbf{L}}(8,13;1,1)$, appears in the denominators of $\hat{\mathbf{L}}(8,13;3)$ and $\hat{\mathbf{L}}(8,13;5)$. This phenomenon is closely related to the above conjecture. We will discuss this topic more precisely in a subsequent paper.

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