Multiply Quasiplatonic Riemann Surfaces

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CONTENTS

The aim of this article is the study of the circumstances under which a compact Riemann surface may contain two regular dessin d'enfants of different types. In terms of Fuchsian groups, an equivalent condition is the uniformizing group being normally contained in several different triangle groups.

The question is answered in a graph-theoretical way, providing algorithms that decide if a surface that carries a regular dessin (a quasiplatonic surface) can also carry other regular dessins.

The multiply quasiplatonic surfaces are then studied depending on their arithmetic character. Finally, the surfaces of lowest genus carrying a large number of nonarithmetic regular dessins are computed.

1. INTRODUCTION

It is well known that compact Riemann surfaces given by algebraic curves defined over $\bar{\mathbb{Q}}$ correspond exactly with Belyi surfaces. A Belyi function, and hence the complex structure of the surface, is completely determined by the associated dessin d'enfant. The resulting function $\mathcal{R}$ from dessins to the moduli spaces of compact Riemann surfaces is not surjective, and it turns out that it is also not injective, as nonisomorphic dessins may be defined in the same underlying surface. In [Singerman 01] and [Singerman and Syddall 01], the noninjectivity locus for the restriction of $\mathcal{R}$ to the special class of regular clean dessins was studied (platonic surfaces). In this article, we study the noninjectivity of the restriction of $\mathcal{R}$ to a wider class, namely that of all regular dessins (quasiplatonic surfaces). We explore thus how several nonisomorphic regular dessins can be found in the same surface.

The paper is organized as follows. In Section 2, we give a very brief introduction to Belyi surfaces and dessins. Section 3 deals with what we call surgery on uniform dessins—the algorithms relating different regular dessins which can be found in the same surface. The surgery tests, those that actually determine if a given regular

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dessin is obtained from another one embedded in the same surface, are introduced in Section 4. We turn our attention to quasiplatonic surfaces in Section 5, while Section 6 deals with the multiply quasiplatonic case in the two different cases that may occur (arithmetic or nonarithmetic types). We finish by providing the lowest genus examples of the different multiply quasiplatonic surfaces with the largest number of regular dessins possible.

2. BELYI SURFACES AND DESSINS D'ENFANTS

Abstract compact Riemann surfaces, e.g., two-dimensional compact manifolds with complex analytic structure, have been studied from very different points of view. On the one hand, they arise out of complex algebraic curves, and on the other hand as quotient spaces by the action of groups of Möbius transformations.

These two settings are related in a highly obscure way, but the connection can sometimes be shown more explicitly, as in the case of Belyi surfaces. We shall just provide a brief introduction to them in this section, but the interested reader can find the details in [Cohen et al. 94] or [Jones and Singerman 96] and in the references given there. Note that Jones and Singerman often use the language of *hypermaps* instead of that of *dessins* that we employ here.

In the abstract setting, a *Belyi surface* is defined to be a compact Riemann surface X for which a holomorphic function $\beta : X \to \hat{\mathbb{C}}$ with at most three branch values can be defined. Such a β is called a *Belyi function*, and the branch values can be supposed to be contained in $\{0, 1, \infty\}$ after normalization. The following famous result makes clear why this class of surfaces is so interesting (for the proof, see [Belyi 80] or [Wolfart 97]).

Theorem 2.1. (Belyi.) *X is a Belyi surface if and only if the corresponding algebraic curve can be defined over $\bar{\mathbb{Q}}$.*

Suppose $\beta : X \to \hat{\mathbb{C}}$ is a Belyi function. We associate to β an embedded graph in X by considering $\beta^{-1}\{t \in \mathbb{R} \mid 0 \le t \le 1\}$. This is a bipartite graph (its vertices being $\beta^{-1}\{0, 1\}$), since we can colour the preimages of 0 in black, and the preimages of 1 in white, and then every two adjacent vertices have different color. This motivates the following:

Definition 2.2. A *dessin d'enfant* is a bipartite graph $\mathcal{D}$ embedded in a compact Riemann surface X, such that each component of $X \setminus \mathcal{D}$ is simply connected. Those components are called the *faces* of $\mathcal{D}$.

The combinatorial structure of a given dessin $\mathcal{D}$ can be encoded in the following way: Label the edges of $\mathcal{D}$ with numbers $1, 2, \ldots, N$. Now, if a black vertex is fixed, several edges are adjacent to it, and the anticlockwise orientation of the surface gives a cyclic permutation of them. Hence, if $\mathcal{D}$ contains B black vertices, we get a permutation r_b that is a product of B disjoint cycles, and the length of each cycle is the valency of the corresponding black vertex. In the same way, we construct a permutation r_w looking at the white vertices, and we find that the cycles of the permutation $r_f = (r_w r_b)^{-1}$ give information about the faces, since every cycle describes half the edges going around a face. Thus, a cycle of length k of r_f corresponds to a $2k$-gonal face.

We say that $\mathcal{D}$ is of type (l, m, n) if l (respectively, m) is the least common multiple of the valencies of the black vertices (respectively, the white vertices), and n is half the least common multiple of the face valencies. These are, of course, just the orders of r_b, r_w, and r_f. The subgroup $G_{\mathcal{D}}$ of $\mathbb{S}_N$ generated by these three permutations is called the *monodromy group* of the dessin.

It is not difficult to reconstruct the dessin from its monodromy, since r_b, r_w, and r_f carry all the combinatorial information.

In fact, the complex structure of a Belyi surface is determined by its dessin. More precisely, the combinatoric data of the dessin determines a Fuchsian group that gives as quotient space the Belyi surface. Given the integers l, m, and n, let $T(l, m, n)$ be a hyperbolic triangle from angles $\pi/l, \pi/m$, and π/n. Construct the group $\tilde{\Delta}(l, m, n)$ generated by the reflections across the three sides of $T(l, m, n)$. Let $\Delta(l, m, n)$ be the index two subgroup formed by the orientation preserving elements of $\tilde{\Delta}(l, m, n)$ (these are the words of even length in the three reflections). The Fuchsian group $\Delta(l, m, n)$ is called a *triangle group*, and it has the well-known presentation $< \gamma_b, \gamma_w, \gamma_f; \gamma_b^l = \gamma_w^m = \gamma_f^n = \gamma_f \gamma_w \gamma_b = 1 >$ (the three generators giving this presentation can be chosen in some geometrical way, as explained at the beginning of Section 4.1).

There is a natural group homomorphism determined by every dessin $\mathcal{D}$ of type (l, m, n), going from the triangle group $\Delta(l, m, n)$ onto some group of permutations, its definition being simply

$$\begin{array}{rcl} \theta : \Delta(l, m, n) & \longrightarrow & G_{\mathcal{D}} \\ \gamma_i & \longmapsto & r_i \end{array}$$

for $i = b, w, f$.

Now, if G_k is the stabilizer of some k in $G_{\mathcal{D}}$, let us denote $\Gamma = \theta^{-1}\{G_k\}$. This Γ is a cocompact Fuchsian group inside $\Delta(l, m, n)$, its index being the number N of sides of $\mathcal{D}$; Γ is called the *fundamental group of $\mathcal{D}$ inside* $\Delta(l, m, n)$. It is well defined, in terms of $\mathcal{D}$, up to conjugacy in $\Delta(l, m, n)$. Thus $\mathbb{D}/\Gamma$ is *the* Riemann surface underlying the dessin $\mathcal{D}$, and the Belyi function corresponds just to the natural projection $\mathbb{D}/\Gamma \to \mathbb{D}/\Delta(l, m, n)$.

An important characterization of Belyi surfaces in terms of Fuchsian groups is then:

Theorem 2.3. *X is a Belyi surface if and only if it is isomorphic to $\mathbb{D}/\Gamma$, where Γ is a subgroup of a Fuchsian triangle group.*

Remark 2.4. The roles played by the vertices and faces of a dessin may always be interchanged. For instance, coloring the vertices with the opposite color, keeping incidence relations unchanged, gives again a dessin. In terms of Belyi functions, this corresponds just to the fact that $1 - \beta$ is again a Belyi function when β is.

We could also keep the role of black vertices, and interchange those of the white vertices and the faces, by doing the following: Mark one point in the interior of each face, and remove the former white vertices. These marked points will be the new set of white vertices. The incidence relations are obtained by looking around each former white vertex. Suppose v is one of them: Around it, we find alternatively new white vertices (as many as the number of closed faces that contained v) and black vertices (exactly those that were incident with v), that in the new dessin form a circular subgraph. Obviously, the former white vertices correspond to the faces of the new dessin and vice versa. In terms of Belyi functions, what we have done is replace β with $\beta/(\beta - 1)$.

Of course, the interchange of the role of black vertices and faces can be done in a similar way, leading the replacement of β by $1/\beta$.

All the dessins that are obtained from a given one by this procedure carry no additional information. Thus we shall consider them equivalent. Nevertheless, in the sequel, we will make few explicit references to this equivalence relation. We will choose a representative following a convention: When we talk about a dessin of type (l, m, n), the three periods will refer respectively to black vertices, white vertices, and faces, in that order. Also, we arrange the periods in increasing order in almost all cases. There will be only one exception to this rule: When two of the periods are equal, it will be very convenient to let the other period refer to the faces, even if the latter is smaller than the first couple (we work, for instance, with dessins of type $(7, 7, 3)$ instead of $(3, 7, 7)$).

Remark 2.5. Fuchsian triangle groups have the remarkable rigidity property of being determined, up to conjugation in $\mathbb{PSL}(2, \mathbb{R})$, by the three periods. In fact, the ordering of the periods is also not relevant. Here we treat triangle groups because of their relation with dessins, so we use the same convention as in Remark 2.4 for the ordering of their periods.

A very interesting class of dessins, and hence of Belyi surfaces, is called *uniform*. These are dessins of type (l, m, n) such that all the cycles of r_b have length l, and accordingly all the cycles of r_w and r_f have length m and n, respectively. This means that all the black vertices (respectively, white vertices or faces) of the dessin have the same valency.

If $\mathcal{D}$ is uniform, it is easy to see that the corresponding Γ acts in the hyperbolic disc without fixing points. If not, let $1 \neq \gamma \in \Gamma$ be an element with a fixed point in $\mathbb{D}$. Then γ has finite order, and therefore it is conjugate in the corresponding triangle group $\Delta(l, m, n)$ to some nontrivial power of γ_i for $i = b, w$, or f. Suppose without loss of generality that it is conjugate to γ_b^t, where $0 < t < l$. All the cycles of $\theta(\gamma)$ have equal length, since $\theta(\gamma)$ is conjugate to r_b^t in $G_{\mathcal{D}}$. Now, as $\theta(\gamma)$ belongs to the stabilizer G_k, it follows that $\theta(\gamma) = 1$, and therefore $\theta(\gamma_b^t) = r_b^t = 1$, which is obviously absurd.

The fundamental group of a uniform dessin inside the group $\Delta(l, m, n)$ is then a uniformizing group for the underlying surface, and then it is isomorphic to the topological fundamental group of the surface (this explains the notation employed). A surface that carries a uniform dessin is in turn called a *smooth* or *uniform Belyi surface*. Since not every dessin is uniform, not every Belyi surface is smooth.

Now, let an *automorphism* of a dessin be simply a bijection of its set of edges that commutes with r_b and r_w, and hence with r_f. It turns out that every automorphism in the dessin sense defines an automorphism (biholomorphic self-mapping) of the underlying Belyi surface, although the group of automorphisms of the surface could still be larger.

When the automorphism group acts transitively on the set of edges, the fundamental group of the dessin inside the triangle group is exactly the kernel of θ, as the stabilizer G_k is in that case trivial. Hence the group Γ turns out to be a normal subgroup of the corresponding triangle group. These types of dessins are called *regular*.

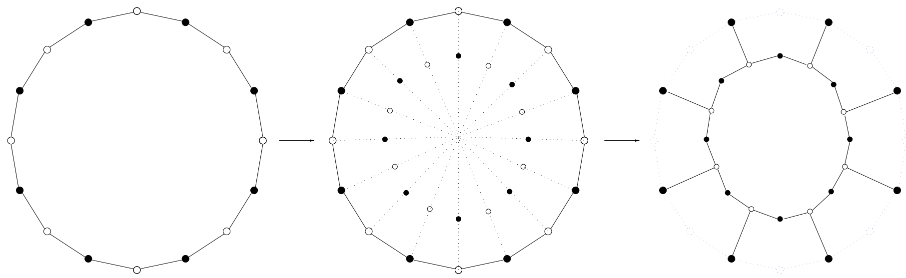

FIGURE 1. The action of the type 1 truncation in a face ($n = 4$).

3. SURGERY ON UNIFORM DESSINS

We introduce a concept that we will use often from now on.

Definition 3.1. A *surgery* is a function S from the class of (l', m', n')-uniform dessins to the class of (l, m, n)-uniform dessins such that:

i) $S(\mathcal{D}')$ is embedded in the same surface as $\mathcal{D}'$;

ii) the set of vertices, and also the incidence relations of $S(\mathcal{D}')$, is defined in terms of those of $\mathcal{D}'$ by an algorithm that depends only on the periods l', m', and n', and not on the particular combinatorial structure of $\mathcal{D}'$, neither on the surface underlying $\mathcal{D}'$;

iii) $S(\mathcal{D}')$ has more edges than $\mathcal{D}'$.

We proceed now to describe the basic surgery procedures. The first four are generic (they are defined in a one- or two-parametric family of types of uniform dessins), and the last four are rigid, in the sense that they can be done just on a unique type of dessins. Through the descriptions, when the vertices around a face are denoted by $b_1, w_1, b_2, w_2, \ldots$, it should be understood that the labels correspond to the (alternate black and white) vertices that lie on the border of that face, according to the orientation of the surface. Other points labeled with a "b" or "B" will always be black, while points labeled with "w" or "W" will be white.

Baricentral surgery. Let $\mathcal{D}$ be a uniform dessin of type (n, n, n). The baricentral dessin $\mathrm{Bar}(\mathcal{D})$ is constructed defining its white and black vertices by the same procedure in all the faces of $\mathcal{D}$. Let $b_1, w_1, \ldots, b_n, w_n$ be the vertices in the border of a face of $\mathcal{D}$, counted according to the orientation of the surface. Choose a point p in the interior of the face, and points p_j (respectively, q_j) for $j = 1, \ldots, n$ in the interior of the edge $[b_j, w_j]$ (respectively, $[w_j, b_{j+1}]$). Mark now one point B_i (B stands for black) in the interior of $[p, p_i]$, the geodesic arc joining p with p_i ($i = 1, \ldots, 2n$), and also a point W_i (white) in the interior of $[p, q_i]$. Each time it occurs that two faces of $\mathcal{D}$ meet in one edge, then the vertices of $\mathrm{Bar}(\mathcal{D})$ we have created corresponding to that edge, one inside each of the faces of $\mathcal{D}$, have opposite color. They become adjacent vertices in $\mathrm{Bar}(\mathcal{D})$. Finally, join B_i with W_i and W_{i-1}. This way, every white vertex, black vertex, or face of $\mathcal{D}$ corresponds now to a face of $\mathrm{Bar}(\mathcal{D})$, that is uniform of type $(3, 3, n)$.

Medial surgery. If $\mathcal{D}$ is uniform of type (n, n, m), then the medial dessin $\mathrm{Med}(\mathcal{D})$ is constructed as follows. Every black or white vertex of $\mathcal{D}$ is turned a white vertex of $\mathrm{Med}(\mathcal{D})$, its black vertices being the midpoints of edges of $\mathcal{D}$. Every black vertex of $\mathrm{Med}(\mathcal{D})$ is now joined just with the two endpoints of the side of $\mathcal{D}$ on which it was, and each white vertex is joined with n black vertices, following the sides of $\mathcal{D}$. Clearly, $\mathrm{Med}(\mathcal{D})$ is uniform and has type $(2, n, 2m)$.

Type 1 truncation surgery. The dessin $\mathrm{Trunc}_1(\mathcal{D})$ is a $(2, 3, 2n)$ uniform dessin constructed from a given $(2, n, 2n)$ uniform dessin $\mathcal{D}$ as follows: The vertices of $\mathrm{Trunc}_1(\mathcal{D})$ are of two kinds: the black vertices of $\mathcal{D}$, that will still be black vertices of $\mathrm{Trunc}_1(\mathcal{D})$, and a set of newly created vertices, in the following way. Let $b_1, w_1, \ldots, b_{2n}, w_{2n}$ be the vertices in the boundary of a face of $\mathcal{D}$. Choose a point p in the interior of the face, and mark a point in $[p, b_j]$ as a white vertex W_j of $\mathrm{Trunc}_1(\mathcal{D})$, and also a point B_j in $[p, w_j]$ as a black vertex of $\mathrm{Trunc}_1(\mathcal{D})$, for $j = 1, \ldots, 2n$. Set the incidence relations as follows: B_j is joined by an edge with W_j and W_{j+1}, and b_j is joined with W_j. This way the faces of $\mathrm{Trunc}_1(\mathcal{D})$ are in one-to-one correspondence with both the faces and the white vertices of $\mathcal{D}$.

Type 2 truncation surgery. Suppose that $\mathcal{D}$ is a $(3, n, 3n)$-uniform dessin. The construction of the type 2 truncation $\mathrm{Trunc}_2(\mathcal{D})$ goes as follows. Let $b_1, w_1, \ldots, b_{3n}, w_{3n}$ be the vertices in the boundary of a face of $\mathcal{D}$. Choose a point p in the interior of the face, and points B_j in $[p, w_j]$ and W_j in $[p, b_j]$ as in the case of the type 1 truncation. Choose also points $\tilde{B}_j$ in $[W_j, b_j]$, $j = 1, \dots, 3n$, and keep the black vertices of $\mathcal{D}$ as white vertices of $\mathrm{Trunc}_2(\mathcal{D})$, changing the label from b_j to W_j^b. Now, the incidence relations that define $\mathrm{Trunc}_2(\mathcal{D})$ are as follows: B_j is joined with W_j and W_{j-1}, whereas $\tilde{B}_j$ is joined with W_j and with W_j^b. It is not difficult to show that the so-created $\mathrm{Trunc}_2(\mathcal{D})$ is uniform, and has type $(2, 3, 3n)$. It has one face for each face or white vertex of $\mathcal{D}$.

Rigid surgery $R_\mathbf{2}$. Let $\mathcal{D}$ be a uniform dessin of type $(7, 7, 2)$. Let f be a face of $\mathcal{D}$, its border being a circular 4-graph in the vertices b_1, w_1, b_2, and w_2. Let us choose the following points: B in the interior of f, p_j in the interior of the side $[b_j, w_j]$, $j = 1, 2$, and q_j in the interior of $[w_j, b_{j+1}]$. Now, let W_j^p be some point in the interior of the line $[B, p_j]$. Choose also two points B_j^b and W_j^b in the line $[W_j^p, b_j]$ with B_j^b closer than W_j^b to W_j^p and, in a similar way, choose B_j^w and W_j^w in the line $[W_j^p, w_j]$. Finally, choose $\tilde{B}_j^w$ (respectively, $\tilde{B}_j^b$) in the interior of the triangle $[W_j^w, w_j, q_j]$ (respectively, the triangle $[W_j^b, q_{j-1}, b_j]$). Repeat the same procedure in every face of $\mathcal{D}$, color the B-points in black and the W-points in white, and define incidence relations among them in the following way. W_j^p is joined by an edge with B, B_j^w, and B_j^b. W_j^w is incident with $B_j^w, \tilde{B}_j^w$, and the $\tilde{B}^w$-type point corresponding to w_j as a vertex of f_j, the other face of $\mathcal{D}$ that meets f at $[b_j, w_j]$. Similarly, W_j^b is incident with $B_j^b, \tilde{B}_j^b$, and the $\tilde{B}^b$-type vertex inside f_j that corresponds to b_j.

It can be easily checked that the so-created dessin is uniform and has type $(2, 3, 7)$.

Rigid surgery $R_\mathbf{3}$. Let $\mathcal{D}$ be a uniform dessin of type $(8, 8, 3)$. Let f be a face of $\mathcal{D}$, its border being a circular 6-graph in the vertices $b_1, w_1, \dots, b_3, w_3$. Let us choose the following points: W in the interior of f, p_j in the interior of the side $[b_j, w_j]$ of $\mathcal{D}$, $j = 1, 2, 3$, and q_j in the interior of $[w_j, b_{j+1}]$. Now, let B_j^q and W_j^{bw} be two ordered points in the (oriented) line $[W, q_j]$ and also B_j^b, W_j^b in the oriented line $[W_j^{bw}, b_{j+1}]$, and B_j^w and W_j^w in the oriented line $[W_j^{bw}, w_j]$. Finally, choose $\tilde{B}_j^w$ (respectively, $\tilde{B}_j^b$) in the interior of the triangle $[W_j^w, w_j, q_j]$ (respectively, the triangle $[W_j^b, b_{j+1}, p_{j+1}]$). Repeat the same procedure in every face of $\mathcal{D}$, color the B-points in black and the W-points in white, and define incidence relations among them in the following way. W is joined by an edge with B_j^q, $j = 1, 2, 3$. W_j^{bw} is incident with B_j^q, B_j^b, and B_j^w. Finally, W_j^b is incident with $B_j^b, \tilde{B}_j^b$, and the $\tilde{B}^b$-type point corresponding to b_{j+1} as a vertex of f_{j+1}, the other face of $\mathcal{D}$ that meets f at $[b_{j+1}, w_{j+1}]$. Similarly, W_j^w is incident with $B_j^w, \tilde{B}_j^w$, and the $\tilde{B}^w$-type vertex inside f_j that corresponds to w_j, where f_j meets f at the side $[b_j, w_j]$.

The newly created dessin is uniform of type $(2, 3, 8)$.

Rigid surgery $R_\mathbf{5}$. Let $\mathcal{D}$ be a uniform dessin of type $(4, 4, 5)$, and let $b_1, w_1, \dots, b_5, w_5$ be the set of vertices of the circular subgraph of $\mathcal{D}$ that is the border of a face f. Choose a point p in the interior of f; let p_j and q_j be an interior point of the side $[b_j, w_j]$ and $[w_j, b_{j+1}]$, respectively. We construct a new set of vertices as follows. Choose points B_j^p in the interior of the line $[p, p_j]$, and W_j^q in $[p, q_j]$. Also, choose B_j^b (respectively, B_j^w) in the interior of the line $[W_j^q, b_{j+1}]$ (respectively, the line $[W_j^q, w_j]$), and color all the B-points in black and all the W-points in white. Also, color the former black or white vertices of $\mathcal{D}$ in white, changing the label from b_j into $W_{j,b}$ and w_j into $W_{j,w}$ to make clear the change.

The incidence relations that define the new dessin are as follows: W_j^q is joined with B_j^w, B_j^b, B_j^p, and B_{j+1}^p. Also B_j^b (respectively, B_j^w) is also incident with $W_{j+1,b}$ (respectively, $W_{j,w}$). The new dessin clearly has type $(2, 4, 5)$, and is uniform.

Rigid surgery $R_\mathbf{7}$. Let $\mathcal{D}$ be a uniform dessin of type $(3, 3, 7)$, and let $b_1, w_1, \dots, b_7, w_7$ be the set of vertices of the circular subgraph of $\mathcal{D}$ that is the border of a face f. Choose a point p in the interior of f, let p_j and q_j be an interior point of the sides $[b_j, w_j]$ and $[w_j, b_{j+1}]$, respectively. We construct a new set of vertices as follows: Choose points B_j in the interior of the line $[p, p_j]$, $j = 1, \dots, 7$, and (ordered in this way) W_j^q, B_j^q, and W_j^{bw} in the directed line from p to q_j. Finally, choose B_j^b (respectively, B_j^w) in the interior of the line $[W_j^{bw}, b_{j+1}]$ (respectively, the line $[W_j^{bw}, w_j]$), and color all the B-points in black and all the W-points in white. Also, color the former black or white vertices of $\mathcal{D}$ in white, changing the label from b_j into $W_{j,b}$ and w_j into $W_{j,w}$ to make clear the change of its role.

The incidence relations that define the new dessin are as follows: W_j^q is joined with B_j, B_{j+1}, and B_j^q. W_j^{bw} is incident with B_j^q, B_j^b, and B_j^w. Finally, B_j^b (respectively, B_j^w) is also incident with $W_{j+1,b}$ (respectively, $W_{j,w}$). The new dessin has type $(2, 3, 7)$, and is uniform.

Two of the eight preceding surgeries already appear in [Singerman 01] and [Singerman and Syddall 01] in a way similar to that employed here. These are the medial and first truncation, introduced by Singerman and Syddall in the context of *maps*.

We may think also about surgery procedures following from the composition of some of these eight basic ones: For instance, we can pass from $\mathcal{D}$ to $\mathrm{Bar}(\mathcal{D})$, and then to $\mathrm{Med}(\mathrm{Bar}(\mathcal{D}))$. Also, it could be sometimes necessary to pass to an equivalent dessin (interchanging the role of vertices and faces) before doing the second surgery. For instance, if $\mathcal{D}$ has type $(8,8,4)$, then $\mathrm{Med}(\mathcal{D})$ has type $(2,8,8)$. After passing to the equivalent $(8,8,2)$-dessin, another medial surgery can be performed.

Proposition 3.2. *Every possible surgery can be expressed as a composition of the eight basic surgeries, modulo dessin equivalence.*

Proof: Note first that a surgery $\mathcal{S}$ from (l',m',n') uniform dessins to (l,m,n) uniform dessins exists if and only if the triangle group $\Delta(l',m',n')$ is contained in $\Delta(l,m,n)$. To see this, let Γ' be any uniformizing group inside $\Delta(l',m',n')$. The application $\mathbb{D}/\Gamma \to \mathbb{D}/\Delta(l',m',n')$ is a Belyi function with associated dessin $\mathcal{D}'$, and $\mathcal{S}(\mathcal{D}')$ determines a uniformizing group Γ for the same surface, with Γ contained in $\Delta(l,m,n)$. Conjugating if necessary, we can suppose $\Gamma' = \Gamma$. It follows that any uniformizing group inside $\Delta(l',m',n')$ is also contained in $\Delta(l,m,n)$, and therefore that $\Delta(l',m',n') < \Delta(l,m,n)$.

In fact, it is not difficult to determine the inclusions of triangle groups that correspond to the eight basic surgeries:

$$\begin{gathered}
\Delta(n,n,n) <_3 \Delta(3,3,n) \equiv \text{Baricentral}\\
\Delta(n,n,m) <_2 \Delta(2,n,2m) \equiv \text{Medial}\\
\Delta(2,n,2n) <_3 \Delta(2,3,2n) \equiv \text{Truncation (1)}\\
\Delta(3,n,3n) <_4 \Delta(2,3,3n) \equiv \text{Truncation (2)}\\
\Delta(7,7,2) <_9 \Delta(2,3,7) \equiv R_2\\
\Delta(8,8,3) <_{10} \Delta(2,3,8) \equiv R_3\\
\Delta(4,4,5) <_6 \Delta(2,4,5) \equiv R_5\\
\Delta(3,3,7) <_8 \Delta(2,3,7) \equiv R_7,
\end{gathered}$$

where the subscript k in the symbol $<_k$ stands for the index.

Now, looking at the list of possible inclusions between triangle groups that was first given in [Singerman 72], it can be seen that any inclusion can be expressed as a chain of inclusions involving just these eight. Therefore, any surgery is a composition of the basic ones. □

4. SURGERY TESTS FOR REGULAR DESSINS

The image of a regular dessin after performing a surgery is certainly uniform, but in most cases the regularity will have been lost. Nevertheless, we are interested in regular dessins related by surgery.

Let $\mathcal{D}',\mathcal{D}$ be regular dessins of types (l',m',n') and (l,m,n), with monodromy homomorphisms $\theta' : \Delta(l',m',n') \longrightarrow G_{\mathcal{D}'}$ and $\theta : \Delta(l,m,n) \longrightarrow G_{\mathcal{D}}$. Suppose further that there exists a surgery that maps $\mathcal{D}'$ into $\mathcal{D}$. Both dessins are then embedded in the same surface, that is uniformized by $\Gamma = \ker(\theta) = \ker(\theta')$.

Let us define $\psi : G_{\mathcal{D}'} \longrightarrow G_{\mathcal{D}}$ by $\psi(x) = \theta(i(\gamma'))$, where i stands just for the inclusion of $\Delta(l',m',n')$ in $\Delta(l,m,n)$, and $\gamma' \in \Delta(l',m',n')$ is any element such that $\theta'(\gamma') = x$.

It is not difficult to show that ψ is a well-defined injective homomorphism that makes commutative the diagram

$$\begin{array}{ccc}
\Delta(l,m,n) & \xrightarrow{\theta} & G_{\mathcal{D}}\\
i\uparrow & & \uparrow\psi\\
\Delta(l',m',n') & \xrightarrow{\theta'} & G_{\mathcal{D}'}.
\end{array}$$

We will try to decide when the previous situation actually occurs. More precisely, let $\mathcal{D}$ be a regular dessin of type (l,m,n) in some surface S, and suppose there exists a surgery from (uniform) dessins of type (l',m',n') to (uniform) dessins of type (l,m,n). We would like to decide if $\mathcal{D}$ is the surgery image of some $\mathcal{D}'$. If so, $\mathcal{D}'$ is necessarily also regular.

We proceed as follows: Starting from the monodromy homomorphism $\theta : \Delta(l,m,n) \longrightarrow G_{\mathcal{D}}$ of $\mathcal{D}$, consider the restriction $\theta|_{\Delta(l',m',n')}$. Now let $\pi : \theta(\Delta(l',m',n')) \longrightarrow \mathbb{S}_{N'}$, where $N' = |\theta(\Delta(l',m',n'))|$, be the permutation representation of $\theta(\Delta(l',m',n'))$ given by products on the right.

Then, if $\mathcal{D}$ is the surgery of the regular dessin $\mathcal{D}'$, the monodromy homomorphism of $\mathcal{D}'$ is precisely $\pi \circ \theta|_{\Delta(l',m',n')}$. On the other hand, if $\pi \circ \theta|_{\Delta(l',m',n')}$ is not a valid homomorphism for a regular dessin of type (l',m',n'), then $\mathcal{D}$ is not the surgery image of a dessin of type (l',m',n') induced by the inclusion $\Delta(l',m',n') < \Delta(l,m,n)$.

4.1 Generators of Triangle Groups

For a practical application of the surgery tests, we need to know the explicit expression of a set of generators of the smaller triangle group in terms of those of the bigger one. We will choose generators for triangle groups always in the same way:

Consider the group $\Delta(l,m,n)$ (note Remark 2.5 about the order in which we write the three periods l,m, and n). It is constructed in terms of a triangle $T(l,m,n)$

$\mathbf{\Delta'}$	$\mathbf{\Delta}$	$\boldsymbol{\gamma'_b}$	$\boldsymbol{\gamma'_w}$	$\boldsymbol{\gamma'_f}$
$\Delta(n,n,n)$	$\Delta(3,3,n)$	γ_f	$\gamma_b^2\gamma_f\gamma_b$	$\gamma_w^2\gamma_f\gamma_w$
$\Delta(n,n,m)$	$\Delta(2,n,2m)$	$\gamma_f\gamma_w^2\gamma_b$	γ_w	γ_f^2
$\Delta(2,n,2n)$	$\Delta(2,3,2n)$	$\gamma_b\gamma_w\gamma_f\gamma_b$	$(\gamma_w^2\gamma_b)^2$	γ_f
$\Delta(3,n,3n)$	$\Delta(2,3,3n)$	$\gamma_w^2\gamma_b\gamma_f^{-1}\gamma_w$	$\gamma_w\gamma_f^3\gamma_w^2$	γ_f
$\Delta(7,7,2)$	$\Delta(2,3,7)$	$\gamma_w^2\gamma_b\gamma_f^5\gamma_w$	$\gamma_f^5\gamma_w^2\gamma_b\gamma_f^2$	γ_b
$\Delta(8,8,3)$	$\Delta(2,3,8)$	$\gamma_f^5\gamma_w^2\gamma_b\gamma_f^3$	$\gamma_f^2\gamma_w^2\gamma_b\gamma_f^6$	γ_w
$\Delta(4,4,5)$	$\Delta(2,4,5)$	$\gamma_b\gamma_f^4\gamma_w\gamma_f\gamma_b$	$\gamma_w^3\gamma_b\gamma_w\gamma_b\gamma_w$	γ_f
$\Delta(3,3,7)$	$\Delta(2,3,7)$	$\gamma_w^2\gamma_f^2\gamma_w\gamma_f^5\gamma_w$	$\gamma_w^2\gamma_b\gamma_f^6\gamma_w\gamma_f\gamma_b\gamma_w$	γ_f

TABLE 1. Relation between the generators of the triangle groups involved in the basic surgeries.

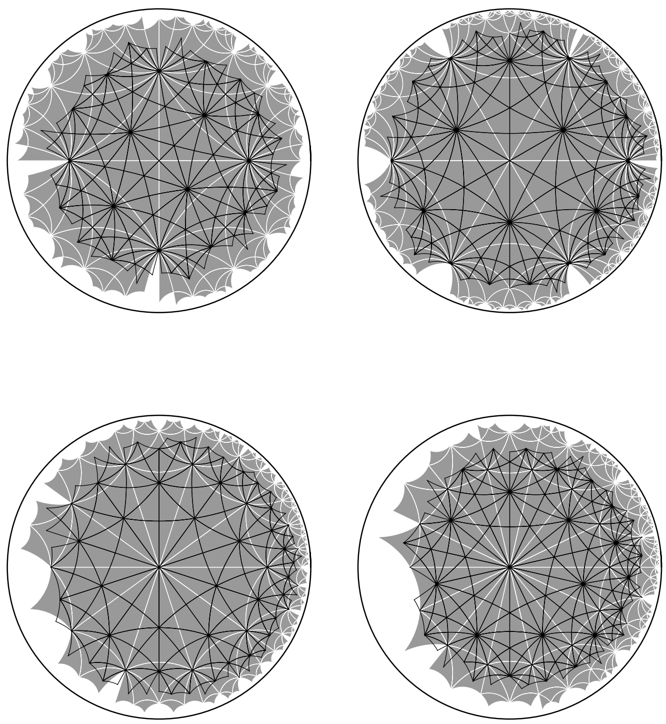

FIGURE 2. Tessellations concerning the inclusions of triangle groups related to the rigid basic surgeries.

whose vertices are, *in clockwise order*, p_l, p_m, p_n (the angle at p_j being π/j). Let γ_b (respectively, γ_w, γ_f) be a noneuclidean turn through an angle $2\pi/l$ around p_l (respectively, $2\pi/m$ around p_m, $2\pi/n$ around p_n) *in anticlockwise order*. This choice of the generators yields

$$< \gamma_b, \gamma_w, \gamma_f; \gamma_b^l = \gamma_w^m = \gamma_f^n = \gamma_f\gamma_w\gamma_b = 1 > \qquad (4\text{–}1)$$

as a presentation of $\Delta(l,m,n)$.

Lemma 4.1. *Table 1 shows a set of generators for a group of the type of Δ' in terms of that of Δ, both sets giving presentations like* (4–1) *for Δ and Δ'. The inclusion $\Delta' < \Delta$ runs over the list of eight inclusions between triangle groups associated with the eight basic surgeries.*

Proof: It can be directly checked after a careful look at the geometry of each inclusion. We can see how a group like Δ' fits inside Δ by looking at a simultaneous picture of both triangulations of the unit disc, and then with a bit of patience one can check the relation between the generators of both groups.

The picture of the triangulations is very easy to guess in the case of the first four inclusions. As for the inclusions associated to the basic rigid surgeries, we refer to Figure 2. It was done using the wonderful package [HTessellate 94] developed in Finland for hyperbolic geometry computations with Mathematica® ([Mathematica 03]). The tessellation by grey triangles in Figure 2 corresponds to the bigger triangle group, and the matrix-like displayed pictures refer to the surgeries

$$\begin{array}{cc} R_2 & R_3 \\ R_5 & R_7. \end{array}$$

□

5. QUASIPLATONIC SURFACES

Definition 5.1. A compact Riemann surface is called *quasiplatonic* if it carries a regular dessin.

Quasiplatonic surfaces are often called *regular Belyi surfaces*. They have a very interesting property: Given a quasiplatonic surface X, if Y is not isomorphic to X but is close enough to it in the topology of the moduli space M_g of compact Riemann surfaces of genus g, then Y has strictly less automorphisms than X. That is the reason why they are also referred to as *surfaces with many automorphisms*: Indeed, every point of M_g with such property corresponds to a quasiplatonic surface, and hence both concepts turn out to be equivalent (see [Wolfart 97]).

By definition, S is quasiplatonic if and only if it is uniformized by a (torsion-free) normal subgroup Γ of some Fuchsian triangle group Δ. The surfaces that will concern us mainly in the sequel are those that are quasiplatonic in more than one way. In terms of groups, this means just requiring the uniformizing group Γ to be also normally contained in a second triangle group Δ', with $\Delta \neq \Delta'$. Recall that Γ is uniquely determined just by the surface X.

We give the following definition in terms of dessins:

Definition 5.2. Let S be a quasiplatonic surface. We will term it *multiply* or *nonmultiply* quasiplatonic according to whether S carries several regular dessins or a unique one.

It should be noted in passing that the number of regular dessins that a compact Riemann surface carries is always finite.

We shall investigate under what circumstances a quasiplatonic surface may, in fact, be multiply quasiplatonic. For that purpose, the following terminology will be very convenient.

Definition 5.3. Let S be a quasiplatonic surface and $\mathcal{D}$ a regular dessin inside S. If $\mathcal{D}$ cannot be obtained by surgery on another regular dessin, we call it a *minimal regular dessin* of S.

Obviously, any nonmultiply quasiplatonic surface carries just one (minimal) regular dessin. For a multiply quasiplatonic surface, minimal regular dessins always exist, but they may or may not be unique: It depends on each particular surface.

Suppose $S = \mathbb{D}/\Gamma$ is quasiplatonic, and let Δ be the triangle group associated with a given minimal regular dessin $\mathcal{D}$. Let Γ be the fundamental group of $\mathcal{D}$ in Δ, and consider $N(\Gamma)$, the normalizer of Γ in $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$. As $N(\Gamma)$ is a Fuchsian group that contains Δ, it follows that it is a triangle group as well. It may occur that $\Delta = N(\Gamma)$, and this means that S is nonmultiply quasiplatonic, since then $\mathcal{D}$ is the only regular dessin in S.

If the inclusion $\Delta < N(\Gamma)$ is proper, then S is multiply quasiplatonic. The inclusion $\Gamma \triangleleft N(\Gamma)$ induces another regular Belyi function, and an associated regular dessin $\mathcal{MD}$ that is in that case never minimal. Of course, $\mathcal{MD}$ is constructed from $\mathcal{D}$ by some surgery.

We then have the following lemma.

Lemma 5.4. *If S is uniformized by Γ, and Δ is the triangle group that corresponds to a minimal regular dessin, then S is multiply quasiplatonic if and only if $\Delta \neq N(\Gamma)$.*

Or, equivalently:

Lemma 5.5. *A compact Riemann surface is multiply quasiplatonic if and only if it carries a regular dessin that is transformed into another regular dessin by some surgery.*

At this point, the following definition is natural:

Definition 5.6. If S is a quasiplatonic surface uniformized by Γ, denote by $\mathcal{MD}$ the regular dessin induced by the inclusion $\Gamma \triangleleft N(\Gamma)$. We will refer to $\mathcal{MD}$ as the *maximal regular dessin* of S.

Note that, contrary to the minimal regular dessins, $\mathcal{MD}$ is always uniquely determined. The term *maximal* in the previous definition refers to the fact that $\mathcal{MD}$ is the regular dessin of the surface that has the most edges, and never produces another dessin by surgery. It does not refer to maximality in terms of Fuchsian groups, since the

Case	K	Minimal	Intermediate	$\mathcal{MD}$
A	5	$\mathcal{D}_1$, type $(2n, 2n, 2n)$ $\mathcal{D}_2$, type $(4n, 4n, n)$	Bar$(\mathcal{D}_1)$ Med$(\mathcal{D}_1)$	Med(Bar$(\mathcal{D}_1)$)
B	4	$\mathcal{D}$, type (n, n, n)	Bar$(\mathcal{D})$ Med$(\mathcal{D})$	Med(Bar$(\mathcal{D}_1)$)
C	4	$\mathcal{D}_1$, type $(2n, 2n, n)$ $\mathcal{D}_2$, type $(4, 4, n)$	Med$(\mathcal{D}_1)$	Med$(\mathcal{D}_2)$
D	4	$\mathcal{D}_1$, type $(4n, 4n, n)$ $\mathcal{D}_2$, type $(3, 3, 2n)$	Med$(\mathcal{D}_1)$	Med$(\mathcal{D}_2)$
E	3	$\mathcal{D}_1$, type $(2n, 2n, m)$ $\mathcal{D}_2$, type $(2m, 2m, n)$	-	Med$(\mathcal{D}_2)$
F	3	$\mathcal{D}_1$, type $(2, n, 2n)$ $\mathcal{D}_2$, type $(3, 3, n)$	-	Med$(\mathcal{D}_2)$
G	3	$\mathcal{D}$, type $(2n, 2n, n)$	Med$(\mathcal{D})$	Med(Med$(\mathcal{D})$)
H	3	$\mathcal{D}$, type $(4n, 4n, n)$	Med$(\mathcal{D})$	Trunc$_1$(Med$(\mathcal{D})$)
I	2	$\mathcal{D}$, type (n, n, n)	-	Bar$(\mathcal{D})$
J	2	$\mathcal{D}$, type (n, n, m)	-	Med$(\mathcal{D})$
K	2	$\mathcal{D}$, type $(2, n, 2n)$	-	Trunc$_1(\mathcal{D})$
L	2	$\mathcal{D}$, type $(3, n, 3n)$	-	Trunc$_2(\mathcal{D})$

TABLE 2. The possibilities for the regular dessins inside a multiply quasiplatonic surface of nonarithmetic type.

triangle group $N(\Gamma)$ may be either a maximal Fuchsian group or not.

The nonmultiply quasiplatonic case occurs obviously when the group Δ corresponding to a minimal regular $\mathcal{D}$ is a maximal triangle group, as then there is no possible surgery. But even if $\Delta < \Delta'$, and hence a surgery could be applied, it could happen that $N(\Gamma) = \Delta$, since normality does not extend automatically from Δ to a larger supergroup (that is, the result of the surgery could be a nonregular uniform dessin).

6. MULTIPLY QUASIPLATONIC SURFACES

In what follows, we suppose S to be a multiply quasiplatonic surface. Let $\mathcal{D}$ be a minimal regular dessin in S, with associated triangle group Δ.

Two very different cases occur, depending on the arithmeticity of $\mathcal{D}$, which is just the arithmeticity of Δ. We treat both situations separately.

6.1 The Nonarithmetic Case

With the exception of a finite number of signatures, triangle groups are nonarithmetic (see [Takeuchi 77a]). The nonarithmetic triangle groups are thus the *generic* ones. The quasiplatonic surfaces carrying a regular dessin corresponding to one of these triangle groups are in turn also called nonarithmetic. We are interested in the different possibilities that may occur for nonarithmetic multiply quasiplatonic surfaces.

Theorem 6.1. *Let S be a multiply quasiplatonic surface of nonarithmetic type. Let us make, as usual, no distinction between isomorphic dessins on S. Then the complete list of regular dessins embedded into S agrees with one of the rows of Table 2, where K stands for the total number of regular dessins (counting the maximal $\mathcal{MD}$, as well as all the minimal and intermediate dessins).*

Proof: Let $\mathcal{D}$ be any minimal regular dessin inside the multiply quasiplatonic surface S. Since, by Lemma 5.4, $N(\Gamma)$ is a larger triangle group, it follows that $\mathcal{D}$ is of type (n, n, m), $(3, n, 3n)$, or $(2, n, 2n)$, and that a composition of basic surgeries of the generic types (baricentral, medial, and truncations) transforms $\mathcal{D}$ into $\mathcal{MD}$. This is because the rigid surgeries involve only arithmetic groups (see [Singerman 72], [Takeuchi 77a], and [Takeuchi 77b]), and therefore they cannot appear in the nonarithmetic surface S.

If there is just one minimal dessin $\mathcal{D}$ and just one basic surgery transforms it into the maximal $\mathcal{MD}$, S corresponds to cases I to L in Table 2.

On the other hand, S could carry more than one minimal dessin, and also some intermediate dessins in between the minimal ones and $\mathcal{MD}$. All the different possibilities follow immediately after considering which are the maximal chains of surgeries that could appear in the same surface when $\mathcal{MD}$ is of some special type.

If the type of $\mathcal{MD}$ is $(2, 3, 4n)$, we obtain case A corresponding to the maximal diagram

$$\begin{tikzcd}
 & (2,3,4n) & \\
(2,2n,4n) \arrow[ur, "\mathrm{Trunc}_1"] & & (3,3,2n) \arrow[ul, "\mathrm{Med}"'] \; ; \\
(4n,4n,n) \arrow[u, "\mathrm{Med}"'] & (2n,2n,2n) \arrow[ul, "\mathrm{Med}"] \arrow[ur, "\mathrm{Bar}"'] &
\end{tikzcd}$$

also for $\mathcal{MD}$ of type $(2, 3, 2n)$, we get case B, since we have

$$\begin{tikzcd}
 & (2,3,2n) & \\
(2,n,2n) \arrow[ur, "\mathrm{Trunc}_1"] & & (3,3,n) \arrow[ul, "\mathrm{Med}"'] \; ; \\
 & (n,n,n) \arrow[ul, "\mathrm{Med}"] \arrow[ur, "\mathrm{Bar}"'] &
\end{tikzcd}$$

whereas case C arises if the type of $\mathcal{MD}$ is $(2, 4, 2n)$, as we see

$$\begin{tikzcd}
 & (2,4,2n) & \\
(2n,2n,2) \arrow[ur, "\mathrm{Med}"] & & (4,4,n) \arrow[ul, "\mathrm{Med}"'] \; ; \\
(2n,2n,n) \arrow[u, "\mathrm{Med}"'] & &
\end{tikzcd}$$

and finally for $\mathcal{MD}$ of type $(2, 2n, 2m)$, we deduce case E from

$$\begin{tikzcd}
 & (2,2n,2m) & \\
(2n,2n,m) \arrow[ur, "\mathrm{Med}"] & & (2m,2m,n) \arrow[ul, "\mathrm{Med}"'] \; .
\end{tikzcd}$$

Note that the rest of the cases in Table 2 arise when the full diagram of dessins and surgeries inside S is a subdiagram of those we have just given. □

Recall that all the types appearing in Theorem 6.1 are nonarithmetic (see [Takeuchi 77a] to check the values of n and m that are not involved in Table 2).

6.2 Multiply Quasiplatonic Arithmetic Surfaces

All possible surgeries involving arithmetic dessins can be read in the inclusion diagrams among arithmetic triangle groups in [Takeuchi 77b] or [Maclachlan and Rosenberger 92]. We find immediately:

Proposition 6.2. *A multiply quasiplatonic surface of arithmetic type carries no more than seven regular dessins of different types.*

Proof: It follows from the fact that the maximum possible number of surgeries into a fixed arithmetic type of dessins equals six. It can be attained only when the maximal regular dessin has type $(2, 3, 8)$ (see [Takeuchi 77b]). □

An interesting question could be to check when this upper bound really occurs. Or, more generally:

Suppose (l, m, n) is a maximal and arithmetic type of regular dessins (or, equivalently, of triangle groups), and let k be the number of different surgeries into dessins of type (l, m, n). Let S be a surface carrying a regular dessin $\mathcal{MD}$ of type (l, m, n): Then the number of regular dessins of different types that may be found inside S is at most $k + 1$, the rest of dessins being preimages of $\mathcal{MD}$ under different surgeries. Is this upper bound attained in low genus for every choice of the maximal type?

The following theorem gives the answer for all the cases with the only exception being, precisely, the case of maximal dessins of type $(2, 3, 8)$.

Theorem 6.3. *Table 3 shows the lowest genus surfaces that contain dessins of all possible types for each maximal arithmetic type listed.*

Some explanations are needed for a complete understanding of the table:

The first column makes reference to the label of the diagrams of inclusions between arithmetic triangle groups as they appear in [Takeuchi 77b]: We add a second label in the cases when more than one maximal group occurs in the same diagram, as in VI.1 or VI.2 (accordingly, the case corresponding to maximal dessins of type $(2, 3, 8)$ would be III.2).

The fifth column (**S**) shows the number of different surfaces of that minimal genus that carry all regular dessins. The total number of different surfaces that carry a regular dessin of that maximal type is found in brackets. Finally, the last column shows some additional information of the so-determined surfaces. The simplest algebraic equations are obtained (some of them already appeared in [Wolfart 00]), whereas in the remaining cases the quote refers to hyperellipticity. The polynomial F appearing in case VII is given by $F(x) = x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1$.

Proof: We obtained the data in Table 3 in the following way. First, fix a maximal type (l, m, n) and a genus g. Then, using GAP [GAP 02], determine the monodromy representation of all the regular dessins of that type. This procedure can be done only up to some g, the limit depending on the maximal type. The reason is

Case	$\mathcal{MD}$	**Rest of dessins**	**g**	**S**	**Equation**
II	$(2,4,6)$	$(6,6,2),(6,6,3),$ $(4,4,3)$	2	1 (1)	$y^2 = x^6 - 1$
III.1	$(2,6,8)$	$(6,6,4),(8,8,3)$	6	1 (2)	Non hyp.
IV	$(2,3,12)$	$(3,4,12),(3,3,6),$ $(6,6,6),(2,6,12),$ $(12,12,3)$	13	1 (1)	Non hyp.
V	$(2,4,12)$	$(12,12,2),(12,12,6),$ $(4,4,6)$	5	1 (1)	$y^2 = x^{12} - 1$
VI.1	$(2,4,5)$	$(5,5,2),(4,4,5)$	4	1 (1)	$\sum_{i=1}^5 x_i^n = 0,$ $n = 1,2,3$
VI.2	$(2,4,10)$	$(4,4,5),(10,10,2),$ $(10,10,5)$	4	1 (1)	$y^2 = x^{10} - 1$
VII	$(2,5,6)$	$(5,5,3)$	9	2 (2)	$y^2 = F(x)$ Non hyp.
VIII	$(2,3,10)$	$(3,3,5),(5,5,5),$ $(2,5,10)$	6	1 (1)	Non hyp.
X.1	$(2,4,7)$	$(7,7,2)$	19	1 (1)	Non hyp.
X.2	$(2,3,7)$	$(7,7,2),(3,3,7),$ $(7,7,7)$	1009	1	Non hyp.
X.3	$(2,3,14)$	$(3,3,7),(7,7,7),$ $(2,7,14)$	15	1 (1)	Non hyp.
XI.1	$(2,3,9)$	$(3,3,9),(9,9,9)$	10	1 (1)	Non hyp.
XI.2	$(2,3,18)$	$(3,3,9),(9,9,9),$ $(2,9,18),(3,6,18)$	37	1 (2)	Non hyp.
XII	$(2,4,18)$	$(18,18,2),(18,18,9),$ $(4,4,9)$	8	1 (1)	$y^2 = x^{18} - 1$
XIII	$(2,3,16)$	$(3,3,8),(8,8,8),$ $(2,8,16),(16,16,4)$	21	2 (2)	Non hyp. Non hyp.
XIV	$(2,5,20)$	$(5,5,10)$	31	1 (1)	Non hyp.
XV	$(2,3,24)$	$(3,8,24),(3,3,12),$ $(12,12,12),(2,12,24),$ $(24,24,6)$	37	2 (3)	Non hyp. Non hyp.
XVI	$(2,5,30)$	$(5,5,15)$	81	2 (2)	Non hyp. Non hyp.
XVII	$(2,3,30)$	$(3,10,30),(3,3,15),$ $(15,15,15),(2,15,30)$	121	1 (1)	Non hyp.
XVIII	$(2,5,8)$	$(5,5,4)$	22	1 (1)	Non hyp.

TABLE 3. The multiply quasiplatonic surfaces of arithmetic type that contain all the regular dessins possible (lowest genus).

that we have to run through all existing groups of order $s = \frac{2g-2}{1-1/l-1/m-1/n}$ (which would be the number of edges of the dessin, or equivalently the index of a torsion-free group of genus g inside $\Delta(l,m,n)$ if such a group exists). Thus, we will eventually run out from the range of orders covered by the existing libraries of finite groups.

Last, apply all possible surgery tests to the regular dessins so obtained, checking if we have found for genus g a regular dessin that is in the image of all the possible surgeries into the class of (l,m,n) dessins.

By this procedure we solve all cases except X.2 (as we soon see, we would need to deal with permutations of size 84672—too high for the libraries of small groups in GAP!). This is, in fact, a very interesting case, since it corresponds to Hurwitz curves, namely Riemann surfaces that have the maximal possible number of automorphisms $84(g-1)$. Fortunately, Hurwitz groups have been widely studied (see, for instance, [Conder 87]); this allows us to study case X.2 from a different point of view. According to Conder, we can argue as follows:

Suppose that S is a surface of type X.2, that is S is a Hurwitz curve that also carries regular dessins of types $(7,7,7)$, $(3,3,7)$, and $(7,7,2)$. It follows that the surface S uniformized by Γ is of type X.2 if and only if Γ

is contained in the core in $\Delta(2,3,7)$ of the three triangle subgroups $\Delta(7,7,2)$, $\Delta(7,7,7)$, and $\Delta(3,3,7)$. The core of the first one is H_1, the unique normal subgroup of $\Delta(2,3,7)$ of index 504 (which has $\Delta(2,3,7)/H_1 \simeq \mathbb{P}\mathrm{SL}(2,8)$), while the cores of the other two agree with H_2, the unique normal subgroup of $\Delta(2,3,7)$ of index 168 (with quotient $\Delta(2,3,7)/H_2 \simeq \mathbb{P}\mathrm{SL}(2,7)$). Thus, Γ is contained in $H = H_1 \cap H_2$, which is a normal subgroup of index 84672. It follows that $\Delta(2,3,7)/\Gamma$ must have $\mathbb{P}\mathrm{SL}(2,7) \times \mathbb{P}\mathrm{SL}(2,8)$ among its quotients, so according to [Conder 87], the order of $\Delta(2,3,7)/\Gamma$ must be divisible by 84672, and hence the genus g has to be of the form $g = 1008m + 1$ for some m.

The unique Hurwitz curve of genus 1009 is actually the type of surface we are looking for in X.2, since its uniformizing group is precisely H (again see [Conder 87]). □

As explained in the proof above, the monodromy homomorphisms of the maximal regular dessins of the table were computed explicitly (with the obvious exception of case X.2): For more details, check the author's web page [Girondo 03]. The explicit form of these homomorphisms, which is not shown here for obvious reasons of space, was used to determine if the corresponding curve is or is not hyperelliptic. In the hyperelliptic cases, the equation can be easily obtained from the combinatorial data, as in II, VI.2, XII, and one of the surfaces that solve case VII. This latter has a different geometric meaning: The roots of the polynomial $F(x)$ (hence the projection to the Riemann sphere of the Weierstrass points) are exactly the centers of the faces of the icosahedral.

We can also recognize Bring's curve as that one appearing in case VI.1.

As for the remaining case, III.2, corresponding to surfaces with the largest number (7) of different regular dessins, we only find the following:

Proposition 6.4. *Let S be a quasiplatonic surface with a maximal regular dessin $\mathcal{MD}$ of type $(2,3,8)$, such that $\mathcal{MD}$ is a surgery image of regular dessins of types $(8,8,3), (3,3,4), (4,4,4), (2,4,8), (8,8,2)$, and $(8,8,4)$. Then the genus g of S is at least 51. More precisely, $g = 10k + 1$, where $k \geq 5$.*

Proof: The expression $g = 10k + 1$ is a necessary consequence of the existence of all those dessins. Nevertheless, we find that for $1 \leq k \leq 4$, there exists no quasiplatonic surface of type $(2,3,8)$. The computations, similar to those used in the proof of Theorem 6.3, were done using GAP ([GAP 02]). It should be mentioned that the higher genus studied ($g = 41$) needed a run-through of the 241004 different finite groups that exist of order 1920. The next case ($g = 51$) corresponds to groups of order 2400, larger than the maximum order currently available in the finite group libraries. □

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