Geometry and Dynamics of Quadratic Rational Maps

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CONTENTS

This article is an expository description of quadratic rational maps from the Riemann sphere to itself.

1. INTRODUCTION

This article has two main parts. The first, comprising Sections 2–6, is concerned with the geometry and topology of quadratic rational maps. The space Rat_2 of all quadratic rational maps from the Riemann sphere to itself is a smooth complex five-manifold, having the homotopy type of an SO(3)-bundle over the real projective plane. However, the "moduli space" $\mathcal{M}_2$, consisting of all holomorphic conjugacy classes of maps in Rat_2, has a much simpler structure, being biholomorphic to the coordinate space $\mathbf{C}^2$. (More precisely, $\mathcal{M}_2$ can be described as an orbifold whose underlying space is isomorphic to $\mathbf{C}^2$.) The locus $\mathrm{Per}_n(\mu)$ consisting of conjugacy classes with a periodic point of period n and multiplier μ is an algebraic curve in $\mathcal{M}_2 \cong \mathbf{C}^2$. For the special cases $n = 1$ and $n = 2$ this curve is a straight line. The moduli space $\mathcal{M}_2$ has a natural compactification $\hat{\mathcal{M}}_2$, isomorphic to the projective plane $\mathbf{CP}^2$. We also consider quadratic maps together with a marking of the critical points, or of the fixed points. As an example, the moduli space $\mathcal{M}_2^{\mathrm{cm}}$ for maps with marked critical points is an orbifold with one essentially singular point, and has the homotopy type of a two-sphere.

The second part, comprising Sections 7–10, surveys some topics from the dynamics of quadratic rational maps. There are few proofs. Maps which are hyperbolic on their Julia set give rise to "hyperbolic components" in moduli space, as studied

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in [Rees 1990a]. If we work in the compactified moduli space $\hat{\mathcal{M}}_2$, every hyperbolic component is a topological four-cell with a preferred center point. However, if we work in $\mathcal{M}_2 \cong \mathbf{C}^2$, there is one exceptional component that has a more complicated topology, namely the "escape component", consisting of maps with totally disconnected Julia set. Section 9 attempts to explore and visualize moduli space by means of complex one-dimensional slices: compare [Rees 1990b, 1992]. Section 10 describes the theory of real quadratic rational maps.

For convenience in exposition, some technical details have been relegated to appendices. Appendix A outlines some classical algebra. Appendix B describes the topology of the space of rational maps of degree d. Appendix C outlines several convenient normal forms for quadratic rational maps, and computes relations between various invariants. Appendix D describes some geometry associated with the curves $\mathrm{Per}_n(\mu) \subset \mathcal{M}_2$. Appendix E describes totally disconnected Julia sets containing no critical points. Appendix F, written in collaboration with Tan Lei, describes an example of a connected quadratic Julia set for which no two components of the complement have a common boundary point. Finally, Appendix G addresses the problems involved in making adequate pictures.

2. THE SPACE Rat_2 OF QUADRATIC RATIONAL MAPS

This section will set the stage by giving a brief description of the space of all quadratic rational maps

It will be convenient to identify the compactified plane $\hat{\mathbf{C}} = \mathbf{C} \cup \infty$ with the unit sphere S^2 via stereographic projection, and to call either one the *Riemann sphere*. Let Rat_d be the space consisting of all holomorphic maps of degree d from S^2 to itself. Recall that any such map can be expressed as a ratio $p(z)/q(z)$ of two relatively prime polynomials, with $d = \max(\deg p, \deg q)$. Information about Rat_d may be found in Appendix B; see also [Segal 1979]. For example, Rat_d is a smooth connected complex manifold of dimension $2d+1$, and the fundamental group $\pi_1(\mathrm{Rat}_d)$ is cyclic of order $2d$ for $d \geq 1$. For $d = 1$, note that Rat_1 can be identified with the group $\mathrm{PSL}(2, \mathbf{C})$ of all Möbius transformations from the Riemann sphere to itself.

We now specialize to the case $d = 2$. Each map f in the space Rat_2 of all *quadratic rational maps* can be expressed as a ratio

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2},$$

where a_0 and b_0 are not both zero and p, q have no common root. It follows easily that Rat_2 can be identified with the Zariski open subset of complex projective 5-space consisting of all points

$$(a_0 : a_1 : a_2 : b_0 : b_1 : b_2) \in \mathbf{CP}^5$$

for which the *resultant*

$$\mathrm{res}(p, q) = \det \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix}$$

is nonzero (see Appendix A). The topology of this space can be described roughly as follows.

Theorem 2.1. *The space* Rat_2 *contains a compact nonorientable manifold* M^5 *as deformation retract. This manifold can be described as the unique nontrivial principal* $\mathrm{SO}(3)$*-bundle over the projective plane* $\mathbf{RP}^2$.

Outline of proof (for details, see Appendix B). Every quadratic rational map has two distinct critical points $\omega_1 \neq \omega_2$ in the Riemann sphere S^2, and two distinct critical values $f(\omega_1) \neq f(\omega_2)$. Let $M^5 \subset \mathrm{Rat}_2$ be the subspace consisting of quadratic rational maps such that:

- ω_1 and ω_2 are "antipodal" in the sense that $\omega_2 = -1/\bar{\omega}_1$,
- $f(\omega_1)$ and $f(\omega_2)$ are also antipodal, and
- every point on the "equator" midway between the critical points maps to a point midway between the critical values.

It is not hard to check that M^5 is indeed a smooth manifold, embedded in Rat_2 as a deformation retract, and that the continuous map $f \mapsto \{\omega_1, \omega_2\}$ from M^5 to the real projective plane is the projection map of a principal fibration, with fiber equal to the group $SO(3) \cong PSU(2) \subset PSL(2, \mathbf{C})$ consisting of all rotations of the two-sphere. Using results of Graeme Segal, we will show in Appendix B that the fundamental group $\pi_1(M^5) \cong \pi_1(\mathrm{Rat}_2)$ is cyclic of order 4, and conclude that this bundle must be nontrivial. □

As one immediate consequence of Theorem 2.1, we see that M^5 (or Rat_2) has the rational homology of a three-sphere. However, the twofold orientable covering manifold of M^5 is homeomorphic to the product $SO(3) \times S^2$ (hence the universal covering of M^5 is homeomorphic to $S^3 \times S^2$). In Section 6 we will discuss the corresponding two-sheeted covering manifold of Rat_2. This covering manifold can be identified with the space of *critically marked* quadratic rational maps, denoted by $\mathrm{Rat}_2^{\mathrm{cm}}$. Its elements can be described as ordered triples (f, ω_1, ω_2) where $f \in \mathrm{Rat}_2$ and $\omega_1 \neq \omega_2$ are the two critical points of f.

3. THE SPACE $\mathcal{M}_2$ OF HOLOMORPHIC CONJUGACY CLASSES

The group $\mathrm{Rat}_1 \cong PSL(2, \mathbf{C})$ of Möbius transformations acts on the space Rat_2 of quadratic rational maps by conjugation: $g \in \mathrm{Rat}_1$ and $f \in \mathrm{Rat}_2$ yield $g \circ f \circ g^{-1} \in \mathrm{Rat}_2$. Two maps in Rat_2 are said to be *holomorphically conjugate* if they belong to the same orbit.

Definition. The quotient space of Rat_2 under this action will be denoted by $\mathcal{M}_2$, and called the *moduli space* of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f.

This action of $PSL(2, \mathbf{C})$ is not free. For example, the Möbius transformation $g(z) = -z$ acts trivially on any odd function, such as $f(z) = a(z + z^{-1})$. Hence we might expect the quotient space $\mathcal{M}_2$ to have singularities. In fact, however, we will see that it has the simplest possible description, and can be identified with the complex affine space $\mathbf{C}^2$. (On the other hand, since it is defined as a nontrivial quotient space, $\mathcal{M}_2$ does have a natural orbifold structure that reflects the complications of the group action: see Section 5.)

In order to describe this affine structure, we will study fixed points. Every map $f \in \mathrm{Rat}_2$ has three, not necessarily distinct, fixed points $z_1, z_2, z_3 \in S^2$. Let μ_i be the *multiplier* of f at z_i—that is, the first derivative of f at z_i, suitably interpreted in the special case when z_i is the point at infinity—and let

$$\begin{aligned} \sigma_1 &= \mu_1 + \mu_2 + \mu_3, \\ \sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3, \\ \sigma_3 &= \mu_1\mu_2\mu_3 \end{aligned}$$

be the elementary symmetric functions of these multipliers. Note that $\mu_i = 1$ if and only if z_i is a multiple fixed point, that is, if and only if $z_i = z_j$ for some $j \neq i$.

Lemma 3.1. *These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction*

$$\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0, \tag{3–1}$$

or, equivalently,

$$\sigma_3 = \sigma_1 - 2. \tag{3–2}$$

Hence the moduli space $\mathcal{M}_2$ is canonically isomorphic to $\mathbf{C}^2$, with coordinates σ_1 and σ_2.

We sometimes use the notation $\langle f \rangle = \langle \mu_1, \mu_2, \mu_3 \rangle$ for the conjugacy class of a map f whose fixed points have multipliers μ_1, μ_2 and μ_3. If $\mu_1\mu_2 \neq 1$, we can solve equation (3–1) to obtain

$$\mu_3 = \frac{2 - \mu_1 - \mu_2}{1 - \mu_1\mu_2}. \tag{3–3}$$

On the other hand, if $\mu_1\mu_2 = 1$, it follows easily from (3–1) that $\mu_1 = \mu_2 = 1$, so that $z_1 = z_2$ is a double fixed point. In this case μ_3 can be arbitrary.

Proof of 3.1. We first prove (3–1). First suppose that the μ_i are all different from 1, so that there is no double fixed point. Then the classical formula

$$\sum \frac{1}{1-\mu_i} = 1$$

is proved by integrating $dz/(z-f(z))$. (See, for example, [Milnor 1990, § 9].) Clearing denominators, we obtain (3–1).

On the other hand, if $\mu_1 = 1$ then z_1 is a double fixed point, with say $z_1 = z_2$ and $\mu_1 = \mu_2 = 1$. Then (3–1) is true for any value of μ_3.

To show that the holomorphic conjugacy class is determined by $\{\mu_1, \mu_2, \mu_3\}$, we first consider a map f that has at least two distinct fixed points, which we can assume are 0 and ∞, after conjugating by a Möbius transformation. It follows easily that f has the form

$$f(z) = z\,\frac{az+b}{cz+d},$$

where $a \neq 0$, $d \neq 0$, and $ad - bc \neq 0$ since f has degree two. After multiplying numerator and denominator by a constant, we may assume that $d = 1$. If we replace $f(z)$ by $f(kz)/k$, the effect is to multiply both a and c by k. Thus there is a unique choice of k that has the effect of replacing a by 1. This yields the normal form

$$f(z) = z\,\frac{z+b}{cz+1} \qquad \text{with } 1 - bc \neq 0, \tag{3–4}$$

where $b = \mu_1$ and $c = \mu_2$ are evidently equal to the multipliers at zero and infinity. Thus f is uniquely determined, up to holomorphic conjugacy, by the multipliers μ_1 and μ_2 associated with any two distinct fixed points. The determinant $1 - \mu_1\mu_2$ cannot vanish, but there are no other restrictions on μ_1 and μ_2. The multiplier at the third fixed point is then determined by (3–3). For further information, see Appendix C.

Now suppose there is only one fixed point, $z_1 = z_2 = z_3$. After a Möbius conjugation, we may assume that $z_1 = \infty$, and that $f^{-1}(\infty) = \{0, \infty\}$. This implies that f has the form $f(z) = p(z)/z$ for some quadratic polynomial $p(z)$. The difference $f(z) - z = (p(z) - z^2)/z$ can have no zeros in the finite plane, hence $p(z) - z^2$ is constant. It follows that $f(z) = z + c/z$, with critical points $\pm\sqrt{c}$. If we normalize so that the critical points of f are ± 1, then $c = 1$ and

$$f(z) = z + \frac{1}{z}. \tag{3–5}$$

In this case, the multipliers at the unique fixed point are given by $\mu_1 = \mu_2 = \mu_3 = 1$, and again the conjugacy class is uniquely determined by these multipliers.

Evidently we can realize any triple $\{\mu_1, \mu_2, \mu_3\}$ that satisfies (3–1). Finally, note that the unordered collection $\{\mu_i\}$ of multipliers is determined by the three elementary symmetric functions $\sigma_n = \sigma_n(\mu_1, \mu_2, \mu_3)$. Since (3–2) shows that σ_3 is determined by σ_1, the lemma follows. □

Remark 3.2: Cubic polynomial maps. There is a strong analogy between the theory of quadratic rational maps and that of cubic polynomial maps [Milnor 1992a; Milnor]. In both cases there are three fixed points and two critical points. In both cases the moduli space of holomorphic conjugacy classes has dimension two, and can be identified with $\mathbf{C}^2$, with the elementary symmetric functions of the multipliers at the fixed points, subject to a single linear relation, as coordinates. In the cubic polynomial case, this linear relation is $\sigma_2 - 2\sigma_1 + 3 = 0$.

Remark 3.3: Affine structure. Since the complex manifold $\mathcal{M}_2 \cong \mathbf{C}^2$ has many holomorphic automorphisms, it is not immediately clear that the affine structure imposed by taking the σ_i as affine coordinates has any preferred status. However, the following three lemmas show that this affine structure does indeed have very special properties. For a different coordinate system, which would impose a different and less useful affine structure, see Remark 6.3.

Definition. For each $\eta \in \mathbf{C}$, let $\mathrm{Per}_1(\eta) \subset \mathcal{M}_2$ be the set of all conjugacy classes $\langle f \rangle$ of maps f that have a fixed point with multiplier equal to η.

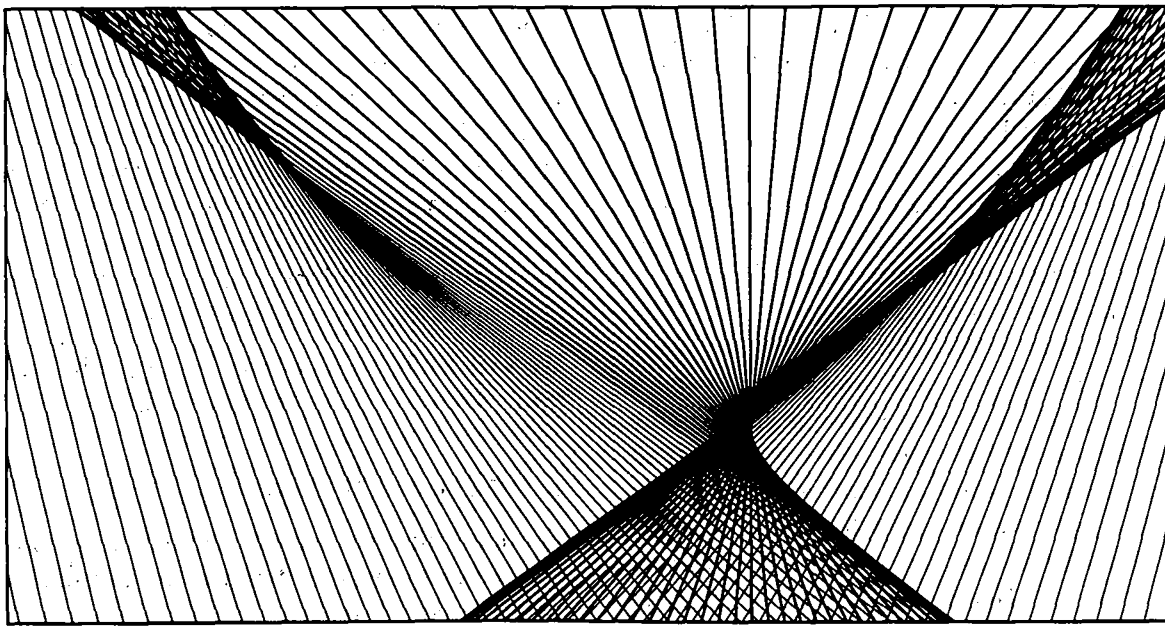

FIGURE 1. The (real) lines $\mathrm{Per}_1(\eta)$ in the real (σ_1, σ_2)-plane. (Compare Section 10 and Figures 15 and 16.) The region $(\sigma_1, \sigma_2) \in [-12, 10] \times [-10, 22]$ is shown, with the horizontal scale exaggerated. The thicker lines represent $-1 \le \eta \le 1$, corresponding to attracting or parabolic real fixed points. The envelope of the family $\{\mathrm{Per}_1(\eta)\}$ consists of the symmetry locus $\mathcal{S}$ of § 5, together with the line $\mathrm{Per}_1(1)$. This envelope cuts the real plane into three regions (cross-hatched) where the real quadratic map f has three distinct real fixed points, and two regions where f has only one real fixed point.

Lemma 3.4. *For each $\eta \in \mathbf{C}$, the locus $\mathrm{Per}_1(\eta) \subset \mathcal{M}_2$ is a straight line with respect to the coordinates σ_1, σ_2, with slope $d\sigma_2/d\sigma_1 = \eta + \eta^{-1}$. For $\eta \ne 0$ it is given by the equation*

$$\sigma_2 = (\eta + \eta^{-1})\sigma_1 - (\eta^2 + 2\eta^{-1}),$$

while $\mathrm{Per}_1(0)$ is the vertical line $\sigma_1 = +2$.

Figure 1 shows the situation in the real case.

Proof of 3.4. The multipliers at the three fixed points are the roots of the equation

$$\eta^3 - \sigma_1\eta^2 + \sigma_2\eta - \sigma_3 = 0.$$

Substituting $\sigma_3 = \sigma_1 - 2$ and solving for σ_2, we obtain the required equation. □

Definition. More generally, for any integer $n \ge 1$ and any number $\eta \ne 1$ in $\mathbf{C}$, let $\mathrm{Per}_n(\eta)$ be the set of $\langle f \rangle \in \mathcal{M}_2$ having a periodic point of period n and multiplier η. (For the special case $\eta = 1$, the definition needs more care, as discussed in Appendix D. One possibility is simply to define $\mathrm{Per}_n(1)$ as the limit of $\mathrm{Per}_n(\eta)$ as $\eta \to 1$, $\eta \ne 1$.)

The following assertion is part of Theorem 4.2:

Lemma 3.5. *Each $\mathrm{Per}_n(\eta)$ is an algebraic curve in $\mathcal{M}_2$ with degree equal to the number of hyperbolic components of period n in the Mandelbrot set.*

Thus, for $n = 1$ and $n = 2$ the curve $\mathrm{Per}_n(\eta)$ is a straight line, but for $n = 3$ it is a cubic curve. For period $n = 2$ we have the following simple description:

Lemma 3.6. *The curves $\mathrm{Per}_2(\eta)$ are parallel straight lines of slope -2, given by the equation*

$$2\sigma_1 + \sigma_2 = \eta.$$

As noted above, the case $\eta = +1$ is exceptional. In fact the proof will show that there is no quadratic rational map having an orbit of period 2 and multiplier $+1$.

Proof of 3.6. The fixed points of the fourth-degree map $f^{\circ 2}$ consist of the fixed points of f together with the period 2 orbits (if any) of f. First consider a map $f \in \mathrm{Rat}_2$ with fixed-point multipliers $\{\mu_i\}$ such that no μ_i is equal to ± 1. Then we will show that the five fixed points of $f^{\circ 2}$ are all distinct. In fact, three of these are the three distinct fixed points of f, and they have multipliers $\mu_1^2, \mu_2^2, \mu_3^2 \neq +1$ when considered as fixed points of $f^{\circ 2}$. The remaining two must constitute a period-two orbit for f. Neither of these points can coincide with a fixed point of f, since such a multiple fixed point of $f^{\circ 2}$ would have to have multiplier $+1$, and the two cannot coincide with each other, since they would then constitute an extra fixed point for f. It follows that the multiplier η for this period-two orbit cannot be $+1$. Hence the rational fixed point formula for $f^{\circ 2}$ takes the form

$$\frac{1}{1-\mu_1^2}+\frac{1}{1-\mu_1^2}+\frac{1}{1-\mu_1^2}+\frac{1}{1-\eta}+\frac{1}{1-\eta}=1$$

(compare [Milnor 1990] or the proof of Lemma 3.1) We can solve this equation so as to express η as a certain rational function of the elementary symmetric functions $\sigma_i(\mu_1, \mu_2, \mu_3)$. In fact, making use of (3–2) and carrying out the division (preferably by computer), we find the required formula $\eta = 2\sigma_1 + \sigma_2$.

Now suppose that $\langle f\rangle$ belongs to $\mathrm{Per}_1(-1)$, or, in other words, that f has a fixed point of multiplier $\mu_i = -1$. Then $f^{\circ 2}$ has a multiple fixed point, with multiplier $\mu_i^2 = +1$. If f has m distinct fixed points, where $m = 2$ or $m = 3$, it follows easily that $f^{\circ 2}$ has at most $m+1$ distinct fixed points. *Hence f cannot have any orbit of period two.* In fact, as $\eta \to 1$, the unique orbit of period two for f degenerates to the fixed point z_i of multiplier -1. (By (3–1) there cannot be two fixed points of multiplier -1.) Note that the equation for the locus $\mathrm{Per}_1(-1)$, as given by Lemma 3.4, coincides precisely with the locus $\eta = 2\sigma_1 + \sigma_2 = +1$. Thus it is convenient to define

$$\mathrm{Per}_2(1) = \mathrm{Per}_1(-1) = \{\langle f\rangle \in \mathcal{M}_2 : 2\sigma_1 + \sigma_2 = 1\}$$

(compare Figure 6). Finally, suppose that f has a double fixed point $z_1 = z_2$ with multiplier $\mu_1 = \mu_2 = 1$, and that the third fixed point z_3 has multiplier $\mu_3 \neq -1$. Then a straightforward argument by continuity shows that the formula $\eta = 2\sigma_1 + \sigma_2$ for the multiplier of the orbit of period two remains true. In this case, a brief computation shows that the multiplier $\eta = 2\sigma_1 + \sigma_2$ for the orbit of period two is equal to $5 + 4\mu_3 \neq +1$. □

We will study these curves $\mathrm{Per}_n(\eta)$ further in Sections 4 and 9, and in Appendix D.

4. THE COMPACTIFICATION $\hat{\mathcal{M}}_2 \cong \mathbf{CP}^2$

The coordinate plane $\mathbf{C}^2$ embeds naturally in the projective plane $\mathbf{CP}^2$. Since $\mathcal{M}_2$ is isomorphic to $\mathbf{C}^2$ with coordinates σ_1 and σ_2, there is a corresponding compactification $\hat{\mathcal{M}}_2 \cong \mathbf{CP}^2$, consisting of $\mathcal{M}_2$ together with a two-sphere of *ideal points* at infinity. Elements of this two-sphere can be thought of very roughly as limits of quadratic rational maps as they degenerate towards a fractional linear or constant map. However, caution is needed, since such a limit cannot be uniform over the entire Riemann sphere.

In terms of the multipliers at the fixed points, the two-sphere at infinity can be described as follows. If at least one of the elementary symmetric functions $\sigma_i(\mu_1, \mu_2, \mu_3)$ tends to infinity, at least one of the μ_i must tend to infinity. If only μ_3, for example, tends to infinity, it follows from (3–3) that the product $\mu_1\mu_2$ must tend to $+1$. On the other hand, if two of the μ_i tend to infinity, we see form (3–3) that the third must tend to zero. Thus *the collection of ideal points in $\hat{\mathcal{M}}_2$ can be identified with the set of unordered triples of the form $\langle \mu, \mu^{-1}, \infty\rangle$, with $\mu \in \hat{\mathbf{C}} = \mathbf{C} \cup \infty$.* It seems appropriate to use the notation $\widehat{\mathrm{Per}_1}(\infty) \subset \hat{\mathcal{M}}_2$ for this two-sphere of points at infinity. A useful parameter on $\widehat{\mathrm{Per}_1}(\infty)$ is the sum $\mu + \mu^{-1} \in \hat{\mathbf{C}}$, which can be identified with the limiting ratio

$$\frac{\sigma_2}{\sigma_3} = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}.$$

Except for the case $\mu = 1$, the dynamics of a representative of a point in $\mathcal{M}_2$ "close" to the ideal point $\langle \mu, \mu^{-1}, \infty \rangle$ can be described as follows. (For $\mu = 1$, a useful description would be more complicated, involving the theory of Écale cylinders [Lavaurs 1989].) As in (3–4) or in (C–1) of Appendix C, we use the normal form

$$f(z) = \frac{z(z+\mu_1)}{\mu_2 z + 1} \quad \text{with } \mu_1\mu_2 \neq 1. \tag{4–1}$$

First suppose that $\mu \neq 0, \infty$, and let $\mu_1 \approx \mu$ and $\mu_2 \approx \mu^{-1}$, so that $\mu_1\mu_2 \approx 1$. It turns out that, over most of the Riemann sphere, this map f is uniformly close to the linear map $z \mapsto z/\mu_2$, or equivalently to $z \mapsto \mu z$. However, the behavior is quite different in a small neighborhood of the point $z = -1/\mu_2$: this neighborhood, which includes both critical points, maps over the entire Riemann sphere. The case $\mu = 0$ is similar. Any $\langle f \rangle \in \mathcal{M}_2$ that is close to the ideal point $\langle 0, \infty, \infty \rangle$ has a convenient representative that is uniformly close to a constant throughout most of the sphere. We will make these statements more precise as part of the proof of the following result:

Lemma 4.1. *For any period $n \geq 2$ and any multiplier $\eta \in \mathbf{C}$, the only possible limit points of the curve $\mathrm{Per}_n(\eta) \subset \mathcal{M}_2$ on the two-sphere at infinity are ideal points of the form $\langle \mu, \mu^{-1}, \infty \rangle$, where μ is a q-th root of unity, with $q \leq n$.*

In particular, the limiting ratio $\sigma_2/\sigma_3 = \mu + \mu^{-1}$ is necessarily a point in the real interval $[-2, 2]$. For example, if $\mu = e^{2\pi i m/n}$, then

$$\frac{\sigma_2}{\sigma_3} = 2\cos\frac{2\pi m}{n}$$

(compare Figures 16 and 17). It is conjectured that the case $q = 1$ cannot occur, and that the set of all limit points of $\mathrm{Per}_n(\eta)$ is precisely the set of $\langle \mu, \mu^{-1}, \infty \rangle$ such that μ is a q-th root of unity with $1 < q \leq n$.

Note added in proof. This conjecture has been verified by Stimson [1993], at least for the case $\eta = 0$. In fact Stimson gives a much more precise description of the intersections of the completed curve $\widehat{\mathrm{Per}}_n(0)$ with the sphere at infinity, showing that a neighborhood of this intersection in $\widehat{\mathrm{Per}}_n(0)$ is made up of finitely many smooth branches, each associated with either one or two components of period n in the Mandelbrot set.

Proof of 4.1. First suppose that $\mu \neq 0, 1, \infty$. Using the normal form (4–1), set

$$\delta = 1 - \mu_1\mu_2, \qquad l(z) = \mu_2 z + 1,$$

and assume that the determinant δ is very close to zero. Note that the linear function $l(z)$ is close to zero if and only if z is close to $-1/\mu_2$. A brief computation shows that

$$\mu_2 \frac{f(z)}{z} = \mu_2 \frac{z + \mu_1}{\mu_2 z + 1} = 1 - \frac{\delta}{l(z)},$$

and that

$$\mu_2 f'(z) = 1 - \frac{\delta}{l(z)^2}.$$

We now partition the z-plane into three disjoint regions D, A, C, according to whether the number $|l(z)|^3$ is less than $|\delta|^2$, between $|\delta|^2$ and $|\delta|$, or greater than $|\delta|$. Thus D is a very small disk centered at the pole $-1/\mu_2$, while A is a small annulus surrounding this disk, and the complementary region C is everything else, including zero and infinity. Note that

$$\left| \mu_2 \frac{f(z)}{z} - 1 \right| \leq |\delta|^{1/3} \qquad \text{for } z \in A \cup C$$

and

$$|\mu_2 f'(z) - 1| \leq |\delta|^{1/3} \qquad \text{for } z \in C,$$

but

$$|\mu_2 f'(z) - 1| \geq |\delta|^{-1/3} \qquad \text{for } z \in D.$$

Thus the value of $f(z)/z$ is uniformly close to the constant $1/\mu_2 \approx \mu$ everywhere outside of the small disk D. The derivative $f'(z)$ is very large throughout D, and is uniformly close to $1/\mu_2 \approx \mu$ and hence bounded away from zero throughout the outside region C. It follows that both critical points belong to the annulus A.

Now consider a periodic point of period $n \geq 2$. If its orbit is disjoint from the disk D, we have $1 = f^{\circ n}(z)/z \approx \mu^n$. If the orbit touches both D and A, we may assume that $z \in A$, and that $f^{\circ q}(z) \in D$ for some $1 \leq q < n$, which we take to be minimal. In this case, it follows that $\mu^q \approx 1$. Finally, if an orbit touches D but not A, its multiplier must tend to infinity as $\delta \to 0$. Thus, in the limit as $\delta \to 0$ and $\mu_1 \to \mu$, we can have an orbit of period $n \geq 2$ with bounded multiplier only if μ is a q-th root of unity with $q \leq n$. (Note that this argument allows the possibility that $\mu = 1$.) This completes the proof for the case $\mu \neq 0, \infty$.

To handle the case $\mu = \infty$ (or $\mu = 0$, which is equivalent), we need a slightly different argument. Again we use the normal form (4–1), but now assume that the multiplier μ_1 at the origin is very large in absolute value, and that the multiplier μ_2 at infinity is very close to zero. It then follows from (3–3) that the multiplier μ_3 at the third fixed point $z_3 = (\mu_1 - 1)/(\mu_2 - 1) \approx -\mu_1$ is also very large in absolute value. We write $\mu_1, \mu_3 \approx \infty$ but $\mu_2 \approx 0$. For z in the disk $|z| < 2$, it then follows from the computation

$$f'(z) = \frac{\mu_2 z^2 + 2z + \mu_1}{(\mu_2 z + 1)^2}$$

that the derivative $f'(z)$ has bounded distance from μ_1. Hence this disk maps diffeomorphically onto a region U, which is approximately the disk of radius $2\,|\mu_1|$ enclosing both finite fixed points. Let $D \subset U$ be the disk centered at the midpoint of the two finite fixed points, with radius equal to the distance between them. Since the two finite fixed points play a symmetric role, it follows that the preimage $f^{-1}(D) \subset D$ splits up as a neighborhood N_1 of $z_1 = 0$, throughout which $f' \approx \mu_1$, and a neighborhood N_3 of z_3 throughout which $f' \approx \mu_3$. It is now easy to check that the Julia set $J(f)$ is a Cantor set, contained in the union $N_1 \cup N_3$, and that every orbit outside of the Julia set converges to the fixed point at infinity. (Compare Section 8.) Thus $f'(z)$ is approximately equal to either μ_1 or μ_3 throughout the Julia set; hence the multiplier of any orbit of period ≥ 2 tends to infinity as $\mu_1, \mu_3 \to \infty$. □

Theorem 4.2. *For any $\eta \in \mathbf{C}$, the degree of the curve* $\mathrm{Per}_n(\eta) \subset \mathcal{M}_2$ *is $\frac{1}{2}\nu_2(n)$, where the numbers $\nu_2(n)$ are defined inductively by the formula*

$$2^n = \sum_{m|n} \nu_2(m),$$

to be summed over all positive integers m which divide n. Equivalently, this degree is equal to the number of hyperbolic components of period n in the Mandelbrot set.

(Compare Lemma 3.5 and Appendix D.) Here are the first few values of this degree:

n	1	2	3	4	5	6	7	8
degree	1	1	3	6	15	27	63	120

Proof of 4.2. By Lemma 3.4, it suffices to consider the case $n > 1$. Since the definition of $\mathrm{Per}_n(\eta)$ is purely algebraic, it is not difficult to check that it is an algebraic curve in $\mathcal{M}_2 \cong \mathbf{C}^2$ (see Appendix D). In fact it is useful to consider its closure $\widehat{\mathrm{Per}}_n(\eta)$ in the projective space $\hat{\mathcal{M}}_2 \cong \mathbf{CP}^2$. By definition, the degree of a curve in $\mathbf{CP}^2$ is equal to its number of intersections with any straight line, counted with multiplicity. As a test line we choose the closure of the locus $\mathrm{Per}_1(0)$, with equation $\sigma_1 = 2$, or, equivalently, $\sigma_3 = \mu_1\mu_2\mu_3 = 0$. This line can be identified with the set of all quadratic polynomial maps $f(z) = z^2 + c$ having a fixed point of multiplier zero at infinity. (Here $\sigma_2 = 4c$.) The closure $\widehat{\mathrm{Per}}_1(0)$ within the compactified space $\hat{\mathcal{M}}_2$ contains just one point at infinity $\langle 0, \infty, \infty\rangle$. By Theorem 4.1, $\widehat{\mathrm{Per}}_n(\eta)$ does not contain this point. Thus it suffices to consider intersections in the finite plane.

First consider the case $\eta = 0$. The points of $\mathrm{Per}_1(0) \cap \mathrm{Per}_n(0)$ can be described as the conjugacy classes of maps $f_c(z) = z^2 + c$ for which the finite critical point 0 has period exactly n under f_c. By definition, these are exactly the center points of the various components of period n of the Mandelbrot set. Furthermore, it follows easily from [Douady and Hubbard 1982, § 3] that such a c cannot be a multiple solution to the equation $f_c^{\circ n}(0) = 0$. In other words, the intersection is always transverse, with intersection multiplicity 1. More generally, whenever $|\eta| < 1$ a similar argument shows that the curves $\mathrm{Per}_1(0)$ and $\mathrm{Per}_n(\eta)$ intersect transversally, with exactly one intersection point in each component of period n of the Mandelbrot set.

To show that this number of intersection points is $\frac{1}{2}\nu_2(n)$, note that the equation $f_c^{\circ n}(0) = 0$ has degree 2^{n-1}. In other words, the number of centers in the Mandelbrot set with period dividing n is equal to 2^{n-1}. After discarding all of those centers corresponding to proper divisors of n, we obtain the required number.

For the case $|\eta| \geq 1$, it is convenient to introduce algebraic curves $Q_n^* \supset Q_n \to P_n$ as follows. Let $Q_n^* \subset \mathbf{C}^2$ be the set of all pairs (c, z) satisfying the polynomial equation $f_c^{\circ n}(z) = z$. (Thus the point z must be periodic with period m dividing n under the map f_c.) Let $Q_n \subset Q_n^*$ be the union of those irreducible components of the curve Q_n^* for which a generic point (c_0, z_0) has the property that z_0 has period exactly n under f_{c_0}. (By [Bousch 1992], there is exactly one such irreducible component; in other words, Q_n is irreducible.) Evidently we can write

$$Q_n^* = \bigcup_{m|n} Q_m,$$

taking the union over all divisors m of n. The cyclic group of order n operates on Q_n by the transformation $(c, z) \mapsto (c, f_c(z))$. Let P_n be the quotient variety of Q_n under this action. Thus a point of P_n can be described as a pair $(c, \{z_i\})$ consisting of a parameter value c and a periodic orbit $z_1 \mapsto z_2 \mapsto \cdots$ under f_c which, at least for a generic point of P_n, has period exactly n.

For each fixed value of z, the defining equation $f_c^{\circ n}(z) = z$ has degree 2^{n-1} in c. In other words, the projection map $(c, z) \mapsto z$ from Q_n^* to the z-plane has degree 2^{n-1}. If we restrict to the subvariety Q_n, it follows that the corresponding projection map $(c, z) \mapsto z$ has degree $\frac{1}{2}\nu_2(n)$. If $z = z_1 \mapsto z_2 \mapsto \cdots$ is the orbit of z under f_c, it follows that each projection $(c, z) \mapsto z_i$ from Q_n to $\mathbf{C}$ also has degree $\frac{1}{2}\nu_2(n)$. Now the multiplier $\eta = (2z_1)(2z_2)\ldots(2z_n)$ of such a periodic orbit is, up to a constant factor, just the product $z_1 z_2 \ldots z_n$. It follows easily that the projection $(c, z) \mapsto \eta$ from Q_n to the η-plane has degree $\sum_1^n \frac{1}{2}\nu_2(n) = \frac{1}{2}n\nu_2(n)$. This can be proved, for example, by considering Q_n as a curve of degree $\frac{1}{2}\nu_2(n)$ in the space $\mathbf{C}^n \subset \mathbf{CP}^n$, consisting of n-tuples $(z_1, \ldots, z_n)$. (We can easily solve for c as a function of the z_i.) The locus $z_1 \ldots z_n = \text{constant} \neq 0$ is a hypersurface of degree n in this same space. The total number of intersections of Q_n with this locus, counted with multiplicity, is equal to the product $\frac{1}{2}n\nu_2(n)$. All of these intersections lie within the finite space $\mathbf{C}^n$, since as any z_i tends to infinity within Q_n the remaining z_i must also tend to infinity.

Finally, since the projection $Q_n \to P_n$ has degree n, this implies that the projection $(c, \{z_i\}) \mapsto \eta$ from P_n to the η-plane has the required degree $\frac{1}{2}\nu_2(n)$. In other words, for a generic choice of η there are $\frac{1}{2}\nu_2(n)$ corresponding points in the curve P_n, which map to $\frac{1}{2}\nu_2(n)$ distinct points of the c-plane. Of course, for particular values of η there may be coincidences, but this will not affect the count with multiplicity. Now using the argument above, we see that the curve $\mathrm{Per}_n(\eta) \subset \mathcal{M}_2$ has degree $\frac{1}{2}\nu_2(n)$.

This proves the theorem and Lemma 3.5. □

Remark 4.3: The curves $\mathrm{Per}_3(\eta)$. We look at the special case $n = 3$ as an example (without proofs). Theorem 4.2 says that each $\mathrm{Per}_3(\eta)$ is a curve of degree three. For most values of η, this curve is nonsingular of genus one, and has two ends corresponding to the two intersection points of its projective completion with the line at infinity (namely a double intersection at $\langle \omega, \bar{\omega}, \infty \rangle$ where ω is a primitive cube root of unity, and a single intersection at $\langle -1, -1, \infty \rangle$). However, there are three special values of η that behave differently. For

$\eta = 0$, the curve $\mathrm{Per}_3(0)$ has genus zero, with a transverse self-intersection point corresponding to the map for which both critical points lie in a single orbit of period 3. (This parameter curve is shown in Figure 9, and the Julia set corresponding to the self-intersection point in Figure 2.)

Similarly, for $\eta = -8$ the curve $\mathrm{Per}_3(-8)$ has genus zero, with a single transverse self-intersection point corresponding to the map $z \mapsto 1/z^2$, which has two distinct orbits of period 3 with multiplier -8. Finally, for $\eta = 1$ the cubic curve $\mathrm{Per}_3(\eta)$ degenerates into a union of three straight lines:

$$\mathrm{Per}_3(1) = \mathrm{Per}_1(\omega) \cup \mathrm{Per}_1(\bar{\omega}) \cup \mathrm{Per}_2(-3),$$

where ω is a primitive cube root of unity (compare Figures 7 and 10). The first two lines, with slope -1, correspond to maps for which one orbit of period 3 degenerates to a fixed point of multiplier ω or $\bar{\omega}$, while the third straight line, with equation $\sigma_2 = -2\sigma_1 - 3$, corresponds to maps for which the two period-three orbits coincide. For some reason, which I do not understand, this locus is precisely equal to the line $\mathrm{Per}_2(-3)$. This third line is visible as part of the boundary of the hyperbolic component labeled 3 in Figure 16. Thus the curve $\mathrm{Per}_3(1)$ has two finite self-intersections—corresponding to the map $z \mapsto \omega(z + z^{-1})$ and its complex conjugate—and one self-intersection at infinity.

5. MAPS WITH SYMMETRIES

By an *automorphism* of a quadratic rational map f we will mean a Möbius transformation g that commutes with f, so that $g \circ f \circ g^{-1} = f$. The collection of all automorphisms of f forms a finite group

$$\mathrm{Aut}\, f \subset \mathrm{Rat}_1 \cong \mathrm{PSL}(2, \mathbf{C})$$

which measures the extent to which the action of Rat_1 on Rat_2 by conjugation fails to be free at f.

Theorem 5.1. *A quadratic rational map possesses a nontrivial automorphism if and only if it is conjugate to a map in the unique normal form*

$$f(z) = k\left(z + \frac{1}{z}\right) \tag{5–1}$$

with $k \in \mathbf{C} \smallsetminus \{0\}$. For f in this normal form, $\mathrm{Aut}\, f$ is cyclic of order two (consisting of the maps $z \mapsto \pm z$) if $k \neq -\frac{1}{2}$, but is nonabelian of order six for $k = -\frac{1}{2}$.

(Compare [Doyle and McMullen 1989; McMullen 1988].) Figure 12 shows a picture of the k-plane.

Remark 5.2. For a map in this normal form, the point at infinity is fixed with multiplier $\mu = k^{-1}$. There are two other fixed points at

$$z = \pm\sqrt{\frac{k}{1-k}},$$

both with multiplier $2k - 1$. Thus the fixed-point multipliers of f are

$$\{\mu_i\} = \{k^{-1},\, 2k - 1,\, 2k - 1\}. \tag{5–2}$$

There are two special values of k for which all three multipliers are equal: the exceptional point $k = -\frac{1}{2}$ of Theorem 5.1, with $\mu_1 = \mu_2 = \mu_3 = -2$, and the point $k = 1$ with $\mu_1 = \mu_2 = \mu_3 = 1$. In the latter case, all three fixed points coincide with the point at infinity, as discussed in the proof of Lemma 3.1.

Proof of 5.1. First consider an automorphism of order two. Any element of order two in $\mathrm{PSL}(2, \mathbf{C})$ is conjugate to the map $g(z) = -z$, so it suffices to look at quadratic rational maps f which commute with $z \mapsto -z$. In other words, it suffices to look at odd functions, $f(-z) = -f(z)$. Writing $f(z)$ as a quotient $p(z)/q(z)$ of two polynomials, we see easily that f is odd if and only if one of these two polynomials is odd and the other is even. If $p(z)$ is even and $q(z)$ is odd, we can write

$$f(z) = \frac{kz^2 + l}{z} = kz + lz^{-1}$$

with $kl \neq 0$. Choosing $\lambda = \pm\sqrt{l/k}$, we see that the conjugate map $f(\lambda z)/\lambda$ has the required form

$z \mapsto k(z+z^{-1})$. On the other hand, if $p(z)$ is odd and $q(z)$ is even, the conjugate map $1/f(1/z)$ will have the form even/odd, so the above argument applies.

For f in the normal form (5–1), the critical points ± 1 are interchanged by the automorphism $z \mapsto -z$. In fact, for any quadratic rational map f and automorphism g, it is clear that the set of critical points $\{\omega_1, \omega_2\}$ must be mapped into itself by g. Hence the automorphism group $\operatorname{Aut} f$ contains a subgroup $\operatorname{Aut}^0 f$ of index at most two, consisting of automorphisms that fix each of the two critical points. Suppose that this subgroup contains a nontrivial automorphism g. After a Möbius change of coordinates, we may assume that the two critical points are 0 and ∞. Thus the nontrivial automorphism g fixing these two points must have the form $g(z) = \lambda z$ for some $\lambda \neq 0, 1$. The equation $f(\lambda z) = \lambda f(z)$ then implies that $f(0) \in \{0, \infty\}$, and similarly that $f(\infty) \in \{0, \infty\}$. If f also fixed both critical points, then evidently we would have $f(z) = \alpha z^2$ for some constant $\alpha \neq 0$, and the equation $f(\lambda z) = \lambda f(z)$ would imply that $\lambda = 1$, contrary to our hypothesis. Since the two critical values must be distinct, the only other possibility is that f interchanges the two critical points. Thus f must have the form $f(z) = \alpha/z^2$, and after a scale change we may assume that $\alpha = 1$, so that

$$f(z) = \frac{1}{z^2}. \tag{5–3}$$

A brief computation then shows that the group $\operatorname{Aut}^0 f$ of automorphisms that fix zero and infinity is the cyclic group of order three, consisting of all maps $g(z) = \lambda z$ with $\lambda^3 = 1$. The full group of automorphisms for (5–3) is generated by this subgroup, together with the involution $z \mapsto 1/z$. From the discussion above, or by direct computation, we see that the map (5–3) is holomorphically conjugate to the special case $z \mapsto -\frac{1}{2}(z + z^{-1})$ of (5–1). Further details of the proof are straightforward. □

In terms of the fixed points of f, we can reformulate this result as follows:

Case 1. If f has three distinct fixed points (in other words, if $\mu_i \neq 1$), then $\operatorname{Aut} f$ coincides with the group consisting of all multiplier-preserving permutations of the fixed points. Thus it has order 1, 2 or 6 according to whether the μ_i are distinct, two are equal, or all three are equal.

Case 2. If f has only two distinct fixed points, $\operatorname{Aut} f$ is trivial.

Case 3. If f has only one fixed point, $\operatorname{Aut} f$ is cyclic of order two.

Definition. Let $\mathcal{S} \subset \mathcal{M}_2$ be the *symmetry locus* consisting of all conjugacy classes $\langle f \rangle$ of quadratic maps that admit a nontrivial automorphism. (For other characterizations of this set, see Remarks 5.4 and 6.4.)

Using (5–2), we easily prove the following.

Corollary 5.3. *The symmetry locus $\mathcal{S}$ is a curve of degree three and genus zero in $\mathcal{M}_2 \cong \mathbf{C}^2$. It can be defined parametrically by the equations*

$$\sigma_1 = 4k - 2 + k^{-1},$$
$$\sigma_2 = 4k^2 - 4k + 5 - 2k^{-1}$$

as k varies over $\mathbf{C} \smallsetminus \{0\}$. This curve is nonsingular, except for a cusp at the point $\langle z \mapsto z^{-2} \rangle$, with $k = -\frac{1}{2}$, $\sigma_1 = -6$ and $\sigma_2 = 12$.

See Figure 15, which shows the intersection of $\mathcal{S}$ with the real (σ_1, σ_2)-plane, and compare Figure 1.

Remark 5.4: Orbifold structure. Since the action

$$g, f \mapsto g \circ f \circ g^{-1}$$

of the group $\mathrm{PSL}(2, \mathbf{C})$ on the space Rat_2 is proper and locally free, it follows that the quotient space $\mathcal{M}_2$ has an associated orbifold structure. (Compare [Thurston].) In fact, the manifold Rat_2 is smoothly foliated by the orbits under this action, and if $U \subset \mathrm{Rat}_2$ is a local complex two-manifold transverse to the orbit $\mathcal{O}(f_0) = \{g \circ f_0 \circ g^{-1}\}$, then the fundamental group $\pi_1(\mathcal{O}(f_0))$ acts by holonomy on U in a neighborhood U_0 of f_0. It is easy to check that this action factors through the homomorphism $\pi_1(\mathcal{O}(f_0)) \to \operatorname{Aut} f_0$. The quotient of U_0

by this action of the finite group Aut f_0 is precisely a coordinate neighborhood of $\langle f_0\rangle$ for the required orbifold structure on $\mathcal{M}_2$.

Note that we can describe the symmetry locus $\mathcal{S}$ as the set of points of $\mathcal{M}_2$ at which this natural orbifold structure is nontrivial. Evidently this structure is especially nontrivial at the cusp point $\langle z \mapsto 1/z^2\rangle$.

6. MAPS WITH MARKED POINTS

Marked Critical Points

Recall from Section 2 that a *critically marked* quadratic rational map (f, ω_1, ω_2) is a map $f \in \mathrm{Rat}_2$ together with an ordered list of its critical points. The space $\mathrm{Rat}_2^{\mathrm{cm}}$ of all critically marked quadratic rational maps is a smooth two-sheeted covering manifold of Rat_2. The Möbius group $\mathrm{PSL}(2, \mathbf{C})$ acts on $\mathrm{Rat}_2^{\mathrm{cm}}$ by conjugation:

$$g \cdot (f, \omega_1, \omega_2) = (g \circ f \circ g^{-1},\ g(\omega_1),\ g(\omega_2)).$$

The quotient space of $\mathrm{Rat}_2^{\mathrm{cm}}$ by this action will be denoted by $\mathcal{M}_2^{\mathrm{cm}}$, and called the *critically marked moduli space*. Following [Rees 1990a], we will show that this moduli space is a smooth complex manifold except at one singular point, corresponding to the special map $f(z) = z^{-2}$ of (5–3). To understand $\mathcal{M}_2^{\mathrm{cm}}$ it is convenient to use the normal form

$$f(z) = \frac{\alpha z^2 + \beta}{\gamma z^2 + \delta} \qquad \text{with } \alpha\delta - \beta\gamma = 1, \tag{6–1}$$

so that the two marked critical points are zero and infinity, in this order (see also Appendix C). Maps in this form satisfy $f(z) = f(z')$ if and only if $z' = \pm z$, so the Julia set $J(f)$ is invariant under the involution $z \leftrightarrow -z$.

This normal form is unique except for the scale change that replaces $f(z)$ by $\lambda^{-2} f(\lambda^2 z)$. This acts on the unimodular matrix of coefficients by the transformation

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha\lambda & \beta\lambda^{-3} \\ \gamma\lambda^3 & \delta\lambda^{-1} \end{pmatrix} \tag{6–2}$$

for $\lambda \in \mathbf{C} \smallsetminus \{0\}$. (Note that we can change the signs of all coefficients by taking $\lambda = -1$.) It is easy to check that this action of the group $\mathbf{C} \smallsetminus \{0\}$ on the manifold $\mathrm{SL}(2, \mathbf{C})$ of complex unimodular matrices is free with a single exception: The cube roots of unity act trivially on the orbit $\alpha = \delta = 0$, which corresponds to the special mapping (5–3).

We can introduce three expressions that are invariant under the action (6–2):

$$A = \alpha\delta = 1 + \beta\gamma, \quad B = \alpha^3\beta, \quad C = \gamma\delta^3. \tag{6–3}$$

It seems difficult to interpret these quantities geometrically, but they are quite convenient to work with.

Lemma 6.1. *We can identify the moduli space $\mathcal{M}_2^{\mathrm{cm}}$ for critically marked quadratic rational maps with the hypersurface W consisting of all those triples $(A, B, C) \in \mathbf{C}^3$ that satisfy the equation*

$$A^3(A - 1) = BC. \tag{6–4}$$

This algebraic surface is nonsingular except at the point $A = B = C = 0$, corresponding to $\langle z \mapsto z^{-2}\rangle$, where it has an essential singularity. The deck transformation

$$(f, \omega_1, \omega_2) \leftrightarrow (f, \omega_2, \omega_1)$$

of $\mathcal{M}_2^{\mathrm{cm}}$ over $\mathcal{M}_2$, which interchanges the two critical points, corresponds to the map $(A, B, C) \leftrightarrow (A, C, B)$.

Proof. It is clear that A, B and C are indeed invariant under (6–2), and that they satisfy (6–4). Conversely, given A, B, C satisfying (6–4), we can find $\alpha, \beta, \gamma, \delta$ satisfying (6–3), unique up to the action of $\mathbf{C} \smallsetminus \{0\}$, as follows. If $B \neq 0$ or $A \neq 0$, we can set $\alpha = 1$, and solve uniquely for $\beta = B$, $\delta = A$ and either $\gamma = (A-1)/B$ or $\gamma = C/A^3$. The case $C \neq 0$ is similar, and the case $A = B = C = 0$ reduces to the map $z \mapsto z^{-2}$. Finally, note that the conjugacy that replaces $f(z)$ by

$$\frac{1}{f(1/z)} = \frac{\delta z + \gamma}{\beta z + \alpha}$$

interchanges the roles of the two critical points, and interchanges the invariants B and C. □

Remark 6.2: Singularities. In order to understand the singularity of the hypersurface (6–4) at the origin, it is convenient to introduce the following terminology:

Definition. A surface in $\mathbf{C}^3$ has a *singularity of type* (p, q, r), where $p, q, r > 1$, if it can be reduced to the form $z_1^p + z_2^q + z_3^r = 0$ by a local holomorphic change of variable.

Such a point is indeed always singular. In fact, the surface is locally homeomorphic to the cone over a three-manifold with nontrivial fundamental group; see, for example, [Milnor 1975]. In particular, for a singularity of type $(2, 2, r)$ this fundamental group is cyclic of order r. Similarly, we will say that a curve in $\mathbf{C}^2$ has a *singularity of type* (p, q) if it can be reduced to the form $z_1^p + z_2^q = 0$.

Clearly the hypersurface (6–4) has a singularity of type $(2, 2, 3)$ at the origin. Hence a neighborhood of the origin is homeomorphic to the cone over a three-dimensional lens space whose fundamental group has order three. We can resolve this singularity locally by passing to a three-sheeted cover ramified at this single singular point (compare Lemma 6.6).

Remark 6.3: $\mathcal{M}_2 \cong \mathbf{C}^2$. Lemma 6.1 provides a quite different proof that the moduli space $\mathcal{M}_2$ is isomorphic to $\mathbf{C}^2$. Evidently we can obtain $\mathcal{M}_2$ from the algebraic surface (6–4) by identifying each triple (A, B, C) with (A, C, B). Let us introduce the sum $\Sigma = B + C$, which is invariant under this involution. Given any pair $(A, \Sigma) \in \mathbf{C}^2$, we can solve the equations $A^3(A - 1) = BC$ and $\Sigma = B + C$ uniquely for the unordered pair $\{B, C\}$. Thus the quotient surface is isomorphic to $\mathbf{C}^2$, with coordinates A and Σ. (However, these new coordinates are not compatible with the compactification introduced in Section 4.) Of course this proof immediately raises a question: How are these new coordinates (A, Σ) related to the coordinates (σ_1, σ_2) of Section 3? This question will be answered in Corollary C.4 (Appendix C).

Remark 6.4: $\mathcal{M}_2^{\mathrm{cm}}$ **as two-sheeted covering.** Evidently the critically marked moduli space $\mathcal{M}_2^{\mathrm{cm}} \cong W$ can be considered as a two-sheeted ramified covering space of $\mathcal{M}_2 \cong \mathbf{C}^2$. The covering is ramified precisely over the symmetry locus $\mathcal{S}$ of Section 5. For there exists an automorphism of f interchanging the two critical points if and only if $\langle f \rangle \in \mathcal{S}$. The ramification locus or symmetry locus corresponds to the set of points $(A, B, C) \in W$ that satisfy $B = C = \frac{1}{2}\Sigma$, and hence are fixed by the involution $B \leftrightarrow C$. In terms of the coordinates (A, Σ) on $\mathcal{M}_2$, this locus can be described by the fourth-degree equation

$$4A^3(A - 1) = \Sigma^2.$$

(Thus $\mathcal{S}$ is a fourth-degree curve in the coordinates (A, Σ) for $\mathcal{M}_2 \cong \mathbf{C}^2$, but a cubic curve in the more natural coordinates (σ_1, σ_2).) As noted already in Section 5, $\mathcal{S}$ can be described geometrically as a curve of genus zero in $\mathbf{C}^2$ with a single cusp point at the origin.

Remark 6.5: Homotopy type. The critically marked moduli space $\mathcal{M}_2^{\mathrm{cm}}$ has the homotopy type of the two-sphere. In fact, the correspondence

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \frac{\alpha^3}{\gamma} = \frac{B}{A-1} = \frac{A^3}{C}$$

is a smooth map from $\mathcal{M}_2^{\mathrm{cm}}$ onto $S^2 = \mathbf{C} \cup \infty$, with the property that the inverse image of any point is isomorphic to $\mathbf{C}$. (This map is of course not a fibration: local triviality fails about the singular point $A = B = C = 0$, which maps to 0.) The topological two-sphere

$$0 \le A \le 1, \quad |B| = \sqrt{A^3(1 - A)}, \quad C = -\bar{B}$$

is embedded in $\mathcal{M}_2^{\mathrm{cm}}$ as a deformation retract, and maps homeomorphically onto $\bar{\mathbf{C}}$. (This two-sphere provides a natural example of a “teardrop orbifold” [Thurston], which is simply connected but has one point with nontrivial orbifold structure.)

Marked Fixed Points

Instead of numbering the critical points of a rational map f, we can equally well number the fixed points. Let $\mathrm{Rat}_2^{\mathrm{fm}}$ denote the space of *fixed-point marked* quadratic rational maps, that is, ordered quadruples (f, z_1, z_2, z_3), where $z_1, z_2, z_3 \in S^2$ are the fixed points of f, with repetitions in the case of a double or triple fixed point. Let $\mathcal{M}_2^{\mathrm{fm}}$ be the quotient of $\mathrm{Rat}_2^{\mathrm{fm}}$ by the group Rat_1 acting by conjugation

Indeed we can go further and mark both the fixed points and the critical points, thus producing a space $\mathrm{Rat}_2^{\mathrm{tm}}$ of *totally marked* rational maps $(f, z_1, z_2, z_3, \omega_1, \omega_2)$, whose quotient moduli space we denote by $\mathcal{M}_2^{\mathrm{tm}}$.

Lemma 6.6. *The space* $\mathrm{Rat}_2^{\mathrm{fm}}$ *is a smooth complex five-dimensional manifold, and* $\mathrm{Rat}_2^{\mathrm{tm}}$ *is an unramified twofold covering manifold of it. The action of* Rat_1 *on* $\mathrm{Rat}_2^{\mathrm{tm}}$ *by conjugation is free, so* $\mathcal{M}_2^{\mathrm{tm}}$ *is a smooth complex two-manifold. The action of* Rat_1 *on* $\mathrm{Rat}_2^{\mathrm{fm}}$ *has one nonfree orbit, so* $\mathcal{M}_2^{\mathrm{fm}}$ *has one singular point. It corresponds to the conjugacy class of the map* $z \mapsto z + z^{-1}$, *which has just one triple fixed point.*

Proof. To see that the space $\mathrm{Rat}_2^{\mathrm{tm}}$ of totally marked maps is a smooth complex five-manifold, we will identify it with an open subset of the product $(S^2)^5$. More precisely, we will show that a point

$$(f, z_1, z_2, z_3, \omega_1, \omega_2) \in \mathrm{Rat}_2^{\mathrm{tm}}$$

is uniquely determined by its ordered quintuple of fixed and critical points, and that a given quintuple actually occurs if and only if

(a) $\omega_1 \neq \omega_2$, and
(b) for each $i \neq j$, the cross-ratio

$$\frac{(z_i - \omega_1)(z_j - \omega_2)}{(z_j - \omega_1)(z_i - \omega_2)} \in \mathbf{C} \cup \infty$$

is well defined and different from -1.

As in (6–1), it is convenient to consider the special case $\omega_1 = 0$, $\omega_2 = \infty$, so that

$$f(z) = \frac{\alpha z^2 + \beta}{\gamma z^2 + \delta}.$$

With this choice of critical points, condition (b) simply says that $z_i \neq -z_j$. In fact, the two points z_i and $-z_i$ cannot be distinct and both fixed since they have the same image under f, and we cannot have $z_i = z_j \in \{0, \infty\}$, since a critical fixed point cannot also be a double fixed point. The fixed points of f are the roots of the equation $\gamma z^3 - \alpha z^2 + \delta z - \beta = 0$ (a root at infinity corresponding, as usual, to a polynomial equation of reduced degree). The set of roots determines and is determined by the point $(\alpha : \beta : \gamma : \delta) \in \mathbf{CP}^3$, which is subject only to the determinant inequality $\alpha\delta - \beta\gamma \neq 0$. If we express these coefficients in terms of the fixed points, a brief computation shows that this determinant inequality is equivalent to condition (b).

Thus $\mathrm{Rat}_2^{\mathrm{tm}}$ is a smooth complex five-manifold. Since $\mathrm{Rat}_2^{\mathrm{fm}}$ is the quotient space of $\mathrm{Rat}_2^{\mathrm{tm}}$ by the holomorphic involution

$$(z_1, z_2, z_3, \omega_1, \omega_2) \leftrightarrow (z_1, z_2, z_3, \omega_2, \omega_1),$$

which has no fixed points, $\mathrm{Rat}_2^{\mathrm{fm}}$ is also a smooth complex manifold. Similarly, since the quintuple $(z_1, z_2, z_3, \omega_1, \omega_2)$ must contain at least three distinct points, the action of Rat_1 on $\mathrm{Rat}_2^{\mathrm{tm}}$ by conjugation is necessarily free, and the quotient space $\mathcal{M}_2^{\mathrm{tm}}$ is a smooth complex manifold.

On the other hand, the action of Rat_1 on $\mathrm{Rat}_2^{\mathrm{fm}}$ is not free, since the involution $z \mapsto -z$ fixes the point $(f, \infty, \infty, \infty) \in \mathrm{Rat}_2^{\mathrm{fm}}$, where the map $f : z \mapsto z + z^{-1}$ has just one triple fixed point at infinity. In fact, it follows immediately from Lemma 3.1 that the moduli space $\mathrm{Rat}_2^{\mathrm{fm}}$ can be identified with the hypersurface consisting of points $(\mu_1, \mu_2, \mu_3) \in \mathbf{C}^3$ satisfying the polynomial equation

$$\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0.$$

It is easy to check that this surface has a singular point of type $(2,2,2)$ at the point $\mu_1 = \mu_2 = \mu_3 = 1$, corresponding to a map having a triple fixed point. (Compare Remark 6.2.) □

To summarize, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Rat}_2^{\mathrm{tm}} & \xrightarrow{6} & \mathrm{Rat}_2^{\mathrm{cm}} \\ {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\ \mathrm{Rat}_2^{\mathrm{fm}} & \xrightarrow{6} & \mathrm{Rat}_2 \end{array}$$

of holomorphic maps, where the vertical maps are unramified two-sheeted coverings. The horizontal maps are six-sheeted coverings, ramified along the double fixed-point locus $\prod(\mu_i - 1) = 0$, and with a more complicated ramification along the triple fixed point orbit $\mu_1 = \mu_2 = \mu_3 = 1$. Similarly, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_2^{\mathrm{tm}} & \xrightarrow{6} & \mathcal{M}_2^{\mathrm{cm}} \\ {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\ \mathcal{M}_2^{\mathrm{fm}} & \xrightarrow{6} & \mathcal{M}_2 \end{array}$$

of holomorphic maps, where now all four maps have ramification points. However, the projection $\mathcal{M}_2^{\mathrm{tm}} \to \mathcal{M}_2^{\mathrm{fm}}$ ramifies only at the singular point $\mu_1 = \mu_2 = \mu_3 = 1$, so that $\mathcal{M}_2^{\mathrm{tm}}$ can be considered as a desingularization of $\mathcal{M}_2^{\mathrm{fm}}$. Similarly, the projection $\mathcal{M}_2^{\mathrm{tm}} \to \mathcal{M}_2^{\mathrm{cm}}$ has an isolated ramification point at the unique singular point $\mu_1 = \mu_2 = \mu_3 = -2$, so that it can be considered at least locally as a desingularization.

Remark 6.7: Topology. I know almost nothing about the homology or homotopy of the spaces $\mathrm{Rat}_2^{\mathrm{tm}}$ and $\mathrm{Rat}_2^{\mathrm{fm}}$ and $\mathcal{M}_2^{\mathrm{tm}}$ and $\mathcal{M}_2^{\mathrm{fm}}$, or about the fiber bundle $\mathrm{Rat}_1 \hookrightarrow \mathrm{Rat}_2^{\mathrm{tm}} \twoheadrightarrow \mathcal{M}_2^{\mathrm{tm}}$. Any information would be appreciated.

Remark 6.8: Compactifications. Each of these moduli spaces has an appropriate compactification. Thus $\mathcal{M}_2^{\mathrm{cm}}$ can be compactified by adding a two-sphere $\widehat{\mathrm{Per}_1}(\infty)$ of ideal points. The resulting variety $\hat{\mathcal{M}}_2^{\mathrm{cm}}$ can be described as the disjoint union of an open subset consisting of maps

$$z \mapsto 1 + \frac{\alpha}{z} + \frac{\beta}{z^2}$$

with marked critical point zero, where $(\alpha, \beta) \in \mathbf{C}^2 \smallsetminus \{(0,0)\}$, together with a two-sphere $\widehat{\mathrm{Per}_2}(0)$ of limit points as $|\alpha| + |\beta| \to \infty$, and a two-sphere $\widehat{\mathrm{Per}_1}(0)$ of limit points as $(\alpha, \beta) \to (0,0)$. (Compare [Stimson].) It has the homology (but not the cohomology ring) of $S^2 \times S^2$. It has three singular points, at $\langle -2, -2, -2 \rangle$ as before, at $\langle -1, -1, \infty \rangle$ and at $\langle 0, \infty, \infty \rangle$. Locally these singularities can be described as cones over lens spaces with fundamental group of order 3, 2 and 4, respectively.

Similarly, $\mathcal{M}_2^{\mathrm{fm}}$ can be compactified by adding three pairwise intersecting copies of $\widehat{\mathrm{Per}_1}(\infty)$. The resulting variety $\hat{\mathcal{M}}_2^{\mathrm{fm}}$ is singular only at the finite point $\langle 1, 1, 1 \rangle$. It has a cell subdivision with this singular point as vertex, with three copies of $\widehat{\mathrm{Per}_1}(1)$ as two-cells, and with a single four-cell parametrized by the numbers $(1 - \mu_i)^{-1}$ with sum 1. The singular point can be resolved by passing to the twofold branched covering space $\hat{\mathcal{M}}_2^{\mathrm{tm}}$, but this process will introduce new singularities at the three copies of the point $\langle 0, \infty, \infty \rangle$.

Remark 6.9: One marked fixed point. Sometimes it is convenient to consider maps with just one distinguished fixed point. (Compare the normal form (C-2) in Appendix C.) The multiplier μ at this fixed point is then an invariant, and the product τ of the multipliers at the other two fixed points is also an invariant. The pair $(\mu, \tau) \in \mathbf{C}^2$ determines the map up to conjugacy. For we can solve for $\sigma_1 = \sigma_3 + 2 = \mu\tau + 2$, and the sum of the multipliers at the other two fixed points is equal to $\sigma_1 - \mu = \mu\tau + 2 - \mu$. We can easily solve for $\sigma_2 = \mu(\mu\tau + 2 - \mu) + \tau$. Thus τ is an affine parameter along the line $\mathrm{Per}_1(\mu) \subset \mathcal{M}_2$.

Remark 6.10: Marked cubic polynomials. There is a completely analogous concept of markings for cubic polynomial maps (compare Remark 3.2). A cubic polynomial map is uniquely determined by its fixed points z_i and its critical points ω_j, which are subject only to the equality $\frac{1}{3}(z_1 + z_2 + z_3) = \frac{1}{2}(\omega_1 + \omega_2)$ of barycenters. and to the inequality

$$z_1 z_2 + z_1 z_3 + z_2 z_3 \neq 3\omega_1\omega_2.$$

(The equality $z_1z_2 + z_1z_3 + z_2z_3 = 3\omega_1\omega_2$, together with the equality of barycenters, would characterize triples $\{z_1, z_2, z_3\}$ that have a common image under every cubic map with critical points ω_1, ω_2.) From this description, it is not difficult to check that the space of all totally marked cubic polynomial maps is a manifold having the homotopy type of the nonorientable two-sphere bundle over a circle. The corresponding moduli space, consisting of conjugacy classes of totally marked cubic polynomial maps, is a complex manifold that can be identified with the complement of a quadratic curve in $\mathbf{CP}^2$. This moduli space has the homotopy type of $\mathbf{RP}^2$.

These descriptions seem rather complicated. In fact, in the polynomial case it is usually much more convenient to work with *monic centered polynomials*, or sometimes with *critically marked monic centered polynomials*, rather than getting into the complications of a fixed-point marking. However, for quadratic rational maps there does not seem to be any correspondingly convenient normal form.

7. HYPERBOLIC JULIA SETS AND HYPERBOLIC COMPONENTS IN MODULI SPACE

We now turn to the dynamics of quadratic rational maps, starting with a brief description of results due to Mary Rees and Tan Lei. Recall that a rational map is *hyperbolic* (that is, hyperbolic on its Julia set) if and only if the orbit of every critical point converges to some attracting periodic orbit. Hyperbolic maps form an open subset of moduli space, whose connected components are called *hyperbolic components*. Rees works with the critically marked moduli space $\mathcal{M}_2^{\mathrm{cm}}$ of Section 6. However, we can work equally well with the unmarked moduli space $\mathcal{M}_2$ of Section 3, or with the moduli space $\mathcal{M}_2^{\mathrm{fm}}$ or $\mathcal{M}_2^{\mathrm{tm}}$ of Section 6.

Rees shows that the hyperbolic components can be divided into four classes, listed below. (The names are my own.) Recall that the *immediate basin* of an attracting periodic point $z_0 = f^{\circ n}(z_0)$ is the component of z_0 in the open set consisting of all points whose orbit under $f^{\circ n}$ converges to z_0, or, equivalently, the component of z_0 in the Fatou set $\hat{\mathbf{C}} \smallsetminus J(f)$.

Type B: Bitransitive. Each of the two critical points belongs to the immediate basin of some attracting periodic point, with the two basins distinct but belonging to the same orbit. Evidently the period must be two or more.

Type C: Capture. Only one critical point belongs to the immediate basin of a periodic point, but the orbit of the other critical point eventually falls into this immediate basin. Again the period must be two or more (compare Lemma 8.2).

Caution. The word "capture" is used in a quite different sense in [Wittner 1988], as a construction for passing from quadratic polynomial maps to quadratic rational maps. I have only recently been able to get a copy of this important work.

Type D: Disjoint attractors. The two critical points belong to the attracting basins of two disjoint attracting periodic orbits. One particularly interesting and important class of examples are those obtained by *mating* two quadratic polynomial maps: see the next subsection.

Type E: Escape. Both critical orbits converge to the same attracting fixed point. This implies the Julia

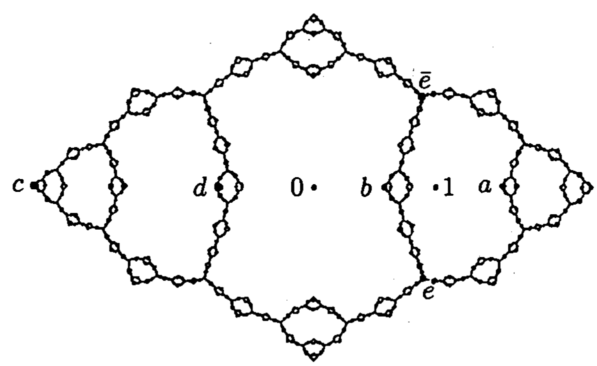

FIGURE 2. Type B: Julia set for $z \mapsto 1 - 1/z^2$, with both critical points in the period three orbit $0 \mapsto \infty \mapsto 1 \mapsto 0$. The fixed points are $d, e, \bar{e}$, and the other period three orbit is $a \mapsto b \mapsto c \mapsto a$.

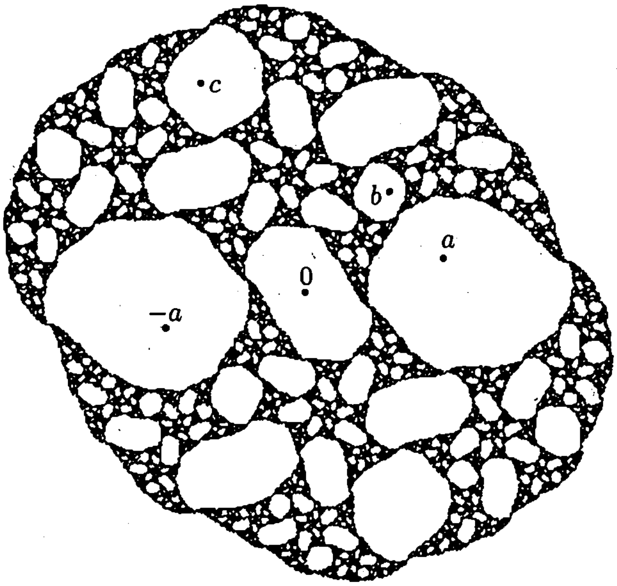

FIGURE 3. Type C: Julia set for $z \mapsto a+1/(z^2-a^2)$, with $a \approx .7467 + .2286i$. Here the periodic critical orbit $\infty \leftrightarrow a$ captures the orbit

$$0 \mapsto b \mapsto c \mapsto -a \mapsto \infty$$

of the other critical point.

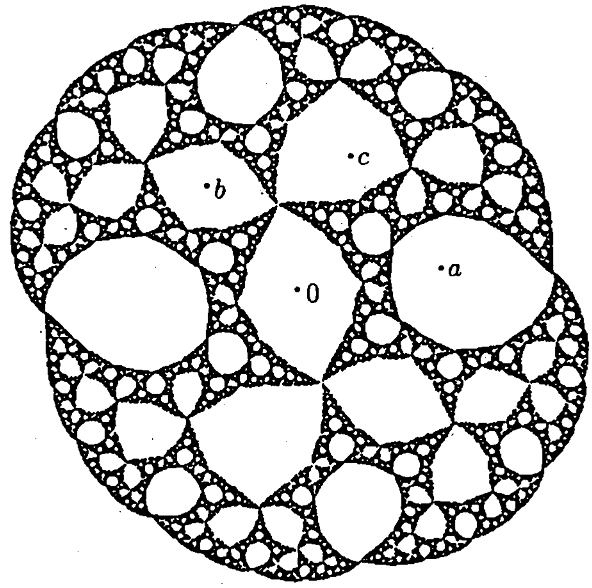

FIGURE 4. Type D: Julia set for $z \mapsto a+1/(z^2-a^2)$, with $a \approx .8200 + .1446i$. (See [Bielefeld 1992].) Here there are two disjoint periodic critical orbits

$$\infty \leftrightarrow a \quad \text{and} \quad 0 \mapsto b \mapsto c \mapsto 0.$$

This example is constructed by mating.

set is totally disconnected. To fix our ideas, we will take this fixed point to be the point at infinity, and say that both critical orbits *escape* to infinity. There is just one such hyperbolic component: it will be discussed in Section 8.

For an analogous discussion of polynomial maps, see [Milnor 1992a, 1992b].

There are infinitely many hyperbolic components of each of the types B, C and D, and they share many similarities. For every hyperbolic map f of one of these types, the Julia set $J(f)$ is connected and locally connected (see Figures 2–4, and compare Lemmas 7.3 and 8.2 below). It follows that each component of its complement $\hat{\mathbb{C}} \setminus J(f)$ is conformally isomorphic to the unit disk. Following McMullen, Rees shows that each hyperbolic component of type B, C or D contains precisely one map f_0, called its *center point*, which is *postcritically finite* (this means that the orbit of each critical point is periodic or eventually periodic). Furthermore, with one trivial exception, she shows that each hyperbolic component of type B, C or D is a topological four-cell. The unique exception is the hyperbolic component $H_*^{\text{cm}} \subset \mathcal{M}_2^{\text{cm}}$ of type B centered at the unique singular point $\langle z \mapsto 1/z^2 \rangle$ of the space $\mathcal{M}_2^{\text{cm}}$. Even in this exceptional case, we can get rid of the singularity and obtain a topological cell simply by working in one of the other versions of moduli space. In fact the corresponding hyperbolic component $H_* \subset \mathcal{M}_2$ is a topological cell, and its six-sheeted ramified covering $H_*^{\text{fm}} \subset \mathcal{M}_2^{\text{fm}}$ is also a topological cell. The corresponding set in $\mathcal{M}_2^{\text{tm}}$ is a disjoint union of two copies of H_*^{fm}.

Some representative examples of Julia sets for postcritically finite hyperbolic maps of types B–D are shown in Figures 2–4. (It is probably impossible to distinguish these three types just by looking at the Julia set.) In each case, the critical points have been placed at zero and infinity, so that the symmetry of the Julia set is evident.

Mating

Suppose given two quadratic polynomials $f_a(z) = z^2 + a$ and $f_b(z) = z^2 + b$, both with connected Julia set. It is often possible to paste together their filled Julia sets so as to obtain a Riemann sphere that decomposes into two halves, one half with the dynamics of f_a and the other with the dynamics of f_b: see [Rees 1992; Tan 1987; Shishikura 1990; Wittner 1988]. According to Tan Lei, *in the postcritically finite case, this is possible if and only if f_a and f_b do not belong to limbs of the Mandelbrot set that are complex conjugate to one another.* (We say that a map belongs to the *t-limb* of the Mandelbrot set M, where t is any angle in $\mathbf{R}/\mathbf{Z}$, if it either has a fixed point of multiplier $e^{2\pi i t}$, or is separated from the central region of M by the map that has such a fixed point.)

Mating yields many examples of quadratic rational maps with two distinct superattractive cycles. However, not every component of type D can be obtained in this way. Wittner has described a real quadratic map with attracting cycles of period three and four that cannot be obtained by mating (Appendix F).

For maps that are not postcritically finite, Tan Lei's result can be conjecturally generalized as follows:

Quadratic Mating Conjecture. If f_a and f_b do not belong to complex conjugate limbs of the Mandelbrot set, it is possible to define a mating $f_a \amalg f_b$, unique as an element of $\mathcal{M}_2^{\mathrm{cm}}$, and depending continuously on a and b.

Here is one way of trying to construct such a mating. Let $U(f_a, \varepsilon)$ be the neighborhood of the filled Julia set $K(f_a)$ consisting of all points for which the Green's function (that is, the canonical potential function) G_a of $K(f_a)$ takes values $G_a(z) < \varepsilon$. Let $M = M(f_a, f_b, \varepsilon)$ be the compact Riemann surface obtained from the disjoint union $U(f_a, \varepsilon) \sqcup U(f_b, \varepsilon)$ by identifying the open subset $U(f_a, \varepsilon) \smallsetminus K(f_a)$ with $U(f_b, \varepsilon) \smallsetminus K(f_b)$ under the correspondence $(G, t) \leftrightarrow (\varepsilon - G, -t)$, where G is the Green's function and t is the external angle. By virtue of the Uniformization Theorem, M is conformally isomorphic to the Riemann sphere $\hat{\mathbf{C}}$. More explicitly, there is a unique conformal isomorphism that takes the critical points of f_a and f_b to $+1$ and -1, respectively, and takes the point with coordinates $G = \varepsilon/2$, $t = 0$ to the point at infinity in $\hat{\mathbf{C}}$. Now f_a and f_b fit together to yield a holomorphic map from $M(f_a, f_b, \varepsilon)$ to $M(f_a, f_b, 2\varepsilon)$. Identifying each of these manifolds with $\hat{\mathbf{C}}$, we get a holomorphic map from $\hat{\mathbf{C}}$ to itself that has critical points at ± 1 and a fixed point at infinity. Thus it is a quadratic rational map of the form

$$\amalg(f_a, f_b, \varepsilon) : w \mapsto \frac{1}{\lambda}\Big(w + \frac{1}{w} + c\Big),$$

where λ is the multiplier at infinity; compare (C–2). If the coefficients c and λ tend to well-defined finite limits as ε tends to zero, the limiting rational map may be called the *mating* $f_a \amalg f_b$. For other, more standard, forms of the definition, see [Shishikura 1990; Tan 1987; Bielefeld 1992].

Remark 7.1: Common Fatou boundary points. In the Julia sets illustrated in Figures 2–4, there are many pairs of Fatou components that have a common boundary point, or (in Figure 2) even a Cantor set of common boundary points. However, this need not be the case. Appendix F, written in collaboration with Tan Lei, describes an example of a hyperbolic map for which no two Fatou components have a common boundary point. The corresponding Julia set is a "Sierpiński carpet".

Remark 7.2: A compactness question. How can one decide whether some given hyperbolic component has compact closure within the moduli space $\mathcal{M}_2 \cong \mathbf{C}^2$, or whether it is unbounded? Any hyperbolic component with an attracting fixed point is certainly unbounded, and Figures 8–10, 16 and 17 show many other examples of unbounded components. On the other hand, consider a component obtained by mating two hyperbolic components of period ≥ 2 of the Mandelbrot set. If the Quadratic Mating Conjecture is true, it is easy to show that such a component has compact closure, canonically homeomorphic to $\bar{D} \times \bar{D}$ (at least in the critically marked case). McMullen [1988] has conjectured that whenever a Julia set $J(f)$ of degree d is a Sierpiński carpet (Appendix F) the corresponding hyperbolic component in $\mathcal{M}_d$ has compact closure.

Lemma 7.3. *If the Julia set of a hyperbolic map is connected, it is locally connected.*

Outline of proof (suggested by Douady). It suffices to consider the postcritically finite case because, according to [McMullen 1988], every hyperbolic map with connected Julia set can be deformed through hyperbolic maps to a postcritically finite one, and according to [Mañé et al. 1983] or [Lyubich 1984] the topology of the Julia set, and the isotopy class of its embedding into $\hat{\mathbf{C}}$, does not change under such a deformation. In the case of a periodic Fatou component U, every internal ray from the superattracting point lands at a well defined boundary point, which depends continuously on the initial angle. (Compare the argument in [Douady and Hubbard 1984, p. 24].) Since the continuous image of a locally connected space is locally connected, it follows that the boundary of U is not only connected but also locally connected. Since every Fatou component is eventually periodic [Sullivan 1985], it follows that *the boundary of every Fatou component is connected and locally connected.*

Let $V \subset \hat{\mathbf{C}}$ be the complement of the postcritical set for f. We will use the Poincaré metric on V and on its universal covering manifold $\tilde{V}$. Note that f^{-1} lifts to a well defined contracting map $\widetilde{f^{-1}}$ on $\tilde{V}$. Each Fatou component U that contains no postcritical point lifts homeomorphically to a subset $U' \subset \tilde{V}$, and $\widetilde{f^{-1}}$ maps U' to a set of strictly smaller diameter in the Poincaré metric. In fact, a straightforward compactness argument shows that the diameter shrinks by a factor bounded away from 1. Thus, *for any $\varepsilon > 0$, there are only finitely many Fatou components of diameter more than ε.* From this it follows easily that J is locally connected. In fact, if two points of J are close to each other, we can find a connected subset of J of small diameter containing both points as follows. Draw a straight line segment between them, and replace its intersection with each Fatou component U by the entire boundary of U if U is small, or by a suitable small connected subset of ∂U otherwise. □

8. THE "ESCAPE LOCUS": TOTALLY DISCONNECTED JULIA SETS

We start with a lemma about rational maps of arbitrary degree, which follows from results that will be proved in Appendix E.

Lemma 8.1. *If all the critical values of a rational map are contained in a single component of the Fatou set $\hat{\mathbf{C}} \smallsetminus J$, then the Julia set J is totally disconnected, and every orbit in the Fatou set converges to an attracting fixed point or to a parabolic fixed point of multiplicity two.*

(The *multiplicity* $m \geq 1$ of a (finite) fixed point $f(z_0) = z_0$ is defined as the degree of the first nonzero term in the Taylor expansion about z_0,

$$f(z) - z = \alpha(z - z_0)^m + \text{higher terms},$$

with $\alpha \neq 0$.) In the quadratic case, there is an explicit dichotomy, and an effective criterion, expressed in the following lemma, which we will prove at the end of this section. See also [Yin 1992; Makienko; Przytycki 1989; Rees 1990a].

Lemma 8.2. *The Julia set J of a quadratic rational map is either connected, or totally disconnected and homeomorphic, with its dynamics, to the one-sided shift on two symbols. It is totally disconnected if and only if either*

(*a*) *both critical orbits converge to a common attracting fixed point, or*
(*b*) *both critical orbits converge to a common fixed point of multiplicity two, but neither critical orbit actually lands on this point.*

Example 8.3: Quadratic maps. For the map $f(z) = z + 2 + z^{-1}$, one critical orbit $1 \mapsto 4 \mapsto 6.25 \mapsto \cdots$ converges to the parabolic fixed point at infinity, while the other critical orbit $-1 \mapsto 0 \mapsto \infty$ actually lands at this fixed point. Thus the Julia set is connected. In fact $J(f)$ is the interval $[-\infty, 0]$. For $f(z) = \pm(z + z^{-1})$, both critical orbits $\pm 1 \mapsto \pm 2 \mapsto \pm 2.5 \mapsto \cdots$ converge to the parabolic point at infinity. Since the multiplicity of the fixed point at infinity is either three (if the multiplier is $+1$) or one (if the multiplier is -1), the Julia set is again connected. It equals the imaginary axis plus the point at infinity. On the other hand, for $z \mapsto z + c + z^{-1}$ with $c > 2$, the Julia set is totally disconnected.

Example 8.4: Cubic maps. The situation for maps of higher degree is more complicated. The rational map

$$z \mapsto 2 + \frac{2z^3}{27(2-z)}$$

has connected Julia set, although the critical points $0, 0, 3, \infty$ are all contained in an orbit $3 \mapsto 0 \mapsto 2 \mapsto \infty$, which lands at a superattracting fixed point. The Julia set of the polynomial map $f(z) = z^3 - 12z + 12$ is a Cantor set, contained in the interval $[-4, 3]$, although the critical point $+2$ belongs to it. The Julia set for the map $z \mapsto \frac{5}{36}(z-3)^2(z+4)$ is disconnected, but contains the connected interval $[0, 5]$. For a thorough analysis of such polynomial examples, see [Branner and Hubbard 1992].

Definition: Escape locus. There is just one hyperbolic component of type E in the moduli space $\mathcal{M}_2$. We will call it the *hyperbolic escape locus*, and denote it by $E \subset \mathcal{M}_2$. (Perhaps a better term would be "shift locus" or "totally disconnected locus"?) If we work with critically marked conjugacy classes, there is a corresponding unique component $E^{\mathrm{cm}} \subset \mathcal{M}_2^{\mathrm{cm}}$.

The escape locus is very different from the other hyperbolic components. For a map f of type E, the Julia set $J(f)$ is a Cantor set, and its complement is a connected open set of infinite connectivity. Such a map can never be postcritically finite, so this hyperbolic component E does not have any preferred center point within $\mathcal{M}_2$.

Here is a well-known collection of examples. We consider the one-parameter family of polynomials $z \mapsto z^2 + c$, which can be identified with the one-dimensional slice $\sigma_3 = 0$ through the moduli space $\mathcal{M}_2$. The intersection of this family with the escape locus E is precisely the complement of the Mandelbrot set. This complement is conformally isomorphic to $\mathbf{C} \smallsetminus \bar{D}$, with free cyclic fundamental group. The corresponding hyperbolic escape locus for polynomials of higher degree has been studied by Blanchard, Devaney and Keen, who show that it has a very rich fundamental group.

Using the critically marked moduli space, Rees shows that the escape locus E^{cm} has a rather complicated topology. In particular, her description implies that E^{cm} has the homotopy type of a Klein bottle, and hence nonabelian fundamental group. However, the unmarked escape locus $E \subset \mathcal{M}_2$ has a much simpler description:

Lemma 8.5. *The hyperbolic escape locus $E \subset \mathcal{M}_2$ is homeomorphic to the product $D \times (\mathbf{C} \smallsetminus \bar{D})$, where D is the open unit disk.*

More explicitly, if $\mu = \mu(f) \in D$ is the multiplier at the unique attracting fixed point of f, we will show that the correspondence $\langle f \rangle \mapsto \mu(f)$ yields a fibration of E over D, with fiber E_μ homeomorphic to $\mathbf{C} \smallsetminus \bar{D}$ (or equivalently to $D \smallsetminus \{0\}$). As in Remark 6.9, it will be convenient to work with the coordinates (μ, τ), where μ is the multiplier at the preferred (attracting) fixed point and τ is the product of the multipliers at the other two (repelling) fixed points. Thus each line of constant $\mu \in D$ in the (μ, τ)-plane corresponds to a straight line $\mathrm{Per}_1(\mu) \subset \mathcal{M}_2$, consisting of all conjugacy classes of maps that have an attracting fixed point of multiplier μ. (Compare Section 3. These straight lines $\mathrm{Per}_1(\mu)$ intersect each other within $\mathcal{M}_2$, but not within the escape locus, since for $\langle f \rangle \in E$ the map f has only one attracting fixed point.)

Proof of 8.5. The proof will be based on [Goldberg and Keen 1990]. For each fixed $\mu \in D$, these authors show, using the theory of polynomial-like mappings [Douady and Hubbard 1985], that the complement M_μ of the escape locus in $\mathrm{Per}_1(\mu) \cong \mathbf{C}$ is canonically homeomorphic to the Mandelbrot set $M = M_0$. (This statement is quite likely true even in the limiting case $\mu = 1$; see Figure 5.) For $\langle f \rangle \in M_\mu$ the map f restricted to a suitable neighborhood of its Julia set is polynomial-like of degree two with connected Julia set, and hence is hybrid-equivalent to a unique conjugacy class in M_0. Furthermore, by [Douady and Hubbard 1985, § 7], the resulting map from $\bigcup_\mu M_\mu \subset \mathcal{M}_2$ to M_0 is continuous.

Caution. This statement is formulated rather differently by Goldberg and Keen, since the authors work with marked critical points and hence study a twofold branched covering space $\mathrm{Per}_1(\mu)^{\mathrm{cm}}$, which contains two disjoint copies of the Mandelbrot set for $\mu \neq 0$. Also, some notations (such as Rat_2) used in [Goldberg and Keen 1990] are incompatible with ours.

It follows from the Riemann Mapping Theorem that the complement

$$E_\mu = \mathrm{Per}_1(\mu) \cap E \cong \mathbf{C} \smallsetminus M_\mu$$

is conformally isomorphic to the region $\mathbf{C} \smallsetminus \bar{D}$. In fact, we can choose a canonical conformal isomorphism

$$\psi_\mu : \mathbf{C} \smallsetminus \bar{D} \to \mathbf{C} \smallsetminus M_\mu,$$

normalized so that the multiplier at infinity, $\lambda(\mu) = 1/\lim_{z\to\infty} \psi'_\mu(z)$, is real and positive. We must show that ψ_μ depends continuously on μ, using the topology of locally uniform convergence, so that the correspondence

$$(\mu, z) \mapsto (\mu, \psi_\mu(z)) \in \mu \times E_\mu$$

yields the required homeomorphism between $D \times (\mathbf{C} \smallsetminus \bar{D})$ and the escape locus E.

Note that this multiplier $\lambda(\mu)$ provides an invariant measure of the "size" of the open set $E_\mu = \psi_\mu(\mathbf{C} \smallsetminus \bar{D})$. More generally, if $U_1 \subset U_2$ are simply connected neighborhoods of infinity in $\hat{\mathbf{C}}$ and if λ_1 and λ_2 are the multipliers at infinity of the corresponding Riemann maps $\mathbf{C} \smallsetminus \bar{D} \to U_i$, it follows easily from the Schwarz Lemma that $\lambda_1 \le \lambda_2$, with equality if and only if $U_1 = U_2$.

Assume, for a contradiction, that the correspondence $\mu \mapsto \psi_\mu$ is not continuous. Then we can choose a sequence of points $\mu_i \in D$ converging to a limit $\hat{\mu} \in D$ so that the corresponding sequence of functions ψ_{μ_i} on $\mathbf{C} \smallsetminus \bar{D}$ does not converge locally uniformly to $\psi_{\hat{\mu}}$. However, since the ψ_{μ_i} belong to a normal family, we may assume after passing to a subsequence that these functions do converge locally uniformly to some univalent limit $\hat{\psi} \neq \psi_{\hat{\mu}}$. It is not hard to check that the image $\hat{\psi}(\mathbf{C} \smallsetminus \bar{D})$ must be a proper subset of the open set $\psi_{\hat{\mu}}(\mathbf{C} \smallsetminus \bar{D})$. Therefore, as noted above, the multiplier $\hat{\lambda}$ of $\hat{\psi}$ at infinity must be strictly less than the multiplier $\lambda(\hat{\mu})$ of $\psi_{\hat{\mu}}$, say $\hat{\lambda} < \lambda(\hat{\mu})/(1+\varepsilon)$ with $\varepsilon > 0$. Now the circle of radius $1+\varepsilon$ in $\mathbf{C} \smallsetminus \bar{D}$ corresponds under $\psi_{\hat{\mu}}$ to a loop that encloses a small neighborhood of the compact set $M_{\hat{\mu}}$. For μ_i sufficiently close to $\hat{\mu}$ the compact set M_{μ_i} must be contained in this neighborhood, and it follows easily that

$$\lambda(\mu_i) \ge \frac{\lambda(\hat{\mu})}{1+\varepsilon}.$$

Passing to the locally uniform limit as $i \to \infty$, since $\lambda(\mu_i) \to \hat{\lambda}$, we obtain a contradiction. □

Remark 8.6: The escape locus in $\hat{\mathcal{M}}_2$. If we work with the compactified moduli space $\hat{\mathcal{M}}_2$ of Section 4, the appropriate hyperbolic escape locus $E^* \subset \hat{\mathcal{M}}_2$ is a topological four-cell with a preferred center point, just like all other hyperbolic components. In fact, let E^* consist of E together with the two-cell consisting of all ideal points $\{\mu, \mu^{-1}, \infty\}$ for which $|\mu| < 1$. Then E^* fibers over the open disk with an open disk as fiber; and each fiber contains one and only one ideal point. By definition, the "center" of this filled-in component is the ideal point $\langle 0, \infty, \infty \rangle$. This center can perhaps be identified with the improper map $\langle z \mapsto z^2 + \infty \rangle$, or with the limit of a sequence of quadratic maps "tending" to a constant map.

Next we study the escape locus $E^{\text{cm}} \subset \mathcal{M}_2^{\text{cm}}$, with marked critical points. We will prove the following result; for a more precise description of E^{cm}, see [Rees 1990a].

Lemma 8.7. *The critically marked escape locus E^{cm} contains a Klein bottle as retract. Hence the fundamental group of E^{cm} is nonabelian.*

Proof. E^{cm} can be described as a twofold branched covering of E, branched along the symmetry locus $\mathcal{S} \cap E$, which consists of all pairs (μ, τ) with $\mu \in D \setminus \{0\}$ and $\tau = (2-\mu)^2/\mu^2$. However, it seems somewhat easier to take a different approach. We will first work with fixed-point markings, and describe the corresponding escape locus $E^{\text{fm}} \subset \mathcal{M}_2^{\text{fm}}$. Recall that a point in $\mathcal{M}_2^{\text{fm}}$ is specified by a triple (μ_1, μ_2, μ_3) satisfying the cubic equation (3–1). For a map in the escape locus, one of the three fixed points must be attracting and the other two must be repelling. Thus E^{fm} actually splits into three distinct components, depending on which of the three marked fixed points is attracting. To fix our ideas, suppose that $|\mu_1|, |\mu_2| > 1 > |\mu_3|$. We will take the pair $(\mu_1, \mu_2) \in (\mathbf{C} \setminus \bar{D}) \times (\mathbf{C} \setminus \bar{D})$ as independent parameters, solving for μ_3 by means of (3–3). Of course, not every such pair determines a map in the escape locus, or even a map with $|\mu_3| < 1$, but pairs with sufficiently large $|\mu_1|$ and $|\mu_2|$ do:

Lemma 8.8. *If $|\mu_1| > 6$ and $|\mu_2| > 6$, the associated map f belongs to the escape locus.*

This estimate is probably far from sharp. The largest values of $|\mu_1|$ and $|\mu_2|$ known to me for which f lies outside the escape locus occur for $f(z) = -(z + z^{-1})$, with $\mu_1 = \mu_2 = -3$.

Proof of 8.8. We will use the fixed-point normal form

$$f(z) = \frac{z(z + \mu_1)}{\mu_2 z + 1} = \frac{\mu_1 + z}{\mu_2 + z^{-1}}$$

of (3–4) (see also Appendix C). If $|\mu_1| > 6$ and $|\mu_2| > 6$, a brief computation shows that f maps the annulus $2/|\mu_2| < |z| < |\mu_1|/2$ into a compact subset of itself. Furthermore, the polynomial $\mu_2 z^2 + 2z + \mu_1$, whose roots are the critical points of f, is strictly nonzero outside of this annulus, so both critical points are contained in the annulus. Using the Poincaré metric for this annulus, we see that both critical orbits must converge to a common attracting fixed point. The conclusion then follows by Lemma 8.2. □

We return to the proof of Lemma 8.7. We can easily construct a mapping from our preferred component of E^{fm} to the torus by the correspondence

$$(\mu_1, \mu_2) \mapsto (\arg \mu_1, \arg \mu_2).$$

The subset $|\mu_1| = |\mu_2| = 7$ maps homeomorphically to the torus, and hence is embedded in E^{fm} as a retract.

Now we mark also the critical points, thus passing to the two-sheeted covering manifold E^{tm}. The single ramification point for the map $\mathcal{M}_2^{\text{tm}} \to \mathcal{M}_2^{\text{fm}}$ does not belong to the escape locus, and hence causes no difficulty. It is not difficult to show that a choice of critical point for f is equivalent to a choice of sign for $\pm\sqrt{1 - \mu_1\mu_2}$. (Compare the discussion following (C–1) in Appendix C.) Since the ratio $(1 - \mu_1\mu_2)/(\mu_1\mu_2)$ always lies in the left half-plane when $|\mu_1\mu_2| > 1$, this is equivalent to making a choice of sign for the geometric mean $\eta = \pm\sqrt{\mu_1\mu_2}$. In other words, our component of E^{tm} can be identified with an open subset of the manifold consisting of all $(\mu_1, \mu_2, \eta) \in (\mathbf{C} \setminus \bar{D})^3$ for which $\eta^2 = \mu_1\mu_2$. In particular, we obtain an explicit retraction onto the torus

$$T = \{(\mu_1, \mu_2, \eta) : |\mu_1| = |\mu_2| = |\eta| = 7,\ \eta^2 = \mu_1\mu_2\}.$$

In order to obtain the hyperbolic component E^{cm}, as described by Rees, we must pass to a quotient space by identifying under the involution of E^{tm} which interchanges the role of the two repelling fixed points. A brief computation shows that this corresponds to the fixed point free involution

$$(\mu_1, \mu_2, \eta) \leftrightarrow (\mu_2, \mu_1, -\eta).$$

The quotient space E^{cm} is still a smooth manifold. If we collapse the torus T under this involution, we obtain a Klein bottle. Thus this argument proves that $E^{\rm cm}$ contains a Klein bottle as retract, and hence that $\pi_1(E^{\rm cm})$ retracts onto the fundamental group of a Klein bottle. □

If we try to carry out the same argument for the unmarked escape locus $E \subset \mathcal{M}_2$, we must simply work with the torus $|\mu_1| = |\mu_2| = 7$ and the orientation-reversing involution $(\mu_1, \mu_2) \mapsto (\mu_2, \mu_1)$. In this case, the quotient space is a Möbius band, which has the homotopy type of a circle. Thus we would conclude only that E contains a circle as retract.

Recall that $\mathcal{M}_2^{\rm cm}$ has the homotopy type of a two-sphere. It is conjectured that the inclusion map $E^{\rm cm} \to \mathcal{M}_2^{\rm cm}$ is homologically nontrivial, or, more explicitly, that it induces an isomorphism of two-dimensional homology with modulo 2 coefficients.

Proof of 8.2. We first prove that every quadratic Julia set J is either connected or totally disconnected. J is connected if and only if every component of the Fatou set $\hat{\mathbf{C}} \smallsetminus J$ is simply connected. According to Sullivan, every component of $\hat{\mathbf{C}} \smallsetminus J$ is eventually periodic, and every periodic component is either a Siegel disk, a Herman ring, or an immediate basin for some attracting or parabolic point. According to [Shishikura 1987], there are no Herman rings in the quadratic case.

We will frequently use the fact that a ramified covering of a simply connected region that has only one ramification point is again simply connected.

First suppose that every periodic Fatou component is simply connected. Any cycle of Fatou components must either contain a critical point (in the attracting or parabolic cases), or have a critical orbit that is dense in its boundary (in the Siegel case). Thus there is at most one critical point left over. It follows inductively that every Fatou component is simply connected.

Now suppose that some periodic Fatou component is not simply connected. Evidently it must be an immediate basin for some attracting or parabolic periodic point. Such an immediate basin can be reconstructed from a simply connected neighborhood of the fixed point in the attracting case, or from a simply connected petal in the parabolic case, by taking a direct limit of successive ramified coverings, ramified only at the critical values. The result will again be simply connected, unless both critical points belong to the same connected component. But in that case, it follows from Lemma 8.1 that J is totally disconnected. (Compare [Rees 1990a] and Appendix E.) Thus J is connected or totally disconnected.

If both critical orbits converge to the same attracting fixed point or fixed point of multiplicity two, at least one critical point must belong to the immediate basin U. Hence the restriction of f to U is two-to-one. Since f is quadratic, and since $f(U) = U$, this implies that U is fully invariant: $U = f^{-1}(U)$. Hence both critical points belong to U, and again we can apply Lemma 8.1.

Finally, we must prove that a totally disconnected J is homeomorphic to a one-sided two-shift. In the hyperbolic case, this is proved in [Goldberg and Keen 1990]. The key step is the construction of an embedded disk $\Delta \subset \hat{\mathbf{C}} \smallsetminus J$ that contains both critical values and is forward invariant, $f(\Delta) \subset \Delta$. The two components of $\hat{\mathbf{C}} \smallsetminus f^{-1}(\Delta)$ then cover the Julia set, and form the required Bernoulli partition. In the parabolic case, we modify this argument by constructing a simply connected open petal P that contains both critical values and satisfies $f(P) \subset P$. Again the two components of $\hat{\mathbf{C}} \smallsetminus f^{-1}(P)$ cover the Julia set and form the required Bernoulli partition. The proof that a point in the Julia set is uniquely determined by its symbol sequence with respect to this partition is more delicate in this case. However, since a completely analogous argument is carried out in Appendix E, details will be left to the reader. □

9. SOME COMPLEX ONE-DIMENSIONAL SLICES

The moduli space $\mathcal{M}_2$ can be thought of as a kind of table of contents, each point $\langle f \rangle \in \mathcal{M}_2$ corresponding to a different form of dynamic behavior. Since this space is a complex two-manifold, it is difficult to visualize all of it directly. Hence it may be helpful to try to describe what kinds of behavior occur for $\langle f \rangle$ belonging to some one-dimensional slice through M.

As a first example of a dynamically interesting one-dimensional slice, fix some integer $n \geq 1$ and some complex number μ, and consider the curve $\mathrm{Per}_n(\mu) \subset \mathcal{M}_2$ consisting of all conjugacy classes of maps that possess a periodic point of period n and multiplier μ (see Lemma 3.4 and Theorem 4.2). The case $\mu = 0$ is of particular interest: compare [Rees 1990b, 1992; Milnor 1992a; Milnor], and see Figures 8 and 9. Evidently the center point of every hyperbolic component must belong to at least one of these curves $\mathrm{Per}_n(0)$. For any μ in the closed disk $|\mu| \leq 1$, the condition $\langle f \rangle \in \mathrm{Per}_n(\mu)$ imposes very strong restrictions on the dynamics of f. The cases where μ is a root of unity are noteworthy, since these curves contain faces where two or more hyperbolic components of $\mathcal{M}_2$ come together along a common boundary: see Figures 5–7.

We can also consider curves for which one critical orbit is eventually periodic, say

$$f^{\circ t}(\omega_1) = f^{\circ t+n}(\omega_1).$$

See Figure 11 for the case $t = 2$, $n = 1$. These curves also contain common boundary faces between two hyperbolic components. The symmetry locus of Section 5 is another curve of interest (Figure 12).

As a final interesting family of curves, for each integer $t \geq 1$ we can consider all f such that the t-th forward image of one critical point is equal to the other critical point, $f^{\circ t}(\omega_1) = \omega_2$. See Figure 13 for the case $t = 1$. Note that the center point of every hyperbolic component of type B or C must belong to at least one of these curves.

Here is a more detailed discussion of some sample curves. It will be convenient to define the *bifurcation locus* of a parametrized family of maps $\{f_a\}$ as the closed set consisting of parameter values a for which the associated Julia set $J(f_a)$ does not vary continuously under deformation of a; compare [Mañé et al. 1983; Lyubich 1984].

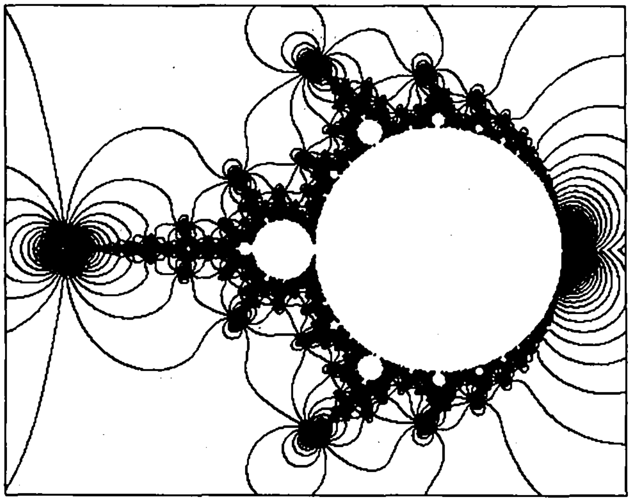

FIGURE 5. The bifurcation locus in $\mathrm{Per}_1(1)$ seems to be a homeomorphic copy of the boundary of the Mandelbrot set, with the cusp straightened out. Here $\mathrm{Per}_1(1)$ is represented as the $-a^2$-plane for the family of maps $f_a(z) = z + a + 1/z$, having a double fixed point of multiplier $+1$ at infinity, and having critical points at ± 1 (see Appendix C). The other fixed point has multiplier $1-a^2$, and the orbit of period two has multiplier $9 - 4a^2$. Level curves of the function $\mathrm{Re}(f_a^{\circ n}(\pm 1)/a) - n$, for large n, are also shown. See also Appendix G.

Period 1

As noted in Lemma 3.4, the curves $\mathrm{Per}_1(\mu)$ are particularly easy to describe: Each locus $\mathrm{Per}_1(\mu)$ is a straight line of slope $\mu + \mu^{-1}$ in the (σ_1, σ_2)-coordinate plane, given for example by the equation

$$\mu^3 - \sigma_1\mu^2 + \sigma_2\mu + 2 - \sigma_1 = 0.$$

First consider the case $\mu = 0$. Putting the fixed critical point at infinity, we can use the normal form $z \mapsto z^2 + c$, with $\sigma_1 = 2$ and $\sigma_2 = 4c$. The corresponding bifurcation locus in the c-plane is the boundary of the familiar Mandelbrot set. The hyperbolic components that intersect $\mathrm{Per}_1(0)$ all have type D or E.

(If we lift to the critically marked moduli space $\mathcal{M}_2^{\mathrm{cm}}$, the locus $\mathrm{Per}_1(0)$ lifts to a union of two lines that intersect transversally. In fact, in the coordinates (A, B, C) of (6–3), the locus $\mathrm{Per}_1^{\mathrm{cm}}(\mu)$ splits as the union of a line $A = 1$, $B = 0$ for which

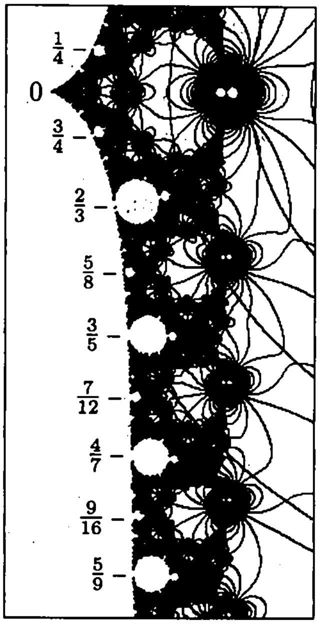

FIGURE 6. The bifurcation locus in $\mathrm{Per}_1(-1)$ is a distorted copy of the Mandelbrot set boundary with the $\frac{1}{2}$-limb missing (collapsed to the point at infinity). This picture shows part of the $-a^2$-plane for the family of maps $z \mapsto -(z+a+z^{-1})$, having a fixed point of multiplier -1. The notation p/q indicates that the map also has a fixed point of multiplier $e^{2\pi ip/q}$; the case $\frac{1}{2}$ would correspond to the point at infinity.

the first critical point is fixed and a line $A = 1$, $C = 0$ for which the second critical point is fixed. These two lines intersect transversally at $A = 1$, $B = C = 0$, corresponding to the map $z \mapsto z^2$ with both critical points fixed.)

As noted in the proof of Lemma 8.5, the bifurcation locus in the line $\mathrm{Per}_1(\mu)$ varies by a continuous isotopy as μ varies within the open unit disk. As shown in Figure 5, it even seems to retain the same topology as μ tends nontangentially to $+1$. On the other hand, as $\mu \to -1$ nontangentially, the central region of the Mandelbrot set opens out so as to contain a full half-plane, and the entire $\frac{1}{2}$-limb of the Mandelbrot set disappears to infinity. Also, a number of the tentacles of the Mandelbrot set join together, so as to enclose new regions (Figure 6).

Similarly, as μ tends to $e^{-2\pi ip/q}$ nontangentially, the entire p/q-limb disappears to infinity. This dis-

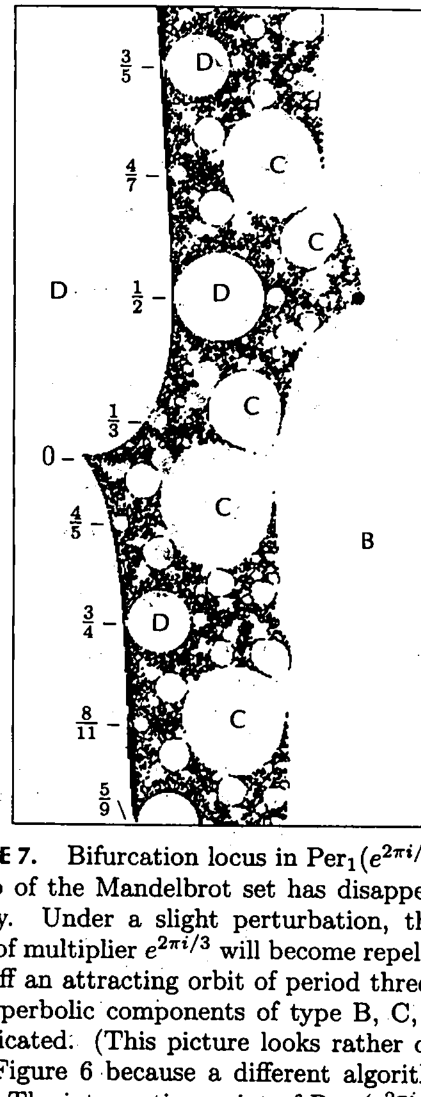

FIGURE 7. Bifurcation locus in $\mathrm{Per}_1(e^{2\pi i/3})$. The $\frac{2}{3}$-limb of the Mandelbrot set has disappeared to infinity. Under a slight perturbation, the fixed point of multiplier $e^{2\pi i/3}$ will become repelling but split off an attracting orbit of period three, yielding hyperbolic components of type B, C, and D, as indicated. (This picture looks rather different from Figure 6 because a different algorithm was used.) The intersection point of $\mathrm{Per}_1(e^{2\pi i/3})$ with the line $\mathrm{Per}_2(-3)$ of Figure 10 is indicated by a heavy dot (upper right).

appearance is one aspect of Tan Lei's observation that mating between two quadratic polynomials is never possible when the two belong to complex conjugate limbs of the Mandelbrot set. The case $p/q = \frac{2}{3}$ is illustrated in Figure 7: again, various tentacles of the Mandelbrot set have come together to enclose new regions.

A more precise statement can be made. Suppose that $f \in \mathrm{Rat}_2$ has an attracting fixed point of multiplier $\mu = \mu_1$ and a repelling fixed point with

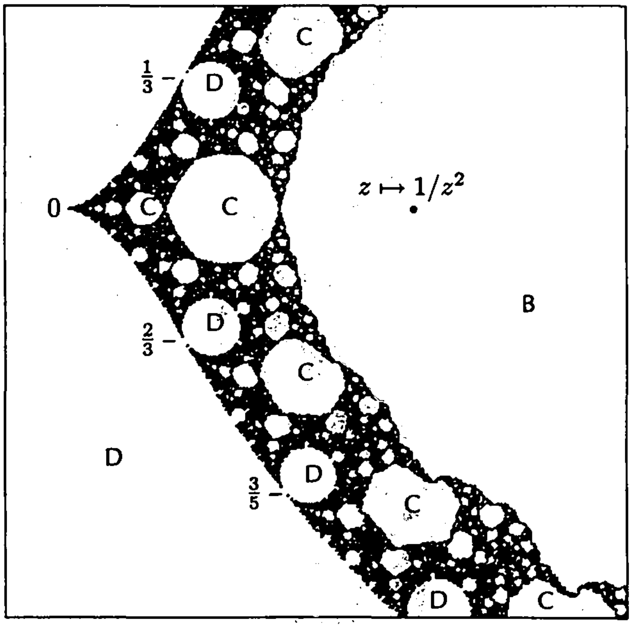

FIGURE 8. Bifurcation locus in $\mathrm{Per}_2(0)$, showing part of the $-a^3$-plane for the family of maps

$$z \mapsto a/z + 1/z^2,$$

with a critical point of period two at $z = 0$. This appears to be a homeomorphic copy of Figure 6. The types of some of the more conspicuous hyperbolic components are indicated.

combinatorial rotation number p/q and multiplier μ_2. Then it is shown in [Petersen] that a suitable choice of $\log \mu_2$ is contained in the disk bounded by the circle tangent to the imaginary axis at $2\pi ip/q$ and passing through $\log(1/\mu_1)$. Hence, as μ_1 tends to $e^{-2\pi ip/q}$ nontangentially, μ_2 tends to $e^{2\pi ip/q}$. If $0 < p/q < 1$, it follows from (3–3) that the multiplier μ_3 at the third fixed point tends to infinity. Thus *the entire p/q-limb of the Mandelbrot set must be missing in the locus* $\mathrm{Per}_1(e^{-2\pi ip/q})$.

Period 2

According to Lemma 3.6, the curve $\mathrm{Per}_2(\mu)$ is also a straight line in the (σ_1, σ_2)-coordinate plane, with equation $2\sigma_1 + \sigma_2 = \mu$. In the case $\mu = 1$, we have $\mathrm{Per}_2(1) = \mathrm{Per}_1(-1)$: see Figure 6. As μ varies from 1 to 0, the left-hand region of Figure 6 opens up even further, as shown in Figure 8.

There is some difficulty in finding a good normal form in the case $\mu = 0$, since the curve $\mathrm{Per}_2(0)$ includes the special point $\langle z \mapsto 1/z^2 \rangle$, which has an anomalously large group of automorphisms. If we place the critical point of period two at the origin and its image at infinity, and normalize by an appropriate scale change, we get the formula

$$f(z) = \frac{a}{z} + \frac{1}{z^2}.$$

This normal form is not unique, since for any cube root of unity λ the map $f(\lambda z)/\lambda$ will have the same form, but with the parameter a replaced by λa. Thus to obtain an invariant we must pass to a^3, which ranges over $\mathbf{C}$. In fact it turns out that $a^3 - 6 = \sigma_1$.

Period 3

The curve $\mathrm{Per}_3(0)$ also has genus zero, being conformally isomorphic to a punctured plane or to a cylinder. We may put the periodic critical point at the origin, with orbit $0 \mapsto \infty \mapsto 1 \mapsto 0$. This yields the unique normal form

$$z \mapsto 1 - \frac{1+c}{z} + \frac{c}{z^2},$$

where now the parameter c ranges over $\mathbf{C} \smallsetminus \{0\}$.

Figure 9 shows the situation in the $(\log c)$-plane. The lower left-hand portion of this picture can perhaps be understood as a perturbation of Figure 7, and the upper left portion as a perturbation of a complex conjugate figure. In fact, as μ varies from 0 to $1 - \varepsilon$ the bifurcation locus in $\mathrm{Per}_3(\mu)$ varies continuously. However, in the limit, as $\mu \to 1$, the locus $\mathrm{Per}_1(\mu)$ degenerates into a union of three straight lines (compare Remark 4.3). Two of these lines correspond to the locus $\mathrm{Per}_1(e^{2\pi i/3})$ and its complex conjugate, while the third line can be identified with the locus $\mathrm{Per}_2(-3)$. Thus, in order to understand the right half of Figure 9, we must also study the bifurcation locus in $\mathrm{Per}_2(-3)$, as shown in Figure 10. Unfortunately, it seems to require considerable imagination to see how to

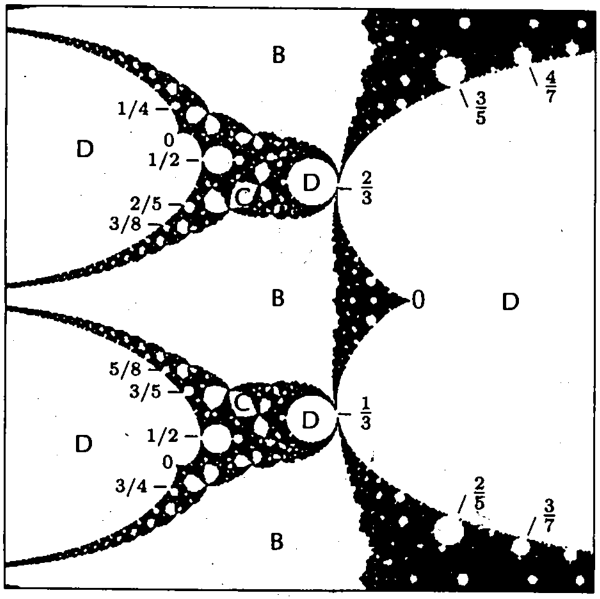

FIGURE 9. Picture in $\mathrm{Per}_3(0)$, showing the $(\log c)$-plane for the family of maps $z \mapsto 1-(1+c)/z+c/z^2$ with a critical point of period 3 at $z = 0$. (The top and bottom are to be identified, with a bit of overlap.) There are only two components of type B in the cylinder $\mathbf{C}/2\pi i\mathbf{Z}$, but many components of type C and D. For the three most prominent components of type D, corresponding to the three components of period three in the Mandelbrot set, internal angles have been indicated. For the large component on the right, the point at infinity has internal angle $\frac{1}{2}$; for the upper and lower D components on the left, it has internal angle $\frac{1}{3}$ and $\frac{2}{3}$, respectively.

cut and paste Figure 7, its complex conjugate, and Figure 10 so as to obtain Figure 9.

For more detailed studies of $\mathrm{Per}_3(0)$ and of the associated dynamics, see [Rees 1990b, 1992; Wittner 1988].

Period 4

The curve $\mathrm{Per}_4(0)$ is isomorphic to a three-times punctured plane. Again we place the periodic critical point at the origin, and now suppose that it has orbit $0 \mapsto \infty \mapsto 1 \mapsto \rho \mapsto 0$. The parameter ρ can be described as the cross-ratio of the points in this orbit. It is not difficult to check that ρ determines the quadratic map uniquely, and that it ranges over

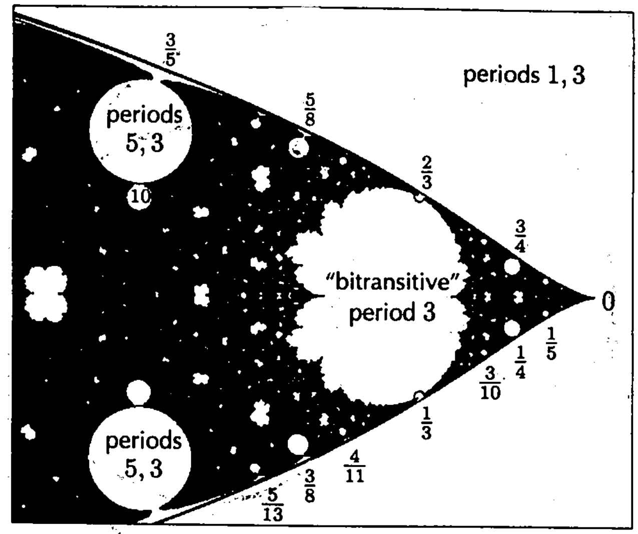

FIGURE 10. Bifurcation locus in the line $\mathrm{Per}_2(-3)$, which coincides with the locus of $\langle f\rangle$ for which f has a nondegenerate orbit of period 3 and multiplier $+1$. (The algorithm used to draw this figure is not too satisfactory. In fact, the boundary of the outer region looks rather ragged because of very slow convergence.)

the set $\mathbf{C} \smallsetminus \{0, \frac{1}{2}, 1\}$. The value $\frac{1}{2}$ is excluded since as $\rho \to \frac{1}{2}$ the associated quadratic map degenerates to a fractional linear map $z \mapsto 1-(2z)^{-1}$, and the class $\langle f\rangle$ tends to the ideal point $\langle i, -i, \infty\rangle$.

Preperiodic critical orbits

Another important class of parameter slices are those for which one critical point is preperiodic. Since a nonperiodic critical point of a quadratic map cannot map directly to a periodic point, the simplest possibility is that the second forward image $f^{\circ 2}(\omega_1)$ is fixed. If we place the critical points at ± 1 and this fixed point at infinity, this leads to the normal form

$$w \mapsto \lambda\Big(w + 2 + \frac{1}{w}\Big),$$

with $-1 \mapsto 0 \mapsto \infty$. Note that λ^{-1} is the multiplier of the fixed point at infinity. A corresponding picture in the λ-plane is shown in Figure 11. Values of λ outside of the unit disk correspond to maps in the escape locus E. However, points within the

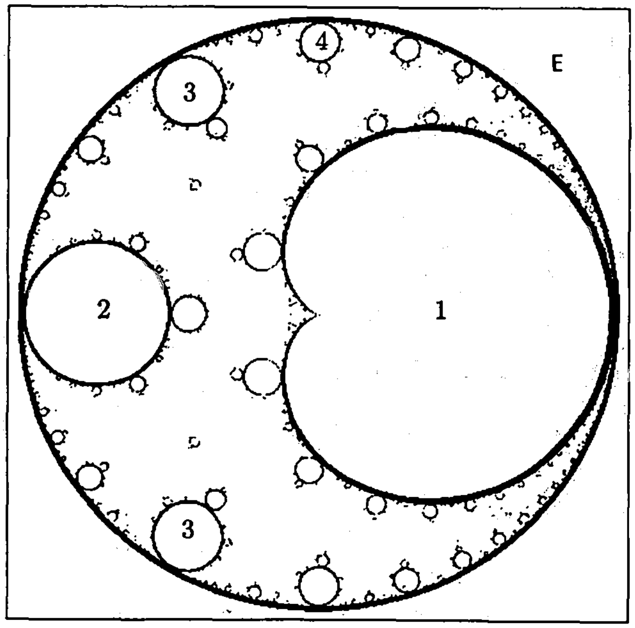

FIGURE 11. Picture for the λ-plane for the family of maps $z \mapsto \lambda(z+2+z^{-1})$ with preperiodic critical orbit $-1 \mapsto 0 \mapsto \infty \mapsto \infty$. In the more prominent regions with an attracting orbit, the period is indicated.

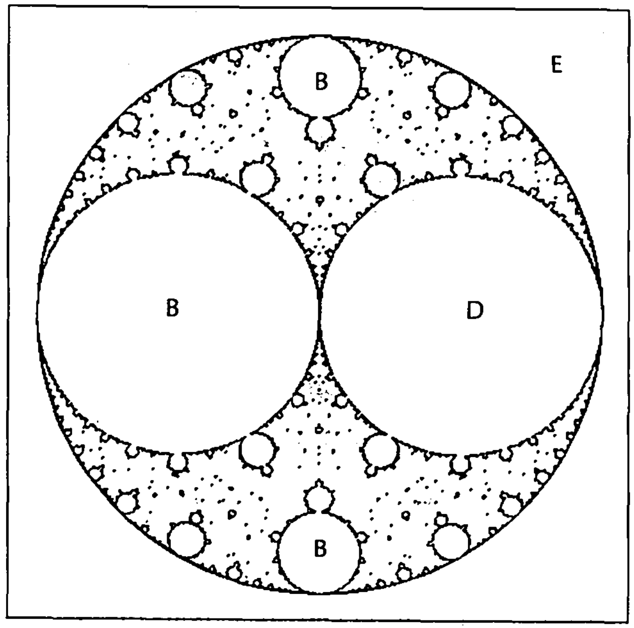

FIGURE 12. Picture of the symmetry locus $\mathcal{S}$, represented as the k-plane for the family of maps $z \mapsto k(z+z^{-1})$. The component of type D on the right is centered at $k=\frac{1}{2}$, with both critical points fixed, while the component of type B on the left is centered at $k=-\frac{1}{2}$, with critical points $-1 \leftrightarrow 1$.

unit disk correspond to nonhyperbolic maps, since one critical orbit lands on a repelling fixed point.

In contrast with the slices considered previously, where each map has at least one attracting or parabolic orbit, so the Julia set is a proper subset of $\hat{\mathbf{C}}$, the present family contains maps that have no attracting or parabolic fixed points. In fact, the orbits of both critical points ± 1 may eventually land on repelling cycles. This happens, for example, for the value $\lambda_0 = -\frac{1}{4}$, with $1 \mapsto -1 \mapsto 0 \mapsto \infty$. The Julia set for such a map is necessarily the entire sphere. In fact, the following much sharper statement follows immediately from a theorem in [Rees 1986]: *Every neighborhood of such a point λ_0 contains a set of parameter values λ of positive area (two-dimensional Lebesgue measure) for which the associated Julia set is the entire sphere.*

The symmetry locus

An apparently quite different one-dimensional slice is the *branch locus* or *symmetry locus* $\mathcal{S}$, consisting of all conjugacy classes $\langle f\rangle$ for which f has a nontrivial automorphism. By Lemma 5.1 this locus can be described parametrically as the set of $\langle f_k\rangle$ with

$$f_k(z) = k\Big(z + \frac{1}{z}\Big),$$

where k ranges over $\mathbf{C} \setminus \{0\}$. A picture of the k-plane is shown in Figure 12; see also [Haeseler and Peitgen 1988].

Note that Figure 12 is centrally symmetric. In fact, from $f_k(-z) = -f_k(z) = f_{-k}(z)$ it follows that $f_k \circ f_k = f_{-k} \circ f_{-k}$. Hence the Julia set $J(f_k)$ is precisely equal to $J(f_{-k})$, and the bifurcation diagram in the k-plane, describing different forms of dynamical behavior, is essentially invariant under the involution $k \mapsto -k$. (This does not mean that f_k has precisely the same dynamics as f_{-k}. For example, f_1 has a triple fixed point while f_{-1} has three distinct fixed points.) As in the previous example, this symmetry locus contains maps for which both critical orbits eventually land on repelling cycles: for example, $z \mapsto \frac{i}{2}(z+z^{-1})$, with $\pm 1 \mapsto \pm i \mapsto 0 \mapsto \infty$.

C. Petersen has pointed out to me that this family of maps is closely related to the locus of Figure 11. In fact, the two-to-one substitution $w = z^2$ semiconjugates the family $z \mapsto k(z + z^{-1})$ to the family

$$w \mapsto \left(k(w^{1/2} + w^{-1/2})\right)^2 = \lambda(w + 2 + w^{-1}),$$

with $\lambda = k^2$. Note that symmetric pairs of hyperbolic components in the k-plane of Figure 12 correspond to "subhyperbolic components", where only one critical orbit converges to a periodic attractor, in the λ-plane of Figure 11.

One critical point mapping eventually to the other

As a last source of interesting sections, we consider the condition that one critical point maps to the other after some number of iterations. The simplest possibility is that one critical point maps directly to the other. Taking the critical points to be $0 \mapsto \infty$, we obtain the normal form $z \mapsto a + 1/z^2$ of Figure 13. Here $\sigma_1 = -6$ and $\sigma_2 = 12 - 4a^3$.

As in the previous two examples, this family contains maps for which the critical orbits eventually land on a repelling cycle. This happens, for example, if $f(z) = a + 1/z^2$ with $a^3 = -0.5$, so that $0 \mapsto \infty \mapsto a \mapsto -a \mapsto -a$. Again, it follows from [Rees 1986] that there is a set of parameter values of positive area for which the associated Julia set is the entire Riemann sphere.

10. REAL QUADRATIC MAPS

Consider a quadratic rational map that has real coefficients, and hence induces a map from the circle $\hat{\mathbf{R}} = \mathbf{R} \cup \infty$ into itself. We will distinguish seven cases, depending on the topology of this map from the circle to itself.

First suppose that the two critical points of f are conjugate complex. Then it is easy to check that f induces a two-to-one covering map from $\hat{\mathbf{R}}$ onto itself. In this case, we must distinguish between the positive derivative case, where f maps this circle with degree $+2$, and the negative derivative case, with degree -2.

On the other hand, suppose both critical points are real. Then f maps the entire circle $\hat{\mathbf{R}}$ onto a closed interval $I = f(\hat{\mathbf{R}})$ bounded by the two critical values. Evidently, in order to study both critical orbits, we need only study the dynamics of f restricted to this interval I.

Definition. A real quadratic map with real critical points will be called *monotone*, *unimodal* or *bimodal*, according to whether the interior of the interval $I = f(\hat{\mathbf{R}})$ contains no critical points, one critical point, or both critical points. In the monotone case, we distinguish between increasing and decreasing. Similarly, in the bimodal case we distinguish two patterns of increasing and decreasing laps: $(+-+)$ and $(-+-)$.

The unimodal case is illustrated in Figure 14: just one critical point $\omega_1 = 0$ belongs to the interior of the interval $I = [-1, 1]$. (The other critical point $\omega_2 = 1$ belongs to the boundary of I. Thus this

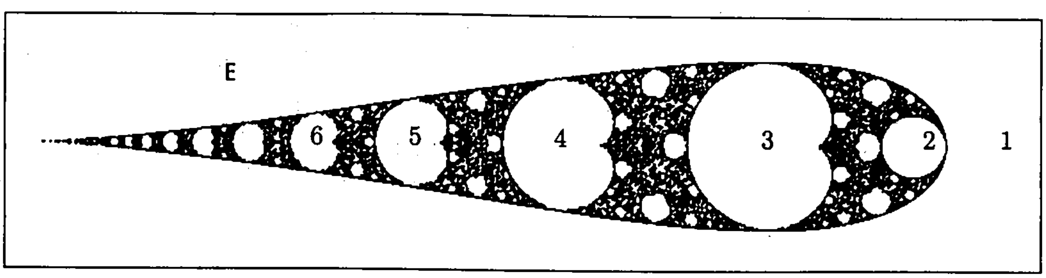

FIGURE 13. The a^3-plane for the family of maps $z \mapsto a + 1/z^2$, with one critical point mapping directly to the other. In the more conspicuous hyperbolic components (type B or E), the period of the attractor is indicated. The dot in the period-two component represents the origin $a = 0$.

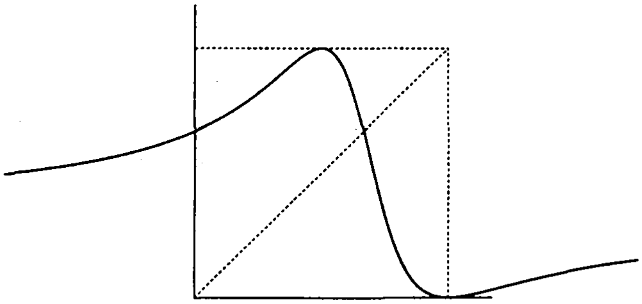

FIGURE 14. Graph of the borderline unimodal real quadratic map $f_0(x) = (1-2x-x^2)/(1-2x+3x^2)$. The square represents the interval $f_0(\hat{\mathbf{R}}) = [-1, 1]$. Both critical orbits land on the repelling fixed point $\frac{1}{3}$, so the (complex) Julia set $J(f_0)$ is the entire Riemann sphere.

particular map lies on the boundary between the unimodal and the $(+-+)$-bimodal regions of the real moduli space.)

It is not difficult to check that a conjugacy class $\langle f \rangle \in \mathcal{M}_2$ possesses a representative with real coefficients if and only if the two invariants σ_1 and σ_2 of Section 3 are both real. Hence we define the *real moduli space* to be simply the real (σ_1, σ_2)-plane. In general, the real representative for $\langle f \rangle \in \mathcal{M}_2$ is unique up to real conjugacy, but not in the special case when $\langle f \rangle$ belongs to the real part of the symmetry locus $\mathcal{S}$ of Section 5. In fact, for the conjugacy class of $f(x) = \lambda(x+x^{-1})$, where $\lambda \in \mathbf{R} \smallsetminus \{0\}$, we can choose f itself as real representative, with critical points ± 1 and degree zero. But we can also take $f(ix)/i = \lambda(x - x^{-1})$ as representative, with critical points $\pm i$ and with degree equal to $2 \operatorname{sgn} \lambda$.

The classification of real quadratic maps into seven topological types corresponds to a partition of real moduli space into seven regions, as indicated in Figure 15. More precisely, the real (σ_1, σ_2)-plane is cut up into seven pieces by the two real branches of the symmetry locus $\mathcal{S}$, and by the vertical lines $\sigma_1 = 2$ and $\sigma_1 = -6$ that correspond to the classes $\langle f \rangle$ satisfying $f(\omega_1) = \omega_1$ and $f(\omega_1) = \omega_2$, respectively.

The bifurcation locus in the real (σ_1, σ_2)-plane is plotted in Figures 16 and 17. Figure 16 shows the same region as Figures 1 and 15. Note that the vertical line $\sigma_1 = 2$ parametrizes the family of real quadratic polynomials $x \mapsto x^2 + \text{constant}$, while the line $\sigma_1 = -6$ parametrizes the family of maps that carry one critical point to the other—that is, the real axis of Figure 13. As in Figure 13, the numbers label hyperbolic components of type B, with attracting period as indicated. Figure 17 shows a much larger region, so that we can begin to see the limiting behavior as (σ_1, σ_2) tends to the line at infinity $\widehat{\mathrm{Per}_1}(\infty)$. (Compare Lemma 4.1.)

Significant features of both figures are the following. The two straight lines $\mathrm{Per}_1(1)$ of slope $+2$ and $\mathrm{Per}_1(-1)$ of slope -2 cut the plane into four quadrants:

Top quadrant. This is part of the escape locus. If we restrict attention to maps in this quadrant with two real critical points, the real dynamics is completely trivial: the successive images of $\hat{\mathbf{R}}$ converge uniformly to the real fixed point.

Right quadrant. These are the maps with two attracting fixed points. They form the real part of the hyperbolic component $\mathrm{D}_{1,1}$ consisting of all maps with two attracting fixed points. It is centered at the point $\langle z \mapsto z^2 \rangle$.

Bottom quadrant. As we pass from the right-hand quadrant to the bottom quadrant, one fixed point bifurcates to an orbit of period two, then to period four, and so on, in the usual Sharkovskii pattern, until we again reach the escape locus. (In fact, the standard quadratic polynomial family is just the section of this picture with the vertical line $\sigma_1 = 2$.) However, this lower component of the real escape locus has much more complicated real dynamics, since the entire Julia set—a Cantor set—is contained in $\hat{\mathbf{R}}$.

Left quadrant. Here we evidently have a much more complicated situation. There are conspicuous hyperbolic components of type B, bordered by hyperbolic components of type C. There are also hyperbolic components of type D, at least within the bimodal region $\sigma_1 < -6$ (compare Appendix F), but these are tiny and are not visible in the pictures.

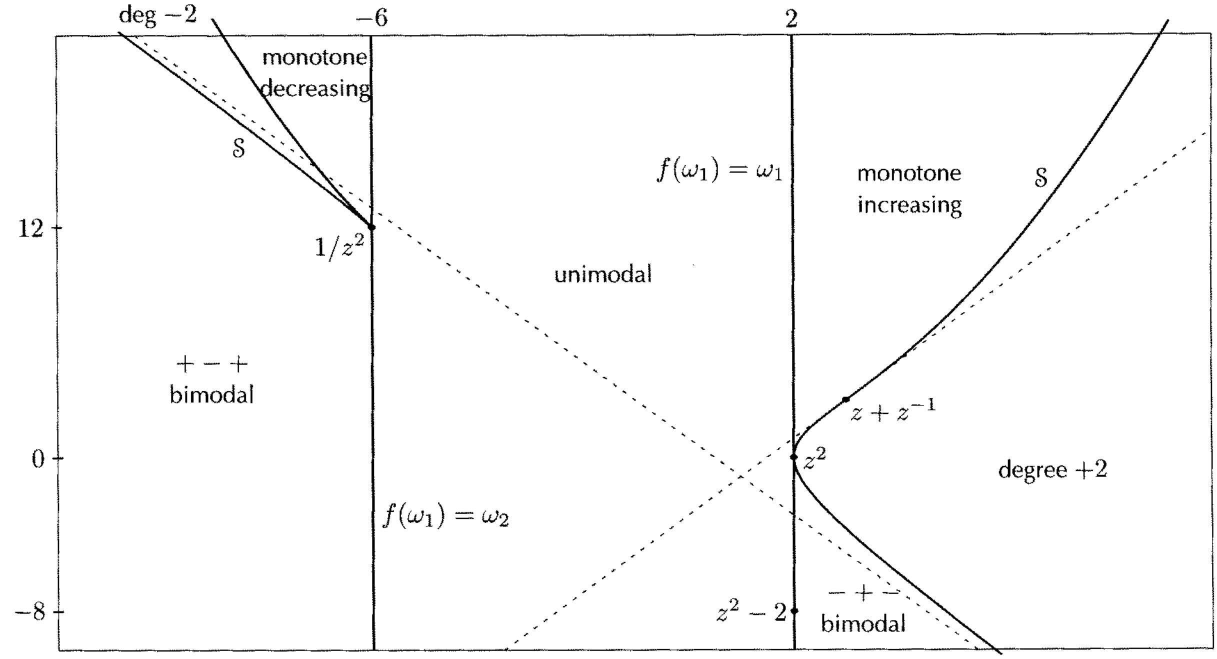

FIGURE 15. The regions in the real (σ_1, σ_2)-plane corresponding to the seven topological descriptions of f on $\hat{\mathbf{R}}$ are bounded by the thick lines, comprised of the symmetry locus $\mathcal{S}$ and of the vertical lines $\sigma_1 = -6$ and $\sigma_1 = 2$. The dotted lines are the loci $\mathrm{Per}_1(\pm 1)$. The rectangles shown is $(\sigma_1, \sigma_2) \in [-12, 10] \times [-10, 22]$, with the horizontal scale exaggerated, as in Figures 1 and 16.

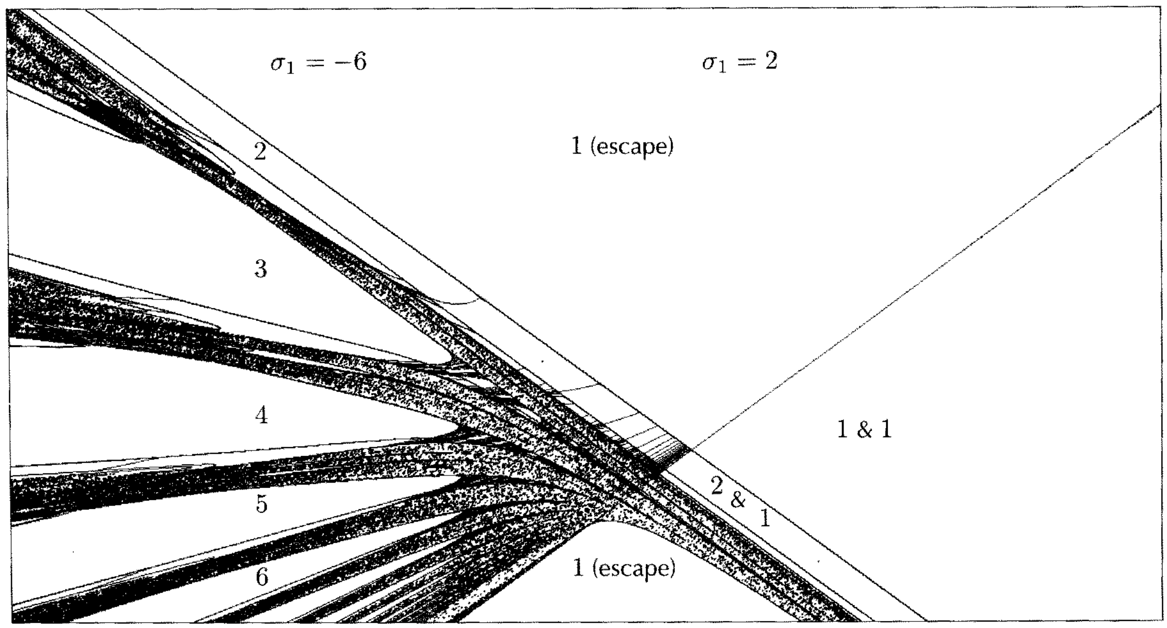

FIGURE 16. Rectangle $[-12, 10] \times [-10, 22]$ in the real (σ_1, σ_2)-plane, emphasizing dynamic behavior. (Horizontal scale exaggerated.) The periods of (possibly complex) attracting orbits for some of the more prominent hyperbolic components are indicated. As we traverse any vertical line with $-6 \leq \sigma_1 \leq 2$, we follow a full family of unimodal maps. For maps within the top component of the escape locus, there are no real periodic orbits other than the attracting fixed point. However, for maps in the bottom component, the entire Julia set is real.

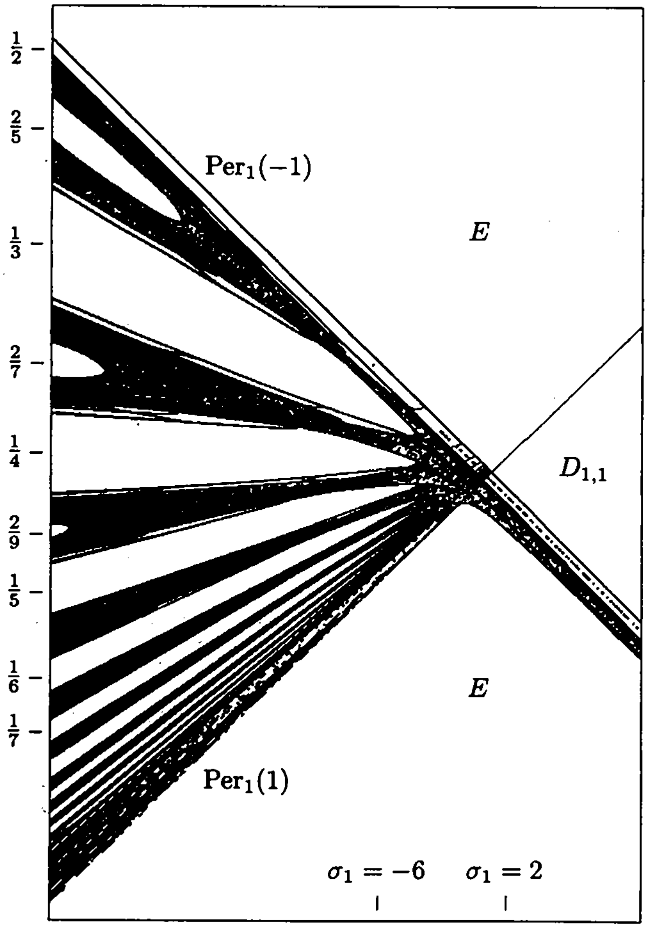

FIGURE 17. Picture of the much larger rectangle $[-25,10] \times [-55,55]$ in the real (σ_1, σ_2)-plane. For σ_1 very negative, with $\sigma_2/\sigma_1 \approx 2\cos(2\pi i p/q)$ we are in a component of type B with attracting orbit of period q. Some representative values of p/q are indicated along the left margin.

For σ_1 very negative, with $\sigma_2/\sigma_1 \approx 2\cos(2\pi i p/q)$, we are in a component of type B with attracting orbit of period q. Some representative values of p/q are indicated along the left margin. For $p = 1$, these are exactly the same as the components occurring in Figure 13. In fact, the real axis in Figure 13 corresponds exactly to the vertical line $\sigma_1 = -6$ in Figures 16 and 17.

The real topological classification of Figure 15 can be compared with the complex dynamic classification of Figures 16 and 17 as follows:

Monotone case. If f is monotone increasing, clearly the two critical orbits must converge, either to a single fixed point or to two distinct fixed points. If neither fixed point is parabolic, this means that $\langle f \rangle$ must belong either to the hyperbolic component of type $D_{1,1}$ with two attracting fixed points, or to the escape component E. Similarly, in the monotone decreasing case the two critical orbits must converge to a common fixed point or period-two orbit. In the nonparabolic case, this means that $\langle f \rangle$ belongs either to the escape component or to the period-two hyperbolic component of type B.

Unimodal case. Here there is a wide range of possible dynamic behavior. In the hyperbolic case, $\langle f \rangle$ may belong to a component of any one of the four types B, C, D, or E. One essential restriction is that there cannot be two attracting orbits of period greater than one.

Bimodal case. Not all bimodal maps can arise as quadratic maps, since the condition that each point has at most two preimages imposes an essential restriction. In particular, the topological entropy of the map $I \to I$ can be at most $\log 2$. In the case $(-+-)$, there is another quite strong restriction:

Lemma 10.1. *Every $(-+-)$-bimodal quadratic map must have an attracting fixed point.*

Proof. Clearly such a map must have three fixed points on the interval $I = f(\hat{\mathbf{R}})$, and two of the three must have negative multipliers, say $\mu_1, \mu_3 < 0$. If these two fixed points were both repelling or parabolic, $\mu_1, \mu_3 \le -1$, the formula

$$\mu_2 = \frac{2 - \mu_1 - \mu_3}{1 - \mu_1 \mu_3}$$

would entail a negative (or infinite) multiplier for the third fixed point as well, which is impossible. □

Case of degree ±2. Here the dynamics is once more sharply restricted:

Lemma 10.2. *If the real quadratic map f has complex conjugate critical points, so that it induces a* *covering mapping of degree* ± 2 *from the circle* $\hat{\mathbf{R}}$ *onto itself, the (complex) Julia set* $J(f)$ *is either the circle* $\hat{\mathbf{R}}$ *or a Cantor set contained in* $\hat{\mathbf{R}}$. *Both critical orbits converge to attracting orbits of period at most two or to a parabolic fixed point:*

The proof is not difficult, and will be omitted.

APPENDIX A. RESULTANT AND DISCRIMINANT

This will be a brief summary of standard material. Proofs may be found, for example, in [van der Waerden 1991]. Fix two integers $m, n \geq 0$. Let p range over the vector space V_m of complex polynomials

$$p(z) = a_0 z^m + \cdots + a_m$$

of degree $\leq m$, and let q range over the space V_n of polynomials

$$q(z) = b_0 z^n + \cdots + b_n$$

of degree $\leq n$. The *resultant* $\operatorname{res}_{m,n}(p,q)$ can be characterized as the unique function from $V_m \times V_n$ to $\mathbf{C}$ that is bimultiplicative,

$$\operatorname{res}_{m_1+m_2,n}(p_1 p_2, q) = \operatorname{res}_{m_1,n}(p_1,q)\operatorname{res}_{m_2,n}(p_2,q)$$
$$\operatorname{res}_{m,n_1+n_2}(p, q_1 q_2) = \operatorname{res}_{m,n_1}(p,q_1)\operatorname{res}_{m,n_2}(p,q_2)$$

and that coincides with the determinant in the case $\deg p = \deg q = 1$:

$$\operatorname{res}_{1,1}(az+b, cz+d) = ad - bc.$$

This function is $(-1)^{mn}$-symmetric:

$$\operatorname{res}_{m,n}(p,q) = (-1)^{mn}\operatorname{res}_{n,m}(q,p),$$

and bihomogeneous of degree (n,m):

$$\operatorname{res}_{m,n}(\lambda p, \mu q) = \lambda^n \mu^m \operatorname{res}_{m,n}(p,q).$$

Note that the subscripts m, n are essential, since we sometimes consider polynomials with leading coefficient zero. However, in practice we will suppress these subscripts whenever they are clear from the context.

The resultant can be expressed as an $(m+n) \times (m+n)$ determinant, for example:

$$\operatorname{res}(az^2+bz+c, pz^2+qz+r) = \det\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ p & q & r & 0 \\ 0 & p & q & r \end{pmatrix}$$

In the special case $m = 0$, note that $\operatorname{res}(a_0, q) = a_0^n$ is independent of q, while, for $m = 1$, we have $\operatorname{res}(p,q) = q(\xi)$ if $p(z) = z - \xi$. More generally, whenever the leading coefficient a_0 is nonzero, we can factor p as $p(z) = a_0(z-\xi_1)\cdots(z-\xi_m)$, and it follows that

$$\operatorname{res}(p,q) = a_0^n \prod_1^m q(\xi_i).$$

(On the other hand, if $a_0 = 0$, then $\operatorname{res}_{m,n}(p,q) = (-1)^n b_0 \operatorname{res}_{m-1,n}(p,q)$.) Similarly, if $b_0 \neq 0$, we can set $q(z) = b_0(z-\eta_1)\cdots(z-\eta_n)$ and write

$$\begin{aligned}\operatorname{res}(p,q) &= (-1)^{mn} b_0^m \prod_1^n p(\eta_j) \\ &= a_0^n b_0^m \prod_1^m \prod_1^n (\xi_i - \eta_j).\end{aligned}$$

The resultant may well be nonzero even if one of the leading coefficients is zero. However, if both leading coefficients are zero, so that $\deg p < m$ and $\deg q < n$, then $\operatorname{res}_{m,n}(p,q) = 0$. In fact, $\operatorname{res}_{m,n}(p,q)$ is zero if and only if one of the following conditions is true: p and q have a root in common, or $\deg p < m$ and $\deg q < n$.

Closely related is the *discriminant* of a polynomial of degree m or less. This is a polynomial function $\operatorname{discr}_m : V_m \to \mathbf{C}$, homogeneous of degree $2m-2$:

$$\operatorname{discr}_m \lambda p = \lambda^{2m-2}\operatorname{discr} p.$$

If the leading coefficient a_0 is nonzero, the discriminant can be defined by the formula

$$\operatorname{discr}_m p = \frac{(-1)^{m(m-1)/2}}{a_0}\operatorname{res}_{m,m-1}(p,p'),$$

where p' is the derivative. Alternatively, setting $p(z) = a_0(z - \xi_1) \cdots (z - \xi_m)$, it can be defined by the formula

$$\operatorname{discr} p = a_0^{2m-2} \prod_{i<h} (\xi_i - \xi_h)^2.$$

(We omit the subscript m whenever it is clear from the context.) Evidently, for such a polynomial with $a_0 \neq 0$, the discriminant is nonzero if and only if p has m distinct roots. On the other hand, if $a_0 = 0$ then $\operatorname{discr}_m p = a_1^2 \operatorname{discr}_{m-1} p$. In particular, for any $m \geq 2$, if both of the leading coefficients a_0 and a_1 are zero, then $\operatorname{discr}_m p$ is zero.

Here are explicit formulas for the cases $m \leq 3$, which will be of interest.

$$\begin{aligned} \operatorname{discr}_1(az+b) &= 1, \\ \operatorname{discr}_2(az^2+bz+c) &= b^2 - 4ac, \\ \operatorname{discr}_3(az^3+bz^2+cz+d) &= b^2c^2 - 4ac^3 - 4b^3d \\ &\quad - 27a^2d^2 + 18abcd. \end{aligned}$$

APPENDIX B. THE SPACE Rat_d OF DEGREE-d RATIONAL MAPS

This appendix describes results from [Segal 1979], as well as some consequences of the work therein.

For each integer $d \geq 0$ we can form the space Rat_d of all rational maps $f(z) = p(z)/q(z)$, where

$$\begin{aligned} p(z) &= a_0 z^d + \cdots + a_d, \\ q(z) &= b_0 z^d + \cdots + b_d \end{aligned} \tag{B-1}$$

are complex polynomials with no common root and with leading coefficients a_0 and b_0 not both zero. Evidently Rat_d is a Zariski open subset of the complex projective space $\mathbf{CP}^{2d+1}$, whose points are ratios $(a_0 : \ldots : a_d : b_0 : \ldots : b_d)$. We will write such ratios briefly as $(p : q) \in \mathbf{CP}^{2d+1}$. More explicitly, Rat_d can be described as the complement

$$\mathbf{CP}^{2d+1} \smallsetminus W_d,$$

where W_d is the algebraic variety consisting of all ratios $(p : q)$ whose resultant $\operatorname{res}(p,q) = \operatorname{res}_{d,d}(p,q)$ is equal to zero (see Appendix A). In particular, it follows easily that Rat_d *is a smooth connected complex manifold of complex dimension* $2d+1$.

If we map each rational map $f \in \mathrm{Rat}_d$ to the image $f(\infty) = a_0/b_0$ in the Riemann sphere $S^2 = \mathbf{C} \cup \infty$, we obtain a fibration

$$\mathrm{Rat}_d^0 \hookrightarrow \mathrm{Rat}_d \twoheadrightarrow S^2,$$

where the fiber Rat_d^0 consists of rational maps that fix the point at infinity in S^2. Let Map_d be the infinite-dimensional space of *all* continuous maps from S^2 to itself of degree d, and let Map_d^0 be the subspace consisting of maps that fix some base point. The following fact is proved in [Segal 1979]:

Assertion B.1. *The two inclusions* $\mathrm{Rat}_d \subset \mathrm{Map}_d$ *and* $\mathrm{Rat}_d^0 \subset \mathrm{Map}_d^0$ *induce homotopy equivalences in dimensions up to and including d. More precisely, the induced map of homotopy groups* π_n *is an isomorphism for* $n < d$ *and is onto for* $n = d$.

As model for the fiber Rat_d^0, Segal uses the space of rational maps $f(z) = p(z)/q(z)$, where p and q are *monic* polynomials of degree d, so that $f(\infty) = +1$. With this normalization, using an argument due to J. D. S. Jones, he proves the following.

Assertion B.2. *For* $d \geq 1$ *the fundamental group* $\pi_1(\mathrm{Rat}_d^0)$ *is free cyclic. In fact, let* (p,q) *range over pairs of monic polynomials of degree d with no common root. Then the correspondence*

$$\frac{p}{q} \mapsto \operatorname{res}(p,q) \in \mathbf{C} \smallsetminus \{0\}$$

defines a fibration

$$\mathrm{Rat}_d^0 \to \mathbf{C} \smallsetminus \{0\}$$

that induces an isomorphism between fundamental groups.

The fiber F_d of this fibration consists of all pairs (p,q) of monic polynomials of degree d having resultant $\operatorname{res}(p,q)$ equal to 1. Thus F_d can be considered as an algebraic hypersurface in the coordinate space $\mathbf{C}^{2d}$. As an immediate consequence:

Corollary B.3. *The fiber F_d is simply connected. The universal covering space $\widetilde{\mathrm{Rat}}^0_d$ is homeomorphic to $F_d \times \mathbf{C}$.*

To develop these ideas further, let

$$E_d = \{(p,q) : \mathrm{res}(p,q) = 1\}$$

be the affine hypersurface in $\mathbf{C}^{2d+2}$ consisting of all pairs of polynomials (p,q) of the form (B–1) and satisfying $\mathrm{res}(p,q) = 1$. We claim there is a natural fibration

$$F_d \hookrightarrow E_d \twoheadrightarrow \mathbf{C}^2 \smallsetminus \{(0,0)\}, \qquad \text{(B–2)}$$

where the projection map from E_d to $\mathbf{C}^2 \smallsetminus \{(0,0)\}$ carries each pair (p,q) of polynomials with resultant 1 to the pair (a_0, b_0) of leading coefficients. In fact, each matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the group $\mathrm{SL}(2,\mathbf{C})$ acts on the total space E_d, carrying each (p,q) with resultant 1 to $(ap+bq, cp+dq)$. Using the fact that $\mathrm{SL}(2,\mathbf{C})$ is generated by elementary matrices, we see easily that this action preserves the resultant. Clearly this same group acts on the base space, and the projection map is equivariant. It follows that we do indeed have a fibration.

Assertion B.4. *The fundamental group $\pi_1(\mathrm{Rat}_d)$ is cyclic of order $2d$. Furthermore, the universal covering $\widetilde{\mathrm{Rat}}_d$ can be identified with this affine hypersurface E_d.*

Proof. Since the resultant of p and q is bihomogeneous of degree (d,d) in the coefficients of p and q, we have

$$\mathrm{res}(\lambda p, \lambda q) = \lambda^{2d} \mathrm{res}(p,q).$$

Thus, whenever the resultant of p, q is nonzero, so that p/q is rational of degree d, there are exactly $2d$ choices of λ for which this expression $\lambda^{2d}\mathrm{res}(p,q)$ is equal to $+1$. Evidently the group of $(2d)$-th roots of unity operates freely on the locus E_d, with quotient space isomorphic to Rat_d. Thus E_d is a $2d$-fold covering manifold of Rat_d. But E_d is simply connected, since it is the total space of a fibration (B–2) having simply connected fiber and base. □

Now we specialize to the case $d = 2$ and prove Theorem 2.1, as promised in Section 2.

Proof of 2.1. Every quadratic rational map has two distinct critical points $\omega_1 \neq \omega_2$ in the Riemann sphere S^2. We will use the notation $\mathcal{DS}_2(S^2)$ for the *deleted symmetric product*, consisting of all unordered pairs of distinct elements in S^2. Evidently the correspondence $f \mapsto \{\omega_1, \omega_2\}$, assigning to each $f \in \mathrm{Rat}_2$ its set of critical points, defines a continuous map $\mathrm{Rat}_2 \to \mathcal{DS}_2(S^2)$.

On the other hand, the group $\mathrm{Rat}_1 \cong \mathrm{PSL}(2,\mathbf{C})$ of all Möbius transformations of S^2 acts freely on Rat_2 by left composition, that is, by the action $\mathrm{Rat}_1 \times \mathrm{Rat}_2 \to \mathrm{Rat}_2$ defined by $(g,f) \mapsto g \circ f$. It is easy to check that two maps in Rat_2 belong to the same orbit under this action if and only if they have the same critical points. Thus *our correspondence $f \mapsto \{\omega_1, \omega_2\}$ is the projection map of a principal fiber bundle, having the group $\mathrm{Rat}_1 \cong \mathrm{PSL}(2,\mathbf{C})$ as fiber.*

If we are interested only in the homotopy theory of this bundle, we may as well pass to a compact principal subbundle

$$\mathrm{SO}(3) \hookrightarrow M^5 \twoheadrightarrow \mathbf{RP}^2 \qquad \text{(B–3)}$$

which is embedded as a deformation retract. To see this, note that the projective unitary group $\mathrm{PSU}(2) \cong \mathrm{SO}(3)$ is embedded in the Möbius group Rat_1 as a deformation retract, and that the projective plane $\mathbf{RP}^2$ consisting of pairs of antipodal points in S^2 is embedded in $\mathcal{DS}_2(S^2)$ as a deformation retract. It is not difficult to check that the corresponding total space M^5 is indeed a smooth manifold, embedded in Rat_2 as a deformation retract.

Such $\mathrm{SO}(3)$-bundles over $\mathbf{RP}^2$ are classified by homotopy classes of mappings $\mathbf{RP}^2 \to \mathrm{BSO}(3)$, or (using obstruction theory or Hopf's Theorem) by cohomology classes in the group

$$H^2(\mathbf{RP}^2; \pi_1(\mathrm{SO}(3))) \cong \mathbf{Z}/2.$$

In fact the bundle (B–3) must be the nontrivial $\mathrm{SO}(3)$-bundle over $\mathbf{RP}^2$. For if M^5 were a product, the fundamental group $\pi_1(M^5) \cong \pi_1(\mathrm{Rat}_2)$ would be the direct sum of two cyclic groups of order two. But according to Assertion B.4, this group is actually cyclic of order four. □

As one immediate consequence of Theorem 2.1, we see that Rat_2 has the rational homology of a three-sphere. Here is another easy consequence:

Corollary B.5. *The twofold orientable covering of M^5 is homeomorphic to the product $\mathrm{SO}(3) \times S^2$. Hence the universal covering of M^5 is homeomorphic to $S^3 \times S^2$.*

Proof. Geometrically, this twofold covering can be described as the space of triples (f, ω_1, ω_2) consisting of a quadratic map in M^5 and its marked critical points (Section 6). The statement that the lifted SO(3)-bundle over S^2 splits as a product means that there exists some cross-section $\omega \mapsto f_\omega$ that assigns to each point $\omega \in S^2$ a quadratic rational map having ω as critical point. This follows from an easy cohomology argument, or can be constructed explicitly as follows. For each ω in the finite plane $\mathbf{C}$, consider the Möbius transformation

$$g_\omega(z) = \frac{\bar{\omega} z + 1}{-z + \omega},$$

belonging to the compact subgroup $\mathrm{SO}(3) \subset \mathrm{Rat}_1$. This map carries ω to ∞ and carries the antipodal point $-1/\bar{\omega}$ to zero. The composition

$$f_\omega(z) = g_{\omega^2}^{-1}(g_\omega(z)^2)$$

is then a quadratic rational map in M^5 having ω and $-1/\bar{\omega}$ as critical points. This construction has been chosen so that $f_\omega(z)$ tends to a well-behaved limiting value $f_\infty(z) = z^2$ as $\omega \to \infty$. □

Remark B.6. Of course the correspondence $\omega \mapsto f_\omega$, assigning to each $\omega \in S^2$ a rational map $f_\omega \in \mathrm{Rat}_2$ having ω as critical point, cannot possibly be holomorphic. For the correspondence assigning to ω the other critical point of f_ω has degree -1, and hence is nonholomorphic.

APPENDIX C. NORMAL FORMS AND RELATIONS BETWEEN CONJUGACY INVARIANTS

We now study three convenient normal forms for quadratic rational maps.

Fixed-Point Normal Form

As in the proof of Lemma 3.1, let

$$f(z) = \frac{z(z + \mu_1)}{\mu_2 z + 1}, \qquad \text{(C–1)}$$

with $\mu_1\mu_2 \neq 1$. The origin is a fixed point with multiplier μ_1, and infinity is fixed with multiplier μ_2. The third fixed point is $(1-\mu_1)/(1-\mu_2)$, with multiplier

$$\mu_3 = \frac{2 - \mu_1 - \mu_2}{1 - \mu_1\mu_2}$$

The critical points for this map are the two roots

$$\omega = \frac{-1 \pm \sqrt{1 - \mu_1\mu_2}}{\mu_2}$$

of the equation $\mu_2\omega^2 + 2\omega + \mu_1 = 0$, with one critical point at infinity when $\mu_2 = 0$. A brief computation shows that the corresponding critical values are given by $f(\omega) = -\omega^2$.

Mixed Normal Form

Now consider the normal form

$$f(z) = \frac{1}{\mu}\Big(z + \frac{1}{z}\Big) + a, \qquad \text{(C–2)}$$

with critical points ± 1 and with a fixed point of multiplier $\mu \neq 0$ at infinity. (Compare Theorem 5.1, Example 8.3, and Figures 5, 6, 12 and 13.) The critical values are $f(\pm 1) = a \pm 2/\mu$. The two finite fixed points z_1 and z_2 can be described as the roots of the equation

$$(1-\mu)z_i^2 + a\mu z_i + 1 = 0.$$

Either by direct computation, or using Lemma C.1 below, one finds that the corresponding multipliers $\mu_i = f'(z_i) = (1 - z_i^{-2})/\mu$ satisfy the relation

$$\sigma_1 = \mu_1 + \mu_2 + \mu = \mu(1 - a^2) - 2 + \frac{4}{\mu}.$$

As in the proof of Lemma 3.4, the symmetric function $\sigma_2 = \mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3$ can then be computed from the equation

$$\begin{aligned} 0 &= \mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_3 \\ &= \mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_1 + 2, \end{aligned}$$

the result being

$$\sigma_2 = \Bigl(\mu + \frac{1}{\mu}\Bigr)\sigma_1 - \Bigl(\mu^2 + \frac{2}{\mu}\Bigr)$$

The parameter $\tau = \mu_1\mu_2$ along the line $\mathrm{Per}_1(\mu)$ is equal to $\sigma_3/\mu = (\sigma_1 - 2)/\mu$, hence

$$\tau = \Bigl(\frac{2-\mu}{\mu}\Bigr)^2 - a^2$$

(compare Remark 6.9). Thus, *for $\mu \neq 0$, the parameter τ along the line $\mathrm{Per}_1(\mu)$ coincides with $-a^2$, up to translation by a constant depending only on μ.*

Critical-Point Normal Form

If we put the critical points at zero and infinity, so that $f(z) = f(-z)$, we get the normal form

$$f(z) = \frac{\alpha z^2 + \beta}{\gamma z^2 + \delta}. \tag{C–3}$$

This form is particularly convenient for computations and for Julia set pictures, such as Figures 2–4. (In fact, Figures 7, 10, 16 and 17 were made by first translating into these coordinates, using Corollary C.4.) As in Lemma 6.1, the quantities

$$\begin{aligned} A &= \frac{\alpha\delta}{\alpha\delta - \beta\gamma} = 1 + \frac{\beta\gamma}{\alpha\delta - \beta\gamma}, \\ B &= \frac{\alpha^3\beta}{(\alpha\delta - \beta\gamma)^2}, \\ C &= \frac{\gamma\delta^3}{(\alpha\delta - \beta\gamma)^2} \end{aligned}$$

are invariant under holomorphic conjugacies that fix the two critical points. Hence A and $\Sigma = B+C$ are holomorphic conjugacy invariants. As noted in Remark 6.3, we can use either the invariants σ_1, σ_2 of Section 3 or these invariants A and Σ as coordinates on the moduli space $\mathcal{M}_2$, in order to show that $\mathcal{M}_2$ is isomorphic to $\mathbf{C}^2$. The coordinates (A, Σ) are apparently easier to compute, since we need only solve a quadratic equation to find the critical points and reduce to the normal form (C–3), while in order to find the fixed points we must solve a cubic equation. In fact we can always compute the invariants without solving any polynomials equations, and it is not too difficult to compute one pair of invariants in terms of the other.

We first prove a result about cross-ratios. Let ω_1, ω_2 be the critical points of f, and let $v_i = f(\omega_i)$ be the corresponding critical values. The relative position of these four points is determined by the cross-ratio

$$\rho(f) = \frac{(v_1 - \omega_1)(v_2 - \omega_2)}{(\omega_1 - \omega_2)(v_1 - v_2)}, \tag{C–4}$$

which is always a finite complex number, since the denominator cannot vanish. Note that $\rho = 0$ if and only if one of the critical points is fixed by f, so that f is conjugate to a polynomial map. Similarly, it is not hard to check that $\rho = 1$ if and only if one critical point maps to the other.

Lemma C.1. *The cross-ratio ρ is related to the invariants σ_i of Lemma 3.1 and to the invariant A of Lemma 6.1 by the linear equation*

$$\rho(f) = -\tfrac{1}{8}\sigma_3 = \tfrac{1}{8}(2 - \sigma_1) = 1 - A.$$

Outline of proof. If we use the normal form (C–3), we have $\omega_1 = 0$, $\omega_2 = \infty$ and $v_1 = \beta/\delta$, $v_2 = \alpha/\gamma$, so the cross-ratio (C–4) reduces to the form

$$\frac{v_1}{(v_1 - v_2)} = -\beta\gamma = 1 - A.$$

On the other hand, if we use the normal form (3–4), computation shows that the critical points and critical values are

$$\omega_\pm = \frac{-1 \pm \sqrt{1 - bc}}{c}, \quad f(\omega_\pm) = -\omega_\pm^2,$$

with $b = \mu_1$ and $c = \mu_2$. The cross-ratio (C–4) then works out to be $bc(b + c - 2)/(8 - 8bc)$. By

(3–3) and (3–2), this reduces to $-\frac{1}{8}\sigma_3 = \frac{1}{8}(2-\sigma_1)$, as required. □

In order to relate the remaining invariants, we use the formulas from Appendix A on the discriminant the resultant. The three fixed points of a quadratic rational function $f(z) = p(z)/q(z)$ are the roots of the polynomial equation $zq(z) - p(z) = 0$. (This equation is understood to have a "root at infinity" if and only if its degree is strictly less than three.) The discriminant of this equation is a homogeneous function of degree four in the coefficients of p and q, which vanishes if and only if there is a multiple root. In order to make this discriminant into an invariant of the rational function $p(z)/q(z)$, we must divide by the resultant of p and q, which is also a homogeneous function of degree four in the coefficients of p and q. This resultant is never zero, since these two polynomials have no common root.

Lemma C.2. *If $f = p/q$ a quadratic rational function, the* discriminant ratio

$$\operatorname{drat} f = \frac{\operatorname{discr}(zq-p)}{\operatorname{res}(p,q)}$$

is a conjugacy invariant that can be expressed in terms of the invariants σ_i of Section 3 as

$$\begin{aligned}\operatorname{drat} f &= \prod(\mu_i - 1) = \sigma_3 - \sigma_2 + \sigma_1 - 1\\ &= 2\sigma_1 - \sigma_2 - 3,\end{aligned}$$

and in terms of the invariants A and $\Sigma = B + C$ of Section 4 as

$$\operatorname{drat} f = -8A^2 + 36A - 4\Sigma - 27.$$

Proof. Clearly this ratio is a well defined complex number depending only of the function $f = p/q$. To show that it is a conjugacy invariant, we first try replacing $f(z)$ by $f(z+c)-c$. This has the effect of replacing p and q by $p(z+c)-cq(z+c)$ and $q(z+c)$, and of replacing the polynomial $F(z) = zq(z) - p(z)$ by $F(z+c)$. It is then easy to check that both the numerator $\operatorname{discr} F$ and the denominator $\operatorname{res}(p,q)$ remain invariant. Similarly, replacing $f(z)$ by $1/f(1/z)$ has the effect of replacing p, q and F by $z^2q(1/z)$, $z^2p(1/z)$ and $-z^3F(1/z)$. Again it is easy to check that both $\operatorname{discr} F$ and $\operatorname{res}(p,q)$ remain invariant. Finally, if we change scale, replacing $p(z)$, $q(z)$ and $F(z)$ by $p(\lambda z)$, $\lambda q(\lambda z)$ and $F(\lambda z)$, it is not hard to check that both numerator and denominator are multiplied by λ^6. This proves that the discriminant ratio is a holomorphic conjugacy invariant.

In both cases, we can now simply make a direct computation. Thus if $p(z) = \alpha z^2 + \beta$ and $q(z) = \gamma z^2 + \delta$ as in (C–3), it is easy to check that

$$\operatorname{res}(p,q) = (\alpha\delta - \beta\gamma)^2 = 1.$$

The classical formula for the discriminant, given at the end of Appendix A, yields

$$\begin{aligned}\operatorname{discr}(zq-p) &= \operatorname{discr}(\gamma z^3 - \alpha z^2 + \delta z - \beta)\\ &= -27\beta^2\gamma^2 - 4\delta^3\gamma + 18\alpha\beta\gamma\delta\\ &\qquad + \alpha^2\delta^2 - 4\alpha^3\beta.\end{aligned}$$

In terms of the invariants (6–3) we can easily reduce this to the expression $-8A^2 + 36A - 4(B + C) - 27$, as required.

Similarly, direct computation using the normal form (3–5), together with (3–4), shows that

$$\operatorname{drat} f = \prod(\mu_i - 1). \qquad □$$

Remark C.3. Lemma C.1 could also be expressed by an analogous explicit formula

$$\sigma_3 = \frac{\operatorname{res}(zq-p,\ p'q-q'p)}{\operatorname{res}(p,q)^2}.$$

Combining Lemmas C.1 and C.2, we obtain:

Corollary C.4. *The invariants σ_i, A and $\Sigma = B + C$ are related by*

$$\sigma_1 = 8A - 6, \qquad \sigma_2 = 8A^2 - 20A + 4\Sigma + 12.$$

APPENDIX D. THE GEOMETRY OF PERIODIC ORBITS

First some elementary number theory. Let $\nu_d(n)$ be the number of periodic points of period n for a generic polynomial map $f : \mathbf{C} \to \mathbf{C}$ of degree $d \geq 2$. Since the fixed points of $f^{\circ n}$, that is, the roots of the equation $f^{\circ n}(z) - z = 0$ of degree d^n, are precisely the periodic points with period m dividing n, it follows that

$$d^n = \sum_{m|n} \nu_d(m),$$

to be summed over all numbers $m \geq 1$ that divide n. We can easily solve inductively for $\nu_d(n)$. For example,

$$\begin{aligned} \nu_d(1) &= d, \\ \nu_d(2) &= d^2 - d, \\ \nu_d(3) &= d^3 - d, \\ \nu_4(d) &= d^4 - d^2 \end{aligned}$$

(Using the Möbius Inversion Formula, the solution can be written as

$$\nu_d(n) = \sum_{m|n} \mu\Big(\frac{n}{m}\Big) d^m,$$

where the Möbius function μ is defined by

$$\mu(p_1 \ldots p_k) = (-1)^k$$

for a product of k distinct primes with $k \geq 0$, and $\mu(m) = 0$ if m is not a product of distinct primes. See, for example, [Hardy and Wright 1979].) It is easy to check that $\nu_d(n)$ is always divisible by both d and n.

Henceforth, we specialize to the case $d = 2$. By the formula, the number of orbits of period n for a generic quadratic polynomial map is $\nu_2(n)/n$. As in the proof of Theorem 4.2, the number of period n hyperbolic components in the Mandelbrot set is $\nu_2(n)/2$. Here are the first few values:

n	1	2	3	4	5	6	7	8	9	10	11	12
$\nu_2(n)/n$	2	1	2	3	6	9	18	30	56	99	186	335
$\nu_2(n)/2$	1	1	3	6	15	27	63	120	252	495	1023	2010

We return to the study of quadratic rational maps. To simplify the discussion, we choose some normal form for which the point at infinity cannot be periodic. For example, any conjugacy class contains a representative of the form

$$f(z) = \frac{1}{a(z + z^{-1}) + b}$$

with $\infty \mapsto 0 \mapsto 0$, so that ∞ is not periodic. Writing $f^{\circ n}(z)$ as a quotient $p_n(z)/q_n(z)$ of two polynomials, where $\deg q = 2^n \geq \deg p$, it follows that the equation

$$zq_n(z) - p_n(z) = 0$$

for fixed points of $f^{\circ n}$ has degree exactly $2^n + 1$. Using an argument similar to the proof of Theorem 4.2, we see that there are unique polynomials $\Phi_n(z)$ such that

$$zq_n(z) - p_n(z) = \prod_{m|n} \Phi_m(z),$$

or, equivalently,

$$\Phi_n(z) = \prod_{m|n} (zq_n(z) - p_n(z))^{\mu(n/m)}$$

$\Phi_n(z)$ has degree $\nu_2(n)$ for $n > 1$, and $\Phi_1(z)$ has degree 3. Let $N = N(n)$ be given by $N = \nu_2(n)/n$ is $n > 1$, and $N = 3$ if $n = 1$. By definition, the Nn roots of $\Phi_n(z)$, counted with multiplicity, will be called the (formal) collection of periodic points of period n. In the generic case, these period-n points will all be distinct, but in special cases it may happen either that two period-n orbits come together, or that a period-n orbit degenerates to an orbit of lower period.

In any case, since the map f carries this collection of Nn points (counted with multiplicity) into itself, we can group the points into N orbits of formal period n (and actual period dividing n), and form the collection of multipliers $\{\eta_1^{(n)}, \cdots, \eta_N^{(n)}\}$ of these period-n orbits. Let $\sigma_1^{(n)}, \cdots, \sigma_N^{(n)}$ be the elementary symmetric functions of these multipliers.

Lemma D.1. *Each $\sigma_i^{(n)}$ can be expressed as a polynomial function of the two invariants σ_1 and σ_2 of Section 3.*

Outline of proof. Since σ_1 and σ_2 form a complete set of conjugacy invariants, we can express $\sigma_i^{(n)}$ as a single-valued function of (σ_1, σ_2). This function is continuous, since the collection of roots of a polynomial depends continuously on the polynomial. It is algebraic (that is, its graph is an affine variety in $\mathbf{C}^3$) since the construction is purely algebraic. But a continuous algebraic function from $\mathbf{C}^2$ to $\mathbf{C}$ is necessarily a polynomial. □

Corollary D.2. *Each locus* $\mathrm{Per}_n(\eta) \subset \mathcal{M}_2$ *is an affine algebraic variety, defined by the polynomial equation*

$$\eta^N - \sigma_1^{(n)}\eta^{N-1} + - \cdots \pm \sigma_N^{(n)} = 0.$$

The proof is immediate.

Example D.3. For $n = 2$ there is just one orbit of period two. According to Lemma 3.6, its multiplier $\eta_1^{(2)}$ is equal to

$$\sigma_1^{(2)} = 2\sigma_1 + \sigma_2.$$

For $n = 3$ there are two orbits of period 3. The sum and product of their multipliers are

$$\sigma_1^{(3)} = \sigma_1(2\sigma_1 + \sigma_2) + 3\sigma_1 + 3,$$
$$\sigma_2^{(3)} = (\sigma_1 + \sigma_2)^2(2\sigma_1 + \sigma_2) - \sigma_1(\sigma_1 + 2\sigma_2) + 12\sigma_1 + 28.$$

In particular, $\sigma_1^{(3)}$ is a quadratic polynomial in σ_1 and σ_2, and $\sigma_2^{(3)}$ is a cubic polynomial. I don't know any neat proof of these formulas. However, they can be verified by first studying the two multipliers $\eta_i^{(3)}$ asymptotically as $\sigma_1, \sigma_2 \to \infty$ in order to determine the degree and the leading terms of these polynomials, and then computing enough explicit examples to uniquely determine the lower-degree terms. Table 1 shows six easily computed examples, which can be used to complete this computation.

$f(z)$	$\{\mu_i\}$	σ_1	σ_2	$\{\eta_i^{(3)}\}$	$\sigma_1^{(3)}$	$\sigma_2^{(3)}$
z^2	$\{0,\ 0,\ 2\}$	2	0	$\{8,\ 8\}$	16	64
$z^2 - 1.75$	$\{0, 1 \pm \sqrt{8}\}$	2	-7	$\{1,\ 1\}$	2	1
$z^2 - 2$	$\{0, -2,\ 4\}$	2	-8	$\{8, -8\}$	0	-64
$1/z^2$	$\{-2, -2, -2\}$	-6	12	$\{-8, -8\}$	-16	64
$z(z+\omega)/(\omega z + 1)$	$\{\omega, \omega, -2\omega\}$	0	$-3\bar{\omega}$	$\{1,\ 1\}$	2	1
$z(z+\bar{\omega})/(\bar{\omega} z + 1)$	$\{\bar{\omega}, \bar{\omega}, -2\bar{\omega}\}$	0	-3ω	$\{1,\ 1\}$	2	1

TABLE 1. Data for Example D.3, where $\omega = e^{2\pi i/3} = \frac{1}{2}(-1 + i\sqrt{3})$.

APPENDIX E. TOTALLY DISCONNECTED JULIA SETS IN DEGREE d

Let f be a rational map of degree $d \geq 2$. This appendix will prove the following two lemmas, which together imply Lemma 8.1.

Lemma E.1. *If all critical values of* f *belong to a single Fatou component, the Julia set* J *of* f *is totally disconnected.*

Remark E.2. A totally disconnected Julia set containing no critical points is homeomorphic, with its dynamics, to the one-sided shift on d symbols [Przytycki; Makienko]. In particular, this is true for rational maps satisfying the condition of the lemma. For the polynomial case compare [Blanchard et al. 1991], and for the quadratic case compare [Goldberg and Keen 1990]. For a study of totally disconnected Julia sets that do contain critical points, see [Branner and Hubbard 1992].

Lemma E.3. *If the Julia set* J *is totally disconnected, all orbits in the Fatou set* $\hat{\mathbf{C}} \smallsetminus J$ *converge either to an attracting fixed point or to a parabolic fixed point of multiplicity two.*

Remark E.4. A fixed point $z_0 \neq \infty$ has multiplicity two if and only if the first two derivatives are $f'(z_0) = 1$ and $f''(z_0) \neq 0$, so that there is exactly one attracting petal in the associated Leau–Fatou flower.

Proof of E.1 (modeled on [Rees 1990a]). Suppose that all critical values belong to the Fatou component U_1. Then U_1 must be an immediate basin for an attracting or parabolic periodic point, and must belong to a cycle

$$U_1 \to U_2 \to \cdots \to U_p \to U_1$$

of Fatou components with period $p \geq 1$. Evidently every Fatou component must eventually fall onto this cycle. (Compare [Sullivan 1985].)

Hyperbolic case. If f is hyperbolic, the closure of the postcritical set X splits as a disjoint union $\bar{X} = \bar{X}_1 \cup \cdots \cup \bar{X}_p$, where $\bar{X}_i$ is a compact subset of U_i. Choose a closed topological disk $D \subset U_1$ with $\bar{X}_1$ contained in the interior of D, and let Γ be the boundary of D. Note that every critical value of f is contained in $X_1 \subset D$, and that Γ separates X_1 from the Julia set.

Step 1. We prove that no component Γ_n of $f^{-n}(\Gamma)$ can separate points of any X_i. If Γ_n is not contained in U_i, then Γ_n certainly cannot separate X_i. For $\Gamma_n \subset U_i$, we work by induction on n. If $1 \leq i < p$, the statement follows from the induction assumption, since f induces a homeomorphism from the pair (U_i, X_i) onto (U_{i+1}, X_{i+1}). Thus we may suppose that $i = p$, so that $\Gamma_n \subset U_p$ with

$$f(\Gamma_n) = \Gamma_{n-1} \subset U_1.$$

Suppose inductively that Γ_{n-1} does not separate X_1. Choose a small disk $\Delta \subset U_1$ containing X_1 and disjoint from Γ_{n-1}, and let Δ' be the closure of the complementary disk $\hat{\mathbf{C}} \smallsetminus \Delta$. Since Δ' contains no critical values, $f^{-1}(\Delta')$ is a union of d disjoint disks, and hence its complement $f^{-1}(\Delta)$ is connected. Therefore the loop $\Gamma_n \subset f^{-1}(\Delta')$ cannot separate points of $X_p \subset f^{-1}(\Delta)$. This proves Step 1.

In particular, since Γ_{n-1} cannot separate two critical values, each Γ_n maps homeomorphically onto $f(\Gamma_n) = \Gamma_{n-1}$.

Step 2. We use the Poincaré metric on the complement of $\bar{X}$. Since the iterated preimages of Γ remain bounded away from $\bar{X}$, and since each component of $f^{-n}(\Gamma)$ maps homeomorphically onto Γ, the Poincaré diameters of the various components of $f^{-n}(\Gamma)$ shrink uniformly to zero as $n \to \infty$. Therefore the diameters of these components in the spherical metric shrink to zero also.

Step 3. If $n \equiv 0 \bmod p$, so that $f^{\circ n}(X_1) \subset X_1$, then $f^{-n}(\Gamma)$ separates X_1 from the Julia set. For if there were a path joining X_1 to a point of J in the complement of $f^{-n}(\Gamma)$, its image under $f^{\circ n}$ would be a path from X_1 to J in the complement of Γ, which is impossible.

Step 4. We now prove that J is totally disconnected and that $U_1 = \hat{\mathbf{C}} \smallsetminus J$. Using the spherical metric, choose any ε smaller than the diameter of X_1, and choose $n \equiv 0 \bmod p$ so large that each component of $f^{-n}(\Gamma)$ has diameter less than ε. Since X_1 cannot fit inside any of these loops, every component of J must fit inside one of them. Similarly, any other Fatou component U' must fit inside one of these loops. Since ε can be arbitrarily small, this proves that J is totally disconnected, and that U_1 is the unique Fatou component. In particular, $p = 1$.

Parabolic case. Here the argument is more delicate, since the postcritical closure $\bar{X}$ intersects the Julia set in a parabolic cycle. Again we can choose a disk D that contains X_1, the set of postcritical points of U_1, in its interior, but now D must have the parabolic point of $\bar{X}_1$ on its boundary. Steps 1, 3 and 4 go through very much as in the hyperbolic case, but Step 2 requires more work. Again we use the Poincaré metric on the complement of $\bar{X}$. This decreases under every branch of f^{-1}, and decreases by a definite factor less than 1 throughout any region bounded away from $\bar{X}$. Let P be the union of suitably chosen repelling petals at the points of our parabolic cycle. Then there is a unique branch of f^{-1} which maps P into itself. For any compact curve segment in P, the spherical lengths of the successive images under this branch of f^{-1} shrink to zero. Now consider an arbitrary sequence of preimages

$$f : \cdots \xrightarrow{\cong} \Gamma_2 \xrightarrow{\cong} \Gamma_1 \xrightarrow{\cong} \Gamma_0 = \Gamma,$$

where each Γ_n is a component of $f^{-1}(\Gamma)$. We must prove that the diameter of Γ_n in the spherical metric shrinks to zero as $n \to \infty$. Let $g_n : \Gamma \xrightarrow{\cong} \Gamma_n$ be the branch of f^{-n} that maps Γ to Γ_n.

For any point $z \in \Gamma$ there are two possibilities. If $g_n(z) \in P$ for all $n \geq n_0$, the same is true, with the same n_0, for all points in some neighborhood N of z. It follows that the spherical length of $g_n(N)$ shrinks to zero as $n \to \infty$. In this way, we control the points belonging to some open subset of Γ.

For points z in the complementary closed subset $Y \subset \Gamma$, there are infinitely many images $g_n(z)$ that do not lie in P, and hence are bounded away from the postcritical closure $\bar{X}$. It is easy to check that the Poincaré arc length of $g_n(Y)$ is finite for large n. Thus the Poincaré length of $g_n(Y)$ can be expressed as the integral over Y of a function that decreases monotonically to zero as $n \to \infty$. Hence this length shrinks to zero as $n \to \infty$, and the spherical arc length of $g_n(Y)$ must also shrink to zero. The rest of the proof proceeds as before. □

Proof of E.3. If J is totally disconnected and contains a parabolic periodic point z_0, this point must be a fixed point of multiplicity two. For otherwise there would be more than one parabolic basin; but if J is totally disconnected, the Fatou set must be connected. □

APPENDIX F. A SIERPIŃSKI CARPET AS JULIA SET

(written in collaboration with Tan Lei)

We call a compact subset of the plane a *Sierpiński carpet* if it is connected, locally connected, nowhere dense, and has the property that the boundaries of the various complementary components are pairwise disjoint Jordan curves. (The term "Sierpiński curve" is commonly used in the literature. However, we have adopted Mandelbrot's term [Mandelbrot 1982], which seems more descriptive and serves to distinguish this object from other examples of fractals due to Sierpiński.) All Sierpiński carpets are homeomorphic [Whyburn 1958]. The standard example is obtained from the unit square by subdividing into nine subsquares, removing the interior of the middle subsquare, then removing the middle ninth of each of the remaining eight subsquares, and continuing inductively.

It seems to be widely known among experts that such a Sierpiński carpet can occur as the Julia set for a rational function, but we are not aware of any explicit example in the literature.

Wittner [1988] noted that, up to conjugacy, there is one and only one real quadratic rational map whose two critical orbits have periods three and four. (He also showed that this map cannot be obtained by mating two quadratic polynomials.) More precisely, consider a real quadratic map

$$f(z) = a\left(z + \frac{1}{z}\right) + b,$$

so that the critical points are at ± 1 and there is a fixed point of multiplier $1/a$ at infinity. A simple computation shows that there are unique values $a = -.138115091$, $b = -.303108805$ such that the critical point $x_0 = 1$ has period four and the critical point $y_0 = -1$ has period three. The critical orbits are arranged along the real line in the order

$$x_3 < y_0 < x_1 < y_1 < x_2 < x_0 < y_2,$$

where $x_0 \mapsto x_1 \mapsto x_2 \mapsto x_3 \mapsto x_0$, for example.

Theorem F.1. *The Julia set of f is homeomorphic to a Sierpiński carpet.*

Proof. It is convenient to conjugate f by the Möbius transformation $z \leftrightarrow (z+i)/(iz+1)$, which interchanges the real axis $\hat{\mathbf{R}}$ and the unit circle S^1. The critical orbits for the resulting map g are shown in Figure 18, and the corresponding Julia set, drawn to the same scale, is shown in Figure 19. Note that the critical values x_1 and y_1 divide the unit circle S^1 into a shorter arc I and a longer arc $J = S^1 \setminus \mathring{I}$. The map g carries both the interval $[-1, 1]$ and the complementary interval $\hat{\mathbf{R}} \setminus (-1, 1)$ homeomorphically onto I, and carries both the upper and lower unit semicircles homeomorphically onto J. It has a repelling fixed point at the bottom point $-i$ of the lower semicircle.

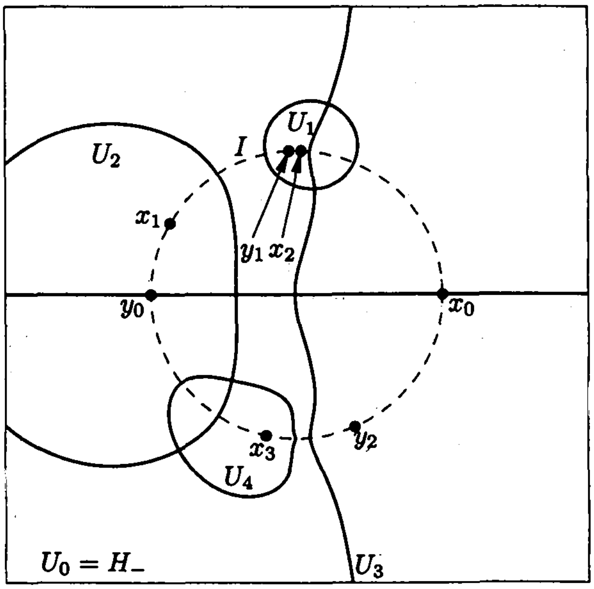

FIGURE 18. Critical orbits for g.

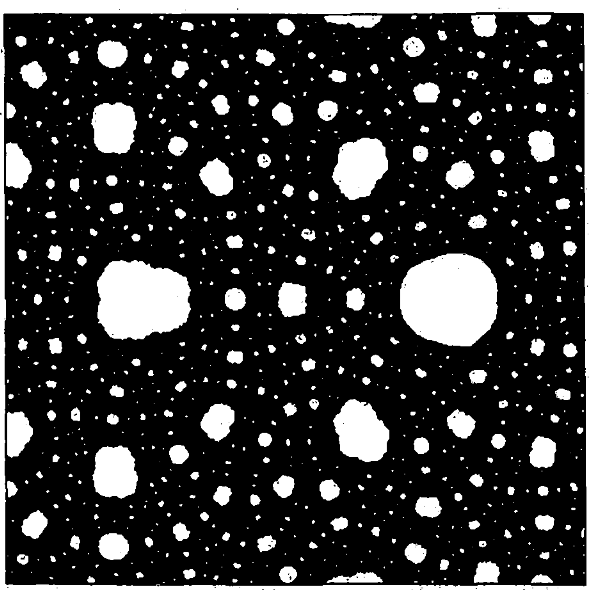

FIGURE 19. The Julia set $J(g)$.

It follows from Lemma 8.2 that the Julia set $J = J(g)$ it connected, so that each Fatou component is simply connected. Since g is hyperbolic on J, it follows from Lemma 7.3 that J is locally connected. We now prove that each Fatou component is bounded by a Jordan curve.

(Incidentally, the property that every Fatou component is bounded by a Jordan curve seems very common among hyperbolic rational maps with connected Julia set. It would be interesting to decide just when it is satisfied. But note that the boundary of a fixed fully invariant Fatou component—for example, the component containing infinity for a polynomial map—is usually not a Jordan curve.)

We will first construct a simply connected neighborhood U_4 of x_3 such that $g^{\circ 4}$ maps U_4 by a polynomial-like map of degree four onto the lower half-space H_-. (For definitions, see [Douady and Hubbard 1985].) In fact, for $0 \le j \le 4$, let U_j be the component of $g^{-j}(H_-)$ that contains the point x_{3-j} (where $x_{-1} = x_3$), so that

$$U_4 \to U_3 \to U_2 \to U_1 \to U_0 = H_-.$$

Note that the set $U_0 = H_-$ is invariant under the inversion $z \mapsto 1/\bar{z}$ in the circle S^1. Since the map g commutes with this inversion, and since each U_j contains a point x_{3-j} of the circle, it follows inductively that each U_j is invariant under inversion. Furthermore, since $g(z) = g(1/z)$, each $g^{-1}(U_j)$ is invariant under $z \mapsto 1/z$, or, equivalently, under complex conjugation. We will also show inductively that each U_j is simply connected. We can distinguish two cases. If U_j contains one critical value x_1 or y_1, then $U_{j+1} = g^{-1}(U_j)$ is a ramified twofold covering of U_j. In this case, U_{j+1} must contain the real point x_0 or y_0, and must be invariant under complex conjugation. On the other hand, if U_j contains no critical value, $g^{-1}(U_j)$ splits into two simply connected components. Neither can intersect the real axis, so one must lie in the upper half-plane and the other must be its complex conjugate in the lower half-plane. Just one of these two is the component U_{j+1} containing $x_{3-(j+1)}$. (If U_j

contained both critical values, then $g^{-1}(U_j)$ would not be simply connected; however, this case does not occur.) It is now easy to check that the sets U_j are situated as illustrated in Figure 18, that they contain postcritical points as shown, and that the proper maps $U_4 \to U_3 \to U_2 \to U_1 \to U_0$ have degrees $1, 2, 2, 1$. Thus the composition is proper of degree four.

Next we show that the region U_4 is compactly contained in $U_0 = H_-$. Note that the successive images of $\hat{\mathbf{R}}$ under g are intervals around the unit circle as follows:

$$\hat{\mathbf{R}} \to I = [y_1, x_1] \xrightarrow{\cong} [y_2, x_2] \to [x_1, x_3] \to [x_0, y_1],$$

where $[\alpha, \beta]$ stands for the interval from α to β in the counterclockwise direction around S^1. Suppose that the closure $\bar{U}_4$ contains some point u belonging to $\partial H_- = \hat{\mathbf{R}}$. Then the third forward image $g^{\circ 3}(u)$ must belong to the interval $[x_1, x_3] \subset S^1$. But it must also belong to $g^{\circ 3}(\bar{U}_4) = \bar{U}_1$, which is contained in the upper half-space, so it must belong to the smaller interval $[x_1, y_0]$. Hence $g^{\circ 4}(u) \in \bar{U}_0$ must belong to $g[x_1, y_0] = [x_2, y_1] \subset H_+$, which is impossible. Thus $g^{\circ 4} : U_4 \to U_0$ is polynomial-like.

Let X_j be the immediate attractive basin for the superattracting point x_j (equivalently, let X_j be the Fatou component containing the point x_j). Each X_j is invariant under the inversion $z \mapsto 1/\bar{z}$. Also, X_1 and X_2 are contained in the upper half-plane H_+, while X_3 is contained in the lower half-plane H_-. For if one of these sets intersected the real axis, its image X_2 or X_3 or X_0 would intersect the interval $g(\mathbf{R}) = I$. Hence this image, which is simply connected and invariant under inversion, would have a nonconnected intersection with the unit circle; which is impossible. Since $X_3 \subset H_- = U_0$, it follows by induction on i that $X_{3-i} \subset U_i$ for $i = 1, 2, 3, 4$ (again, with the convention $X_{-1} = X_3$). In particular, $X_3 \subset U_4$.

Let K be the filled Julia set for the polynomial-like mapping $g^{\circ 4} : U_4 \to U_0$. By definition, K consists of those points whose successive images under $g^{\circ 4}$ remain within U_4. Since $X_3 \subset U_4$, and since $g^{\circ 4}$ maps the basin X_3 onto itself, it follows that $X_3 \subset K \subset U_4$.

(In fact, X_3 is precisely equal to that component of the interior of K that contains the superattractive point x_3. Interior points of K certainly always belong to the Fatou set $\hat{\mathbf{C}} \setminus J$. On the other hand, it is not difficult to check that boundary points of K belong to the Julia set J. It follows that each component of the interior of K is a Fatou component for g, that is, a connected component of $\mathbf{C} \setminus J$.)

Next we prove that X_3 is bounded by a Jordan curve. Since J is locally connected, it follows from a theorem of Carathéodory that the various internal rays from x_3 land at well defined points of ∂X_3, depending continuously on the internal angle. (Compare the discussion in [Douady and Hubbard 1984, p. 20].) Thus, to show that ∂X_3 is a Jordan curve, we need only show that no two internal rays land at a common point. But two rays from x_3 landing at a common point $z \in \partial X_3$ would form a simple closed curve Γ within the closure $\bar{X}_3 \subset K \subset U_4$. Since $\hat{\mathbf{C}} \setminus K$ is connected, it would follow that Γ bounds a region $W \subset K$, necessarily contained in the Fatou set $\hat{\mathbf{C}} \setminus J$. Now some intermediate ray from x_3 must land at a distinct point $z' \neq z$ of ∂X_3. (The reflection principle implies that there cannot be an entire interval of internal angles with a common landing point.) Evidently this landing point z' must belong both to the region $W \subset \hat{\mathbf{C}} \setminus J$ and to the boundary $\partial X_3 \subset J$ which is impossible.

It follows easily that every iterated preimage of X_3 is also bounded by a Jordan curve.

Now let Y_i be the Fatou component containing y_i. The proof that Y_2 is bounded by a Jordan curve is completely analogous, involving a neighborhood V_6 of y_2 such that the map

$$g^{\circ 6} : V_6 \to V_0 = H$$

is polynomial-like of degree 8. In fact, let V_i be the component of $g^{-i}(H_-)$ that contains y_{5-i} (with $y_{-1} = y_5$). Then $V_i = U_i$ for $i \leq 3$. However,

V_4 is the complex conjugate (and the reciprocal) of U_4. The preimage $g^{-1}(V_4) = V_5$ is a neighborhood of y_0 containing no critical values, so its preimage is the union of a component V_6 containing y_2 and a complex conjugate component. A similar argument shows that V_6 is compactly contained in $V_0 = H_-$, and it follows that $Y_2 \subset V_6$ is bounded by a Jordan curve. Since every Fatou component is an iterated preimage of either X_3 or Y_2, this proves that every Fatou component is bounded by a Jordan curve.

Finally, we must prove that the various regions X_i and Y_j have no boundary points in common. We noted above that X_1 and X_2 are contained in the upper half-plane H_+, and a similar argument shows that $Y_1 \subset H_+$. On the other hand, X_3 and Y_2 are compactly contained in H_-. This proves that the closures $\bar{X}_3$ and $\bar{Y}_2$ are disjoint from $\bar{X}_1$ and $\bar{X}_2$ and $\bar{Y}_1$. It then follows easily that all of the sets $\bar{X}_i$ and $\bar{Y}_j$ are pairwise disjoint. For example, if $\bar{X}_2$ intersected $\bar{Y}_1$, then $g^{\circ 4}(\bar{X}_2) = \bar{X}_2$ would have to intersect $g^{\circ 4}(\bar{Y}_1) = \bar{Y}_2$.

Now suppose that any pair of Fatou components F and F' had a boundary point in common. We claim first that $g(F) \neq g(F')$. For if $g(F) = g(F')$, there would be two internal rays from the center point of $g(F)$ landing on a common boundary point. This would imply that the boundary of $g(F)$ is not a Jordan curve, contradicting our previous conclusion. Thus $g(F)$ and $g(F')$ must be distinct Fatou components, with a boundary point in common. Taking successive forward images under g, we would eventually find that two of the X_i and Y_j have a boundary point in common, which is impossible. This completes the proof that $J(g)$ is a Sierpiński carpet. □

APPENDIX G. REMARKS ON GRAPHICS

Several of the illustrations for this article are of poor quality. The primary reason for this is that I don't know any good general algorithm for producing either Julia set pictures or parameter space pictures. Following are more detailed comments.

The easiest Julia sets to plot are those which are hyperbolic, with known attracting periodic points. If there are at least two such points, then one can easily make a color picture with one color assigned to each of the associated attracting basins. From that, it is not difficult to produce a black and white picture which shows the boundaries between these basins, alias the Julia set. This procedure was used in Figures 2, 3, 4, and 19.

Parameter space pictures for a one-parameter family of maps can often be handled in much the same way. As an example, for Figure 8 each map in the family has one critical point of period 2. Thus there is just one "free" critical point, and we need only follow its orbit to understand the map. We can assign one color each to the two known attracting basins, with a supplementary color for the case that the free critical orbit does not converge to either one of these known basins, and then proceed as above. The procedure for Figure 9 is completely analogous.

The case of a parabolic periodic orbit can be handled similarly, although the convergence is very slow, so that the picture quality is poor near any map with a post-critical parabolic point. In Figures 5 and 6, this procedure has been modified slightly by drawing in curves of equal convergence rate towards the parabolic point.

However, in cases where no attracting or parabolic orbit is known the problem is more difficult. One strategy, which is used in Figures 7, 10, 13, 16 and 17, is to follow the critical orbits for many iterations, and then test for approximate periodicity of low period. This procedure has several weaknesses. Near a parabolic bifurcation, convergence is extremely slow, so it is difficult to observe the periodicity or to distinguish the correct period. Further, it does not seem practical to test for high periods.

A different strategy would be to estimate the Liapunov exponents at the two critical values by computing iterated derivatives. This method was used for Figures 11 and 12. It gives reasonably good pictures in these cases since the bifurcation locus is known to have positive measure. However in an example such as the Mandelbrot set, where the bifurcation locus (conjecturally at least) has measure zero, it would produce a useless picture.

Evidently these procedures are only marginally satisfactory for either a general Julia set or a general parameter slice. More effective algorithms are definitely needed. Possibly some combination of the last two methods, with a more careful testing for periodicity, would yield substantially better results.

One other step in constructing some of these pictures is worth noting. In Figures 7, 10, 16 and 17, the quadratic rational map is first described by the invariants σ_1 and σ_2 of Section 3. Some work is then needed to construct an actual map with these invariants. However, using Corollary C.4 and the proof of Lemma 6.1, it is not difficult to find an explicit representative map in the normal form (6–1), which is quite convenient for computation.

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