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Asymptotic independence in large random permutations with fixed descent set

Tarrago Pierre*

Abstract

In [5], Ehrenborg, Levin and Readdy have introduced a new probabilistic approach to the combinatorics of permutations with fixed set of descents. In this paper we extend this approach by introducing a more general probabilistic model. The study of this model yields new estimates on the behavior of a uniform random permutation σ having a fixed descent set. In particular, we find a positive answer to Conjecture 1 of [2] and we show that independently of the shape of the descent set, $\sigma(i)$ and $\sigma(j)$ are almost independent when $i-j$ becomes large.

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1 Introduction

A descent of a permutation σ of $n \in \mathbb{N}^*$ is an integer i such that $\sigma(i) > \sigma(i+1)$. For each permutation σ, the corresponding descent set $D(\sigma)$ is the set of all the descents of σ. Since descents can be located everywhere except on n, a descent set is just a subset of $\{1, \dots, n-1\}$. Let us call a composition of n the data of n and a subset of $\{1, \dots, n-1\}$. A composition D is represented by a ribbon Young diagram λ_D of n cells labelled 1 to n by the following rule : cells i and $i+1$ are neighbors and the cell $i+1$ is right to i if $i \notin D$, below i otherwise. Therefore, the descent set of a permutation σ is D if and only if inserting $\sigma(i)$ in each cell i of λ_D yields a standard ribbon Young tableau. For example, the composition $D = \{10, (3, 5, 9)\}$ gives the ribbon Young diagram displayed in Figure 1:

The permutation $\sigma = (3, 5, 8, 4, 7, 1, 6, 9, 10, 2)$ has the descent set D since the associated filling of λ_D yields a ribbon Young tableau, as shown in figure 2.

The descent statistic of a composition D is the number of standard fillings of the associated ribbon Young tableau λ_D (or, equivalently, the number of permutations having D as descent set). This latter number, denoted by $\beta(D)$, has been intensively studied

*Université de Paris-Est, France and Universität des Saarlandes, Germany.
E-mail: tarrago@math.uni-sb.de

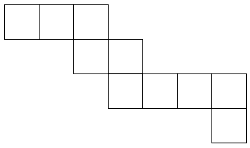

Figure 1: Ribbon Young diagram λ_D of to the composition $D = \{10, (3, 5, 9)\}$

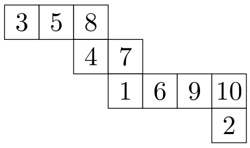

Figure 2: Standard filling of the composition $(3, 2, 4, 1)$

in the last decades (see Viennot [10] and [11] , Niven [8], de Bruijn [3] , ...). Two main questions arose in this study: the first one is to find the compositions of n having a maximum descent statistic, and the second one is to find exact or asymptotic formulae for the descent statistic of large compositions having a given shape. For example, Niven and de Bruijn proved in [8] and [3] that the two compositions of n maximizing the descent statistic are $D_1(n) = \{1, 3, 5, \dots\} \cap [1, n]$ and $D_2(n) = \{2, 4, 6, \dots, \} \cap [1, n]$: permutations having such descent sets are called alternating permutations. Désiré André already gave in [1] an asymptotic formula for the number of alternating permutations by showing that $\beta(D_1)(n) \sim 2(2/\pi)^n n!$ as n goes to infinity.

In order to evaluate the descent statistic of a broad class of compositions, Ehrenborg, Levin and Readdy formalized in [5] a probabilistic approach to the counting problem, by relating each permutation of n to a particular simplex of $[0, 1]^n$. Since the Lebesgue measure yields a probability measure on $[0, 1]^n$, it is possible to use probabilistic tools to get interesting results on descent statistics. Ehrenborg obtained in [4] an asymptotic formula for the descent statistics of the so-called nearly periodic permutations: the latter consist in permutations having the same descent pattern repeated several times, with some local perturbations. As for alternating permutations, the asympotic formula has the shape $K\lambda^n n!$, with K and λ being some constants depending on the situation. Using the approach of [5] with functional analysis tools, Bender, Helton and Richmond extended in [2] the previous results to a broader class of descent sets, and they found asymptotic formulae of the same shape as before.

The factorial term of the asymptotic formula is easy to understand, since it comes from the cardinality of the set of permutations of n elements. However, the term λ^n seems more mysterious. In [2], the authors identified in their examples the phenomenon that makes the term λ^n appear: namely, if we consider a large uniform random permutation with a fixed descent set, then the value of $\sigma(1)$ and $\sigma(n)$ are nearly independent, which causes a factorization in the asymptotic counting. Thus, the natural question is to know which compositions induce this phenomenon; it has been conjectured in [2] that every composition have this property as they become large.

In the present article we construct a family of probabilistic models, called sawtooth models, which extend the probabilistic approach of Ehrenborg, Readdy and Levin. These models are more general than the ones used in [2], but the combinatorial properties of the large descent sets appear more clearly in this broader case; thus, we first study these models in their full generality, before deducing some specific results on descent sets. A main consequence of the latter work is an affirmative answer to Conjecture 1 on

asymptotic independence from Bender, Helton and Richmond ([2]). We are also able to give by the following intuitive result on compositions:
In the random filling of a composition, the contents of two distant cells are almost independent.
In a forthcoming paper, we will use the results of this article to study an analog of the Young lattice that was introduced by Gnedin and Olshanski in [6].

2 Preliminaries and results

2.1 Compositions

This paragraph gives definitions and notations concerning compositions.

Definition 2.1. *Let $n \in \mathbb{N}$. A composition λ of n is a sequence of positive integers $(\lambda_1, \dots, \lambda_r)$ such that $\sum \lambda_j = n$.*

A unique ribbon Young diagram with n cells is associated to each composition: each row j has λ_j cells, and the first cell of the row $j+1$ is just below the last cell of the row j. For example the composition of 10, $(3, 2, 4, 1)$ is represented as in figure 1. This picture shows directly the link between Definition 2.1 and the definition we stated in the introduction : a composition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n yields a subset D_λ of $\{1, \dots, n-1\}$, namely the subset $\{\lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \dots + \lambda_{r-1}\}$. The latter correspondence is clearly bijective.
The size $|\lambda|$ of a composition is the sum of the λ_j. When nothing is specified, λ will always be assumed to have the size n, and n will always denote the size of the composition λ.
A standard filling of a composition λ of size n is a standard filling of the associated ribbon Young diagram: this is an assignement of a number between 1 and n for each cell of the composition, such that every cells have different entries, and the entries are increasing to the right along the rows and decreasing to the bottom along the columns. An example for the composition of figure 1 is shown in figure 2.
In particular, reading the tableau from left to right and from top to bottom associates a permutation σ to each standard filling; moreover, the descent set of such a permutation σ, namely the set of indices i such that $\sigma(i+1) < \sigma(i)$, is exactly the set

$$D_\lambda = \{\lambda_1, \lambda_1 + \lambda_2, \dots, \sum_1^{r-1} \lambda_i\}.$$

There is a bijection between the standard fillings of λ and the permutations of $|\lambda|$ with descent set D_λ. For example the filling in figure 2 yields the permutation $(3, 5, 8, 4, 7, 1, 6, 9, 10, 2)$.

2.2 Runs of a composition

Let λ be a composition. We number the cells as we read them, from left to right and from top to bottom . The cells are identified with integers from 1 to n through this numbering. For example in the standard filling of figure (2), the number 7 is in the cell 5.
We call run any set consisting in all the cells of a given column or row. The set of runs is ordered with the lexicographical order. In the same example as before the runs are

$$s_1 = (1, 2, 3), s_2 = (3, 4), s_3 = (4, 5), s_4 = (5, 6), s_5 = (6, 7, 8, 9), s_6 = (9, 10),$$

where we put in the parenthesis the cells of each run.
Note that inside each run the cells are ordered by the natural order on integers. We call extreme cell a cell that is an extremum in a run with respect to this order, and denote by $\mathcal{E}_\lambda$ the set of extreme cells of λ. Apart from the first and last cells of the composition,

each extreme cell belongs to two consecutive runs. Let P_λ be the set of extreme cells followed by a column, or preceded by a row and V_λ the set of extreme cells followed by a row or preceded by a column. The elements of P_λ are called peaks and the ones of V_λ valleys. The sets V_λ and P_λ are also ordered with the natural order:

$$P_\lambda = \{p_1 < \cdots < p_k\}, V_\lambda = \{q_1 < \cdots < q_{k'}\},$$

with $k-1 \leq k' \leq k+1$.
The first and last cells are always extreme points. A composition is said being of type ++ (resp. +-,-+,- -) if the first cell is a peak and the last cell is a peak (resp peak-valley, valley-peak, valley-valley).
Finally, let $l(s)$, the length of a run s, be the cardinality of s, and $L(\lambda)$, the amplitude of λ, be the supremum of all lengths $l(s)$.

2.3 Result on asymptotic independence

We present here the main results that are proven in the present paper.

Notation 2.2. *Let λ be a composition. Let Σ_λ denote the set of all permutations with descent set D_λ. With the uniform counting measure $\mathbb{P}_\lambda$, it becomes a probability space, and σ_λ denotes the random permutation coming from this probability space. As usual $|\Sigma_\lambda|$ is the cardinality of the set Σ_λ.*

$|\Sigma_\lambda|$ is thus the descent statistic associated to the composition λ.
Denote for each random variable X by $\mu(X)$ its law and by d_X its density, and write $\mu \otimes \nu$ for the independent product of two laws. The goal of the paper is to prove that distant cells in a composition have independent entries, namely:

Theorem 2.3. *Let $\epsilon, r \in \mathbb{N}$. Then there exists $k \geq 0$ such that if λ is a composition of n and $0 < i_1 < \cdots < i_r \leq n$ are indices with $i_{j+1} - i_j \geq k$,*

$$\pi\left(\mu(\frac{\sigma_\lambda(i_1)}{n}, \dots, \frac{\sigma_\lambda(i_r)}{n}), \mu(\frac{\sigma(i_1)}{n}) \otimes \cdots \otimes \mu(\frac{\sigma(i_r)}{n})\right) \leq \epsilon,$$

with π denoting the Levy-Prokhorov metric on the set of measures of $[0,1]^r$.

As it is shown in Section 6, if i_j is in a large run then the law of $\frac{\sigma(i_j)}{n}$ is approximately a dirac mass, which yields directly the approximate independence. Therefore, the interesting cases arise when none of the considered cells are in large runs. In particular, if the first and last runs of λ remain bounded and λ becomes large then the approximate independence of $\frac{\sigma_\lambda(1)}{n}$ and $\frac{\sigma_\lambda(n)}{n}$ can be given with a stronger metric than the Levy-Prokhorov metric. This is the content of Conjecture 1 of [2], which is proven in this paper and formulated in Theorem 6.2.

2.4 The coupling method

In this paragraph we introduce a probabilistic tool called the coupling method, and set the relative notations for the sequel. We refer to [7] for a review on the subject. We will present the notions in the framework of random variables but we could have done the same with probability laws as well.

Definition 2.4. *Let $(E, \mathcal{E})$ be a probability space and X, Y two random variables on E. A coupling of (X, Y) is a random variable (Z_1, Z_2) on $(E \times E, \mathcal{E} \otimes, \mathcal{E})$ such that*

$$Z_1 \sim_{law} X, Z_2 \sim_{law} Y.$$

Such a coupling always exists : it suffices to consider two independent random variables Z_1 and Z_2 with respective law μ_X and μ_Y. However, a coupling is often useful precisely when the resulting random variables Z_1 and Z_2 are far from being independent. In particular, in this article we are mainly interested in the case where Z_1 and Z_2 respect a certain order on the set E. From now on E is a Polish space considered with its Borel $\sigma-$algebra $\mathcal{E}$, and $\triangleleft$ is a partial order on E such that the graph $\mathcal{G} = \{(x,y), x \triangleleft y\}$ is $\mathcal{E}-$measurable.

Definition 2.5. *Let X, Y be two random variables on E. Y stochastically dominates X (denoted $Y \succeq X$) if and only if*

$$\mathbb{P}(X \in A) \leq \mathbb{P}(Y \in A)$$

for any Borel set A such that

$$x \in A \Rightarrow \{y \in E, x \triangleleft y\} \subset A.$$

For example if $E = \mathbb{R}$ with the canonical order $\leq$ and $\sigma-$algebra $\mathcal{B}(\mathbb{R})$, then Y stochastically dominates X if and only if for all $x \in \mathbb{R}$,

$$\mathbb{P}(X \in [x, +\infty[) \leq \mathbb{P}(Y \in [x, +\infty[)$$

or equivalently, if we denote their respective cumulative distribution function by $F_X(t)$ and $F_Y(t)$:

$$F_Y(t) \leq F_X(t) \quad \text{for all } t \in \mathbb{R}.$$

There are several ways to characterize the stochastic dominance:

Proposition 2.6. *The three following statements are equivalent :*

- *Y stochastically dominates X*
- *there exists a coupling (Z_1, Z_2) of X, Y such that $Z_1 \triangleleft Z_2$ almost surely.*
- *for any positive measurable bounded function f that is non-decreasing with respect to $\triangleleft$,*

$$\mathbb{E}(f(X)) \leq \mathbb{E}(f(Y))$$

The proof is straightforward and can be found in [7]. This yields the following intuitive Lemma :

Lemma 2.7. *Let (X_1, X_2, Y_1, Y_2) be a random variable on E^4 such that :*

- *$X_1 \preceq Y_1$ and $Y_2 \preceq X_2$,*
- *(X_1, Y_1) is independent from (X_2, Y_2).*

Then

$$\mathbb{P}(X_1 \triangleleft X_2) \geq \mathbb{P}(Y_1 \triangleleft Y_2).$$

Proof. Let $\ll$ be the partial order on $E \times E$ defined by

$$(x, y) \ll (x', y') \leftrightarrow x \triangleleft x' \text{ and } y' \triangleleft y.$$

Since $Y_1 \succeq X_1$ and $X_2 \succeq Y_2$, there exists a coupling $(\hat{X}_1, \hat{Y}_1)$ (resp. $(\hat{X}_2, \hat{Y}_2)$) of X_1, Y_1 (resp. X_2, Y_2) such that almost surely $\hat{X}_1 \triangleleft \hat{Y}_1$ (resp $\hat{X}_2 \triangleright \hat{Y}_2$). The random variables $(\hat{X}_1, \hat{Y}_1)$ and $(\hat{X}_2, \hat{Y}_2)$ can be chosen independent one from each other. Since (X_1, Y_1) and (X_2, Y_2) are also independent, this implies that $((\hat{X}_1, \hat{X}_2), (\hat{Y}_1, \hat{Y}_2))$ is a coupling of $((X_1, X_2), (Y_1, Y_2))$ with almost surely

$$(\hat{X}_1, \hat{X}_2) \ll (\hat{Y}_1, \hat{Y}_2).$$

But if $\hat{Y}_1 \triangleleft \hat{Y}_2$, then $\hat{X}_1 \triangleleft \hat{Y}_1 \triangleleft \hat{Y}_2 \triangleleft \hat{X}_1$ and thus

$$\mathbb{P}(Y_1 \triangleleft Y_2) = \mathbb{P}(\hat{Y}_1 \triangleleft \hat{Y}_2) \leq \mathbb{P}(\hat{X}_1 \triangleleft \hat{X}_2) = \mathbb{P}(X_1 \triangleleft X_2).$$

□

These results will be concretely applied on $\mathbb{R}^p, p \geq 1$, and thus we need to define a family of partial orders on those sets.

Definition 2.8. *Let $p \geq 1$. The partial order $\leq$ on $\mathbb{R}^p$ is the natural order on $\mathbb{R}$ for $p = 1$, and for $p \geq 2$ if $(x_i)_{1\leq i\leq p}, (y_i)_{1\leq i\leq p} \in \mathbb{R}^p$,*

$$(x_i)_{1\leq i\leq p} \leq (y_i)_{1\leq i\leq p} \Leftrightarrow \forall i \in [1; p], x_i \leq y_i.$$

For any word ϵ of length p in $\{+1, -1\}$ (or simply in $\{+, -\}$), the modified partial order $\leq_\epsilon$ is defined as

$$(x_i)_{1\leq i\leq p} \leq_\epsilon (y_i)_{1\leq i\leq p} \Leftrightarrow \forall i \in [1; p], \epsilon_i x_i \leq \epsilon_i y_i.$$

The easiest way to check the stochastic dominance is to look at the cumulative distribution function. The proof of the following Lemma is a direct application of Proposition 2.6.

Lemma 2.9. *Let $(X_i)_{1\leq i\leq p}$ and $(Y_i)_{1\leq i\leq p}$ be two random variables on $(\mathbb{R}^p, \leq_\epsilon)$. Then $(Y_i)_{1\leq i\leq p}$ stochastically dominates $(X_i)_{1\leq i\leq p}$ if and only if for all $(t_i)_{1\leq i\leq p} \in \mathbb{R}^p$,*

$$F^\epsilon_{(X_i)}(t_1, \ldots, t_p) \geq F^\epsilon_{(Y_i)}(t_1, \ldots, t_p),$$

with $F^\epsilon_{(X_i)}$ being the modified cumulative distribution function defined by

$$F^\epsilon_{(X_i)}(t_1, \ldots, t_p) = \mathbb{P}((X_i) \leq_\epsilon (t_i)).$$

The stochastic dominance in the case $(\mathbb{R}^p, \leq_\epsilon)$ is denoted as $(X_1, \ldots, X_p) \preceq_\epsilon (Y_1, \ldots, Y_p)$. A consequence of the previous result is that if $(Y_1, \ldots, Y_p)$ stochastically dominates $(X_1, \ldots, X_p)$, then for all subsets $I = (i_1, \ldots, i_r)$ of $\{1, \ldots, p\}$, $(Y_{i_1}, \ldots, Y_{i_r})$ also stochastically dominates $(X_{i_1}, \ldots, X_{i_r})$.
Applying Lemma 2.9 to the case $p = 2$ yields the following Lemma:

Lemma 2.10. *Let $(U_1, V_1), (U_2, V_2)$ be two random variables on $[0, 1]$ such that U_2 and V_2 are independent. Suppose that for all $0 \leq t \leq 1$,*

$$F_{V_1}(t) \leq F_{V_2}(t)$$

and for all $v \in [0, 1]$,

$$F_{U_1|V_1=v}(t) \leq F_{U_2}(t).$$

There exists a coupling $((Z_1, \tilde{Z}_1), (Z_2, \tilde{Z}_2))$ of (U_1, V_1) and (U_2, V_2) such that almost surely

$$(Z_1, \tilde{Z}_1) \geq (Z_2, \tilde{Z}_2).$$

3 Sawtooth model

3.1 Definition of the model

In this section we introduce a statistical model of particles in a tube, which is a generalization of the probabilistic approach of Ehrenborg, Levin and Readdy in [5]. The model consists in a sequence of particles, each of them moving vertically in an horizontal two-dimensional tube. Each particle has a repulsive action on the two neighbouring particles, and moreover, the set of particles splits into two groups: the upper particles and the lower particles. The upper particles are always above the lower ones. The model is depicted in Figure 3.
Such a system is called a Sawtooth model in the sequel.

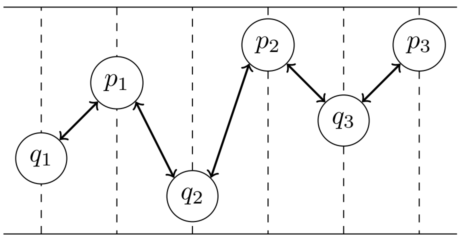

Figure 3: Upper particles $\{p_1, p_2, p_3\}$ and lower particles $\{q_1, q_2, q_3\}$ in a tube.

Remark 3.1. If there are k upper-particles, there must be k' lower particles with $k' \in \{k-1, k, k+1\}$, depending on the type of the first and the last particles. We define therefore the type $\epsilon(\mathcal{S})$ of the model $\mathcal{S}$ as the word $\epsilon_I \epsilon_F$, with $\epsilon_I = +$ (resp. $\epsilon_F = +$) if the first (resp. last) particle is an upper one, and $\epsilon_I = -$ (resp. $\epsilon_F = -$) otherwise.

Unless specified otherwise, the first particle is a lower particle (as in the picture). The particles are ordered from the left, and following this order the upper particles are written $\{p_1 < p_2 < \cdots < p_k\}$ and the lower particles $\{q_1 < \cdots < q_{k'}\}$. Since the nature of our results won't depend on the type of the model, we will also assume that there are $k+1$ lower particles, yielding that the last particle is a lower one too.
Denote by x_i the position of q_i and by y_i the position of p_i: by a rescaling, we can assume that $x_i, y_i \in [0,1]$. These positions are considered as random, and each configuration of positions is weighted according to repulsive interactions between neighbouring particles. This yields the following definition:

Definition 3.2. *A Sawtooth model $\mathcal{S}$ is the union of two families of random variables $\{X_i\}_{1\le i\le k+1}$ and $\{Y_j\}_{1\le j\le k}$ on $[0,1]$ with the multivariate density*

$$\mathbb{P}(\{X_i = x_i, Y_j = y_j\}) = \frac{1}{\mathcal{V}} \prod \mathbf{1}_{x_i \le y_i \ge x_{i+1}} f_i(y_i - x_i) g_i(y_i - x_{i+1}) \prod dx_i \prod dy_j, \quad (3.1)$$

where $\{f_i, g_i\}_{1\le i\le k}$ is a family of increasing positive C^1 functions on $[0,1]$.
The quantity $\mathcal{V}$ is called the volume of $\mathcal{S}$ and is sometimes denoted by $\mathcal{V}(\mathcal{S})$ to avoid confusion.
$\mathcal{S}$ is said normalized if $\int f_i = \int g_i = 1$ for $1 \le i \le k$.

The volume has the following expression:

$$\mathcal{V}(\mathcal{S}) = \int_{[0,1]^{2k+1}} \prod \mathbf{1}_{x_i \le y_i \ge x_{i+1}} f_i(y_i - x_i) g_i(y_i - x_{i+1}) \prod dx_i dy_i. \quad (3.2)$$

In particular, an appropriate rescaling of the functions f_i, g_i can transform any Sawtooth model into a normalized one, without changing the probability space. Thus, from now on and unless stated otherwise, the model is assumed normalized. In case we are considering non-normalized models, we will use the notation f_i, g_i, etc. for the normalized quantities, and $\tilde{f}_i, \tilde{g}_i, etc.$ for the non-normalized ones.
Aiming the results we stated on compositions, we should answer these questions :

1. *As the number of particles goes to infinity, is there some independence between X_1 and X_{k+1} ?*
2. *It is possible to estimate the behavior of a particle X_r by only considering its neighbouring particles ?*

For each subset of particles $\Omega = (q_{i_1}, \ldots, q_{i_r}, p_{j_1}, \ldots, p_{j_{r'}})$ and measurable event $\mathcal{X}$, denote by

$$d_{\Omega|\mathcal{X}}(x_{i_1}, \ldots, x_{i_r}, y_{j_1}, \ldots, y_{j_{r'}})$$

the marginal density of Ω conditioned on $\mathcal{X}$. The subscripts will be dropped when there is no confusion, and we denote by X_I the first variable X_1 and X_F the last variable X_{k+1}. Finally, since the system is fully described by the functions $\{f_i, g_j\}$, we will refer sometimes to a particular system just by mentioning this set of functions.
The definition of a Sawtooth model yields directly two first results which are given in Lemma 3.3 and Lemma 3.4. The first one stresses the Markovian aspect of a Sawtooth model :

Lemma 3.3. *Let $\mathcal{S}$ be a Sawtooth model of size k, and $1 \leq i \leq k$. Let Z be the position of a particle right to X_i (namely $Z = X_j$ for $j > i$ or $Z = Y_j$ for $j \geq i$) and $\mathcal{X}$ be an event depending on the positions of particles right to Z. Then for $0 \leq z \leq 1$,*

$$d_{X_i|Z=z,\mathcal{X}} = d_{X_i|Z=z}.$$

Proof. It suffices to prove that the particles left to Z are independent of the particles right to Z conditionally on the value of Z. This is implied by the form of the density of the model, since the latter splits between the density of the particles left to Z and the ones right to Z. □

The second one is a generalization of Lemma 3-(a) in [2]. :

Lemma 3.4. *Let $1 \leq r \leq k+1$, and let $\mathcal{X}$ be an event depending on the position of all particles except X_r. Then $d_{X_r|\mathcal{X}}(x_r)$ is decreasing in x_r.*

Proof. Let a be in $[0,1]$. By Lemma 3.3,

$$\begin{aligned} d_{X_r|\mathcal{X}}(a) &= \int_{[0,1]^2} d_{(X_r|\mathcal{X})|Y_{r-1}=z,Y_{r+1}=z'}(a) d_{Y_{r-1},Y_{r+1}|\mathcal{X}}(z,z')dzdz' \\ &= \int_{[0,1]^2} d_{X_r|Y_{r-1}=z,Y_{r+1}=z'}(a) d_{Y_{r-1},Y_{r+1}|\mathcal{X}}(z,z')dzdz'. \end{aligned}$$

Thus, it is enough to prove the monotonicity in the case of a conditioning on $Y_{r-1} = z, Y_{r+1} = z'$. In this case

$$d_{X_r|Y_{r-1}=z,Y_{r+1}=z'}(a) = \mathbf{1}_{z \geq a, z' \geq a} \frac{1}{R}(g_{r-1}(z-a)f_r(z'-a)),$$

with R a normalizing constant. Since g_{r-1} and f_r are increasing, this concludes the proof. □

The same result holds for upper particles, but in this case the density is increasing.

3.2 The processes $\mathcal{S}_\lambda$ and Σ_λ

Let us see how these definitions fit into the framework of compositions. The main idea from [5] is to consider the set of all permutations with a given descent set D_λ as a probability space.
$|\Sigma_\lambda|$ can indeed be related to the volume of a polytope in $[0,1]^n$ (see for example the survey of Stanley on alternating permutations, [9]) . For each sequence of distinct elements $\vec{z} = (z_1, \ldots, z_n)$ in $[0,1]$, the ranking permutation of $\vec{z}$ is the permutation $\sigma(\vec{z})$ that assigns to each j the position of z_j in the ordered sequence $(z_{i_1} < \cdots < z_{i_n})$: namely, $\sigma(\vec{z})(j) = k$ if and only if $\#\{1 \leq i \leq n | z_i \leq z_j\} = k$.

Proposition 3.5 ([5])**.** *The law of σ_λ is the law of the ranking permutation for a sequence of independent uniform variables $Z_1, \ldots, Z_n$ in $[0,1]$ conditioned on the event*

$$\{Z_i > Z_{i+1} \text{ if and only if } i \in D_\lambda\}.$$

In particular, the following expression of the number of permutations with descent set D_λ *holds :*

$$|\Sigma_\lambda| = n! \int_{[0,1]^n} \prod_{i \in D_\lambda} \mathbf{1}_{z_i \geq z_{i+1}} \prod_{i \notin D_\lambda} \mathbf{1}_{z_i \leq z_{i+1}} \prod dz_i,$$

with $z_{n+1} = 1$*.*

The proof of the latter proposition is straightforward as soon as we remark that the volume of the polytope $\{0 \leq z_1, \dots, z_n \leq 1\}$ is exactly $\frac{1}{n!}$. The processus $\{Z_i\}_{1 \leq i \leq n}$ in the previous proposition is denoted by $\tilde{\mathcal{S}}_\lambda$. Since the indicator function in the integrand depends on conditions between neighbouring points, this result can be rephrased in terms of Sawtooth model.
Regrouping the inequalities between elements of the same run of λ yields:

$$|\Sigma_\lambda| = n! \int_{[0,1]^n} \mathbf{1}_{z_1 \leq z_2 \leq \dots \leq z_{i_1}} \mathbf{1}_{z_{i_1} \geq z_{i_1+1} \geq \dots \geq z_{i_1+i_2}} \dots \mathbf{1}_{z_{n-i_{2r}} \leq \dots \leq z_n} \prod dz_i, \tag{3.3}$$

and by integrating over all the coordinates that do not correspond to extreme cells, we get

$$\begin{aligned}|\Sigma_\lambda| =& n! \int_{[0,1]^n} \mathbf{1}_{x_1^- \leq x_1^+ \geq x_2^- \leq \dots} \frac{1}{(l(s_1)-2)!} |x_1^+ - x_1^-|^{l(s_1)-2} \\ & \frac{1}{(l(s_2)-2)!} |x_1^+ - x_2^-|^{l(s_2)-2} \dots \frac{1}{(l(s_{2r})-2)!} |x_k^+ - x_{k+1}^-|^{l(s_k)-2} \prod_{i=1}^{k} dx_i^+ \prod_{i=1}^{k+1} dx_i^- .\end{aligned}$$

Let $\mathcal{S}_\lambda$ be the non-normalized Sawtooth model with the non-normalized density functions $\{\tilde{f}_j, \tilde{g}_j\}_{1 \leq i \leq r}$ such that

$$\tilde{f}_j(t) = \frac{1}{(l(s_{2j-1})-2)!} t^{l(s_{2j-1})-2}, \tilde{g}_j(t) = \frac{1}{(l(s_{2j})-2)!} t^{l(s_{2j})-2}.$$

A comparison between the latter expression of $|\Sigma_\lambda|$ and the expression (3.2) of the volume of a Sawtooth model gives

$$|\Sigma_\lambda| = |\lambda|! \mathcal{V}(\mathcal{S}_\lambda)$$

To sum up, three processes are constructed from λ. The first one, σ_λ comes from the uniform random standard filling of the ribbon Young tableau λ, the second one, $\tilde{\mathcal{S}}_\lambda$, comes from the probabilistic approach of [5], and the third one, $\mathcal{S}_\lambda$, is obtained from $\tilde{\mathcal{S}}_\lambda$ by considering only the extreme particles. They are of course intimately related, even if the first one is discrete and the second and third ones are continuous. σ_λ can be recovered from $\tilde{\mathcal{S}}_\lambda$ by the associated ranking permutation, and when $|\lambda|$ goes to infinity $\frac{\sigma_\lambda(i)}{n}$ and Z_i are approximately the same :

Lemma 3.6. *The following inequality always holds for* $0 < A, n \in \mathbb{N}$*:*

$$\mathbb{P}(\max(|\frac{\sigma_\lambda(i)}{n+1} - Z_i| > \frac{A}{\sqrt{n+2}}) \leq \frac{1}{A^2}$$

In particular, if the densities of Z_i *remains bounded by a constant* B*,*

$$\|F_{Z_i} - F_{\frac{\sigma(i)}{n}}\|_\infty \to_{|\lambda| \to +\infty} 0.$$

Proof. Let us evaluate $\mathbb{P}(|\frac{\sigma_\lambda(i)}{n+1} - Z_i| > \frac{A}{n+2})$. Condition the event $\{|\frac{\sigma_\lambda(i)}{n+1} - Z_i| > \frac{A}{n+2}\}$ on a particular realization σ of σ_λ, and suppose that $\sigma(i) = k$. In this case, the conditional density of Z_i is :

$$\begin{aligned} d_{Z_i|\sigma_\lambda=\sigma}(z_i) =& n! \left(\int_{0 \leq z_{\sigma^{-1}(1)} \leq \cdots \leq z_{\sigma^{-1}(k-1)} \leq z_i} \prod_{1 \leq \sigma(j) \leq k-1} dz_j \right) \\ & \left(\int_{z_i \leq z_{\sigma^{-1}(k+1)} \leq \cdots \leq z_{\sigma^{-1}(n)} \leq 1} \prod_{k+1 \leq \sigma(j) \leq 1} dz_j \right) \\ =& \frac{n!}{(k-1)!(n-k)!} z_i^{k-1}(1-z_i)^{n-k}. \end{aligned}$$

Computing the conditional expectation yields $\mathbb{E}(Z_i|\sigma_\lambda = \sigma) = \frac{k}{n+1}$ and

$$Var(Z_i|\sigma_\lambda = \sigma) = \left(\frac{k}{n+1} \frac{n+1-k}{n+1} \right) \frac{1}{n+2} \leq \frac{1}{n+2}.$$

Thus, by the Chebyshev's inequality,

$$\mathbb{P}_{Z_i|\sigma_\lambda=\sigma} \left(|Z_i - \frac{\sigma(1)}{n+1}| > \frac{A}{\sqrt{n+2}} \right) \leq \frac{1}{A^2}.$$

Integrating this inequality on all the disjoint events σ on which Z_i can be conditioned yields the first part of the Lemma. □

From now on, let $\tilde{\gamma}_r$ denote for $r \geq 2$ the function $\tilde{\gamma}_r(t) = \frac{1}{(r-2)!} t^{r-2}$, and $\gamma_r(t) = (r-1)t^{r-2}$ its normalized density function.

4 Convex Sawtooth Model

In this section, we study the behavior of the extreme particles for a Sawtooth model respecting a particular convexity property. The results of this section are much easier to get in the particular case of the Sawtooth models $\mathcal{S}_\lambda$ of the last section, since the density functions $\{f_i, g_i\}$ are explicitly given. We will use this particular Sawtooth models as examples for our more general computations.

4.1 Log-concave densities

To be able to get some results on the behavior of the particles, it is necessary to impose some conditions on the density functions $\{f_i, g_i\}$. Actually the condition we need is quite natural from a physical point of view, since we will require that the repulsive forces in the definition of the Sawtooth model come from a convex potential : the consequence is that the density functions should be log-concave. This motivates the following definition :

Definition 4.1. *A Sawtooth model is called convex if all the functions $(f_i, g_i)_{1 \leq i \leq k}$ are log-concave. This means that for all $1 \leq i \leq k$, $\frac{f_i'(t)}{f_i(t)}$ and $\frac{g_i'(t)}{g_i(t)}$ are decreasing.*

The main advantage of the log-concavity is that the behavior of the particles becomes monotone in a certain sense.
For $1 \leq s \leq k+1$, let $\mathcal{S}_{\to X_s}$ (resp. $\mathcal{S}_{X_s \leftarrow}$) denote the Sawtooth model obtained by keeping only the particles between X_I and X_s (resp. between X_s and X_F) and the functions $\{f_i, g_i\}_{i \leq s}$ (resp. $\{f_i, g_i\}_{i \geq s+1}$). Likewise, let $\mathcal{S}_{\to Y_s}$ (resp. $\mathcal{S}_{Y_s \leftarrow}$) denote the Sawtooth model obtained by keeping only the particles between X_I and Y_s (resp. between Y_s and

X_F) and the functions $\{f_i, g_j\}_{\substack{i\leq s\\ j\leq s-1}}$ (resp. $\{f_i, g_j\}_{\substack{i\geq s+1\\ j\geq s}}$).
In order to emphasize a specific Sawtooth model $\mathcal{S}$, we write $X_i^{\mathcal{S}}$ to denote the particle X_i in $\mathcal{S}$, and $F_{X_i,\mathcal{S}}$ to denote the cumulative distribution function of X_i in $\mathcal{S}$ (and the same for Y_i).

Proposition 4.2. *Let $\{f_i, g_i\}$ be a convex Sawtooth model. Then for $1 \leq s \leq k$, $(X_s|Y_s = y)$ is increasing with y (in terms of stochastic dominance) and $(Y_s|X_{s+1} = x)$ is increasing with x. Moreover,*

$$X_s^{\mathcal{S}\to X_s} \succeq (X_s|Y_s = y), \quad Y_s^{\mathcal{S}\to Y_s} \succeq (Y_s|X_{s+1} = x).$$

Proof. Let $1 \leq s \leq k$. To prove the first part of the proposition, it is enough to show that for $0 \leq t \leq 1$, $F_{X_s|Y_s=y}(t)$ is decreasing in y and $F_{Y_s|X_{s+1}=x}(t)$ is decreasing in x.
Let $d(x)$ be the density of X_s in $\mathcal{S}_{\to X_s}$. Then by the definition of the probability density of $\mathcal{S}$, the density of X_s in $\mathcal{S}$ conditioned on the value of Y_s is $\mathbf{1}_{x\leq y}\frac{d(x)f_s(y-x)}{A}$, with A a normalizing constant. Thus, the cumulative distribution function $F_y(.)$ of X_s conditioned on $Y_s = y$ is

$$F_y(t) = \frac{\int_0^{t\wedge y} d(x)f_s(y-x)dx}{\int_0^y d(x)f_s(y-x)dx}.$$

For $t > y$ it is clear that $\frac{\partial}{\partial y}F_y(t) = 0$, and from now on we only consider $t \leq y$. Since the logarithm function is increasing, it is enough to show that $\frac{\partial}{\partial y}\log(F_y(t)) \leq 0$. This derivative is equal to

$$\frac{\partial}{\partial y}\log(F_y(t)) = \frac{\int_0^t d(x)f_s'(y-x)dx}{\int_0^t d(x)f_s(y-x)dx} - \frac{\int_0^y d(x)f_s'(y-x)dx}{\int_0^y d(x)f_s(y-x)dx} - \frac{d(y)f_s(0)}{\int_0^y d(x)f_s(y-x)dx}.$$

Since $(-\frac{d(y)f_s(0)}{\int_0^y d(x)f_s(y-x)dx}) \leq 0$, the non-positivity of the remaining part of the sum suffices. Denote

$$\Delta = \int_0^t d(x)f_s'(y-x)dx \int_0^y d(x)f_s(y-x)dx - \int_0^y d(x)f_s'(y-x)dx \int_0^t d(x)f_s(y-x)dx.$$

Thus, we have to show that $\Delta \leq 0$. For $t \leq y$,

$$\begin{aligned}\Delta = &\int_0^t d(x)f_s'(y-x)dx \left(\int_0^t d(x)f_s(y-x)dx + \int_t^y d(x)f_s(y-x)dx\right)\\ &- \left(\int_0^t d(x)f_s'(y-x)dx + \int_t^y d(x)f_s'(y-x)dx\right)\int_0^t d(x)f_s(y-x)dx\\ = &\int_0^t d(x)f_s'(y-x)dx \int_t^y d(x)f_s(y-x)dx\\ &- \int_t^y d(x)f_s'(y-x)dx \int_0^t d(x)f_s(y-x)dx.\end{aligned}$$

Expressing products of integrals as double integrals yields

$$\begin{aligned}\Delta = &\int_{\substack{0\leq z_1\leq t\\ t\leq z_2\leq y}} d(z_1)d(z_2)f_s'(y-z_1)f_s(y-z_2)dz_1dz_2\\ &- \int_{\substack{0\leq z_1\leq t\\ t\leq z_2\leq y}} d(z_1)d(z_2)f_s(y-z_1)f_s'(y-z_2)dz_1dz_2\\ = &\int_{\substack{0\leq z_1\leq t\\ t\leq z_2\leq y}} d(z_1)d(z_2)(f_s'(y-z_1)f_s(y-z_2) - f_s(y-z_1)f_s'(y-z_2))dz_1dz_2.\end{aligned}$$

Since $d(z_1)d(z_2)$ is positive and $\frac{f_s'(t)}{f_s(t)}$ is decreasing, $\Delta \leq 0$ and the first part of the proposition is proven.
The second part of the proposition is equivalent to the inequalities

$$F_{X_s|Y_s=y}(t) \geq F_{X_s,\mathcal{S}\to X_s}(t)$$

and

$$F_{Y_s|X_{s+1}=x}(t) \leq F_{Y_s,\mathcal{S}\to Y_s}(t)$$

for all $0 \leq t \leq 1$.
From the first part of the Proposition, it suffices to prove the first inequality only for $y=1$. Since f_s is increasing, there exists a measure μ on $[0,1]$ such that $f_s(x) = \int_0^x d\mu(u)$. Thus,

$$F_1(t) = \frac{\int_0^t d(x)\left(\int_0^{1-x} d\mu(u)\right)dx}{\int_0^1 d(x)\left(\int_0^{1-x} d\mu(u)\right)dx} = \frac{\int_{[0,1]^2} \mathbf{1}_{x\leq t, u\leq 1-x} d(x)d\mu(u)dx}{\int_{[0,1]^2} \mathbf{1}_{u\leq 1-x} d(x)d\mu(u)dx}.$$

The main point is to express the latter quantity as the expectation of a random variable almost surely greater than $\int_0^t d(x)dx$. Changing the order of the integrals yields

$$F_1(t) = \frac{\int_0^1\left(\int_0^{t\wedge(1-u)} d(x)dx\right)d\mu(u)}{\int_0^1\left(\int_0^{1-u} d(x)dx\right)d\mu(u)} = \frac{\int_0^1\left(\int_0^{t\wedge(1-u)} \frac{d(x)}{\int_0^{1-u} d(x)dx}dx\right)\left(\int_0^{1-u} d(x)dx\right)d\mu(u)}{\int_0^1\left(\int_0^{1-u} d(x)dx\right)d\mu(u)}.$$

Let $\tilde{U}$ be a random variable absolutely continuous with respect to μ and having the density

$$d_{\tilde{U}}(u) = \frac{\left(\int_0^{1-u} d(x)dx\right)d\mu(u)}{\int_0^1\left(\int_0^{1-u} d(x)dx\right)d\mu(u)}.$$

Then

$$F_1(t) = \mathbb{E}_{\tilde{U}}\left(\frac{\int_0^{t\wedge(1-\tilde{U})} d(x)dx}{\int_0^{1-\tilde{U}} d(x)dx}\right).$$

Since for each $u \geq 0$

$$\frac{\int_0^{t\wedge 1-u} d(x)dx}{\int_0^{1-u} d(x)dx} \geq \int_0^t d(x)dx,$$

this concludes the proof.
It is exactly the same for $F_{Y_s|X_{s+1}=x}(t)$. □

Remark 4.3. In the case of a Sawtooth model $\mathcal{S}_\lambda$, a simpler proof of the monotonicity result of Proposition 4.2 can be done by induction on the length of the run of λ between x_s^- and x_s^+. Namely, if the run has length 2,

$$F_{X_s|Y_s=y}(t) = \frac{\int_0^{t\wedge y} d_{X_s,\mathcal{S}_\lambda\to X_s}(x)dx}{\int_0^y d_{X_s,\mathcal{S}_\lambda\to X_s}(x)dx},$$

which is decreasing in y. If the run has length $r>2$, the expression of the density in the integral of (3.3) yields

$$F_{X_s|Y_s=y}(t) = \frac{\int_0^y F_{\tilde{X}_s|\tilde{Y}_s=y'}(t)d_{\tilde{Y}_s,\mathcal{S}_{\tilde{\lambda}}\to\tilde{Y}_s}(y')dy'}{\int_0^y d_{\tilde{Y}_s,\mathcal{S}_{\tilde{\lambda}}\to\tilde{Y}_s}(y')dy'},$$

where $\tilde{\lambda}$ is the composition λ with the run between x_s^- and x_s^+ reduced to $r-1$, and $\tilde{X}_s$ and $\tilde{Y}_s$ correspond to the variables x_s^- and x_s^+ in $\mathcal{S}_{\tilde{\lambda}}$. By the induction hypothesis, $F_{\tilde{X}_s|\tilde{Y}_s=y'}(t)$ is decreasing in y', and thus, $\frac{\int_0^y F_{\tilde{X}_s|\tilde{Y}_s=y'}(t)d_{\tilde{Y}_s,\mathcal{S}_{\tilde{\lambda}}\to\tilde{Y}_s}(y')dy'}{\int_0^y d_{\tilde{Y}_s,\mathcal{S}_{\tilde{\lambda}}\to\tilde{Y}_s}(y')dy'}$ is decreasing in y.

In Proposition 4.2, the convexity of the Sawtooth model is essential to get the monotonicity of the conditional law. Suppose for example that $f_1(x) = (2x^2+1)e^{x^2}$. In this case, for $t \le y$,

$$F_{X_1|Y_1=y}(t) = \frac{\int_0^{t\wedge y} f_1(y-x)dx}{\int_0^y f_1(y-x)dx} = \frac{ye^{y^2} - (y-t)e^{(y-t)^2}}{ye^{y^2}}.$$

Thus,

$$\frac{\partial}{\partial y} F_{X_1|Y_1=y}(t) = te^{t^2-2ty}\left(\frac{2y(y-t)-1}{y^2}\right),$$

which is positive for $y > \frac{1}{\sqrt{2}}$ and $0 < t < \frac{2y^2-1}{2y}$. Therefore, $(X_1|Y_1=y)$ is not increasing in y for $y \ge \frac{1}{\sqrt{2}}$.

4.2 Alternating pattern of a convex sawtooth model

Proposition 4.2 yields two main features for the model. The first one is an extension of the previous result.

Proposition 4.4. *Let $1 \le s$, $0 \le t \le 1$. Then for $r \ge s$, $F_{X_s|X_r=x}(t)$ is decreasing in x and $F_{X_s|Y_r=y}(t)$ is decreasing in y. Likewise, $F_{X_s|X_r=0}(t)$ is decreasing in r and $F_{X_s|Y_r=1}(t)$ is increasing in r.*
Moreover,

$$F_{X_s,\mathcal{S}_{\to X_r}}(t) \le F_{X_s|Y_r=y}(t)$$

and

$$F_{X_s,\mathcal{S}_{\to Y_r}}(t) \ge F_{X_s|X_{r+1}=x}(t).$$

Proof. Let $s \ge 1$ and let us prove the monotonicty on x and y by induction on r, starting at $s = r$. $F_{X_s|X_s=x}(t)$ is clearly decreasing in x and from Proposition 4.2, $F_{X_s|Y_s=y}(t)$ is decreasing in y. Thus, the initialization is done.
Suppose the result proved until X_r. Then

$$F_{X_s|X_{r+1}=x}(t) = \int_0^1 F_{X_s|Y_r=y,X_{r+1}=x}(t)d_{Y_r|X_{r+1}=x}(y)dy,$$

and by an integration by part, since from Lemma 3.3 $F_{X_s|Y_r=y,X_{r+1}=x}(t) = F_{X_s|Y_r=y}(t)$,

$$F_{X_s|X_{r+1}=x}(t) = F_{X_s|Y_r=1}(t) - \int_0^1 \frac{\partial}{\partial y} F_{X_s|Y_r=y}(t) F_{Y_r|X_{r+1}=x}(y)dy.$$

Thus,

$$\frac{\partial}{\partial x} F_{X_s|X_{r+1}=x}(t) = -\int_0^1 \frac{\partial}{\partial y} F_{X_s|Y_r=y}(t) \frac{\partial}{\partial x} F_{Y_r|X_{r+1}=x}(y)dy.$$

By induction, $\frac{\partial}{\partial y} F_{X_s|Y_r=y}(t)$ is negative and by Proposition 4.2 $\frac{\partial}{\partial x} F_{Y_r|X_{r+1}=x}(y)$ is negative, thus $\frac{\partial}{\partial x} F_{X_s|X_{r+1}=x}(t)$ is also negative. It is exactly the same for $F_{X_s|Y_{r+1}=y}(t)$.
Let $r \ge s$. $F_{X_s|X_{r+1}=0}(t) = \int_0^1 F_{X_s|X_r=x}(t)d_{X_r|X_{r+1}=0}(x)dx$, thus by Proposition 4.2

$$F_{X_s|X_{r+1}=0}(t) \le \int_0^1 F_{X_s|X_r=0}(t)d_{X_r|X_{r+1}=0}(x)dx \le F_{X_s|X_r=0}(t).$$

The same proof holds to show that $F_{X_s|Y_r=1}(t)$ is increasing in r.
Let us prove the second part of the proposition and let $y \in [0,1]$. Conditioning X_s on X_r in $\mathcal{S}_{\to X_r}$ yields

$$F_{X_s,\mathcal{S}_{\to X_r}}(t) = \mathbb{E}\left(F_{X_s|X_r=\tilde{X}_r}(t)\right),$$

with $\tilde{X}_r$ following the law of q_r in $\mathcal{S}_{\to X_r}$.
On one hand from the first part of the proposition, $F_{X_s|X_r=x}(t)$ is decreasing in x. On the other hand from Proposition 4.2, $\tilde{X}_r$ stochastically dominates $(X_r|Y_r = y)$. Thus, from Proposition 2.6,

$$F_{X_s,\mathcal{S}_{\to X_r}}(t) = \mathbb{E}\left(F_{X_s|X_r=\tilde{X}_r}(t)\right) \leq F_{X_s|Y_r=y}(t).$$

The same pattern proves the second inequality. $\square$

There is an immediate consequence of this Proposition on the behavior of $F_{X_s,\mathcal{S}_{\to X_u}}(t)$ with $u \geq s$.

Corollary 4.5. *The following inequalities hold for $k \geq s$:*

$$F_{X_s,\mathcal{S}_{\to X_s}}(t) \leq \cdots \leq F_{X_s,\mathcal{S}_{\to X_u}}(t) \leq \cdots \leq F_{X_s,\mathcal{S}_{\to Y_u}}(t) \cdots \leq F_{X_s,\mathcal{S}_{\to Y_s}}(t).$$

Proof. The previous Proposition yields directly the following inequalities :

$$F_{X_s,\mathcal{S}_{\to Y_r}}(t) \geq F_{X_s|Y_r=1}(t) \geq F_{X_s,\mathcal{S}_{\to X_r}}(t).$$

Moreover,

$$\begin{aligned} F_{X_s,\mathcal{S}_{\to X_{u+1}}}(t) &= \int_{[0,1]} F_{X_s|Y_u=y}(t) d_{Y_n,\mathcal{S}_{\to X_{u+1}}}(y)dy \\ &\geq \int_{[0,1]} F_{X_s,\mathcal{S}_{\to X_u}}(t) d_{Y_u,\mathcal{S}_{\to X_{u+1}}}(y)dy \\ &\geq F_{X_s,\mathcal{S}_{\to X_u}}(t), \end{aligned}$$

the first inequality being due to Proposition 4.2. By symmetry between X_u and Y_u the general result holds. $\square$

4.3 Estimates on the behavior of extreme particles

As a second consequence of Proposition 4.2 we can get a more accurate estimate on the behavior of the first and last particles of $\mathcal{S}$. In particular, we can achieve a coupling of (X_I, X_F) with two couples of random variables, which only depend on f_1 and g_n and give some bounds on (X_I, X_F) in the sense of the stochastic domination.
In this paragraph we will not assume that the first and last particles are lower ones, and deal with model of any type (refer to Remark 3.1 for the definition of the type of a model). Moreover, to describe the bounding random variables, we introduce two particular transforms Γ^+ and Γ^-:

Definition 4.6. *Let f be a positive function on $[0, 1]$. Then $\Gamma^+(f)$ and $\Gamma^-(f)$ are the functions defined on $[0, 1]$ as :*

$$\Gamma^-(f)(t) = \frac{\int_{1-t}^1 f(u)du}{\int_0^1 f(u)du},$$

and

$$\Gamma^+(f)(t) = \frac{\int_0^t f(u)du}{\int_0^1 f(u)du}.$$

Remark that $\Gamma^-(f)(t)$ (resp. $\Gamma^+(f)(t)$) is the cumulative distribution function of the random variable $1 - Z$ (resp. Z), Z being the random variable with density $\frac{f(x)}{\int_0^1 f(x)dx}$.

Proposition 4.7. *Let $\mathcal{S}$ be a convex Sawtooth model of type ϵ with density functions $\{f_i, g_i\}_{1\leq i\leq k}$ and at least four particles. There exists a probability space and two couples of random variables $(X_+, Y_+), (X_-, Y_-)$ on it, such that :*

- $(X_-, Y_-) \preceq_\epsilon (X_I, X_F) \preceq_\epsilon (X_+, Y_+)$.
- X_+ *and* Y_+ *are independent with distribution function*

$$F_{X_+,Y_+}(s,t) = \Gamma^{\epsilon_1}(f_1)(s)\Gamma^{\epsilon_2}(g_n)(t).$$

- X_- *and* Y_- *are independent with distribution function*

$$F_{X_-,Y_-}(s,t) = \left(\Gamma^{\epsilon_1} \circ \Gamma^{\epsilon_1^*}(f_1)\right)(s)\left(\Gamma^{\epsilon_2} \circ \Gamma^{\epsilon_2^*}(g_n)\right)(t).$$

with $-^* = +$ *and* $+^* = -$.

Proof. We assume without loss of generality that each f_i, g_i is normalized and, since the type of the Sawtooth model doesn't change the pattern of the proof, we assume that $\mathcal{S}$ is of type $--$.
On one hand the conditional law of (X_I, X_F) given the value of $Y_1 = y_1, Y_k = y_k$ has for cumulative distribution function :

$$\begin{aligned} F_{X_I,X_F|Y_1=y_1,Y_k=y_k}(t_1,t_2) =& \frac{\left(\int_0^{t_1\wedge y_1} f_1(y_1-x)dx\right)\left(\int_0^{t_2\wedge y_k} g_k(y_k-y)dy\right)}{(\int_0^{y_1} f_1(x)dx)(\int_0^{y_k} g_k(x)dx)} \\ =& F_{X_I|Y_1=y_1}(t_1)F_{X_F|Y_k=y_k}(t_2). \end{aligned}$$

This together with Proposition 4.2 gives the bound

$$\begin{aligned} F_{X_I,X_F|Y_1=y_1,Y_k=y_k}(t_1,t_2) =& F_{X_I|Y_1=y_1}(t_1)F_{X_F|Y_k=y_k}(t_2) \\ \geq& F_{X_I|Y_1=1}(t_1)F_{X_F|Y_k=1}(t_2). \end{aligned}$$

Since

$$F_{X_I|Y_1=1}(t_1)F_{X_F|Y_k=1}(t_2) = (1-F_{f_1}(1-t_1))(1-F_{g_k}(1-t_2)) = \Gamma^-(f_1)(t_1)\Gamma^-(g_k)(t_2),$$

this gives the upper part of the stochastic bound.
On the other hand, the density of (Y_1, Y_k) conditioned on the value of (X_2, X_k) is

$$\begin{aligned} & d_{Y_1,Y_k|X_2=x_2,X_k=x_k}(y_1,y_k) \\ =& \mathbf{1}_{y_1\geq x_2, y_k\geq x_k} \frac{\left(\int_0^{y_1} f_1(y_1-x)dx\right)g_1(y_1-x_2)}{\int_{x_2}^1\left(\int_0^z f_1(z-x)dx\right)g_1(z-x_2)dz} \frac{\left(\int_0^{y_k} g_k(y_k-x)dx\right)f_k(y_k-x_k)}{\int_{x_k}^1\left(\int_0^z g_k(z-x)dx\right)f_k(z-x_k)dz} \\ =& \mathbf{1}_{y_1\geq x_2, y_k\geq x_k} \frac{F_{f_1}(y_1)g_1(y_1-x_2)}{\int_{x_2}^1 F_{f_1}(z)g_1(z-x_2)dz} \frac{F_{g_k}(y_k)f_k(y_k-x_k)}{\int_{x_k}^1 F_{g_k}(z)f_k(z-x_k)dz}. \end{aligned}$$

Factorizing the latter density yields

$$d_{Y_1,Y_k|X_2=x_2,X_k=x_k}(y_1,y_k) = d_{Y_1|X_2=x_2}(y_1)d_{Y_k|X_k=x_k}(y_k).$$

Let us first consider Y_1. Recall that g_1 is an increasing $\mathcal{C}^1$ function. This means in particular that

$$g_1(x) = \frac{1}{K}\int_0^x d\lambda(u),$$

with λ a probability measure on $[0,1]$ having eventually a dirac mass at 0 and then a continuous density function on $]0,1]$. Thus, the density of Y_1 conditioned on the value of X_2 is

$$d_{Y_1|X_2=x_2}(y_1) = \frac{1}{A}\mathbf{1}_{y_1\geq x_2}F_{f_1}(y_1)\int_{x_2}^{y_1} d\lambda(u-x_2),$$

with A a normalizing constant. Let d_u be the density function defined for $0 \le u \le 1$ by

$$d_u(y) = \frac{1}{A_u} \mathbf{1}_{y \ge u} F_{f_1}(y),$$

with A_u a normalizing constant depending on u and let $F_u(t)$ be the associated cumulative distribution function. On one hand

$$\begin{aligned} F_{Y_1|X_2=x_2}(t) =& \frac{\int_0^t \mathbf{1}_{y_1 \ge x_2} F_{f_1}(y_1) \int_{x_2}^{y_1} d\lambda(u-x_2) dy_1}{\int_0^1 \mathbf{1}_{y_1 \ge x_2} F_{f_1}(y_1) \int_{x_2}^{y_1} d\lambda(u-x_2) dy_1} \\ =& \frac{\int_0^t \int_{x_2}^1 \mathbf{1}_{y_1 \ge u} F_{f_1}(y_1) d\lambda(u-x_2) dy_1}{\int_0^1 \int_{x_2}^1 \mathbf{1}_{y_1 \ge u} F_{f_1}(y_1) d\lambda(u-x_2) dy_1}, \end{aligned}$$

and after changing the order of the integrals, since $F_u(1) = 1$,

$$\begin{aligned} F_{Y_1|X_2=x_2}(t) =& \frac{\int_{x_2}^1 \left(\int_0^t \mathbf{1}_{y_1 \ge u} F_{f_1}(y_1) dy_1 \right) d\lambda(u-x_2)}{\int_{x_2}^1 \left(\int_0^1 \mathbf{1}_{y_1 \ge u} F_{f_1}(y_1) dy_1 \right) d\lambda(u-x_2)} \\ =& \frac{\int_{x_2}^1 A_u F_u(t) d\lambda(u-x_2)}{\int_{x_2}^1 A_u d\lambda(u-x_2)} \\ =& \mathbb{E}_{\tilde{U}}(F_{\tilde{U}}(t)), \end{aligned}$$

with $\tilde{U}$ a random variable with law $d\tilde{U}(u) = \mathbf{1}_{u \ge x_2} \frac{A_u d\lambda(u-x_2)}{\int_{x_2}^1 A_u d\lambda(u-x_2)}$.
On the other hand

$$F_u(t) = \mathbf{1}_{t \ge u} \frac{\int_u^t F_{f_1}(u) du}{\int_u^1 F_{f_1}(u) du} = \mathbf{1}_{t \ge u} \frac{\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(u)}{\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)},$$

with $\mathcal{F}_{f_1}$ being the primitive of F_{f_1} taking the value 0 at 0. This yields

$$\begin{aligned} \frac{\partial}{\partial u} F_u(t) =& \frac{\partial}{\partial u} \left(\mathbf{1}_{u \le t} \frac{\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(u)}{\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)} \right) \\ =& \mathbf{1}_{u \le t} \frac{\partial}{\partial u} \left((\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(1)) \frac{1}{\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)} + 1 \right) \\ =& \mathbf{1}_{u \le t} (\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(1)) \frac{\partial}{\partial u} \left(\frac{1}{\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)} \right) \\ =& \mathbf{1}_{u \le t} (\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(1)) \frac{F_{f_1}(u)}{(\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u))^2} \le 0, \end{aligned}$$

and thus

$$F_u(t) \le F_0(t) = \frac{\mathcal{F}_{f_1}(t)}{\mathcal{F}_{f_1}(1)}.$$

Integrating with respect to $\tilde{U}$ yields

$$F_{Y_1|X_2=x_2}(t) = \mathbb{E}_{\tilde{U}} \left(F_{\tilde{U}}(t) \right) \le \mathbb{E}_{\tilde{U}} \left(F_0(t) \right),$$

and finally, $F_{Y_1|X_2=x_2}(t) \le \frac{\mathcal{F}_{f_1}(t)}{\mathcal{F}_{f_1}(1)}$. We can now integrate this inequality to get a bound on the cumulative distribution function of X_I conditioned on X_2 :

$$\begin{aligned}
F_{X_I|X_2=x_2}(t) =& \int_0^1 F_{X_I|Y_1=y}(t) d_{Y_1|X_2=x_2}(y) dy \\
=& F_{X_I|Y_1=1}(t) - \int_0^1 \frac{\partial}{\partial y} F_{X_I|Y_1=y}(t) F_{Y_1|X_2=x_2}(y) dy \\
\leq& F_{X_I|Y_1=1}(t) - \int_0^1 \frac{\partial}{\partial y} F_{X_I|Y_1=y}(t) \frac{\mathcal{F}_{f_1}(y)}{\mathcal{F}_{f_1(1)}} dy \\
\leq& \int_0^1 F_{X_I|Y_1=y}(t) \frac{F_{f_1}(y)}{\mathcal{F}_{f_1(1)}} dy.
\end{aligned}$$

Note that the direction of the inequality on the third line is due to the negative sign of $\frac{\partial}{\partial y} F_{X_I|Y_1=y}(t)$. Since

$$\begin{aligned}
\int_0^1 F_{X_I|Y_1=y}(t) \frac{F_{f_1}(y)}{\mathcal{F}_{f_1(1)}} dy =& \int_0^1 \frac{\int_0^{t\wedge y} f_1(y-u) du}{F_{f_1}(y)} \frac{F_{f_1}(y)}{\mathcal{F}_{f_1}(1)} dy \\
=& \int_0^t \int_u^1 \frac{f_1(y-u)}{\mathcal{F}_{f_1}(1)} dy du \\
=& \frac{\int_0^t F_{f_1}(1-u) du}{\mathcal{F}_{f_1}(1)} = \Gamma^-(F_{f_1})(t),
\end{aligned}$$

this yields the inequality

$$F_{X_I|X_2=x_2}(t) \leq \Gamma^- \circ \Gamma^+(f_1)(t).$$

Note that the latter inequality is valid even if the model has only three particles (see the next Corollary). Finally, since in our case there are at least four particles, $X_F \neq X_2$, and thus $F_{X_I|X_2=x_2, X_F=y}(t) = F_{X_1|X_2=x_2}(t)$. Therefore

$$F_{X_I|X_F=y}(t) \leq \Gamma^- \circ \Gamma^+(f_1)(t),$$

and by averaging on y,

$$F_{X_I}(t) \leq \Gamma^- \circ \Gamma^+(f_1)(t).$$

Doing the same with X_F gives the bound :

$$F_{X_F}(t) \leq \Gamma^- \circ \Gamma^+(g_k)(t).$$

The result follows from Lemma 2.10. □

Remark 4.8. The case of a Sawtooth model $\mathcal{S}_\lambda$ illustrates the pattern of the proof in the general case. Namely, suppose that λ has a first run of length r which is increasing. Then, conditioning the law of x_1 on the value of the first particle after the first peak (which is x_{r+1} in this case) yields the formula:

$$F_{x_1|x_{r+1}=z}(t) = \frac{\int_0^t (\int_{x\wedge z}^1 (y-x)^r dy) dx}{\int_0^1 (\int_{x\wedge z}^1 (y-x)^r dy) dx}.$$

Computing the integral in the numerator and in the denominator yields

$$F_{x_1|x_{r+1}=z}(t) = \frac{[1-(1-t)^r] - [z^r - ((t \vee z) - t)^r]}{1 - z^r}. \tag{4.1}$$

By Proposition 4.2, $F_{x_1|x_{r+1}=1}(t) \leq F_{x_1|x_{r+1}=z}(t) \leq F_{x_1|x_{r+1}=0}(t)$: therefore, the bounds are given by the cases $z=1$ and $z=0$. By Equation (4.1), $F_{x_1|x_{r+1}=0}(t) = 1-(1-t)^r = \Gamma^-\Gamma^+(\tilde{\gamma}_r)(t)$. Suppose that $z \geq t$: rewriting the right hand side of (4.1) as $\frac{h(1)-h(z^r)}{1-z^r}$ with $h(x) = x-(x^{1/r}-t)$ yields

$$F_{x_1|x_{r+1}=1}(t) = h'(1) = 1-(1-t)^{r-1} = \Gamma^-(\gamma_r)(t).$$

The proof of Proposition 4.7 is actually a generalization of the proof in the case $\mathcal{S}_\lambda$.

In particular, as a corollary of Proposition 4.7 (and as a corollary of the proof in the case $k=2$), the following result holds :

Corollary 4.9. *Let $\mathcal{S}$ be a convex Sawtooth model of type ϵ with density functions $\{f_i, g_i\}_{1\leq i\leq k}$. There exists a couple of random variables $(Z^{(1)}, Z^{(2)})$ such that for $y \in [0,1]$,*

- $Z^{(1)} \preceq_{\epsilon(1)} (X_I|X_F = y) \preceq_{\epsilon(1)} Z^{(2)}$,
- *The cumulative distribution function of $Z^{(2)}$ is :*

$$F_{Z^{(2)}}(t) = \Gamma^{\epsilon(1)}(f_1)(t).$$

- *The cumulative distribution function of $Z^{(1)}$ is*

$$F_{Z^{(1)}}(t) = \Gamma^{\epsilon(1)} \circ \Gamma^{\epsilon(1)^*}(f_1)(t).$$

Proof. For $k \geq 3$, the result is deduced from the latter Proposition. In the case $k=2$, the proof is exactly the same as in the latter Proposition, except that we only deal with the left case, and thus we don't need anymore the fact that $X_2 \neq X_F$. ☐

In the case of a composition λ with first run of length $R+1$, the latter corollary yields that for $\mathcal{S}_\lambda$:

$$1-(1-t)^R \leq F_{X_I}(t) \leq 1-(1-t)^{R+1},$$

if the first run is increasing, and

$$t^{R+1} \leq F_{X_I}(t) \leq t^R,$$

if the first run is decreasing.

5 The independence theorem in a bounded Sawtooth Model

This section is devoted to the proof of the approximate independence of X_I and X_F when the number of particles grows whereas the repulsion forces remain bounded. In this section the Sawtooth model is assumed normalized.

5.1 Decorrelation principle and bounding Lemmas

Definition 5.1. *Let $A > 0$. A Sawtooth model $\mathcal{S}$ with density functions $\{f_i, g_i\}$ is bounded by A if*

$$\max(\|f_i\|_{[0,1]}, \|g_i\|_{[0,1]}) \leq A.$$

The purpose is to prove the following Theorem :

Theorem 5.2. *Let $A > 0$. For all $\epsilon > 0$ there exists $N_A \geq 0$ such that for any Sawtooth model $\mathcal{S}$ bounded by A and with $2k \geq N_A$ particles we have :*

$$\|d_{X_I,X_F}(x,y) - d_{X_I}(x)d_{X_F}(y)\|_\infty \leq \epsilon$$

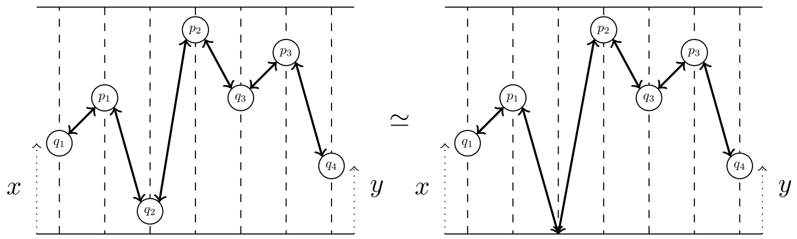

Figure 4: Decorrelation of the process

The pattern of the proof is the following : conditioned on the fact that a particle P - from now on called a splitting particle - is close to the boundary of the domain, the left part $\mathcal{S}_{\to P}$ and the right part $\mathcal{S}_{\leftarrow P}$ of the system are almost not correlated anymore (see Figure 4).
However, we may still not have independence if the law of X_I and X_F depends on which particle splits the system. Thus, we have to find a set of particles that is large enough, so that with probability close to one an element of this set is close to the boundary, and such that nonetheless conditioning on having any particle from this set close to the boundary yields the same law on (X_I, X_F).
Let us first begin by bounding the density of (X_I, X_F).

Lemma 5.3. *Suppose that $\|f_1\|_\infty \le A$ and let $\mathcal{S}$ be a Sawtooth model larger than 2. Then there exists K_A only depending on A such that for any event $\mathcal{X}$ depending on $\{X_i, Y_i\}_{i\ge 2}$,*

$$\|d_{X_I|\mathcal{X}}\|_\infty \le K_A.$$

More precisely $K_A = 4A^2$ fits.

This Lemma was already mentioned in the specific context of compositions in [2]. We provide here a different proof.

Proof. By Lemma 3.3, it suffices to prove it for a conditioning on $\{X_2 = x_2\}$. From Lemma 3.4, $d_{X_I|X_2=x_2}(x)$ is decreasing in x and thus it is enough to bound $d_{X_I|X_2=x_2}(0)$. We have

$$d_{X_I|X_2=x_2}(0) = \frac{\int_{x_2}^1 f_1(z)g_1(z-x_2)dz}{\int_{x_2}^1 F_{f_1}(z)g_1(z-x_2)dz} \le A\frac{\int_{x_2}^1 g_1(z-x_2)dz}{\int_{x_2}^1 F_{f_1}(z)g_1(z-x_2)dz}.$$

Remark that

$$\frac{\int_{x_2}^1 g_1(z-x_2)dz}{\int_{x_2}^1 F_{f_1}(z)g_1(z-x_2)dz} = \frac{1}{\mathbb{E}_{\tilde{Z}}(F_{f_1}(\tilde{Z}))},$$

with $\tilde{Z}$ being a random variable with density $\mathbf{1}_{z\ge x_2}g_1(z-x_2)$. Since $\|F'_{f_1}\| \le A$ and $F_{f_1}(1) = 1$, $F_{f_1}(t) \ge 1/2$ on $[1-1/(2A)]$; moreover, $z \mapsto g_1(z-x_2)$ is increasing, thus $\mathbb{P}(\tilde{Z} \in [1-1/(2A), 1]) \ge \frac{1}{2A}$ and by Markov's inequality $\mathbb{E}_{\tilde{Z}}(F_{f_1}(\tilde{Z})) \ge 1/4A$. Finally,

$$d_{X_I|X_2=x_2}(0) \le 4A^2.$$

□

The next step is to get a bound on the first derivative of d_{X_I}. This is possible only if g_1 is also bounded by A and the model is large enough.

Lemma 5.4. *Suppose that* $\max(\|f_1\|_\infty, \|g_1\|_\infty) \leq A$ *and that* $\mathcal{S}$ *is a Sawtooth model with at least four particles. Then there exists a constant* R_A *only depending on* A *such that for any event* $\mathcal{X}$ *depending on* $\{X_{i+1}, Y_i\}_{i\geq 2}$,

$$\|(d_{X_I|\mathcal{X}})'\|_\infty \leq R_A.$$

Proof. For exactly the same reasons as in the previous proof, it suffices to bound the derivative of the density conditioned on $\mathcal{X} = \{Y_2 = y_2\}$. The expression of the density probability yields

$$d_{X_I|Y_2=y_2}(x) = \frac{\int_x^1 f_1(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1}{\int_0^1 \left(\int_x^1 f_1(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1\right) dx}.$$

Let $\Delta = \int_0^1 \left(\int_x^1 f_1(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1\right) dx$, which is independent of x. Then

$$\begin{aligned}
|\frac{\partial}{\partial x} d_{X_I|Y_2=y_2}(x)| =& \frac{1}{\Delta} |\frac{\partial}{\partial x} \int_x^1 f_1(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1| \\
=& \frac{1}{\Delta} |\int_x^1 (\frac{\partial}{\partial x} f_1(y_1 - x)) d_{Y_1|Y_2=y_2}(y_1) dy_1 - f_1(0) d_{Y_1, \mathcal{S}_{Y_2}\leftarrow|Y_2=y_2}(x)| \\
\leq& \frac{1}{\Delta} \left(|\int_x^1 -(\frac{\partial}{\partial x} f_1)(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1| + |f_1(0) d_{Y_1|Y_2=y_2}(x)| \right),
\end{aligned}$$

Let us first bound the numerator. By the expression of the density of Y_1 conditioned on $Y_2 = y_2$,

$$d_{Y_1|Y_2=y_2}(y_1) = \frac{F_{f_1}(y_1) d_{Y_1, \mathcal{S}_{Y_1}\leftarrow|Y_2=y_2}(y_1)}{\mathbb{E}_{\tilde{Y}_1}(F_{f_1}(\tilde{Y}_1))},$$

with $\tilde{Y}_1$ having the density $d_{Y_1, \mathcal{S}_{Y_1}\leftarrow|Y_2=y_2}$. Since g_1 is bounded by A, from Lemma 5.3, $|d_{Y_1, \mathcal{S}_{Y_1}\leftarrow|Y_2=y_2}| \leq K_A$. From Lemma 3.4, $d_{Y_1, \mathcal{S}_{Y_1}\leftarrow|Y_2=y_2}(y)$ is increasing in y, and $|F'_{f_1}| \leq A$, thus $\mathbb{E}_{\tilde{Y}_1}(F_{f_1}(\tilde{Y}_1)) \geq \frac{1}{4A^2}$ and

$$|f_1(0) d_{Y_1, \mathcal{S}_{Y_2}\leftarrow|Y_2=y_2}(x)| \leq 4A^2 K_A^2.$$

Let us bound also the first term of the sum: f_1 being increasing, $\frac{\partial}{\partial x} f_1(y_1 - x) \leq 0$ and we can thus remove the absolute value in this first term. An other application of Lemma 5.3 yields:

$$\begin{aligned}
\int_x^1 -(\frac{\partial}{\partial x} f_1)(y_2 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1 &\leq K_A(\int_x^1 (\frac{\partial}{\partial x} f_1)(y_2 - x) dy_2) \\
&\leq K_A((f_1(1-x) - f_1(0)) \leq A \times K_A.
\end{aligned}$$

The numerator is thus bounded by $AK_A + 4A^2 K_A^2$.
Changing the order of the integrals in Δ yields :

$$\Delta = \int_0^1 F_{f_1}(y_1) d_{Y_1|Y_2=y_2}(y_1) dy_1.$$

Since F'_{f_1} is bounded by A and $F_{f_1}(1) = 1$, we can conclude as in the previous proof that $F_{f_1}(t) \geq \frac{1}{2A}$ on $[1 - 1/(2A), 1]$. Moreover, Y_1 is an upper particle, and thus by Lemma 3.4, $d_{Y_1|Y_2=y_2}(y_1)$ is increasing in y_2. Since $\int_{[0,1]} d_{Y_1|Y_2=y_2} = 1$, this implies that

$$\int_{1-1/(2A)}^1 d_{Y_1|Y_2=y_2}(y_1) dy_1 \geq \frac{1}{2A},$$

and yields $\Delta \geq \frac{1}{4A^2}$. The bounds on the numerator and on Δ yield :

$$|\frac{\partial}{\partial x} d_{X_I|Y_2=y_2}(x)| \leq 4A^3(K_A + 4AK_A^2).$$

$\square$

As an application of Lemma 5.4, we can also prove that $y \mapsto F_{X_I|X_F=y}(t)$ is Lipchitz :

Proposition 5.5. *Let $\mathcal{S}$ be a Sawtooth model with $k \geq 3$ lower particles. Suppose that $\{f_1, g_1, f_k, g_k\}$ are bounded by $A > 0$. Let R_A be the constant of Lemma 5.4 (with $R_A \geq 1$). Then on a neighbourhood $[0, 1/R_A]$ of 0,*

$$\mathcal{F} : \begin{cases} [0, 1/R_A] & \to \quad (\mathcal{C}([0,1], \mathbb{R}), \|.\|) \\ \quad y & \mapsto \qquad F_{X_I|X_F=y} \end{cases}$$

is Lipschitz with a Lipschitz constant B_A only depending on A.

Proof. It suffices to prove that for $x \in [0,1]$, $y \mapsto d_{X_I|X_F=y}(x)$ is Lipschitz on $[0, 1/R_A]$ with a Lipschitz constant independent of x.
From Lemma 3.4, d_{X_F} is decreasing and thus on $[1/R_A, 1]$, $d_{X_F} \leq d_{X_F}(1/R_A)$. From Lemma 5.4, $|\frac{\partial}{\partial y} d_{X_F}(y)| \leq R_A$ and thus on $[0, 1/R_A]$, $d_{X_F}(y) \leq d_{X_F}(1/R_A) + R_A(1/R_A - y)$. This implies that

$$\begin{aligned}\int_{[0,1]} d_{X_F}(y)dy \leq &\int_0^{1/R_A} d_{X_F}(1/R_A) + R_A(1/R_A - y)dy + \int_{1/R_A}^1 d_{X_F}(1/R_A) \\ \leq &d_{X_F}(1/R_A) + \frac{1}{2R_A}.\end{aligned}$$

Since $\int_{[0,1]} d_{X_F} = 1$, this implies that $d_{X_F}(1/R_A) \geq 1 - \frac{1}{2R_A}$, and thus that $d_{X_F} \geq 1 - \frac{1}{2R_A}$ on $[0, 1/R_A]$.
From Lemma 5.4, $\|\frac{\partial}{\partial y} d_{X_F|X_I=x}\| \leq R_A$. Thus, since $\|f_1\| \leq A$, this yields by applying Lemma 5.3 on $d_{X_I,X_F}(x,y) = d_{X_F|X_I=x}(y) d_{X_I}(x)$:

$$|\frac{\partial}{\partial y} d_{X_I,X_F}(x,y)| \leq K_A R_A.$$

Thus, on $[0, 1/R_A]$,

$$\begin{aligned}|\frac{\partial}{\partial y} d_{X_I|X_F=y}(x)| = &\frac{1}{d_{X_F}(y)} |\frac{\partial}{\partial y} d_{X_I,X_F}(x,y) - \frac{d_{X_I,X_F}(x,y)\frac{\partial}{\partial y} d_{X_F}(y)}{d_{X_F}(y)}| \\ \leq &\frac{1}{1 - 1/(2R_A)}(K_A R_A + \frac{R_A K_A^2}{1 - 1/(2R_A)}).\end{aligned}$$

Set $B_A = \frac{1}{1-1/(2R_A)}(K_A R_A + \frac{R_A K_A^2}{1-1/(2R_A)})$. Then $\mathcal{F}$ is B_A-Lipschitz on $[0, 1/R_A]$. $\square$

5.2 Behavior of $\{X_i\}$ for large models

The purpose of this subsection is to find for a model $\mathcal{S}$ a large set of intermediate particles $\{X_r\}$ for which one of these particles is close to 0 with high probability and such that $F_{X_I|X_r=0}$ is essentially the same for all particles of this set.
The first part is a essentially probability computation :

Proposition 5.6. *Let $\eta > 0, \epsilon > 0$. There exists N_0 such that for any model $\mathcal{S}$ of size N larger than $N_0 + 4$ and for any $2 \leq r \leq N - N_0$, $y_{r+N_0} \in [0,1]$,*

$$\mathbb{P}(\bigcup_{r \leq i \leq r+N_0} \{X_i < \eta\} | Y_{r+N_0} = y_{r+N_0}) \geq 1 - \epsilon.$$

Proof. Let N_0 be an integer to specify later and let $\mathcal{S}, r$ be as in the statement of the Proposition. Let $\tilde{P} = \mathbb{P}(\bigcap_{r \leq i \leq r+N_0} \{X_i \geq \eta\} | Y_{r+N_0} = y_{r+N_0})$.
Let $0 \leq y_{r-1}, \dots, y_{r+N_0} \leq 1$ and condition $(\bigcap_{r \leq i \leq r+N_0} \{X_i \geq \eta\} | Y_{r+N_0} = y_{r+N_0})$ on the event $\bigcap_{r-1 \leq i \leq r+N_0-1} \{Y_i = y_i\}$. We denote by $P_{\vec{y}}$ the probability of this conditioned event. By Lemma 3.3, the random variables $\{X_i\}_{r \leq i \leq r+N_0}$ are conditionally independent given the value of $\{Y_i\}_{r-1 \leq i \leq N_0}$; therefore,

$$P_{\vec{y}} = \prod_{i=r}^{r+N_0} \mathbb{P}(X_i \geq \eta | Y_{i-1} = y_{i-1}, Y_i = y_i).$$

Moreover, Lemma 3.4 yields that $d_{X_i | Y_{i-1} = y_{i-1}, Y_i = y_i}$ is decreasing: thus, $\mathbb{P}(X_i \geq \eta | Y_{i-1} = y_{i-1}, Y_i = y_i) \leq (1 - \eta)$. This yields

$$P_{\vec{y}} \leq (1 - \eta)^{N_0+1}.$$

Integrating $P_{\vec{y}}$ with respect to $y_{r-1}, \dots, y_{N_0-1}$ gives $\tilde{P} \leq (1-\eta)^{N_0+1}$. Let N_0 be such that $(1-\eta)^{N_0+1} \leq \epsilon$. For $N \geq N_0$,

$$\mathbb{P}(\bigcup_{r \leq i \leq r+N_0} \{X_i < \eta\} | Y_{r+N_0} = y_{r+N_0}) \geq 1 - \epsilon.$$

□

As said before, it is also necessary that $F_{X_I | X_r = 0}$ remains almost constant among this subset of particles. This is possible for large Sawtooth models, thank to the monotonicity results of Proposition 4.4 :

Proposition 5.7. *Let $A, \epsilon > 0$, $M \in \mathbb{N}^*$. There exists $N_{\epsilon,A,M}$ such that for any Sawtooth model bounded by A and of size $N \geq N_{\epsilon,A,M}$, there exists $1 \leq r \leq N - M$ such that for $r \leq i, j \leq r + M$,*

$$\|F_{X_I | X_i = 0} - F_{X_I | X_j = 0}\|_\infty \leq \epsilon.$$

Proof. Let $\mathcal{S}$ be a Sawtooth model bounded by A and of size N.
Denote by F_i the function $t \mapsto F_{X_I | X_i = 0}(t)$ for $2 \leq i \leq N$. By Lemma 5.3, all the F_i are K_A-Lipschitz. Let $K = \lfloor \frac{2K_A}{\epsilon} \rfloor$. It suffices to find $r \geq 2$ such that for all $r \leq i, j \leq r + M$, and all $0 \leq k \leq K$,

$$|F_i(\frac{k}{K}) - F_j(\frac{k}{K})| \leq \frac{\epsilon}{3}.$$

Denote by $v_i \in [0,1]^{K+1}$ the vector $(F_i(\frac{k}{K}))_{0 \leq k \leq K}$ and let $N_{\epsilon,A,M} = (M+1)(\lfloor \frac{3}{\epsilon} \rfloor + 1)^{K+1}$. Suppose that $N \geq N_{\epsilon,A,M}$. For $\vec{m} \in [\![0, \lfloor \frac{3}{\epsilon} \rfloor]\!]^{K+1}$, denote by $C_{\vec{m}}$ the hypercube $\{\vec{x} \in [0,1]^{K+1} | \forall 1 \leq i \leq K+1, m_i \frac{\epsilon}{3} \leq x_i < (m_i + 1)\frac{\epsilon}{3}\}$. $\{C_{\vec{m}}\}_{\vec{m} \in [\![0, \lfloor \frac{3}{\epsilon} \rfloor]\!]^{K+1}}$ is a partition of $[0,1]^{K+1}$ in $(\lfloor \frac{3}{\epsilon} \rfloor + 1)^{K+1}$ subsets. If v_i and v_j are both in a same $C_{\vec{m}}$, then for all $0 \leq k \leq K$, $|v_i(k) - v_j(k)| \leq \frac{\epsilon}{3}$.
Since $N \geq (M+1)(\lfloor \frac{3}{\epsilon} \rfloor + 1)^{K+1}$, Dirichlet's principle yields the existence of $\vec{m}_0 \in [\![0, \lfloor \frac{3}{\epsilon} \rfloor]\!]^{K+1}$ such that $\#(\{v_i\}_{1 \leq i \leq N} \cap C_{\vec{m}_0}) \geq M + 1$. Let $i_0 < \dots < i_M$ be such that for all $0 \leq j \leq M$, $v_{i_j} \in C_{\vec{m}_0}$; in particular, $i_M \geq i_0 + M$. From the previous paragraph, for all $0 \leq k \leq K$, $|v_{i_M}(k) - v_{i_0}(k)| \leq \frac{\epsilon}{3}$. By Proposition 4.4, $F_i(\frac{k}{K})$ is decreasing in i; thus, since $v_i(k) = F_i(\frac{k}{K})$, for all $i_0 \leq j \leq i_M$ and all $0 \leq k \leq K$

$$v_{i_0}(k) \geq v_i(k) \geq v_{i_M}(k).$$

Since $i_0 + M \leq i_M$, this yields $\|v_i - v_j\|_\infty \leq \frac{\epsilon}{3}$ for $i_0 \leq i, j \leq i_0 + M$. □

5.3 Proof of Theorem 5.2

Theorem 5.2 is a consequence of the following proposition :

Proposition 5.8. *Let $A > 0$. For all $\epsilon > 0$, there exists a number $N_{A,\epsilon} \geq 0$ such that for any Sawtooth model $\mathcal{S}$ bounded by A and with $2k \geq N_{A,\epsilon}$ particles, the following inequality holds:*

$$|F_{X_I|X_F=y}(t) - F_{X_I}(t)| \leq \epsilon.$$

for all $t, y \in [0,1]$.

Proof. Set $\eta = \inf(\frac{1}{R_A}, \frac{\epsilon}{B_A})$ with R_A, B_A the constants given respectively by Lemma 5.4 and Proposition 5.5. Let N_0 be the constant given for η and ϵ by Proposition 5.6. Finally, set $N_{A,\epsilon} = N_{\epsilon/4,A,N_0} + 4$ given by Proposition 5.7.
Let $\mathcal{S}$ be a Sawtooth model bounded by A of size larger than $N_{A,\epsilon}$. Then by Proposition 5.7, there exists $2 \leq r \leq N_{A,\epsilon} - 2 - N_0$ such that for all $r \leq i, j \leq r + N_0$,

$$\|F_{X_I|X_i=0} - F_{X_I|X_j=0}\|_\infty \leq \epsilon.$$

Denote $t = r + N_0$ and let $y_t \in [0,1]$. For $r \leq i \leq r + N_0$, set $L_i = \{X_i \leq \eta \cap \{\forall s > i, X_s > \eta\}\}$. Note that $L_i \cap L_j = \emptyset$ for all $i \neq j$ and $\bigcup L_i = L$ with $L = \bigcup_{r \leq i \leq r+N_0} \{X_i \leq \eta\}$. Moreover, since L_i is $(X_s, Y_s)_{s \geq i}$−measurable, by Lemma 3.3, conditioning X_I on $\{X_i = u, Y_t = y_t\} \cap L_i$ is the same as conditioning X_I on $\{X_i = u\}$. Thus,

$$\begin{aligned}\|F_{X_I|L_i,Y_t=y_t} - F_{X_I|X_r=0}\|_\infty =& \|\int_0^\eta (F_{X_I|X_i=u} - F_{X_I|X_r=0}) d_{X_i|L_i,Y_t=y_t}(u)du\|_\infty \\ \leq& \int_0^\eta \|F_{X_I|X_i=u} - F_{X_I|X_r=0}\|_\infty d_{X_i|L_i,Y_t=y_t}(u)du \\ \leq& 2\epsilon,\end{aligned}$$

by the choice of η. Recall that if $A = \bigcup A_i$, with A_i disjoint events, then for any event C,

$$\mathbb{P}(C|A) = \sum \mathbb{P}(C|A_i)\mathbb{P}(A_i|A)$$

In particular, for $L = \bigcup_i L_i$ this yields

$$\begin{aligned}\|F_{X_I|L,Y_t=y_t} - F_{X_I|X_r=0}\| =& \|\sum_i (F_{X_I|L_i,Y_t=y_t} - F_{X_I|X_r=0})\mathbb{P}(L_i|L, Y_t = y_t)\|_\infty \\ \leq& \sum_i \|(F_{X_I|L_i,Y_t=y_t} - F_{X_I|X_r=0}\|_\infty \mathbb{P}(L_i|L, Y_t = y_t) \\ \leq& 2\epsilon.\end{aligned}$$

By Proposition 5.6 and the choice of N_0, $\mathbb{P}(L|Y_t = y_t) \geq 1 - \epsilon$, and thus

$$\|F_{X_I|Y_t=y_t} - F_{X_I|X_r=0}\|_\infty \leq 3\epsilon.$$

By averaging on y_t with the density $d_{Y_t|X_F=y}$ we get

$$\|F_{X_I|X_F=y} - F_{X_I}\|_\infty \leq 4\epsilon.$$

□

Let us end the proof of the Theorem 5.2, which consists essentially in a rewriting in terms of densities of the latter Proposition.

Proof. Let $A > 0, \epsilon > 0$. Set $\epsilon_1 = \frac{(\epsilon/K_{A)}^2}{4R_A}$ and let $\mathcal{S}$ be a Sawtooth model bounded by A of size larger than N_{A,ϵ_1} (N_{A,ϵ_1} being given by Proposition 5.8). Then from Proposition 5.8, for $y \in [0,1]$,

$$\|F_{X_I|X_F=y} - F_{X_I}\|_\infty \leq \frac{(\epsilon/K_A)^2}{4R_A}. \tag{5.1}$$

Moreover, the following result holds for C^1-functions on $[0,1]$:

Lemma 5.9. *Let $f, g : [0,1] \to [0,1]$ be two C^1- functions, such that $\|f'\|_\infty, \|g'\|_\infty \leq M$. Then for $\epsilon > 0$ small enough, if F, G are two primitives of f, g and*

$$\|F - G\|_\infty \leq \frac{\epsilon^2}{4M},$$

then $\|f - g\|_\infty \leq \epsilon$.

Proof. This is implied by proving that if $f : [0,1] \longrightarrow \mathbb{R}$ verifies $\|f\|_\infty \leq \frac{\epsilon^2}{4M}$ and $\|f''\|_\infty \leq M$, then $\|f'\|_\infty \leq \epsilon$. But the majoration on f'' yields that if $|f'(x)| \geq \epsilon$,

$$\max(|\int_x^{x+\epsilon/M} f'(x)dx|, |\int_{x-\epsilon/M}^x f'(x)dx|) \geq \frac{\epsilon^2}{2M}.$$

Thus,

$$\max(|f(x+\epsilon/M)|, |f(x)|, |f(x-\epsilon/m)|) \geq \frac{\epsilon^2}{4M}.$$

□

Applying this Lemma to (5.1) yields for $y \in [0,1]$,

$$\|d_{X_I|X_F=y} - d_{X_i}\|_\infty \leq \epsilon/K_A.$$

Finally,

$$|d_{X_I,X_F}(x,y) - d_{X_I}(x)d_{X_F}(y)| = |d_{X_F}(y)||d_{X_I|X_F=y}(x) - d_{X_I}(x))| \leq K_A \frac{\epsilon}{K_A} \leq \epsilon.$$

□

6 Application to compositions

Theorem 5.2 can be applied to the framework of compositions :

Corollary 6.1. *Let $A \geq 0, \epsilon > 0$. There exists $n \geq 0$ such that for any composition λ of size larger than n with every runs bounded by A,*

$$\|d_{\mathcal{S}_\lambda}(x,y) - d_{\mathcal{S}_\lambda}(x)d_{\mathcal{S}_\lambda}(y)\| < \epsilon$$

Proof. Each run of λ of length l yields a density function γ_l in $\mathcal{S}_\lambda$, and $\|\gamma_l\|_\infty = l-1$. Thus, if any run of λ is bounded by A, then all the density functions $\{f_i, g_i\}$ in $\mathcal{S}_\lambda$ are bounded by $A-1$. It suffices then to apply Theorem 5.2. □

The purpose of this section is to strengthen Corollary 6.1 and to prove the following Theorem :

Theorem 6.2. *Let $\epsilon > 0$, $A \geq 0$. There exists $n \geq 0$ such that for any composition λ of size larger than n with first and last run bounded by A,*

$$\|d_{\mathcal{S}_\lambda}(x,y) - d_{\mathcal{S}_\lambda}(x)d_{\mathcal{S}_\lambda}(y)\| < \epsilon. \tag{6.1}$$

This Theorem was Conjecture 1 in [2]. The proof of Theorem 6.2 is followed by some applications.

6.1 Effect of a large run on the law of (X_I, X_F)

From Corollary 6.1, it is enough to prove that the presence of a large run inside the composition disconnects the behaviors of X_I and X_F. The main reason for this is the Lemma below: for each composition λ, denote by λ^+ the composition λ with a cell added on the last run, and by λ^- the composition λ with a cell removed on the last run.

Lemma 6.3. *Let $A > 0$ and let λ be a composition with more than three runs and with the first run smaller than A. If the last run of λ is of size R,*

$$\|d_{X_I,\mathcal{S}_\lambda} - d_{X_I,\mathcal{S}_{\lambda^+}}\|_\infty \leq \frac{K_A}{R-1},$$

where K_A is the bound on the density of X_I as defined in Lemma 5.3.

Proof. Let us prove it in the case where the first run of λ is increasing and the last run decreasing, the other cases having the same proof. The expression (3.3) yields

$$d_{(X_I,X_F),\mathcal{S}_{\lambda^+}}(x,y) = \frac{\int_y^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)dz}{\int_{[0,1]^2}\left(\int_y^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)\right)dxdy}.$$

Thus, by integrating with respect to y and then changing the order of the integrals, this yields

$$\begin{aligned} d_{X_I,\mathcal{S}_{\lambda^+}}(x) =& \frac{\int_0^1\left(\int_0^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)\mathbf{1}_{y\leq z}dy\right)dz}{\int_{[0,1]^2}\left(\int_0^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)\mathbf{1}_{y\leq z}dy\right)dxdz} \\ =& \frac{\int_0^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)zdz}{\int_{[0,1]^2} d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)zdzdx}. \end{aligned}$$

Factorizing by $d_{X_I,\mathcal{S}_\lambda}(x)$ makes a conditional expectation appear and thus

$$d_{X_I,\mathcal{S}_{\lambda^+}}(x) = d_{X_I,\mathcal{S}_\lambda}(x)\frac{\mathbb{E}_{\mathcal{S}_\lambda}(X_F|X_I=x)}{\mathbb{E}_{\mathcal{S}_\lambda}(X_F)}.$$

Moreover, Proposition 4.7 yields

$$F_{Z_1} \leq F_{X_F|X_I=x} \leq F_{Z_2},$$

with $F_{Z_1} = \Gamma^-(F_{\gamma_R})$ and $F_{Z_2} = \Gamma^-(\gamma_R)$. Since $\Gamma^-(F_{\gamma_R})(t) = 1-(1-t)^R$ and $\Gamma^-(\gamma_R)(t) = 1-(1-t)^{R-1}$, by stochastic dominance, applying Proposition 2.6 gives

$$\frac{1}{R} \leq \mathbb{E}_{\mathcal{S}_\lambda}(X_F|X_I=x) \leq \frac{1}{R-1}.$$

Integrating the latter result on x yields $\frac{1}{R} \leq \mathbb{E}_{\mathcal{S}_\lambda}(X_F) \leq \frac{1}{R-1}$, and thus

$$\frac{R-1}{R} \leq \frac{\mathbb{E}_{\mathcal{S}_\lambda}(X_F|X_I=x)}{\mathbb{E}_{\mathcal{S}_\lambda}(X_F)} \leq \frac{R}{R-1}.$$

This yields

$$|d_{X_I,\mathcal{S}_{\lambda^+}}(x) - d_{X_I,\mathcal{S}_\lambda}(x)| \leq |d_{\mathcal{S}_\lambda}(x)|\frac{1}{R-1} \leq \frac{K_A}{R-1}.$$

□

In particular, the previous Lemma can be used to bound the conditional law of the first particle with respect to the last one. For each composition λ, and any cells $i, j \in \lambda$, denote by $\lambda_{\to i}$ (resp $\lambda_{i\to}$, resp $\lambda_{i\to j}$) the composition consisting of the cells of λ from 1 to i (resp. from i to n, resp. from i to j). Moreover, denote by $R_{int}(\lambda)$ the set of all runs of λ except the first and last ones.

Proposition 6.4. *Let $A \geq 0$ and λ a composition with first run bounded by A. Then*

$$\|F_{X_I|X_F=x} - F_{X_I}\|_\infty \leq \frac{K_A}{\max_{s\in R_{int}(\lambda)} l(s) - 2}.$$

Proof. Let $t \in [0,1]$. Let s_0 be the run with maximal length R in R_{int} and let i_0 be the rightest cell of this run. This cell corresponds to a particle X_i or Y_i in $\mathcal{S}_\lambda$. Let us assume without loss of generality that this particle is a lower one. From Proposition 4.2, $F_{X_1|X_r=x}(t)$ is decreasing in x and thus

$$\begin{aligned}
|F_{X_I|X_F=x}(t) - F_{X_I}(t)| =& |F_{X_I|X_F=x}(t) - \int_{X_F} F_{X_I|X_F=x}(t) d_{X_F}(x)dx| \\
\leq& |F_{X_I|X_F=0}(t) - F_{X_I|X_F=1}(t)| \\
\leq& F_{X_I|X_F=0}(t) - F_{X_I|Y_k=1}(t).
\end{aligned}$$

Moreover, from Proposition 4.2 and Proposition 4.4,

$$F_{X_I|X_F=0}(t) \leq F_{X_I,\mathcal{S}_\lambda\to Y_k}(t) \leq F_{X_I,\mathcal{S}_\lambda\to Y_i}(t) \leq F_{X_I|X_i=0},$$

and

$$F_{X_I|Y_k=1}(t) \geq F_{X_I,\mathcal{S}_\lambda\to X_k}(t) \geq F_{X_I,\mathcal{S}_\lambda\to X_i}(t).$$

These inequalities imply

$$|F_{X_I|X_F=x}(t) - F_{X_I}(t)| \leq F_{X_I|X_i=0}(t) - F_{X_I,\mathcal{S}_\lambda\to X_i}(t).$$

From the expression (3.3), $F_{X_I,\mathcal{S}_\lambda\to X_i}(t) = F_{X_I,\mathcal{S}_{\lambda_{\to i_0}}}(t)$ and $F_{X_I|X_i=0}(t) = F_{X_1,\mathcal{S}_{\lambda^-_{\to i_0}}}(t)$. Thus, with the previous Lemma, since the last run of $\lambda^-_{\to i_0}$ is of size $R-1$,

$$|F_{X_I|X_F=x}(t) - F_{X_I}(t)| \leq |F_{X_I,\mathcal{S}_{\lambda_{\to i_0}}}(t) - F_{X_I,\mathcal{S}_{\lambda^-_{\to i_0}}}(t)| \leq \frac{K_A}{R-2}.$$

□

6.2 Proof of Theorem 6.2

The latter Proposition together with Lemma 5.9 yields Theorem 6.2 in case d'_{X_I} remains bounded. However, the bound of the derivative in Lemma 5.4 requires also a bound on the second run, and the latter is not assumed in our case. We should thus deal with this case before getting the general proof. Let us first consider a particular case.

Lemma 6.5. *Let λ_b be the composition with three runs of respective length 2, b and 2, and $d_b(x,y) = d_{X_I,|Y_2=y}(x)$. Then the following convergence holds:*

$$\lim_{b\to\infty} \sup_{[0,1]^2} (d_b(x,y) - (1-x^b)) = 0.$$

In particular, the asymptotic independence :

$$\lim_{b\to\infty} \sup_{x,y,y'} (d_b(x,y) - d_b(x,y')) = 0. \tag{6.2}$$

is valid.

Proof. After integrating in (3.3) the coordinates of the particles inside the composition :

$$d_b(x,y) = \frac{1-x^b-(1-y)^b+((x-y)\wedge 0)^b}{(1-1/(b+1))(1-(1-y)^b)+y/(b+1)(1-y)^b}. \tag{6.3}$$

Let us show that $\lim\limits_{b\to\infty} d_b(x,y)-(1-x^{b+1})=0$ uniformly in x and y. In the denominator of (6.3), letting b go to $+\infty$ yields

$$(1-\frac{1}{b+1})(1-(1-y)^b)+y/(b+1)(1-y)^b \sim_{b\to\infty} 1-(1-y)^b,$$

with the equivalent being uniform in x and y. Indeed

$$\frac{y/(b+1)(1-y)^b}{1-(1-y)^b} = \frac{1}{b+1}\frac{(1-y)^b}{\sum_{k=0}^{b-1}(1-y)^k} \leq \frac{1}{b+1}.$$

Since for $x\in[0,1/2], y\in[1/2,1]$, $d_b(x,y)$ converges uniformly to 1, it suffices to consider in the sequel that $x\in[1/2,1]$ and $y\in[0,1/2]$. Let Δ be defined as

$$\begin{aligned}\Delta(x,y) =& \frac{1-x^b-(1-y)^b+(x-y)^b}{1-(1-y)^b}-(1-x^b)\\ =&(1-\frac{x^b-(x-y)^b}{1-(1-y)^b})-(1-x^b)=\frac{(x-y)^b-(1-y)^bx^b}{1-(1-y)^b}.\end{aligned}$$

A derivative computation shows that $\Delta(x,y)\leq\frac{1}{b}$, which proves the uniform convergence. Since $\lim\limits_{b\to\infty}\|d_b(x,y)-(1-x^{b+1})\|_{\infty,[0,1]^2}=0$,

$$\lim_{b\to\infty}\sup_{y,y',x}(d_b(x,y)-d_b(x,y'))=0.$$

□

From the latter result can be deduced the asymptotic independence with a large second run :

Lemma 6.6. *Let $A,\epsilon>0$. There exist $B_A\in\mathbb{N}$ such that if λ is a composition with at least three runs, the extreme runs bounded by A and the second run larger than B_A, then*

$$\|d_{X_I,X_F}-d_{X_I}d_{X_F}\|_\infty\leq\epsilon$$

Proof. Let λ be a composition with first run of length a and second run of length b. From the definition of the density d_{X_I,X_F} in (3.3), conditioning the law of X_I on the position x_P of the particle $P=a+b$ yields

$$d_{X_I|x_p=y}(x) = \frac{\int_x^1\left(\int_0^{z_1\wedge y}(z_1-x)^{a-2}(z_1-z_2)^{b-2}dz_2\right)dz_1}{\mathcal{Z}}.$$

Let $2\leq a\leq A$. Then

$$d_{X_I|x_p=y}(x) = \frac{\int_x^1(u-x)^{a-3}d_b(u,y)du}{\frac{1}{a-2}\int_0^1u^{a-2}d_b(u,y)du}$$

From the first part of Lemma 6.5, $|d_b(u,y)-(1-u^b)|\to_{b\to\infty}0$ uniformly in u and y, and thus

$$\frac{1}{a-2}\int_0^1u^{a-2}d_b(u,y)du\to_{b\to\infty}\frac{1}{(a-2)(a-1)},$$

uniformly in y. Since a is bounded by A, and from the second part of Lemma 6.5,

$$\|d_{X_I|x_p=y} - d_{X_I|x_p=y'}\|_\infty \leq A^2 \sup_{y,y',x} (d_b(x,y) - d_b(x,y')) \to 0$$

uniformly in y. Thus, for b large enough, $\|d_{X_I|x_p=y} - d_{X_I|x_p=y'}\| < \epsilon/A$ for all y, y'; then averaging on the law of x_p conditioned on $X_F = y$ yields $|d_{X_I|X_F=y} - d_{X_I|X_F=y'}| < \epsilon/A$ for all y, y'. Finally, this implies that

$$\|d_{X_I,X_F} - d_{X_I} d_{X_F}\|_\infty \leq \epsilon.$$

□

The proof of Theorem 6.2 is just a gathering of all the previous results :

Proof. Let $A, \epsilon > 0$. Since the first and last runs are bounded by A, any composition large enough has at least three runs. Let B_A be given by Lemma 6.6, R be the associate constante given by Lemma 5.4 for B_A, and set $C = \frac{4K_A R}{(\epsilon/A)^2}$. Finally, let n be the integer given by Corollary 6.1 for compositions of runs bounded by C. Suppose that λ is a composition larger than n. By Lemma 6.6, if the second run is larger than B_A, (6.1) is verified. Thus, we can suppose that the second run is bounded by B_A.
If λ has a run larger than C, then from Proposition 6.4,

$$\|F_{X_I|X_F=x} - F_{X_I}\|_\infty \leq \frac{K_A}{C-1} \leq \frac{(\epsilon/A)^2}{4R}.$$

But from Lemma 5.4, d'_{X_I} is bounded by R, thus the latter inequality yields with Lemma 5.9 :

$$\|d_{X_I|X_F=y} - d_{X_I}\| \leq \epsilon/A.$$

And d_{X_I} being bounded by A, this yields (6.1).
Thus, we can assume that all the runs of λ are bounded by C. Once again by the choice of n and Corollary 6.1, (6.1) is verified. □

Note that we actually proved something stronger than Theorem 6.2, namely :

Corollary 6.7. *Let $A, \epsilon > 0$. There exists n_0 such that for every composition λ of size larger than n_0 and first run bounded by A, and for all $y, y' \in [0,1]$,*

$$\|d_{X_I|X_F=y} - d_{X_I|X_F=y'}\| \leq \epsilon.$$

6.3 Consequences and proof of Theorem 2.3

Here are some interesting consequences of Theorem 6.2. Let us first remove the constraints on the extreme runs.

Lemma 6.8. *Let $\epsilon > 0$. There exists $n \geq 0$ such that for all compositions larger than n with at least two runs,*

$$\sup_{(y,y')\in[0,1]^2} (\|F_{X_I|X_F=y} - F_{X_I|X_F=y'}\|_\infty) \leq \epsilon.$$

Proof. Let R be the length of the first run of a composition λ. From Proposition 4.7 applied to $\mathcal{S}_\lambda$,

$$1 - (1-t)^R \leq F_{X_I|X_F=y}(t) \leq 1 - (1-t)^{R-1}.$$

Since $\sup_{[0,1]}(u^{R-1} - u^R) \to_{R\to\infty} 0$, there exists A such that for any composition with first run larger than A,

$$\sup_{[0,1]^2} \|F_{X_I|X_F=y} - F_{X_I|X_F=y'}\|_\infty \leq \epsilon.$$

Applying Corollary 6.7 to A, ϵ yields that there exists n such that for any composition larger than n,

$$\sup_{[0,1]^2} \|F_{X_I|X_F=y} - F_{X_I|X_F=y'}\|_\infty \leq \epsilon.$$

□

This result can be adapted to show that the law of the first particle depends only on the neighbouring particles : for any composition λ of size N, and $n \leq N$, denote by $\lambda(n)$ the composition λ containing only the n first cells.

Proposition 6.9. *Let $\epsilon > 0$. There exists $n_0 \geq 1$ such that for any $n \geq n_0$ and any composition λ of size larger than n with first run smaller than n,*

$$\|F_{X_I}^{\mathcal{S}_\lambda} - F_{X_I}^{\mathcal{S}_{\lambda(n)}}\|_\infty \leq \epsilon.$$

The proof consists only in an averaging of the inequality of the previous Lemma.
We will close this paper by proving Theorem 2.3.
Let λ be a composition and let $s = [\![i_1, i_2]\!]$ be a run of λ. For a cell i in s, the position of i in s, denoted by a_i, is the ratio $a_i = \frac{i-i_1}{i_2-i_1}$ (resp. $\frac{i_2-i}{i_2-i_1}$) if the run is increasing (resp. decreasing). When a run is large, the behavior of a cell in this run is approximately frozen:

Lemma 6.10. *Let $\epsilon > 0$. There exists $R_\epsilon > 0$ such that for any composition λ of n and $1 \leq i \leq n$ such that i is in a run s of size larger than R_ϵ,*

$$\mathbb{P}(|\frac{\sigma_\lambda(i)}{n} - a_i| \geq \epsilon) \leq \epsilon,$$

where a_i is the position of i in s as previously defined.

Proof. Let λ be a composition of n, and let $1 \leq i \leq n$ be a cell of λ in a run s of length R. Let $i_1 \leq i_2$ be the extreme cells of the run s and suppose without loss of generality that s is increasing. We use the probabilistic model $\tilde{\mathcal{S}}_\lambda$ of Section 3.2. By Lemma 3.6, it suffices to prove that for R large enough,

$$\mathbb{P}(|Z_i - a_i| \geq \epsilon) \leq \epsilon.$$

Conditioning Z_{i_1} on the value of Z_{i_1-1} and Z_{i_2} gives the conditional expectation:

$$\mathbb{E}(Z_{i_1}|Z_{i_1-1} = z, Z_{i_2} = z') = \frac{\int_0^{z\wedge z'} x(z'-x)^{R-2}dx}{\int_0^{z\wedge z'}(z'-x)^{R-2}dx} \leq \frac{1}{R},$$

where the last bound is given by a computation of the integral. Since the bound is independent of z and z', for R large enough $\mathbb{P}(Z_{i_1} \geq \epsilon) \leq \epsilon$. Likewise, for R large enough, $\mathbb{P}(Z_{i_2} \leq 1-\epsilon) \leq \epsilon$. This gives the result if $i = i_1$ or $i = i_2$. Suppose that $i \neq i_1$ and $i \neq i_2$. Conditioned on the value of Z_{i_1} and Z_{i_2}, the law of Z_i is

$$d_{Z_i|Z_{i_1}=z, Z_{i_2}=z'}(x) = \frac{\mathbf{1}_{z\leq x\leq z'}(z'-x)^{i_2-i-1}(x-z)^{i-i_1-1}}{\int_z^{z'}(z'-x)^{i_2-i-1}(x-z)^{i-i_1-1}dx}.$$

Thus, by a computation, the conditional expectation of $Z_i - z$ is

$$\mathbb{E}\left(Z_i - z|Z_{i_1} = z, Z_{i_2} = z'\right) = (z'-z)\frac{i-i_1}{i_2-i_1},$$

and the conditional variance of $Z_i - z$ is

$$Var\left(Z_i - z|Z_{i_1} = z, Z_{i_2} = z'\right) = (z'-z)^2\frac{i-i_1}{i_2-i_1}\left(\frac{i-i_1+1}{i_2-i_1+1} - \frac{i-i_1}{i_2-i_1}\right) \leq (z'-z)^2\frac{1}{R}.$$

Thus, for R large enough, $\mathbb{P}(|Z_i - (Z_{i_1} + a_i(Z_{i_2} - Z_{i_1}))| \geq \epsilon) \leq \epsilon$.
By the first part of the proof, for R large enough $\mathbb{P}(Z_{i_1} \geq \epsilon) \leq \epsilon$ and $\mathbb{P}(Z_{i_2} \leq 1-\epsilon) \leq \epsilon$; thus, for R large enough,

$$\mathbb{P}(|Z_i - a_i| \geq \epsilon) \leq \epsilon.$$

□

We can improve the result of Corollary 6.8 by considering the case of a cell in the middle of a composition.

Lemma 6.11. *Let $\epsilon > 0, R > 0$. There exists $k_R \geq 1$ such that for any composition λ and $1 \leq j_1 < i < j_2 \leq n$ such that i is in a run bounded by R and $|i - j_1|, |j_2 - i| \geq k_R$, then*

$$\|d_{Z_i|Z_{j_1}=z_1, Z_{j_2}=z_2} - d_{Z_i|Z_{j_1}=z_1', Z_{j_2}=z_2'}\|_\infty \leq \epsilon$$

for all $0 \leq z_1, z_2, z_1', z_2' \leq 1$, where Z_i is the random variable corresponding to the particle i in $\tilde{\mathcal{S}}_\lambda$. Likewise,

$$\|d_{Z_i|Z_{j_1}=z_1} - d_{Z_i|Z_{j_1}=z_1'}\|_\infty \leq \epsilon$$

and

$$\|d_{Z_i|Z_{j_2}=z_2} - d_{Z_i|Z_{j_2}=z_2'}\|_\infty \leq \epsilon$$

for all $0 \leq z_1, z_2, z_1', z_2' \leq 1$.

Proof. We will only prove the first part of the Lemma, since the proof of the second part is a simpler version of the one of the first part.
Let λ be a composition and let $1 \leq j_1 < i < j_2 \leq n$ be three cells of λ. By the expression of the density in (3.3),

$$d_{Z_i|Z_{j_1}=z_1, Z_{j_2}=z_2}(x) = \frac{d_{X_F|X_I=z_1, \mathcal{S}_{\nu_1}}(x) d_{X_I|X_F=z_2, \mathcal{S}\nu_2}(x)}{\int_0^1 d_{X_F|X_I=z_1, \mathcal{S}_{\nu_1}}(x) d_{X_I|X_F=z_2, \mathcal{S}\nu_2}(x)dx},$$

where $\nu_1 = \lambda_{j_1 \to i}$ and $\nu_2 = \lambda_{i \to j_2}$. Since i is in a run bounded by R in λ, i is in a run bounded by R in ν_1 and in ν_2. Therefore by Corollary 6.7, there exists n_ϵ such that if $|\nu_1| \geq n_\epsilon$ and $|\nu_2| \geq n_\epsilon$, then

$$\|d_{X_F|X_I=z_1,\nu_1} - d_{X_F|X_I=z_1',\nu_1}\|_\infty \leq \epsilon$$

and

$$\|d_{X_I|X_F=z_2,\nu_2} - d_{X_I|X_F=z_2',\nu_2}\|_\infty \leq \epsilon,$$

for all $0 \leq z_1, z_2, z_1', z_2' \leq 1$. Moreover, by Lemma 5.3, $d_{X_F|X_I=z_1,\nu_1}$ is bounded by some constant K only depending on R, and the same holds for $d_{X_I|X_F=z_2,\nu_2}$. Therefore

$$\|d_{X_F|X_I=z_1,\nu_1}(x) d_{X_I|X_F=z_2,\nu_2}(x) - d_{X_F|X_I=z_1',\nu_1}(x) d_{X_I|X_F=z_2',\nu_2}(x)\|_\infty \leq 2A\epsilon$$

for $0 \leq z_1, z_1', z_2, z_2' \leq 1$. In particular,

$$|\int_0^1 d_{X_F|X_I=z_1,\nu_1}(x) d_{X_I|X_F=z_2,\nu_2}(x) - d_{X_F|X_I=z_1',\nu_1}(x) d_{X_I|X_F=z_2',\nu_2}(x)dx| \leq 2A\epsilon.$$

Set

$$A_{z_1,z_2} = \int_0^1 d_{X_F|X_I=z_1,\nu_1}(x) d_{X_I|X_F=z_2,\nu_2}(x)dx, B_{z_1,z_2} = d_{X_F|X_I=z_1,\nu_1}(x) d_{X_I|X_F=z_2,\nu_2}(x).$$

By the above computations,

$$\begin{aligned}|\frac{B_{z_1,z_2}}{A_{z_1,z_2}}-\frac{B_{z_1',z_2'}}{A_{z_1',z_2'}}|\leq&|\frac{B_{z_1,z_2}}{A_{z_1,z_2}}-\frac{B_{z_1',z_2'}}{A_{z_1,z_2}}|+|\frac{B_{z_1',z_2'}}{A_{z_1,z_2}}-\frac{B_{z_1',z_2'}}{A_{z_1',z_2'}}|\\ \leq&\frac{1}{A_{z_1,z_2}}(2R\epsilon)+\frac{B_{z_1',z_2'}}{A_{z_1,z_2}A_{z_1',z_2'}}(2R\epsilon).\end{aligned}$$

It remains to show that $\frac{1}{A_{z_1,z_2}}$ and $\frac{B_{z_1',z_2'}}{A_{z_1,z_2}A_{z_1',z_2'}}$ are bounded by a constant only depending on R. Since i is in a run bounded by R in ν_1 and ν_2, $|B_{z_1,z_2}|$ is bounded by K^2, where K is the constant given Lemma 5.3 for a run of size R.

Let us show that A_{z_1,z_2} admits a lower bound only depending on R; suppose without loss of generality that the run of λ containing i is increasing and that i is not an extreme cell. Let R_1 be the length of the run containing i in ν_1 and let R_2 be the length of the run containing i in ν_2; since these both runs are part of the run of i in λ, they are both increasing and $R_1+R_2=R+1$.

By Corollary 4.9, $t^{R_1}\leq F_{X_F|X_I=z_1,\nu_1}(t)\leq t^{R_1-1}$ and $1-(1-t)^{R_2-1}\leq F_{X_I|X_F=z_2,\nu_2}(t)\leq 1-(1-t)^{R_2}$ for $0\leq t\leq 1$. By Lemma 3.4, $d_{X_F|X_I=z_1,\nu_1}$ is increasing and $d_{X_I|X_F=z_2,\nu_2}$ is decreasing, thus $F_{X_F|X_I=z_1,\nu_1}$ is convex and $F_{X_I|X_F=z_2,\nu_2}$ is concave. The convexity of $F_{X_F|X_I=z_1,\nu_1}$ yields that

$$F'_{X_F|X_I=z_1,\nu_1}(t)\geq\frac{F_{X_F|X_I=z_1,\nu_1}(t)-F_{X_F|X_I=z_1,\nu_1}(0)}{t-0}\geq t^{R_1-1}.$$

Likewise, the concavity of $F_{X_I|X_F=z_2,\nu_2}$ yields that

$$F'_{X_I|X_F=z_2,\nu_2}(t)\geq\frac{F_{X_I|X_F=z_2,\nu_2}(1)-F_{X_I|X_F=z_2,\nu_2}(t)}{1-t}\geq(1-t)^{R_2-1}.$$

Therefore,

$$A_{z_1,z_2}\geq\int_0^1 x^{R_1-1}(1-x)^{R_2-1}dx=\frac{(R_1-1)!(R_2-1)!}{(R_1+R_2-1)!}\geq\frac{1}{(R_1+R_2-1)!}.$$

Since $R_1+R_2-1=R$, $A_{z_1,z_2}\geq\frac{1}{R!}$. This yields

$$|\frac{B_{z_1,z_2}}{A_{z_1,z_2}}-\frac{B_{z_1',z_2'}}{A_{z_1',z_2'}}|\leq(2R\epsilon)(R!+K^2(R!)^2).$$

Thus, if $\min(|\nu_1|,|\nu_2|)\geq n_\epsilon$, then

$$\|d_{Z_i|Z_{j_1}=z_1,Z_{j_2}=z_2}-d_{Z_i|Z_{j_1}=z_1',Z_{j_2}=z_2'}\|_\infty\leq(2R\epsilon)(R!+K^2(R!)^2),$$

for all $0\leq z_1,z_2,z_1',z_2'\leq 1$. Setting $k_R=n_{\epsilon/(2R(R!+K^2(R!)^2)}$ gives the appropriate constant for the statement of the Lemma. □

We can now prove Theorem 2.3.

Proof of Theorem 2.3. The proof is done by induction on r.

Let $r=2$. Let $\epsilon>0$ and R_ϵ be the constant from Lemma 6.10. Let λ be a composition of n and let $1\leq i<j\leq n$ be two cells of λ. If i and j are both in runs larger than R_ϵ, then by Lemma 6.10, $\mathbb{P}(|\frac{\sigma_\lambda(i)}{n}-a_i|\geq\epsilon)\leq\epsilon$ and $\mathbb{P}(|\frac{\sigma_\lambda(j)}{n}-a_j|\geq\epsilon)\leq\epsilon$. Therefore,

$$\begin{aligned}\pi\left(\mu\left(\frac{\sigma_\lambda(i)}{n},\frac{\sigma_\lambda(j)}{n}\right),\mu(\frac{\sigma_\lambda(i)}{n})\otimes\mu(\frac{\sigma_\lambda(j)}{n})\right)\leq&\,\pi\left(\mu(\frac{\sigma_\lambda(i)}{n},\frac{\sigma_\lambda(j)}{n}),\delta_{a_i}\otimes\delta_{a_j}\right)\\ &+\pi\left(\delta_{a_i}\otimes\delta_{a_j},\mu(\frac{\sigma_\lambda(i)}{n})\otimes\mu(\frac{\sigma_\lambda(j)}{n})\right)\leq 2\epsilon.\end{aligned}$$

Suppose without loss of generality that i is in a run smaller than R_ϵ. On the one hand, for $0 \leq t_1, t_2 \leq 1$,

$$F_{Z_i,Z_j}(t_1,t_2) - F_{Z_i}(t_1)F_{Z_j}(t_2) = \int_0^{t_2} \left(\int_0^{t_1} d_{Z_i|Z_j=y}(x) - d_{Z_i}(x)dx \right) d_{Z_j}(y)dy.$$

On the other end, by Lemma 6.11, there exists k such that if $|j-i| \geq k$,

$$\|d_{Z_i|Z_j=z} - d_{Z_i|Z_j=z'}\|_\infty \leq \epsilon$$

for any $0 \leq z, z' \leq 1$. Therefore, for $|j-i| \geq k$, $\|d_{Z_i|Z_j=y} - d_{Z_i}\|_\infty \leq \epsilon$ for $0 \leq y \leq 1$. This yields

$$|F_{Z_i,Z_j}(t_1,t_2) - F_{Z_i}(t_1)F_{Z_j}(t_2)| \leq \int_0^{t_2} t_1 \epsilon d_{Z_j}(y)dy \leq \epsilon.$$

In particular,

$$\pi\left(\mu(Z_i,Z_j), \mu(Z_i)\otimes\mu(Z_j)\right) \leq \epsilon.$$

Lemma 3.6 concludes the case $r=2$.
Suppose that $r>2$. Let λ be a composition and let $1 \leq i_1, \ldots, i_r \leq n$ be distinct cells of λ. If $i_1, \ldots, i_r$ are all in runs larger than R_ϵ, by the same reason as before,

$$\pi\left(\mu\left(\frac{\sigma_\lambda(i_1)}{n}, \ldots, \frac{\sigma_\lambda(i_r)}{n}\right), \mu(\frac{\sigma_\lambda(i_1)}{n}) \otimes \cdots \otimes \mu(\frac{\sigma_\lambda(i_r)}{n})\right) \leq 2\epsilon.$$

Suppose without loss of generality that i_r is in a run bounded by R_ϵ, and let k be the constant associated to R_ϵ in Lemma 6.11. By the induction hypothesis, there exists k_1 such that if $i_j - i_{j-1} \geq k_1$ for $2 \leq j \leq r-1$, then

$$\pi\left(\mu(Z_{i_1}, \ldots, Z_{i_{r-1}}), \mu(Z_{i_1} \otimes \cdots \otimes \mu(Z_{i_{r-1}})\right) \leq \epsilon.$$

On the one hand for $\vec{t} \in [0,1]^r$,

$$\begin{aligned} &F_{(Z_i)_{1\leq i\leq r}}(\vec{t}) - F_{Z_{i_r}}(t_r)F_{(Z_{i_s})_{s<r}}((t_s)_{s<r}) \\ &\qquad = \int_{x_s\in[0,t_s]} \left(d_{Z_{i_r}|Z_{i_s}=x_s, s<r}(x_r) - d_{Z_{i_r}}(x_r)\right) d_{(Z_{i_s})_{s<r}}((x_s)_{s<r}) \prod_{s=1}^r dx_s. \end{aligned}$$

By Formula (3.3), $d_{Z_{i_r}|Z_{i_1}=x_1,\ldots Z_{i_r-1}=x_{r-1}}(x_r) = d_{Z_{i_r}|Z_{i_a}=x_a, Z_{i_b}=x_b}(x_r)$, where a and b are such that i_a is the cell of $\{i_1, \ldots, i_{r-1}\}$ directly before i_r and i_b is the cell of $\{i_1, \ldots, i_{r-1}\}$ directly after i_r. By Lemma 6.11, if $i_r - i_a \geq k$ and $i_b - i_r \geq k$, then

$$\|d_{Z_{i_r}|Z_{i_a}=x_a, Z_{i_b}=x_b} - d_{Z_{i_r}}\|_\infty \leq \epsilon.$$

Thus,

$$|F_{(Z_{i_s})_{1\leq s\leq r}}(\vec{t}) - F_{Z_{i_r}}(t_r)F_{(Z_{i_s})_{s<r}}((t_i)_{i<r})| \leq \int_{x_s\in[0,t_s], s<r} \epsilon d_{(Z_{i_s})_{s<r}}((x_s)_{s<r}) \prod_{s<r} dx_s \leq \epsilon,$$

which yields

$$\pi(\mu((Z_{i_1}, \ldots, Z_{i_r}), \mu(Z_{i_r}) \otimes \mu((Z_{i_s})_{s<r})) \leq \epsilon.$$

Finally,

$$\begin{aligned} \pi\left(\mu\left(Z_{i_1}, \ldots, Z_{i_r}\right), \mu(Z_{i_1}) \otimes \cdots \otimes \mu(Z_{i_r})\right) &\leq \pi\left(\mu\left(Z_{i_1}, \ldots, Z_{i_r}\right), \mu(Z_{i_r}) \otimes \mu((Z_{i_s})_{s<r}\right) \\ &+ \pi\left(\mu(Z_{i_r}) \otimes \mu((Z_{i_s})_{s<r}), \mu(Z_{i_1}) \otimes \cdots \otimes \mu(Z_{i_r})\right) \leq \epsilon + \epsilon \leq 2\epsilon. \end{aligned}$$

□

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