Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 91, 1–28.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-3879

# Asymptotic behaviour of first passage time distributions for subordinators*

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**Abstract**

In this paper we establish local estimates for the first passage time of a subordinator under the assumption that it belongs to the Feller class, either at zero or infinity, having as a particular case the subordinators which are in the domain of attraction of a stable distribution, either at zero or infinity. To derive these results we first obtain uniform local estimates for the one dimensional distribution of such a subordinator, which sharpen those obtained by Jain and Pruitt [9]. In the particular case of a subordinator in the domain of attraction of a stable distribution our results are the analogue of the results obtained by the authors in [5] for non-monotone Lévy processes. For subordinators an approach different to that in [5] is necessary because the excursion techniques are not available and also because typically in the non-monotone case the tail distribution of the first passage time has polynomial decrease, while in the subordinator case it is exponential.

**Keywords:** Subordinators; first passage time distribution; local limit theorems; Feller class.
**AMS MSC 2010:** 62E17; 60G51; 60F10.
Submitted to EJP on October 21, 2014, final version accepted on July 20, 2015.

## 1 Introduction and overview of main results

Let $X$ be a subordinator, a stochastic process with non-decreasing càdlàg paths with independent and stationary increments, with Laplace exponent $\psi$, given by

$$-\frac{1}{t}\log\left(\mathbb{E}(\exp\{-\lambda X_t\})\right) =: \psi(\lambda) = b\lambda + \int_{(0,\infty)} (1-e^{-\lambda x})\Pi(dx), \qquad \lambda \geq 0,$$

where $b$ denotes the drift and $\Pi$ the Lévy measure of $X$. We will write $\psi_*$ for the exponent of $\{X_t - bt, t \geq 0\}$, so that

$$\psi_*(\lambda) := \psi(\lambda) - b\lambda, \qquad \lambda \geq 0.$$

Also, we will write $\overline{\Pi}(x) := \Pi(x,\infty)$, for $x > 0$.

*Research funded by the CONACYT Project *Teoría y aplicaciones de procesos de Lévy*

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We are interested in determining the **local** asymptotic behaviour of the distribution of $T_x = \inf\{t > 0 : X_t > x\}$. More precisely, we would like to establish estimates for the density function $h_x(t)$, (if it exists: it does if $b = 0$, see Lemma 3.3 below), or more generally of

$$\mathbb{P}(T_x \in (t, t + \Delta]),$$

uniformly for $\Delta$ in bounded sets and uniformly for $x$ in certain regions, both as $t \to \infty$ or as $t \to 0$. Knowing the behaviour of the first passage time distribution of a subordinator is of central importance because of its applications in stochastic modeling and theoretical probability, see for instance [2].

This paper is a continuation of recent research in [5], where the same problem, in the $t \to \infty$ case, has been solved for Lévy processes, excluding subordinators, that are in the domain of attraction of a stable law without centering. The reasons for excluding subordinators from that research were that the techniques used there rely heavily on excursion theory for the reflected process, which in this case does not make sense, and that in the subordinators case the rate of decrease of the tail distribution of the first passage time is typically exponential, while for other Lévy processes in the domain of attraction of a stable law, without centering, it is polynomial. This polynomial behaviour in the tail distribution of the first passage time reflects the asymptotic behaviour of the tail Lévy measure at infinity, which in general is closely related to that of the characteristic exponent at $0$. In the paper [10], under mild assumptions, the behaviour at infinity of the tail distribution of the first passage time is related to the behaviour of the characteristic exponent at $0$.

As can be seen in the paper [5], and in the present case, the distribution of the first passage time has different behaviour according to whether the process first crosses the barrier by a jump or continuously, that is by *creeping*. So, our results will describe the contributions of these events to the first passage time distribution separately. Naturally, if a subordinator has zero drift, by Theorem III.5 in [1] it cannot creep, and moreover the distribution of $T_x$ is absolutely continuous, so our results become somewhat simpler in that case.

Before we give a precise general statement, we start by looking at the illustrative case where $X$ is a stable subordinator of index $\alpha = 1/2$. This means that $b = 0$, $\Pi(dx) = \frac{1}{2\sqrt{\pi}} x^{-\frac{3}{2}} dx$, and $\psi(\lambda) = \sqrt{\lambda}$. In this case we know that the law of $X_t$ is absolutely continuous with density

$$f_t(x) = \frac{t}{2\sqrt{\pi x^3}} e^{-\frac{t^2}{4x}},$$

and hence

$$h_x(t) = \partial_t \mathbb{P}(T_x \le t) = \partial_t \mathbb{P}(X_1 > xt^{-2}) = 2xt^{-3} f_1(xt^{-2}) = \frac{1}{\sqrt{\pi x}} e^{-\frac{t^2}{4x}}.$$

Straightforward calculations allow us to verify the identity

$$h_x(t) = \frac{1}{\sqrt{\pi}} \int_0^x \mathbb{P}(X_t \in dy) \frac{1}{\sqrt{(x-y)}} = \int_0^x \mathbb{P}(X_t \in dy) \overline{\Pi}(x-y), \tag{1.1}$$

and we deduce that

$$\mathbb{P}(T_x \in (t, t+\Delta]) = h_x(t) \int_0^\Delta e^{-ut/2x - u^2/4x} du \sim \frac{2x}{t}(1 - e^{-\frac{t\Delta}{2x}}) h_x(t), \tag{1.2}$$

uniformly as $\frac{x}{t^2} \to 0$. Even in this simple example there is something surprising: if $t\Delta/x \nrightarrow 0$ the RHS of (1.2) is not asymptotic to $\Delta h_x(t)$.

The condition $\frac{t^2}{x} \to \infty$ is equivalent to $\mathbb{P}(T_x > t) = \mathbb{P}(X_t \le x) \to 0$, and it was shown in [9] that the corresponding condition for a general subordinator is that $tH(\rho) \to \infty$, where $H(u) = \psi(u) - u\psi'(u)$, and $\rho$ is the unique solution of $\psi'(\rho) = x/t$, so that in our example $\rho = (\frac{t}{2x})^2$ and $H(\rho) = t/4x$.

We will see later in (3.15) in Subsection 3.2 that $\rho$ is the parameter in an exponential measure change which is essential to our proofs and reminiscent of the large deviations techniques. Taking $\sigma^2(u) = \psi''(u)$, we have $\sigma^2(\rho) = 2x^3/t^3$ in the example.

We therefore see that the above are special cases of

$$f_t(x) \sim \sqrt{\frac{1}{2\pi t\sigma^2(\rho)}} e^{-tH(\rho)}, \tag{1.3}$$

$$h_x(t) \sim \frac{\psi(\rho)}{\rho} f_t(x), \tag{1.4}$$

and

$$\mathbb{P}(T_x \in (t, t+\Delta]) \sim \frac{1}{\rho}(1 - e^{-\Delta\psi(\rho)}) f_t(x). \tag{1.5}$$

Suppose now that $X$ is any driftless subordinator having $\overline{\Pi}(x)$ regularly varying as $x \to 0$ with index $-\alpha$ satisfying $0 < \alpha < 1$, so that $X_t$ is asymptotically stable as $t \to 0$. Then $X$ is absolutely continuous, and our first main result shows that (1.3) is valid as $x/t \to 0$, both as $t \to 0$ and as $t \to \infty$. (This significantly improves a result about the behaviour of $\mathbb{P}(X_t \le x)$ in [9].) Using the representation for $h_x(t)$ in (1.1) it is then straightforward to verify that (1.4) holds. In principle, this must imply (1.5), but it turns out to be easier to deduce (1.5) from (1.3), using the formula

$$\begin{aligned} h_x^J(t, \Delta) : &= \mathbb{P}(T_x \in (t, t+\Delta], X_{T_x} > x) \\ &= \int_{[0,x)} \mathbb{P}(X_t \in x - dy) \int_{[0,y]} U_\Delta(dz) \overline{\Pi}(y - z). \end{aligned}$$

(Here $U_\Delta(dz) = \int_0^\Delta \mathbb{P}(X_s \in dz) ds$, see Lemma 3.3). Moreover if in this situation the subordinator has drift $b > 0$, it is a consequence of the results in [8] that

$$\mathbb{P}(T_x \in dt, X(T_x) = x) = b f_t(x) dt \text{ for } x > bt, \tag{1.6}$$

see Lemma 3.3. Since $X$ jumps over $x$ at time $t$ if $X - bt$ jumps over $x - bt$ at time $t$, we see that (1.4) holds for $x > bt$ with $\psi(\rho)$ replaced by $\psi(\rho) - b\rho$, and it follows that

$$\mathbb{P}(X \text{ creeps over } x | T_x = t) \sim \frac{b\rho}{\psi(\rho)} \to \begin{cases} 1, & \text{if } \rho \to \infty, \\ \frac{b}{\psi'(0+)}, & \text{if } \rho \to 0. \end{cases} \tag{1.7}$$

The case $t \to \infty$ and $x/t \to \infty$ is slightly more difficult: here the natural assumption is that $X_t$ is asymptotically stable as $t \to \infty$, or equivalently that $\overline{\Pi}(x)$ is regularly varying as $x \to \infty$ with index $-\alpha$ satisfying $0 < \alpha < 1$ and $b \ge 0$. Here $X$ is **not** automatically absolutely continuous, and although we can get an estimate for $\mathbb{P}(X_t \in (x, x+\Delta])$ analogous to (1.3), this is only valid for $\Delta$ in compact sub-intervals of $(0, \infty)$. The possible singularity of $\overline{\Pi}(x - y)$ at $y = x$ then seems to make a direct calculation based on (1.1) impossible. Instead we exploit the fact that the RHS of (1.1) for fixed $t$ is a convolution and use the inversion theorem for characteristic functions to establish (1.4) by an indirect method. This trick requires us to make an additional assumption which involves the behaviour of $\Pi$ near zero, (see (H) below), but crucially this assumption does not necessitate $X$ being absolutely continuous.

When dealing with the creeping component we therefore cannot rely on (1.6), and instead we look for an estimate of

$$h_x^C(t, \Delta) := \mathbb{P}(T_x \in (t, t+\Delta], X_{T_x} = x),$$

for which we use the formula

$$\mathbb{P}(T_x \in (t, t+\Delta], X_{T_x} = x) = b \int_{[0,x)} \mathbb{P}(X_t \in dy) u_\Delta(x-y).$$

(Here $u_\Delta(z)dz = U_\Delta(dz)$; see Lemma 3.3). In the case that $X$ is stable-$1/2$ plus a drift $b > 0$ one can check directly that the result is

$$h_x^C(t,\Delta) \sim \frac{b\rho}{\psi(\rho)} \mathbb{P}(T_x \in (t, t+\Delta]) \text{ and} \tag{1.8}$$

$$h_x^J(t,\Delta) \sim \left(1 - \frac{b\rho}{\psi(\rho)}\right) \mathbb{P}(T_x \in (t, t+\Delta]), \tag{1.9}$$

where the asymptotic behaviour of $\mathbb{P}(T_x \in (t, t+\Delta])$ is given by the RHS of (1.2) evaluated with $x$ replaced by $x - bt$. As we will see, (1.8) and (1.9) hold in the general case, as does (1.7). We also have analogous results for the regime where $\rho$ is bounded away from zero and infinity and $t \to \infty$, where we make no assumptions about $X$ other than (H), to be introduced below, and that it is a strongly non-lattice subordinator.

## 2 Main results

Before we state our main results we introduce our basic assumptions. We will say that the condition (H) is satisfied whenever the following condition on the small jumps of $X$ is satisfied:

(H) there exists a $t_0 > 0$ such that

$$\int_1^\infty \exp\left\{-t_0 \int_0^\infty (1 - \cos(zy))\Pi(dy)\right\} \frac{1 + |\psi_*(-iz)|}{z} dz < \infty.$$

As we mentioned before this condition will be useful to obtain estimates in the case where the process is in the domain of attraction of a stable distribution at infinity. Further details about this assumption are given in the following remarks.

**Remark 2.1.**

(i) Using the elementary inequalities $1 - \cos(y) \geq \frac{1}{\pi} y^2$, for $-1 < y < 1$, and $|\sin(y)| \leq 1 \wedge |y|$, for $y \in \mathbb{R}$, it can be verified that (H) holds whenever there exists a $t_0$ such that

$$\int_1^\infty \exp\left\{-t_0 z^2 \int_0^{1/z} b^2 \Pi(db)\right\} \left[\frac{1}{z} + \int_0^{1/z} \overline{\Pi}(a) da\right] dz < \infty.$$

(ii) The stronger condition

$$\int_1^\infty \exp\left\{-t_0 \int_0^\infty (1 - \cos(zy))\Pi(dy)\right\} dz < \infty \tag{2.1}$$

would imply that $X_t$ has an absolutely continuous distribution for each $t \geq t_0$, see Proposition 2.5 in [14], but it is easy to see using the above remark that (H) can hold without (2.1) holding.

(iii) In the compound Poisson case it is clear that this condition cannot hold, but in that case $\Pi$ is integrable at zero, and so we can use a method similar to that we use to deal with the case where $X$ is in the domain of attraction of a stable distribution at 0.

Define a function $c$ by $t\overline{\Pi}(c(t)) = 1$, $t > 0$. We will say that we are working in the framework $(RV_0)$, $(RV_\infty)$ or $(G)$, respectively, if the following happens:

$(RV_0)$ either $t \to \infty$ and $\frac{x}{t} \to b$, or $t \to 0$ and $(x - tb)/c(t) \to 0$; when $x \mapsto \overline{\Pi}(x)$, $x > 0$, is regularly varying at 0 with index $-\alpha$, for some $\alpha \in (0, 1)$;

$(RV_\infty)$ $t \to \infty$, $x/t \to \infty$, $x/c(t) \to 0$; when $x \mapsto \overline{\Pi}(x)$, $x > 0$, is regularly varying at $\infty$, with an index $-\alpha$ for some $\alpha \in (0, 1)$, and (H) holds;

$(G)$ $t \to \infty$ and $b < \liminf_{t\to\infty} \frac{x}{t} \le \limsup_{t\to\infty} \frac{x}{t} < \mu$, $X$ is strongly non-lattice, and (H) holds.

We start by providing some local estimates of the distribution of $X$ in the $(RV_0)$ cases. These are a refinement of the pioneering work by Jain and Pruitt [9], which is one of the main sources of this research, where estimates for $\mathbb{P}(T_x > t) = \mathbb{P}(X_t \le x)$ are obtained. The technique we use is different to that of Jain and Pruitt, though both techniques involve normal approximations. It seems more flexible as it allows us to avoid the stronger condition (2.1). Throughout this paper $\phi : \mathbb{R} \to \mathbb{R}^+$, will denote the standard normal density, that is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\{-x^2/2\}, \qquad x \in \mathbb{R}.$$

**Theorem 2.2.** *Suppose that $X$ is a subordinator which has drift $b \ge 0$ and Lévy measure $\Pi$, such that $x \mapsto \overline{\Pi}(x)$, $x > 0$, is regularly varying at 0 with index $-\alpha$, for some $\alpha \in (0, 1)$. For $b < x/t < \mu := \mathbb{E}(X_1)$, define $x_t := x/t$ and $\rho_t := \rho(x/t)$, that is $\psi'(\rho_t) = x/t$, $H(u) = \psi(u) - u\psi'(u)$, and $\sigma^2(u) = -\psi''(u) = \int_0^\infty y^2 e^{-uy}\Pi(dy)$.*

*(i) The unidimensional law of $X$ admits a density, say $\mathbb{P}(X_t \in dy) = f_t(y)dy$, $y \ge 0$, such that $f_t \in C^\infty(\mathbb{R})$ for each $t > 0$.*

*(ii) In the setting $(RV_0)$ we have the estimate*

$$\sqrt{t}\sigma(\rho_t) f_t(z) = \Big(\phi((z - x)/\sqrt{t}\sigma(\rho_t)) + o(1)\Big) e^{-tH(\rho_t)} e^{\rho_t(z-x)}, \tag{2.2}$$

*uniformly in $z > 0$ and $x$.*

(To be clear, the uniformity here means, in, for example, the $RV_0$ case with $t \to \infty$, that given arbitrary $\varepsilon > 0$ there exists $t_0$ and $\delta_0$ such that the $o(1)$ term is less in absolute value than $\varepsilon$ for all $z > 0, t > t_0$ and $x$ satisfying $|\frac{x}{t} - b| < \delta_0$.)

We now turn to the results for the passage time. We are interested in the probability of $X$ passing above level $x$ in the time interval $(t, t + \Delta]$, either by a jump or by creeping. The latter is positive only when the drift $b > 0$. The latter and former probabilities will be denoted by

$$h_x^C(t, \Delta) := \mathbb{P}(T_x \in (t, t + \Delta], X_{T_x} = x),$$

and

$$h_x^J(t, \Delta) := \mathbb{P}(T_x \in (t, t + \Delta], X_{T_x} > x), \qquad t > 0.$$

First, in the settings $(RV_0)$ we have by the forthcoming Lemma 3.3 that $h_x^C(t, \Delta) = \int_t^{t+\Delta} h_x^C(s)ds$ and

$$h_x^C(t) = bf_t(x), \qquad t, x > 0,$$

and by Theorem 2.2,

$$h_x^C(t) = bf_t(x) = \frac{b}{\sqrt{2\pi t}\sigma(\rho_t)} e^{-tH(\rho_t)}(1 + o(1))$$

uniformly in $x$. Furthermore, in all cases the expression

$$h_x^J(t) = \int_0^x \mathbb{P}(X_t \in dy)\overline{\Pi}(x - y), \qquad t > 0,$$

is the density function of the first passage time on the event $X_{T_x} > x$, see [5] Lemma 1. Then our main result is the following.

**Theorem 2.3.** *Assume we are in the settings ($RV_0$), ($RV_\infty$) or ($G$). Put $\psi_*(\lambda) = \psi(\lambda) - \lambda b = \int_0^\infty (1-e^{-\lambda y})\Pi(dy)$. We have the following estimates*

$$h_x^J(t) = \frac{\psi_*(\rho_t)}{\sqrt{2\pi t}\rho_t\sigma(\rho_t)} e^{-tH(\rho_t)} \left(1+o(1)\right), \tag{2.3}$$

$$h_x^J(t,\Delta) = \frac{\psi_*(\rho_t)\left(1-e^{-\Delta\psi(\rho_t)}\right)}{\psi(\rho_t)\sqrt{2\pi t}\rho_t\sigma(\rho_t)} e^{-tH(\rho_t)} \left(1+o(1)\right), \tag{2.4}$$

$$h_x^C(t,\Delta) = \frac{b\left(1-e^{-\Delta\psi(\rho_t)}\right)}{\psi(\rho_t)\sqrt{2\pi t}\sigma(\rho_t)} e^{-tH(\rho_t)} \left(1+o(1)\right), \tag{2.5}$$

*uniformly in $x$ and uniformly for $0 < \Delta \le \Delta_0$, for any fixed $\Delta_0 > 0$.*

In the above estimates if $\Delta$ is bounded away from zero the term $(1-e^{-\Delta\psi(\rho_t)})/\psi(\rho_t)$ can be replaced by $\Delta$ or $1/\psi(\rho_t)$, according as $\rho_t \to 0$ or $\infty$.

**Remark 2.4.** In the frameworks $RV_0$ and $RV_\infty$, using Karamata's theorem and the monotone density theorem for regularly varying functions, see e.g. [3] Chapter 1, it can be verified that the relevant quantities in the above theorems have the following behaviour. There is a slowly varying function $\ell$ such that

$$\begin{aligned}
\rho_t &\sim \left(\frac{x}{t}\right)^{\frac{1}{\alpha-1}} \ell(x/t), \\
\psi(\rho_t) &\sim \frac{1}{\alpha}\rho_t\psi'(\rho_t) = \frac{1}{\alpha}\frac{x}{t}\rho_t = \frac{1}{\alpha}\left(\frac{x}{t}\right)^{\frac{\alpha}{\alpha-1}} \ell(x/t), \\
H(\rho_t) &\sim \frac{1-\alpha}{\alpha} x\rho_t, \\
\sigma^2(\rho_t) &\sim (2-\alpha)\frac{1}{\rho_t}\psi'(\rho_t) \sim (2-\alpha)\frac{1}{\left(\frac{x}{t}\right)^{\frac{2-\alpha}{\alpha-1}}\ell(x/t)}.
\end{aligned}$$

Our forthcoming final result shows that, in the framework $RV_\infty$, when we remove the condition $x/c(t) \to 0$, it is possible for polynomial rather than exponential decay to occur. This is because we do not necessarily have $tH(\rho_t) \to \infty$.

**Proposition 2.5.** *Suppose now that $X$ is a strongly non-lattice subordinator which has drift $b \ge 0$ and $\overline{\Pi}(\cdot)$ is regularly varying at infinity with an index $-\alpha$ for some $\alpha \in (0,1)$, and (H) holds. Define $c$ by $t\overline{\Pi}(c(t)) = 1$, so that the process $(X(tu)/c(t), u \ge 0)$ converges weakly to $(S_u, u \ge 0)$, a stable subordinator of index $\alpha$. Let $\tilde{g}_t(\cdot)$ and $\tilde{h}_x(\cdot)$ denote the density functions of $S_t$ and $T_x^S := \inf\{t : S_t > x\}$ respectively. Then uniformly for $y_t := x/c(t) > 0$*

$$th_x^J(t) = \tilde{h}_{y_t}(1) + o(1) \text{ as } t \to \infty. \tag{2.6}$$

*and, if $b > 0$, uniformly for $y_t > 0$ and $0 < \Delta < \Delta_0$,*

$$c(t)h_x^C(t,\Delta) = b\Delta\left(\tilde{g}_1(y_t) + o(1)\right) \text{ as } t \to \infty. \tag{2.7}$$

The assumption that $\overline{\Pi}$ varies regularly at zero, or infinity respectively, implies that the process $(X(tu)/c(t), u \ge 0)$ converges in the Skorohod topology to $(S_u, u \ge 0)$, a stable subordinator of index $\alpha$, as $t$ tends towards 0 or infinity respectively, and it is well known this is equivalent to having the one dimensional convergence with $u = 1$. Three generalisations of this setting may be suitable for applications. The first is allowing for a centering function $(\mathbf{b}_t, t \ge 0)$, so that $(X_t - \mathbf{b}_t)/c_t$ converges weakly. This arises for instance when $x \mapsto \overline{\Pi}(x)$ is regularly varying at infinity with an index $-\alpha$, and $\alpha > 1$. The second emerges when instead of having $\overline{\Pi}$ varying regularly this is only bounded above and below by functions which are regularly varying with the same index. Finally, it may occur that the latter convergence holds only along subsequences. These three generalisations can be handled in the single setting of subordinators in the *Feller class* as defined in Maller and Mason in [12] and [11]. Namely, our main results and its proof hold verbatim in this general setting, but for technical reasons we have chosen to describe this class in detail in the Subsection 3.1. The reader interested only in the case where $\overline{\Pi}$ varies regularly can skip this section with out harm and just read $RV_0$ where it reads $SC_0$ and $RV_\infty$ for $SC_\infty$, thereafter.

The rest of this paper is organised as follows. Section 3 is intended to provide the preliminaries for the proof of the main theorems. We start by describing the Feller class in Subsection 3.1. Then in Subsection 3.2 we gather some useful formulas related to first passage time of subordinators and also some other general facts. Then in Subsection 4 we prove Theorem 2.2 and Theorem 2.3 in the case $(SC_0)$. Then in Section 5 we prove the Theorem 2.3 under the assumptions $(SC_\infty)$ or (G).

## 3 Preliminaries

### 3.1 The Feller class

As we already mentioned, in the present work we allow a more general behaviour than that of being in the domain of attraction of a stable law, namely for most of our results we only require $X$ to be in the *Feller class*, said otherwise to be *stochastically compact*, either at infinity or at zero depending on whether $x/t$ tends to $b$ from above, or to $\mathbb{E}(X_1)$ from below. (This class includes subordinators for which $\overline{\Pi}(\cdot)$ is O-regularly varying at zero and infinity, see e.g. [3].) For background on stochastic compacteness we refer to Section 14.6 of [7].

This carries a further difference from our work in [5], namely, the results here obtained apply equally to subordinators which are stochastically compact with or without centering, while in [5] the assumption that the Lévy process is in the domain of attraction of a stable law **without** centering is in force.

In order to provide precise definitions of these notions we start by introducing some notation. We will write

$$H(u) = \psi(u) - u\psi'(u), \qquad \sigma^2(u) = \int_0^\infty y^2 e^{-uy}\Pi(dy), \qquad u \geq 0, \tag{3.1}$$

and for $x > 0$,

$$\overline{\Pi}(x) = \Pi(x,\infty), \quad K_\Pi(x) = x^{-2}\int_{y\in(0,x)} y^2\Pi(dy), \tag{3.2}$$

$$Q_\Pi(x) = \overline{\Pi}(x) + K_\Pi(x). \tag{3.3}$$

An integration by parts shows that

$$Q_\Pi(z) = 2z^{-2}\int_0^z y\overline{\Pi}(y)dy = 2\int_0^1 y\overline{\Pi}(zy)dy, \qquad z > 0, \tag{3.4}$$

and that $Q_\Pi$ is a non-increasing function. We define $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ via the relation

$$\psi'(\rho(s)) = s, \qquad 0 \leq b = \psi'(\infty) < s < \psi'(0+) \leq \infty.$$

It is worth recalling that $\psi'$ is strictly decreasing because $\Pi$ is assumed to be non-degenerate, and also $\psi'(0+)$ determines the mean of $X$,

$$\psi'(0+) = b + \int_0^\infty y\Pi(dy) = \mathbb{E}(X_1) =: \mu.$$

From the former relation it is easily seen that $\rho(\cdot)$ is a non-increasing function. For notational convenience for $b < x_t < \mu$, we will write $\rho_t := \rho(x_t)$. Note that $\rho_t \downarrow 0$ when $x_t \uparrow \mu$ and $\rho_t \uparrow \infty$ when $x_t \downarrow b$.

We will say that $X$ is in a *Feller class* or is *stochastically compact* at infinity, respectively at $0$, if

**[SC]** $\limsup \frac{\overline{\Pi}(y)}{K_\Pi(y)} < \infty$ as $y \to \infty$, respectively as $y \to 0+$.

It is known that this condition is equivalent to

**[SC′]** $\exists \alpha \in (0,2]$ and $c \geq 1$ such that $\limsup \frac{\int_0^{\lambda z} y\overline{\Pi}(y)dy}{\int_0^z y\overline{\Pi}(y)dy} \leq c\lambda^{2-\alpha}$ for $\lambda > 1$, as $z \to \infty$, respectively as $z \to 0+$;

see Lemma 1 in [11] for a proof of this equivalence and background on the study of the Feller class for general Lévy processes. In this case we will say that the condition $SC_\infty$, respectively $SC_0$, holds. We next quote some facts from the work by Maller and Mason in [12], Theorem 2.1, and [11], Theorem 1. In the case where $X$ is stochastically compact at infinity (respectively at zero), Maller and Mason proved that there exist non-decreasing functions $\mathrm{c} : [0,\infty) \to (0,\infty)$ and $\mathbf{b} : [0,\infty) \to [0,\infty)$ such that for any sequence $(t_k, k \geq 0)$ tending towards infinity (respectively, towards 0) there is a subsequence $(t'_k, k \geq 0)$ such that

$$\frac{X_{t'_k} - \mathbf{b}(t'_k)}{\mathrm{c}(t'_k)} \xrightarrow[k\to\infty]{\text{Law}} Y', \tag{3.5}$$

where $Y'$ is a real valued non-degenerate random variable, whose law may depend on the subsequence taken. A standard representation of the functions $\mathrm{c}$ and $\mathbf{b}$ are

$$tQ_\Pi(\mathrm{c}(t)) = 1, \qquad \mathbf{b}(t) = t\left(b + \int_0^{\mathrm{c}(t)} y\Pi(dy)\right), \qquad t > 0. \tag{3.6}$$

If in addition to the condition $SC_\infty$ (respectively $SC_0$) the condition

$$\limsup_{y\to\infty(y\to 0)} \frac{y(b + \int_0^y z\Pi(dz))}{\int_0^y z^2\Pi(dz)} < \infty, \tag{3.7}$$

holds, then the above defined functions satisfy

$$\limsup_{t\to\infty(t\to 0)} \frac{\mathbf{b}(t)}{c(t)} < \infty, \tag{3.8}$$

so that the normalizing function $\mathbf{b}$ is not needed and hence can be assumed to be $0$. In this case it is said that the process $X$ is stochastically compact at zero (respectively at infinity) without centering. In all other cases,

$$\limsup_{t\to\infty(t\to 0)} \frac{\mathbf{b}(t)}{c(t)} = \infty. \tag{3.9}$$

Throughout the rest of the paper we will work in one of the following more general frameworks on $\Pi$, $t$ and $x$: always $b < x_t := x/t < \mu$ and

($SC_0$-I) the Lévy measure $\Pi$ satisfies the condition $SC_0$, $t \to \infty$, $x_t \to b$;

($SC_0$-II) the Lévy measure $\Pi$ satisfies the condition $SC_0$, $t \to 0$, $x_t \to b$, and

$$\frac{x - \mathbf{b}(t)}{c(t)} \xrightarrow[t \to 0]{} -\infty$$

if (3.7) fails; or

$$\frac{x - bt}{c(t)} \xrightarrow[t \to 0]{} 0$$

if (3.7) holds.

($SC_\infty$) the Lévy measure $\Pi$ satisfies the condition $SC_\infty$, $t \to \infty$, $x_t \to \mu$ and

$$\frac{x - \mathbf{b}(t)}{c(t)} \xrightarrow[t \to \infty]{} -\infty$$

if (3.7) fails; or

$$\frac{x - bt}{c(t)} \xrightarrow[t \to \infty]{} 0$$

if (3.7) holds; and we assume also that (H) holds.

The following Lemma is an elementary consequence of Karamata's theorem for regularly varying functions.

**Lemma 3.1.**

*(i) If $x \mapsto \overline{\Pi}(x)$ is regularly varying at $0$ with an index $-\alpha$, for some $\alpha \in (0,1)$, then the condition $SC_0$ is satisfied. Moreover, the frameworks $(SC_0 - I)$ and $(SC_0 - II)$ include the framework $(RV_0)$.*

*(ii) If $x \mapsto \overline{\Pi}(x)$ is regularly varying at $\infty$ with an index $-\alpha$, for some $\alpha > 0$, then the condition $SC_\infty$ is satisfied. Moreover, the framework $(SC_\infty)$ includes the framework $(RV_\infty)$.*

This being said we can now state the more general version of our main Theorems.

**Theorem 3.2.**

*(i) The assertion in (i) in Theorem 2.2 hold in the framework $(SC_0)$.*

*(ii) The assertion in (ii) in Theorem 2.2 hold in the frameworks $(SC_0 - I)$ or $(SC_0 - II)$.*

*(iii) The estimates in Theorem 2.3 hold in the frameworks $(SC_0 - I)$, $(SC_0 - II)$, $(SC_\infty)$ and (G).*

### 3.2 Some useful facts

For sake of conciseness we gather some useful formulas in the following Lemma.

**Lemma 3.3.** *Let $X$ be a subordinator with Laplace exponent $\psi$, drift $b \geq 0$, and Lévy measure $\Pi$. We have the following facts.*

*(i) $X$ creeps, viz. $\mathbb{P}(X_{T_x} = x) > 0$ for some, and hence for all, $x > 0$, if and only if $b > 0$. In that case, for any $0 < t \leq \infty$, the occupation measure*

$$U_t(dy) := \mathbb{E}\left(\int_0^t ds 1_{\{X_s \in dy\}}\right), \qquad y \geq 0,$$

*has a continuous and bounded density on $(0,\infty)$, $u_t(y), y > 0$. The formula*

$$\mathbb{P}(T_x \in (t, t+\Delta], X_{T_x} = x) = b \int_{[0,x)} \mathbb{P}(X_t \in dy) u_\Delta(x - y), \tag{3.10}$$

*holds for $x > 0$, $t \geq 0$, $\Delta > 0$.*

*(ii) On the event of non-creeping,* $\{X_{T_x} > x\}$*, the first passage time distribution has a density given by*

$$\mathbb{P}(T_x \in dt, X_{T_x} > x) = \left( \int_{[0,x)} \mathbb{P}(X_t \in dy) \overline{\Pi}(x-y) \right) dt.$$

*The formula*

$$\begin{aligned} &\mathbb{P}(T_x \in (t, t+\Delta], X_{T_x} > x) \\ &= \int_{[0,x)} \mathbb{P}(X_t \in dy) \int_{[0,x-y]} U_\Delta(dz) \overline{\Pi}(x-y-z), \end{aligned} \tag{3.11}$$

*holds for* $x > 0$, $t \geq 0$, $\Delta > 0$.

*Proof.* The first claim in (i) in Lemma 3.3 follows from Theorem III.5 in [1] which ensures that if the drift $b > 0$, the potential measure $U_\infty$ is absolutely continuous with a continuous and bounded density $u_\infty$,

$$U_\infty(dy) := \mathbb{E}\left( \int_0^\infty ds 1_{\{X_s \in dy\}} \right) = u_\infty(y) dy, \qquad y \geq 0,$$

and the following identity holds

$$\mathbb{P}(X_{T_x} = x) = b u_\infty(x), \qquad x > 0. \tag{3.12}$$

The absolute continuity of $U_t$ on $(0, \infty)$ and the bound $b u_t \leq 1$, for any $t > 0$, follow from the fact that $U_t$ is dominated by above by $U_\infty$. We have furthermore the identity

$$u_t(y) = u_\infty(y) - \int_{[0,y]} \mathbb{P}(X_t \in dz) u_\infty(y-z), \qquad y > 0;$$

from where the continuity of $u_t$ in $(0, \infty)$ is deduced using the continuity of $u_\infty$ and the dominated convergence theorem.

In [8] (see Remark 3.1 (iii) therein) it has been proved that

$$\mathbb{P}(X_{T_x} = x, T_x \leq t) = b \frac{\partial}{\partial y} U_t[0, y]|_{y=x}.$$

The latter together with Lebesgue's derivation theorem ensures that for a.e. $x > 0$

$$\mathbb{P}(X_{T_x} = x, T_x \leq t) = b u_t(x).$$

Now, the claim will be obtained once we prove that for any $t > 0$, the function

$$x \mapsto \mathbb{P}(X_{T_x} = x, T_x \leq t), \qquad x > 0, \tag{3.13}$$

is right-continuous. For that end we observe the identity for $x > 0, t \geq 0$,

$$\begin{aligned} \mathbb{P}(X_{T_x} = x, T_x \leq t) &= \mathbb{P}(X_{T_x} = x) - \mathbb{P}(X_{T_x} = x, T_x > t) \\ &= bu(x) - b \int_{[0,x]} \mathbb{P}(X_t \in dy) u(x-y). \end{aligned}$$

Let $x_n \downarrow x$. Thanks to the equality

$$\begin{aligned} &\mathbb{P}(X_{T_x} = x, T_x \leq t) - \mathbb{P}(X_{T_{x_n}} = x_n, T_{x_n} \leq t) \\ &= u(x) - u(x_n) + b \int_{[0,x]} \mathbb{P}(X_t \in dy)(u(x-y) - u(x_n - y)) \\ &\quad - b \int_{(x,x_n]} \mathbb{P}(X_t \in dy) u(x_n - y), \end{aligned}$$

and the continuity of $u$, the claim is easily deduced from the following facts. Since $u$ is a continuous and bounded function the monotone convergence theorem allows us to infer the limit

$$b\int_{[0,x]}\mathbb{P}(X_t\in dy)(u(x-y)-u(x_n-y))\to 0.$$

Moreover, thanks to the bound $bu(z)\leq 1$, for $z>0$, we deduce the majoration

$$\begin{aligned}0\leq b\int_{(x,x_n]}\mathbb{P}(X_t\in dy)u(x_n-y)&\leq \mathbb{P}(x<X_t\leq x_n)\\ &=\mathbb{P}(X_t\leq x_n)-\mathbb{P}(X_t\leq x);\end{aligned}$$

and the right most term in the above equation tends to zero by the right continuity of $y\mapsto\mathbb{P}(X_t\leq y)$. The claim follows putting the pieces together.

The proof of the formula (3.10) is obtained from the identity

$$\begin{aligned}&\mathbb{P}(T_x\in(t,t+\Delta],X_{T_x}=x)=\mathbb{P}(X_t<x,T_x\circ\theta_t\in(0,\Delta],X_{T_x}=x)\\ &=\int_{[0,x)}\mathbb{P}(X_t\in dy)\mathbb{P}(T_{x-y}\in(0,\Delta],X_{T_{x-y}}=x-y),\end{aligned}$$

where $\theta_t$ denotes the shift operator, and we applied the simple Markov property at time $t$ to get the second equality. Also, in the case where for any $s>0$, $\mathbb{P}(X_s\in dy)=f_s(y)dy$, with $f_s(y)$ a continuous function in $y$, we can take

$$u_t(y)=\int_0^t ds f_s(y).$$

It follows from the formulas above that $T_x$ has a density on the event of creeping and

$$\mathbb{P}(T_x\in dt,X_{T_x}=x)=bf_t(x),\qquad x>0,t>0.$$

The result in (ii) follows from the fact

$$h_x^J(t)=\int_0^x\mathbb{P}(X_t\in dy)\overline{\Pi}(x-y),\qquad t>0,$$

proved in [5] Lemma 11, together with an application of the Markov property as above. □

Most of our calculations involve an exponential change of measure, introduced on page 93 in [9], which we now recall. For $\psi'(\infty)=b<\frac{x}{t}=:x_t<\mu=\psi'(0+)$ we denote by $(Y_s,s\geq 0)$, a subordinator whose Laplace exponent is

$$\psi_{\rho_t}(\lambda)=\psi(\rho_t+\lambda)-\psi(\rho_t)=b\lambda+\int_{(0,\infty)}(1-e^{-\lambda y})e^{-\rho_t y}\Pi(dy),\qquad\lambda\geq 0.\tag{3.14}$$

In particular, identifying the Laplace transforms and using $\psi'(\rho_t)=x_t$, we have the following relation:

$$\mathbb{P}(Y_t\in dy)=e^{tH(\rho_t)}e^{-\rho_t(y-tx_t)}\mathbb{P}(X_t\in dy),\qquad y\in\mathbb{R}^+.\tag{3.15}$$

Observe that in the above definition of $Y$ we are deliberately excluding the dependence in $x_t$ of $Y$. We do this for notational convenience and also because we will mainly use the equality of measures in (3.15).

The proof of our main results rely on the following technical results. The first of them relates various quantities we will consider.

**Lemma 3.4.** *We have the following relations*

*(a)* $\frac{1}{2e}Q_{\Pi}(1/u) \le H(u) \le Q_{\Pi}(1/u)$, *for* $u > 0$.

*(b)* $u^2\sigma^2(u) \le 2H(u)$ *for* $u \ge 0$.

*(c)* $\frac{u^2\sigma^2(u)}{H(u)} \ge \frac{e^{-1}}{\left(1+\frac{\overline{\Pi}(1/u)}{K_{\pi}(1/u)}\right)}$, *for* $u > 0$. *In particular, if* $X$ *is stochastically compact at infinity, respectively at* $0$, *then*

$$\liminf \frac{u^2\sigma^2(u)}{H(u)} > 0,$$

*as* $u \to 0$, *respectively as* $u \to \infty$.

*Proof.* (a) is (5.4) in [9], (b) is (5.5) in [9], and (c) is (5.6) and (5.7) in [9]. □

**Lemma 3.5.** *For* $t > 0$, $b < x_t < \mu$, *we have for any* $s > 0$

$$\mathbb{E}(Y_s) = sx_t =: \mu_s, \tag{3.16}$$

$$\mathbb{E}(Y_s - \mu_s)^2 = s\int_0^{\infty} y^2 e^{-\rho_t y}\Pi(dy) = s\sigma^2(\rho_t), \tag{3.17}$$

$$\mathbb{E}(Y_s - \mu_s)^3 = s\int_0^{\infty} y^3 e^{-\rho_t y}\Pi(dy), \tag{3.18}$$

$$\mathbb{E}\left(|Y_s - \mu_s|^3\right) \le 6s(\rho_t)^{-3}Q_{\Pi}(1/\rho_t) + 2\mu_s s\sigma^2(\rho_t). \tag{3.19}$$

*Proof.* The first three identities are proved by bare hands calculations on the Laplace transform, while the claimed upper bound is obtained as follows:

$$\begin{aligned}
\mathbb{E}\left(|Y_s-\mu_s|^3\right) &= \mathbb{E}\left((Y_s-\mu_s)^3\right) + 2\mathbb{E}\left((\mu_s - Y_s)^3 : Y_s \le \mu_s\right) \\
&\le \mathbb{E}\left((Y_s-\mu_s)^3\right) + 2\mu_s\mathbb{E}\left((\mu_s - Y_s)^2 : Y_s \le \mu_s\right) \\
&\le s\int_0^{\infty} y^3 e^{-\rho_t y}\Pi(dy) + 2\mu_s s\sigma^2(\rho_t) \\
&= s\int_{\{y\rho_t \le 1\}} y^3 e^{-\rho_t y}\Pi(dy) + s\int_{\{y\rho_t > 1\}} y^3 e^{-\rho_t y}\Pi(dy) + 2\mu_s s\sigma^2(\rho_t) \\
&\le s(\rho_t)^{-1}\int_{\{y\rho_t \le 1\}} y^2\Pi(dy) + 6s(\rho_t)^{-3}\overline{\Pi}(1/\rho_t) + 2\mu_s s\sigma^2(\rho_t) \\
&\le 6s(\rho_t)^{-3}Q_{\Pi}(1/\rho_t) + 2\mu_s s\sigma^2(\rho_t).
\end{aligned}$$

□

**Lemma 3.6.** *In the settings* ($SC_0$-*(I-II))*, $(SC_{\infty})$ *and* $(G)$, *we have that* $tH(\rho_t) \to \infty$ *uniformly in* $x$.

*Proof.* The proof of the case $(G)$ is a straightforward consequence of the fact that in this setting $t \to \infty$ and

$$0 < \liminf_{t\to\infty} H(\rho_t) \le \limsup_{t\to\infty} H(\rho_t) < \infty,$$

because $b < \liminf_{t\to\infty} x_t \le \limsup_{t\to\infty} x_t < \mu$, and hence

$$0 < \liminf_{t\to\infty} \rho_t \le \limsup_{t\to\infty} \rho_t < \infty.$$

To deal with the cases ($SC_0$-(I-II)), ($SC_\infty$) we use Theorem 5.1 of Jain and Pruitt [9] which establishes that the condition $tH(\rho_t) \to \infty$ is equivalent to $\mathbb{P}(X_t \le x) \to 0$, as $t \to \infty$ or $t \to 0$. For the case $(SC_0 - I)$ when (3.7) fails, the equality

$$\mathbb{P}(X_t \le x) = \mathbb{P}\left(\frac{X_t}{t} - b \le x_t - b\right),$$

and an application of the weak law of large numbers for subordinators gives the result because $\Pi$ is non-degenerate. To deal with the case $(SC_0 - II)$, we use the equality

$$\mathbb{P}(X_t \le x) = \mathbb{P}\left(\frac{X_t - \mathbf{b}(t)}{c(t)} \le \frac{x - \mathbf{b}(t)}{c(t)}\right),$$

which together with the sequential convergence in (3.5) and the assumption that $\frac{x-\mathbf{b}(t)}{c(t)} \to -\infty$ as $t \to 0$, lead to $\mathbb{P}(X_t \le x) \to 0$ as $t \to 0$. The case when (3.7) holds as well as the cases ($SC_\infty$) are proved with a similar argument. To show the uniformity observe that the function

$$\lambda \mapsto H(\lambda) = \psi(\lambda) - \lambda\psi'(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda x} - \lambda x e^{-\lambda x})\Pi(dx),$$

is increasing because the function $z \mapsto 1 - e^{-z} - ze^{-z}$ is. This implies that the function $\lambda \mapsto H(\rho(\lambda))$ is decreasing. The uniformity in the cases $(G)$ and ($SC_0$-(I-II)) follows easily from this fact. Indeed, it is enough to observe that $tH(\rho_t)$ tends towards $\infty$ as soon as we take a $x_0$ such that $x_0 > x$ and $tH\left(\rho\left(\frac{x_0}{t}\right)\right) \to \infty$. To establish the uniformity in the case $(SC_\infty)$ when (3.7) holds, we observe that the hypotheses imply that there is a function $D$ such that $x \le \mathbf{b}(t) - D(t)$, and $D(t)/c(t) \to \infty$ as $t \to \infty$. The function $D$ is such that

$$tH(\rho_t) \ge tH\left(\rho\left(\frac{\mathbf{b}(t) - D(t)}{t}\right)\right) \xrightarrow[t\to\infty]{} \infty,$$

because by the assumption of stochastic compactness at $\infty$ we have that

$$\mathbb{P}(X_t < \mathbf{b}(t) - D(t)) = \mathbb{P}\left(\frac{X_t - \mathbf{b}(t)}{c(t)} \le -\frac{D(t)}{c(t)}\right) \xrightarrow[t\to\infty]{} 0.$$

In the case $(SC_\infty)$, when (3.7), does not hold we proceed as above but using that there is a function $j$ such that $x \le bt + j(t)$ and $j(t)/c(t) \to 0$ as $t \to \infty$. □

**Lemma 3.7.** *If (H) holds then*

$$\int_{|z|>1} |\exp\{-t\left(\psi(\lambda - iz) - \psi(\lambda)\right)\}| \left|\frac{\psi_*(\lambda - iz)}{\lambda - iz}\right| dz$$
$$\le e^{2t\psi_*(\lambda)} \int_{|z|>1} |\exp\{-t\left(\psi(-iz)\right)\}| \frac{\psi_*(\lambda) + |\psi_*(-iz)|}{z} dz < \infty,$$

*for any $\lambda > 0$ and $t > t_0$.*

*Proof.* The proof of this result is an easy consequence of the two inequalities:

$$\begin{aligned}
&\mathcal{R}\left(\psi(\lambda - iz) - \psi(\lambda)\right)\\
&= \int_{(0,\infty)} (1 - \cos(zy))e^{-\lambda y}\Pi(dy)\\
&= \int_{(0,\infty)} (1 - \cos(zy))\Pi(dy) - \int_{(0,\infty)} (1 - \cos(zy))\left(1 - e^{-\lambda y}\right)\Pi(dy)\\
&\ge \int_{(0,\infty)} (1 - \cos(zy))\Pi(dy) - 2\int_{(0,\infty)} \left(1 - e^{-\lambda y}\right)\Pi(dy)\\
&= \mathcal{R}\psi(-iz) - 2\int_{(0,\infty)} \left(1 - e^{-\lambda y}\right)\Pi(dy);
\end{aligned}$$

and

$$
\begin{aligned}
&|\,(\psi_*(\lambda - iz) - \psi_*(-iz))\,| \\
&= |\int_{(0,\infty)} (1-e^{-\lambda y})e^{izy}\Pi(dy)| \leq \int_{(0,\infty)} (1-e^{-\lambda y})\Pi(dy) = \psi_*(\lambda).
\end{aligned}
$$

□

# 4 Proof of Theorem 2.2 and Theorem 2.3 in the $SC_0$ cases

## 4.1 Proof of (i) in Theorem 2.2

Let $U(x) = \int_0^x y\overline{\Pi}(y)dy$, $x \geq 0$, and

$$
\widehat{U}(s) = \int_0^\infty e^{-sy}U(dy) = s\int_0^\infty e^{-sy}U(y)dy, \quad s > 0,
$$

be its Laplace transform. By hypothesis we have that the condition ($SC_0'$) is satisfied, which implies that $U$ has bounded increase at $0$, see [3] page 68 and 71. So, by the proof of the inclusion $(i) \Rightarrow (ii)$ in Theorem 2.10.2 in [3], we know that there are constants $0 < c_1 \leq c_2 < \infty$ such that for small $s$, $c_1U(s) \leq \widehat{U}(1/s) \leq c_2U(s)$. From this it follows that $\widehat{U}$ has bounded decrease at infinity. Indeed, taking $\alpha$ as in ($SC_0'$) we have for $\lambda > 1$

$$
\liminf_{s\to\infty} \frac{\widehat{U}(s\lambda)}{\widehat{U}(s)} \geq c_3 \liminf_{s\to\infty} \frac{U\left(\frac{1}{s\lambda}\right)}{U\left(\frac{1}{s}\right)} = c_3\left(\limsup_{v\to 0+} \frac{U(\lambda v)}{U(v)}\right)^{-1} \geq c_3\frac{1}{\lambda^{2-\alpha}}.
$$

Proposition 2.2.1 in [3] implies that for any $\beta < -(2-\alpha)$ there exist constants $c_4 > 0$ and $\widetilde{\ell} > 0$, such

$$
\frac{\widehat{U}(y)}{\widehat{U}(x)} \geq c_4\left(\frac{y}{x}\right)^\beta, \qquad y \geq x \geq \widetilde{\ell}. \tag{4.1}
$$

Also, an easy integration by parts implies the identity

$$
\widehat{U}(s) = -\left(\frac{\psi(s)}{s}\right)' = \frac{H(s)}{s^2}, \qquad s > 0. \tag{4.2}
$$

So, by (4.1) we have

$$
\frac{H(y)}{H(x)} \geq c_4\left(\frac{y}{x}\right)^{2+\beta}, \qquad y \geq x \geq \widetilde{\ell}, \tag{4.3}
$$

where $2+\beta < \alpha \leq 2$. Since $0 < \alpha$ there exists $\beta_0$ and positive constants $c_6$ and $\widetilde{\ell}$ such that $0 < 2+\beta_0 < \alpha \leq 2$ and

$$
H(y) \geq y^{2+\beta_0}c_6, \qquad y \geq \widetilde{\ell}.
$$

To conclude we observe that the following inequalities hold

$$
\begin{aligned}
\int_0^\infty (1-\cos(\theta y))\Pi(dy) &\geq c_7\theta^2\int_0^{1/\theta} y^2\Pi(dy) \\
&= c_7K_\Pi(1/\theta) = c_7\frac{K_\Pi(1/\theta)}{Q_\Pi(1/\theta)}Q_\Pi(1/\theta) \\
&\geq c_8Q_\Pi(1/\theta) \\
&= 2c_8\theta^2U(1/\theta) \\
&\geq c_9H(\theta)
\end{aligned} \tag{4.4}
$$

for $\theta$ large enough; here we used the assumption $(SC_0)$ and the equality (3.4). We infer that for $\theta > 0$ large enough

$$\Re(\psi(i\theta)) = \int_0^\infty (1-\cos(\theta y))\Pi(dy) \geq c_{10}\theta^{\beta_0+2}.$$

As a consequence, for $n \geq 0$

$$\int_{\mathbb{R}} |\theta|^n |\mathbb{E}(e^{i\theta X_t})| d\theta = \int_{\mathbb{R}} |\theta|^n \exp\{-t\Re(\psi(i\theta))\} d\theta < \infty,$$

and the conclusion follows from Proposition 28.1 in [14].

### 4.2 Proof of (ii) in Theorem 2.2

Before we start with the proof we state a further auxiliary theorem. This is a consequence of Lemma 5.1 in page 147 in [13].

**Lemma 4.1.** *Let $Z_1, Z_2, \cdots Z_n$ be independent rvs having finite 3rd moments, write $\mathbb{E}(Z_r) = \mu_r, Var(Z_r) = \sigma_r^2$, and $\mathbb{E}\left(|Z_r - \mu_r|^3\right) = \nu_r$, and put $W = \sum_1^n Z_r, m = \mathbb{E}(W) = \sum_1^n \mu_r$, and $s^2 = VarW = \sum_1^n \sigma_r^2$. Assume further that $\int_{-\infty}^{\infty} |\Psi(u)| du < \infty$, where $\Psi(u) = \mathbb{E}\left(e^{iuW}\right)$, and denote by $f$ and $\phi$ the pdf of $W$ and the standard Normal pdf. Then there is an absolute constant $A$ such that*

$$\sup_y \left| f(y) - s^{-1}\phi\left(\frac{y-m}{s}\right)\right| \leq AL + d, \tag{4.5}$$

*where $L = \sum_1^n \nu_r/s^4$ and, with $l = (4Ls^2)^{-1}$,*

$$d = 2\int_l^\infty |\Psi(u)| du.$$

*Proof.* Use Fourier inversion as in Lemma 3 of [4]. □

**Remark 4.2.** Our use of this result exploits the fact that, for any Lévy process, any $t > 0$, and any $n \geq 1$, $X_t$ is the sum of $n$ independent and identically distributed summands.

*Proof of (ii) in Theorem 2.2.* We observe first that the assumption that $b < x_t < \mu$ and $x_t \to b$ implies that $\rho_t \to \infty$, irrespective of whether $t \to 0$ or $t \to \infty$. We next establish that these conditions on $x_t$, the fact that $tH(\rho_t) \to \infty$, and the stochastic compactness at $0$, imply that $x\rho_t \to \infty$, again irrespective of whether $t \to 0$ or $t \to \infty$. Indeed, the identities

$$\frac{tH(\rho_t)}{x\rho_t} = \frac{t\psi(\rho_t) - t\rho_t\psi'(\rho_t)}{t\rho_t\psi'(\rho_t)} = \frac{\psi(\rho_t)}{\rho_t\psi'(\rho_t)} - 1, \tag{4.6}$$

show that it is enough to justify that $0 < \liminf_{z\to\infty} \frac{z\psi'(z)}{\psi(z)}$. If the drift of $X$ is positive this is straightforward. If the drift is zero this holds whenever $\limsup_{z\to 0} \frac{z\overline{\Pi}(z)}{\int_0^z y\Pi(dy)} < \infty$, which in turn holds by stochastic compactness at zero,

$$\limsup_{z\to 0} \frac{z\overline{\Pi}(z)}{\int_0^z y\Pi(dy)} \leq \limsup_{z\to 0} \frac{z^2\overline{\Pi}(z)}{\int_0^z y^2\Pi(dy)} < \infty.$$

The former claim is an easy consequence of the following inequalities

$$\begin{aligned}\frac{\lambda\psi'(\lambda)}{\psi(\lambda)} &= \frac{\int_0^\infty ye^{-\lambda y}\Pi(dy)}{\int_0^\infty e^{-\lambda y}\overline{\Pi}(y)dy} \\ &\geq \frac{\int_0^{1/\lambda} ye^{-1}\Pi(dy)}{\int_0^{1/\lambda}\overline{\Pi}(y)dy + (1/\lambda)\overline{\Pi}(1/\lambda)} \\ &= \frac{e^{-1}\int_0^{1/\lambda} y\Pi(dy)}{\int_0^{1/\lambda} y\Pi(dy) + (2/\lambda)\overline{\Pi}(1/\lambda)}\end{aligned}$$

which are obtained by barehand calculations. It is important to remark that the above facts and the Lemma 3.6 imply that $x\rho_t \to \infty$ uniformly in $x$. Furthermore, our previous remarks allow us to provide a unified proof of the cases $t \to 0$ or $t \to \infty$.

We will apply Lemma 4.1 with $n = [x\rho_t]$ and $W = Y_t = \sum_{k=1}^n Z_k$ with $Z_k = Y_{\frac{tk}{n}} - Y_{\frac{t(k-1)}{n}} \overset{\text{Law}}{=} Y_{\frac{t}{n}}$, for $k \in \{1, \dots, n\}$. We use the estimate (3.16) with $s = t/n$, thus $\mu_s = x/n$, which together with our choice of $n$ lead to the approximation

$$\begin{aligned} n\nu &:= n\mathbb{E}\left(|Z_1 - x/n|^3\right) \le t\left\{6(\rho_t)^{-3}Q_\Pi(1/\rho_t) + 2\frac{x}{n}\sigma^2(\rho_t)\right\} \\ &= t\left\{6(\rho_t)^{-3}Q_\Pi(1/\rho_t) + 2\frac{\rho_t^2\sigma^2(\rho_t)}{\rho_t^3}(1+o(1))\right\}, \end{aligned} \tag{4.7}$$

for $\rho_t$ large enough; here the term

$$1 \le 1 + o(1) = \frac{x\rho_t}{[x\rho_t]} \le \frac{1}{1 - \frac{1}{x\rho_t}} \to 1,$$

and the convergence is uniform in $x$. It is then immediate from the definition of $L$ that for $\rho_t$ large enough

$$\sqrt{t}\sigma(\rho_t)L = \frac{n\nu}{\{t\sigma^2(\rho_t)\}^{\frac{3}{2}}} \le \frac{1}{\left(t\rho_t^2\sigma^2(\rho_t)\right)^{1/2}}\left(6\frac{Q_\Pi(1/\rho_t)}{\rho_t^2\sigma^2(\rho_t)} + 2(1+o(1))\right). \tag{4.8}$$

Which because of the assumption of stochastic compactness at 0 and Lemma 3.4 imply that for $x\rho_t$ and $\rho_t$ large enough there is a constant $k_1$ such that

$$\sqrt{t}\sigma(\rho_t)L \le \frac{k_1}{\sqrt{tH(\rho_t)}}. \tag{4.9}$$

So the lemma tells us that (2.2) holds provided that

$$\gamma := \sqrt{t}\sigma(\rho_t)\int_l^\infty e^{-t\Re(\psi_{\rho_t}(i\theta))}d\theta \to 0.$$

To prove that this is indeed the case, observe that the above estimate for $L$ gives

$$\ell = \frac{1}{4Ls^2} = \frac{t\sigma^2(\rho_t)}{4n\nu} \gtrsim k_2\frac{\left(t\rho_t^2\sigma^2(\rho_t)\right)^{1/2}}{\left(t\sigma^2(\rho_t)\right)^{1/2}} = k_2\rho_t,$$

for $\rho_t$ large enough. Applying the inequality (4.4) we obtain that for $\theta \ge \ell \ge k_2\rho_t$

$$\begin{aligned} \Re(\psi_{\rho_t}(i\theta)) &= \int_0^\infty (1 - \cos(\theta y))e^{-\rho_t y}\Pi(dy) \\ &\ge k_3 e^{-1/k_2}\theta^2\int_0^{1/\theta} y^2\Pi(dy) \ge k_4 H(\theta). \end{aligned}$$

It follows from the above and the estimate (4.3) that for any $0 < \alpha_0 < \alpha \le 2$, with $\alpha$ as in $(SC_0')$, and for $\rho_t$ large enough

$$\begin{aligned} \sqrt{t}\sigma(\rho_t)\int_l^\infty e^{-t\Re(\psi_\rho(i\theta))}d\theta &\le \sqrt{t}\sigma(\rho_t)\int_l^\infty \exp\{-k_4 tH(\rho_t)\frac{H(\theta)}{H(\rho_t)}\}d\theta \\ &\le \sqrt{t}\sigma(\rho_t)\int_\ell^\infty \exp\left\{-k_5 tH(\rho_t)\left(\frac{\theta}{\rho_t}\right)^{\alpha_0}\right\}d\theta \\ &\le \sqrt{t}\rho_t\sigma(\rho_t)\int_{k_2}^\infty \exp\left\{-k_5 tH(\rho_t)\theta^{\alpha_0}\right\}d\theta \\ &\le k_6\frac{\sqrt{t}\rho_t\sigma(\rho_t)}{\left(tH(\rho_t)\right)^{1/\alpha_0}} \\ &\le k_7\frac{1}{\left(tH(\rho_t)\right)^{1/\alpha_0 - 1/2}} \to 0, \end{aligned}$$

where in the last inequality we used Lemma 3.4 (b). Observe that the uniformity follows from Lemma 3.6 and the fact that $\rho_t$ tends to infinity uniformly as well because it is non-increasing. □

### 4.3 Proof of Theorem 2.3 in the $SC_0$ cases

*Proof of estimate in (2.3) in the $SC_0$ case.* With $s_t := \sqrt{t}\sigma(\rho_t)$ we start by observing that

$$\begin{aligned}
&\sqrt{2\pi} s_t e^{tH(\rho_t)} h_x^J(t) \\
&= \sqrt{2\pi} s_t e^{tH(\rho_t)} \int_0^x \overline{\Pi}(y) f_t(x-y) dy \\
&= \sqrt{2\pi} \int_0^x \overline{\Pi}(y) e^{-\rho_t y} \left( \phi\left(\frac{y}{s_t}\right) + o(1) \right) dy \\
&\leq \frac{\psi_*(\rho_t)}{\rho_t}(1+o(1))
\end{aligned} \tag{4.10}$$

where we have used the identity

$$\int_0^\infty e^{-\rho_t y}\overline{\Pi}(y)dy = \frac{\psi_*(\rho_t)}{\rho_t}.$$

To establish a lower bound, we use that for $\varepsilon > 0$, there exists a $\delta > 0$ such that if $v \leq \delta s_t$ we have $\sqrt{2\pi}\phi\left(\frac{v}{s_t}\right) \geq 1-\varepsilon$. Put $x^* := x \wedge \delta s_t$ and write

$$\begin{aligned}
\sqrt{2\pi}\int_0^x \overline{\Pi}(y)e^{-\rho_t y}\phi\left(\frac{y}{s_t}\right) dy &\geq (1-\varepsilon)\int_0^{x^*} \overline{\Pi}(y)e^{-\rho_t y} dy \\
&= (1-\varepsilon)\left(\frac{\psi_*(\rho_t)}{\rho_t} - \int_{x^*}^\infty \overline{\Pi}(y)e^{-\rho_t y}dy\right) \\
&= (1-\varepsilon)\left(\frac{\psi_*(\rho_t)}{\rho_t}\right)(1+o(1)).
\end{aligned}$$

Here we use

$$\begin{aligned}
\frac{\int_{x^*}^\infty \overline{\Pi}(y)e^{-\rho_t y}dy}{\frac{\psi(\rho_t)}{\rho_t} - b} &\leq \frac{\overline{\Pi}(x^*)e^{-\rho_t x^*}}{\int_0^\infty (1-e^{-\rho_t y})\Pi(dy)} \\
&\leq \frac{\overline{\Pi}(x^*)e^{-\rho_t x^*}}{(1-e^{-\rho_t x^*})\overline{\Pi}(x^*)} = \frac{1}{e^{\rho_t x^*}-1}
\end{aligned}$$

and the fact that $x^*\rho_t \to \infty$, uniformly either as $t \to \infty$ or $t \to 0$, which follows from the fact that $s_t\rho_t = \sqrt{t}\rho_t\sigma(\rho_t) \geq tH(\rho_t)$, in Lemma 3.4, and the Lemma 3.6; together with the fact that $x\rho_t \to \infty$, uniformly, which was proved in (4.6). □

Essentially the same arguments, together with the formulas in (3.10) and (3.11) allow us to prove the estimates in (2.4) and (2.5).

*Proof of (2.5) in the $SC_0$ case.* From the identity (3.10) and (2.2) we deduce the following upper bound

$$\begin{aligned}
&\sqrt{2\pi}s_t e^{tH(\rho_t)} h_x^C(t,\Delta) \\
&= b\sqrt{2\pi}s_t e^{tH(\rho_t)} \int_{[0,x)} dy f_t(y) u_\Delta(x-y) \\
&= b\int_{[0,x)} dy \left[\sqrt{2\pi}\phi\left(\frac{y}{s_t}\right) + o(1)\right] e^{-\rho_t y} u_\Delta(y) \\
&\le b(1+o(1)) \int_{[0,\infty)} dy e^{-\rho_t y} u_\Delta(y) \\
&= b(1+o(1)) \frac{1-e^{-\Delta\psi(\rho_t)}}{\psi(\rho_t)}.
\end{aligned}$$

To get a lower bound, we proceed as in the previous proof and bound by below the expression in the 3rd line above by

$$\begin{aligned}
&b(1-\varepsilon+o(1)) \int_{[0,x^*)} dy e^{-\rho_t y} u_\Delta(y) \\
&= b(1-\epsilon+o(1)) \left( \frac{1-e^{-\Delta\psi(\rho_t)}}{\psi(\rho_t)} - \int_{[x^*,\infty)} dy e^{-\rho_t y} u_\Delta(y) \right).
\end{aligned}$$

The conclusion follows from the bound

$$\begin{aligned}
\int_{[x^*,\infty)} dy e^{-\rho_t y} u_\Delta(y) &\le e^{-\rho_t x^*/2} \int_{[0,\infty)} dy e^{-\rho_t y/2} u_\Delta(y) \\
&= e^{-\rho_t x^*/2} \frac{1-e^{-\Delta\psi(\rho_t/2)}}{\psi(\rho_t/2)} = o\left(\frac{1-e^{-\Delta\psi(\rho_t)}}{\psi(\rho_t)}\right),
\end{aligned}$$

where we have used the fact that $\psi(\rho_t) \sim b\rho_t$, which is in turn a well known property of the Laplace exponent $\psi$, see Proposition 2 in Chapter I in [1], and the fact that $\rho_t \to \infty$. □

*Proof of (2.4) in the $SC_0$ case.* From the identity (3.11) and (2.2) we deduce the following upper bound

$$\begin{aligned}
&\sqrt{2\pi}s_t e^{tH(\rho_t)} h_x^J(t,\Delta) \\
&= \sqrt{2\pi}s_t e^{tH(\rho_t)} \int_{[0,x)} dy f_t(x-y) \int_{[0,y)} U_\Delta(dz)\overline{\Pi}(y-z) \\
&= \int_{[0,x)} dy \left[\sqrt{2\pi}\phi\left(\frac{y}{s_t}\right) + o(1)\right] e^{-\rho_t y} \int_{[0,y)} U_\Delta(dz)\overline{\Pi}(y-z) \\
&\le [1+o(1)] \int_{[0,\infty)} dy e^{-\rho_t y} \int_{[0,y)} U_\Delta(dz)\overline{\Pi}(y-z) \\
&= (1+o(1)) \left[\int_0^\Delta ds \mathbb{E}(e^{-\rho_t X_s})\right] \left[\int_{(0,\infty)} dz \overline{\Pi}(z) e^{-\rho_t z}\right] \\
&= (1+o(1)) \left[\frac{1-e^{-\Delta\psi(\rho_t)}}{\psi(\rho_t)}\right] \left[\frac{\psi_*(\rho_t)}{\rho_t}\right].
\end{aligned}$$

To get a lower bound, we proceed as in the previous proof and bound by below the expression in the 3rd line above by

$$\begin{aligned}&(1-\varepsilon+o(1))\int_{[0,x^*)} dye^{-\rho_t y}\int_{[0,y)} U_\Delta(dz)\overline{\Pi}(y-z)\\&=(1-\varepsilon+o(1))\left(\left[\frac{1-e^{-\Delta\psi(\rho_t)}}{\psi(\rho_t)}\right]\left[\frac{\psi_*(\rho_t)}{\rho_t}\right]\right.\\&\quad\left.-\int_{[x^*,\infty)} dye^{-\rho_t y}\int_{[0,y)} U_\Delta(dz)\overline{\Pi}(y-z)\right).\end{aligned}$$

The conclusion follows from the bound

$$\begin{aligned}&\int_{[x^*,\infty)} dye^{-\rho_t y}\int_{[0,y)} U_\Delta(dz)\overline{\Pi}(y-z)\\&\le e^{-\rho_t x^*/2}\int_{[0,\infty)} dye^{-\rho_t y/2}\int_{[0,y)} U_\Delta(dz)\overline{\Pi}(y-z)\\&=e^{-\rho_t x^*/2}\left[\frac{1-e^{-\Delta\psi(\rho_t/2)}}{\psi(\rho_t/2)}\right]\left[\frac{\psi_*(\rho_t/2)}{\rho_t/2}\right],\end{aligned}$$

since it is easy to see that there is some $K$ with

$$\left[\frac{1-e^{-\Delta\psi(\rho_t/2)}}{\psi(\rho_t/2)}\right]\left[\frac{\psi_*(\rho_t/2)}{\rho_t/2}\right]\le K\left[\frac{1-e^{-\Delta\psi(\rho_t)}}{\psi(\rho_t)}\right]\left[\frac{\psi_*(\rho_t)}{\rho_t}\right].$$

Indeed, this is a consequence of the results in Lemma 2 of Chapter 2 in [6] and the fact that the function $x\mapsto U_\Delta[0,x]$ is subadditive, which in turn follows from the identity

$$U_\Delta[0,x]=\mathbb{E}(T_x\wedge\Delta),$$

and the strong Markov property of $X$. □

## 5 Proof of Theorem 2.3

We have already proved Theorem 2.3 in the $(SC_0)$ cases, so we will hereafter omit that case. For the $(SC_\infty)$ and $(G)$ cases we will use the following result instead of Lemma 4.1.

**Lemma 5.1.** *Let $\Lambda$ be defined by*

$$\Lambda:=s_tL=\frac{[x\rho_t]\mathbb{E}\left(|Y_1-\mu_1|^3\right)}{(t\sigma^2(\rho_t))^{3/2}}.$$

*We have the inequality*

$$|\mathbb{E}(e^{-iz(Y_t-\mu_t)/s_t})-e^{-z^2/2}|\le 16\Lambda|z|^3e^{-z^2/3}\text{ for all }|z|\le 1/4\Lambda,$$

*and under the assumptions of Theorem 2.3 there exists a constant $K\in(0,\infty)$, such that for large $t$,*

$$\Lambda\le\frac{K}{\sqrt{tH(\rho_t)}}\to 0. \tag{5.1}$$

*uniformly in $x$.*

*Proof.* The claimed inequality is a consequence of the Esséen-like inequality in Lemma 5.1 page 147 in [13], applied to the sum of i.i.d. random variables

$$\sum_{k=1}^n\left(Y_{\frac{tk}{n}}-Y_{\frac{t(k-1)}{n}}\right)=Y_t,$$

with $n = [x\rho_t]$. The common mean and variance are $\mu_{t/n} = \frac{t}{n}x_t$ and $s^2_{t/n} = \frac{t}{n}\sigma^2(\rho_t)$, respectively. As in the proof of (ii) in Theorem 2.2 it is proved that $x\rho_t \to \infty$ uniformly. Using this fact it is easy to verify that the arguments used to obtain (4.7) and (4.8) can be extended to show that

$$\sqrt{t}\sigma(\rho_t)L = \frac{[x\rho_t]\mathbb{E}\left(|Y_1 - \mu_1|^3\right)}{(t\sigma^2(\rho_t))^{3/2}} \lesssim \frac{1}{\left(t\rho_t^2\sigma^2(\rho_t)\right)^{1/2}}\left(6\frac{Q_\Pi(1/\rho_t)}{\rho_t^2\sigma^2(\rho_t)} + 2\right). \tag{5.2}$$

In the $(SC_\infty)$ case, the above inequality together with the Lemma 3.4 gives the result. In the $(G)$ case, the result also follows immediately from the latter inequality but one needs to recall that because $b < \liminf_{t\to\infty} x_t \le \limsup_{t\to\infty} x_t < \mu$, then

$$0 < \liminf_{t\to\infty} \rho_t \le \limsup_{t\to\infty} \rho_t < \infty,$$

$$0 < \liminf_{t\to\infty} H(\rho_t) \le \limsup_{t\to\infty} H(\rho_t) < \infty,$$

and

$$0 < \liminf_{t\to\infty} \sigma^2(\rho_t) \le \limsup_{t\to\infty} \sigma^2(\rho_t) < \infty.$$

□

*Proof of estimate in (2.3).* We carry out this proof in several steps.
**Step 1: A useful representation for $h^J_x$.** Let $\lambda \ge 0$, fixed. We know that

$$h^J_z(t) = \int_0^z \mathbb{P}(X_t \in dy)\overline{\Pi}(z-y) \tag{5.3}$$

$$= e^{\lambda z}\int_0^z e^{-\lambda y}\mathbb{P}(X_t \in dy)e^{-\lambda(z-y)}\overline{\Pi}(z-y), \tag{5.4}$$

and we note that for $\beta \in \mathbb{C}$, such that $\Re\beta \ge 0$,

$$\int_0^\infty e^{-\beta y}\overline{\Pi}(y)dy = \frac{\psi_*(\beta)}{\beta};$$

when $\beta = 0$ the above inequality is understood in the limiting sense. It follows that

$$\int_0^\infty e^{iyz}e^{-\lambda y}h^J_y(t)dy = \left(\frac{\psi_*(\lambda - iz)}{\lambda - iz}\right)e^{-t\psi(\lambda - iz)}.$$

For each $t > 0$ and $\lambda > 0$ define a probability density function by the relation

$$g^\lambda_t(y) = \frac{\lambda}{\psi_*(\lambda)}e^{-\lambda y + t\psi(\lambda)}h^J_y(t), \qquad y \ge 0.$$

This probability density equals that of the convolution of $\mathbb{P}(Y^\lambda_t \in dy) := e^{-\lambda y + t\psi(\lambda)}\mathbb{P}(X_t \in dy)$ and $\mathbb{P}(Z_\lambda \in dy) = \frac{\lambda}{\psi_*(\lambda)}e^{-\lambda y}\overline{\Pi}(y)dy$. It follows from the above calculations that for any $z \in \mathbb{R}$

$$\widehat{g}^\lambda_t(z) := \int_0^\infty dy e^{iyz}g^\lambda_t(y)$$
$$= \exp\{-t\left(\psi(\lambda - iz) - \psi(\lambda)\right)\}\frac{\psi_*(\lambda - iz)}{\lambda - iz}\frac{\lambda}{\psi_*(\lambda)}.$$

As a consequence of the hypothesis (H) it is proved in the Lemma 3.7 that we always have $\widehat{g}^\lambda \in L_1$. Then by the inversion theorem for Fourier transforms we get the key expression

$$g^\lambda_t(y) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-izy}\widehat{g}^\lambda_t(z)dz, \qquad y \in \mathbb{R}. \tag{5.5}$$

Now take $y = x$, and $\lambda = \rho_t = \rho(x/t)$, and denote $\mu_t = \mathbb{E}(Y_t) = x$. Recalling that $\rho_t x = t\rho_t\psi'(\rho_t)$, we rewrite the above formula as

$$\begin{aligned}\frac{\rho_t}{\psi(\rho_t)}e^{tH(\rho_t)}h_x^J(t) = g_t^{\rho_t}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-izx}\widehat{g}_t^{\rho_t}(z)dz \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}(\exp\{iz\left((Y_t-\mu_t)+Z_{\rho_t}\right)\})dz.\end{aligned} \tag{5.6}$$

**Step 2: Taking $s_t := \sqrt{t}\sigma(\rho_t)$ we prove the uniform convergence**

$$\frac{s_t}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}(\exp\{iz\left((Y_t-\mu_t)+Z_{\rho_t}\right)\})dz \to \frac{1}{\sqrt{2\pi}} \tag{5.7}$$

To this end, we first notice that by a change of variables (5.7) amounts to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}\left(\exp\left\{iz\left(\frac{Y_t-\mu_t}{s_t}\right)\right\}\right)\mathbb{E}\left(\exp\left\{iz\frac{Z_{\rho_t}}{s_t}\right\}\right)dz \to \frac{1}{\sqrt{2\pi}} \tag{5.8}$$

**Step 2.1: We show that $\mathbb{E}\left(Z_{\rho_t}/s_t\right) \to 0$ uniformly.** We write

$$\begin{aligned}\mathbb{E}(Z_{\rho_t}) &= \frac{\rho_t}{\psi_*(\rho_t)}\int_0^{\infty} ye^{-\rho_t y}\overline{\Pi}(y)dy = \frac{\rho_t}{\psi_*(\rho_t)}\int_{(0,\infty)}\Pi(dz)\int_0^z ye^{-\rho_t y}dy \\ &= \frac{1}{\rho_t\left(\psi_*(\rho_t)\right)}\left(\psi_*(\rho_t)-\rho_t\psi_*'(\rho_t)\right) \le \frac{1}{\rho_t},\end{aligned}$$

and the result follows since we know $\rho_t s_t \to \infty$ uniformly.

**Step 2.2.** We now split the integral in (5.8) in four terms

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathbb{E}\left(\exp\left\{iz\left(\frac{Y_t-\mu_t}{s_t}\right)\right\}\right)\mathbb{E}\left(\exp\left\{iz\frac{Z_{\rho_t}}{s_t}\right\}\right)dz \\ &= \int_{|z|>1/4\Lambda}\mathbb{E}\left(\exp\left\{iz\left(\frac{Y_t-\mu_t}{s_t}\right)\right\}\right)\mathbb{E}\left(\exp\left\{iz\frac{Z_{\rho_t}}{s_t}\right\}\right)dz + \int_{|z|\le 1/4\Lambda} e^{-z^2/2}dz \\ &+ \int_{|z|\le 1/4\Lambda}\left[\mathbb{E}\left(\exp\left\{iz\left(\frac{Y_t-\mu_t}{s_t}\right)\right\}\right) - e^{-z^2/2}\right]\mathbb{E}\left(\exp\left\{iz\frac{Z_{\rho_t}}{s_t}\right\}\right)dz \\ &- \int_{|z|\le 1/4\Lambda} e^{-z^2/2}\left[1-\mathbb{E}\left(\exp\left\{iz\frac{Z_{\rho_t}}{s_t}\right\}\right)\right]dz \\ &= I + II + III + IV\end{aligned} \tag{5.9}$$

By Lemma 5.1 we know that $\Lambda \to 0$ uniformly in $x$. It follows that $II \to \sqrt{2\pi}$. Also by the inequality in Lemma 5.1 it is straightforward that $III \to 0$. We can bound $IV$ in modulus by

$$\frac{\mathbb{E}\left(Z_{\rho_t}\right)}{s_t}\int_{|z|\le 1/4\Lambda}|z|e^{-z^2/2}dz,$$

so by the previous result it remains only to verify that

$$\left|\int_{|z|>1/4\Lambda}\mathbb{E}\left(\exp\left\{iz\left(\frac{Y_t-\mu_t}{s_t}\right)\right\}\right)\mathbb{E}\left(\exp\left\{iz\frac{Z_{\rho_t}}{s_t}\right\}\right)dz\right| \to 0, \tag{5.10}$$

**Step 2.3: In the frameworks ($SC_\infty$) or $(G)$, the estimate (5.10) holds.** From Lemma 5.1 and Lemma 3.4 we know that there is a constant $k_1$ such that $s_t\Lambda \le k_1/\rho_t$.

Using this fact and making elementary manipulations we deduce the following upper bound

$$\begin{aligned}
&\left|\int_{|z|>1/4\Lambda}\mathbb{E}\left(\exp\left\{iz\left(\frac{Y_t-\mu_t}{s_t}\right)\right\}\right)\mathbb{E}\left(\exp\left\{iz\frac{Z_{\rho_t}}{s_t}\right\}\right)dz\right|\\
&\leq s_t\int_{|z|>1/4s_t\Lambda}\exp\left\{-t\Re\left\{\psi_{\rho_t}\left(-iz\right)\right\}\right\}\left|\frac{\psi_*(\rho_t-iz)}{\rho_t-iz}\frac{\rho_t}{\psi_*(\rho_t)}\right|dz\\
&\leq s_t\int_{k_2\rho_t<|z|<1}\exp\left\{-t\Re\left\{\psi_{\rho_t}\left(-iz\right)\right\}\right\}\left|\frac{\psi_*(\rho_t-iz)}{\rho_t-iz}\frac{\rho_t}{\psi_*(\rho_t)}\right|dz\\
&\quad+s_t\int_{|z|>1}\exp\left\{-t\Re\left\{\psi_{\rho_t}\left(-iz\right)\right\}\right\}\left|\frac{\psi_*(\rho_t-iz)}{\rho_t-iz}\frac{\rho_t}{\psi_*(\rho_t)}\right|dz\\
&=:A+B
\end{aligned}\tag{5.11}$$

**Step 2.3.1:** $(SC_\infty)$ **case.** In order to prove that $A\to 0$ uniformly, we start by observing that the hypothesis of stochastic compactness at infinity $(SC_\infty)$, and Proposition 2.2.1 in [3] imply that for any $\alpha_0\in(2-\alpha,2)$ there are constants $k_4$ and $k_5$ such that

$$\frac{\int_0^u z\overline{\Pi}(z)dz}{\int_0^v z\overline{\Pi}(z)dz}\leq k_4\left(\frac{u}{v}\right)^{\alpha_0},\qquad u\geq v\geq k_5$$

and thus

$$\frac{Q_\Pi(u)}{Q_\Pi(v)}\leq k_4\left(\frac{u}{v}\right)^{\alpha_0-2},\qquad u\geq v\geq k_5.\tag{5.12}$$

We fix $\alpha_0\in(2-\alpha,2)$, take $\overline{\rho}>\sup_{\{t>1\}}\rho_t$, and choose $v_0>1$ such that $k_5\overline{\rho}\vee\left(\frac{1}{k_3}\right)<v_0$. Next, we bound $A$ above as follows

$$\begin{aligned}
A&=s_t\int_{k_2\rho_t<|z|<1}\exp\left\{-t\Re\left\{\psi_{\rho_t}\left(-iz\right)\right\}\right\}\left|\frac{\psi_*(\rho_t-iz)}{\rho_t-iz}\frac{\rho_t}{\psi_*(\rho_t)}\right|dz\\
&\leq s_t\rho_t\int_{k_2<|\theta|<1/\rho_t}\exp\left\{-t\Re\left\{\psi_{\rho_t}\left(-i\theta\rho_t\right)\right\}\right\}d\theta=:A_1.
\end{aligned}\tag{5.13}$$

To describe the behaviour of $A_1$ we start by bounding from below the exponent of the integrand as follows. For $\theta\in((v_0)^{-1},(k_5\rho_t)^{-1})$, or equivalently $k_5<(\theta\rho_t)^{-1}<v_0/\rho_t$

$$\begin{aligned}
\Re\psi_{\rho_t}(-i\theta\rho_t)&=\int_0^\infty(1-\cos(\theta\rho_t y))e^{-\rho_t y}\Pi(dy)\\
&\geq k_6e^{-1/\theta}K_\Pi(1/(\theta\rho_t))\\
&\geq k_6e^{-v_0}\inf_{u\geq 1/(v_0\rho_t)}\left\{\frac{K_\Pi(u)}{Q_\Pi(u)}\right\}\frac{Q_\Pi(1/(\theta\rho_t))}{Q_\Pi(v_0/\rho_t)}Q_\Pi(v_0/\rho_t)\\
&\geq k_7\inf\left\{\frac{K_\Pi(u)}{Q_\Pi(u)},u\geq(v_0\overline{\rho})^{-1}\right\}(\theta v_0)^{2-\alpha_0}Q_\Pi(v_0/\rho_t),
\end{aligned}$$

where in the last inequality we used (5.12). The latter together with the inequality

$$v_0^2Q_\Pi(v_0/\rho_t)=\rho_t^2\int_0^{v_0/\rho_t}y\overline{\Pi}(y)dy\geq\rho_t^2\int_0^{1/\rho_t}y\overline{\Pi}(y)dy=Q_\Pi(1/\rho_t)\geq H(\rho_t),$$

imply

$$t\Re\psi_{\rho_t}(-i\theta\rho_t)\geq k_8\theta^{2-\alpha_0}(v_0)^{-\alpha_0}tH(\rho_t),$$

for $t$ large enough, uniformly in $x$. Applying this in $A_1$ and the results from Lemma 3.4 we obtain

$$\begin{aligned}A_1 &\le \sqrt{t}\sigma(\rho_t)\rho_t \int_{1/v_0}^{1/\rho_t k_5} \exp\{-k_8\theta^{2-\alpha_0}(v_0)^{-\alpha_0} tH(\rho_t)\}d\theta \\ &\le \sqrt{2tH(\rho_t)} \int_{1/v_0}^{1/\rho_t k_5} \exp\{-k_9\theta^{2-\alpha_0} tH(\rho_t)\}d\theta,\end{aligned}$$

where $k_9 = k_8 v_0^{-\alpha}$. Recall that $tH(\rho_t) \to \infty$, so that putting $\theta^{2-\alpha_0} tH(\rho_t) = z^{2-\alpha_0}$ gives

$$A_1 \le (\sqrt{2}tH(\rho_t))^{\frac{-\alpha_0}{2(2-\alpha_0)}} \int_{(tH(\rho_t)/v_0)^{\frac{1}{2-\alpha_0}}}^{\infty} \exp\{-k_9 z^{2-\alpha_0}\}dz \to 0.$$

**Step 2.3.2:** $(SC_\infty)$ **case.** We next prove that $B \to 0$. Proceeding as above we easily get that for $\theta > 1/\rho_t k_5$

$$\Re\psi_{\rho_t}(-i\theta\rho_t) \ge k_{10}e^{-v_0}Q_\Pi(1/\theta\rho_t) \ge k_{10}e^{-v_0}Q_\Pi(k_5) := k_{11}.$$

Now, we apply this estimate to $B$ to get that for $t > t_0$

$$B \le s_t e^{-(t-\widetilde{t}_0)k_{11}} \frac{\rho_t}{\psi_*(\rho_t)} \int_{|z|>1} \exp\left\{-\widetilde{t}_0 \Re\left\{\psi_{\rho_t}(-iz)\right\}\right\} \left|\frac{\psi_*(\rho_t - iz)}{\rho_t - iz}\right| dz.$$

Observe that by Lemma 3.7 the latter integral, as a function of $\rho_t$, is uniformly bounded. This will be enough to conclude the argument because we already know that $\frac{\rho_t}{\psi_*(\rho_t)} \to 1/(\mathbb{E}(X_1) - b) < \infty$ and it is easy to verify that $s_t$ grows at most as a power function of $t$. The latter is actually true because Lemma 3.4 allows us to ensure that

$$s_t \le \frac{1}{\rho_t}\sqrt{tH(\rho_t)} \le \frac{1}{\rho_t}\sqrt{tH(\overline{\rho})},$$

with $\overline{\rho} = \sup_t \rho_t < \infty$; and moreover given that $tH(\rho_t) \to \infty$ we deduce from (5.12) that for all large enough $t$

$$t\rho_t^{2-\alpha_0} \ge k_{13}tQ_\Pi(1/\rho_t) \ge k_{13}tH(\rho_t) \ge k_{13} > 0,$$

that is

$$\rho_t \ge k_{14}t^{-1/(2-\alpha_0)}, \qquad \text{for } t \text{ large enough.}$$

Which implies the claimed fact.

**Step 2.3.3: case** $(G)$. We will prove that the term

$$s_t \int_{|z|>1/4s_t\Lambda} \exp\left\{-t\Re\left\{\psi_{\rho_t}(-iz)\right\}\right\} \left|\frac{\psi_*(\rho_t - iz)}{\rho_t - iz}\frac{\rho_t}{\psi_*(\rho_t)}\right| dz$$

tends to $0$ uniformly in $x$. Recall that in this setting we have

$$0 < \underline{\rho} := \liminf_{t\to\infty} \rho_t \le \limsup_{t\to\infty} \rho_t := \overline{\rho} < \infty.$$

Using this, the definition of $s_t\Lambda$, and the calculations used in the proof of (3.16) it is easy to check that $s_t\Lambda$ is bounded by below by a strictly positive constant, say $l^*$. Also, as we required $X$ to be strongly non-lattice, and this is a property that is preserved under change of measure, we have that

$$\begin{aligned}\liminf_{\theta\to\infty} \Re(\psi_{\rho_t}(-i\theta)) &= \liminf_{\theta\to\infty} \int_0^\infty (1-\cos(\theta y))e^{-\rho_t y}\Pi(dy) \\ &\ge \liminf_{\theta\to\infty} \int_0^\infty (1-\cos(\theta y))e^{-\overline{\rho}y}\Pi(dy) > 0.\end{aligned} \tag{5.14}$$

We denote $\widetilde{\psi}_{\overline{\rho}}(\theta) = \int_0^\infty (1-\cos(\theta y))e^{-\overline{\rho}y}\Pi(dy)$, and $m(s) = \inf_{\theta\geq s}\widetilde{\psi}_{\overline{\rho}}(\theta)$. The above observations and the continuity of $\widetilde{\psi}_{\overline{\rho}}(\theta)$ imply that $m(s) > 0$, for all $s > 0$. It follows that for $t > t_0$

$$\begin{aligned} & s_t \int_l^\infty e^{-t\Re(\psi_{\rho_t}(-i\theta))} \left| \frac{\psi_*(\rho_t - i\theta)}{\rho_t - i\theta} \frac{\rho_t}{\psi_*(\rho_t)} \right| d\theta \\ & \leq \sqrt{t}\sigma(\rho_t) e^{-(t-t_0)m(l^*)} \int_{l^*}^\infty e^{-t_0\widetilde{\psi}_{\overline{\rho}}(\theta)} \left| \frac{\psi_*(\rho_t - i\theta)}{\rho_t - i\theta} \frac{\rho_t}{\psi_*(\rho_t)} \right| d\theta. \end{aligned}$$

By Lemma 3.7 the right most term tends to $0$ uniformly in $x$. □

The proofs of the estimates (2.4) and (2.5) use arguments very similar to those used in the previous proof, and hence in the forthcoming lines we will only outline the keys facts needed to adapt that proof.

*Proof of the estimate (2.4).* We proceed as before, using Lemma 3.3 we define a probability density

$$\begin{aligned} \mathcal{Q}_t^\lambda(y) &= \frac{\lambda\psi(\lambda)e^{t\psi(\lambda)}}{\psi_*(\lambda)(1-e^{-\Delta\psi(\lambda)})} e^{-\lambda y} h_y^J(t,\Delta) \\ &= \frac{\lambda\psi(\lambda)e^{t\psi(\lambda)}}{\psi_*(\lambda)(1-e^{-\Delta\psi(\lambda)})} e^{-\lambda y}\mathbb{P}(T_y \in (t, t+\Delta], X_{T_y} > y) \\ &= \int_0^y \mathbb{P}(X_t \in da)e^{-\lambda a} e^{t\psi(\lambda)} \frac{\psi(\lambda)}{(1-e^{-\Delta\psi(\lambda)})} \int_0^{y-a} U_\Delta(dz)e^{-\lambda z} \\ &\quad \times \frac{\lambda}{\psi_*(\lambda)} e^{-\lambda(y-a-z)}\overline{\Pi}(y-a-z). \end{aligned} \tag{5.15}$$

We easily verify from the above expression that this is the density of the sum of the three independent random variables, $Y_t^\lambda$, $Z_\lambda$, and $W_\lambda$, with $Y_t^\lambda$, and $Z_\lambda$, as defined in the proof of estimate (2.3), and $W_\lambda$ that follows the probability law

$$\mathbb{P}(W_\lambda \in dy) = \frac{\psi(\lambda)}{(1-e^{-\Delta\psi(\lambda)})} U_\Delta(dy)e^{-\lambda y}.$$

We can therefore proceed as in the proof of estimate (2.3) replacing $Z_{\rho_t}$ by $Z_{\rho_t} + W_{\rho_t}$. But for that end we should first prove that the Fourier transform of $\mathcal{Q}_t^\lambda$ is integrable. This is a straightforward consequence of the fact that

$$\left|\mathbb{E}(\exp\{i\beta(Y_t^\lambda + Z_\lambda + W_\lambda)\})\right| \leq \left|\mathbb{E}(\exp\{i\beta(Y_t^\lambda + Z_\lambda)\})\right|$$

and that we already proved that the rightmost term in the above inequality is integrable. We should now prove that $\mathbb{E}\left(W_{\rho_t}/s_t\right)$ tend to zero , uniformly in $x$ and in $\Delta$. We have

$$\left|\mathbb{E}\left(\exp\left\{i\beta\frac{W_{\rho_t}}{s_t}\right\}\right) - 1\right| \leq \mathbb{E}\left(\frac{|\beta|W_{\rho_t}}{s_t}\right), \tag{5.16}$$

and

$$\begin{aligned}\mathbb{E}\left(\frac{W_{\rho_t}}{s_t}\right) &= \frac{1}{s_t}\frac{\psi(\rho_t)}{1-e^{-\Delta\psi(\rho_t)}}\int_0^\Delta ds\mathbb{E}\left(X_s e^{-\rho_t X_s}\right)\\ &= \frac{1}{s_t}\frac{\psi(\rho_t)}{1-e^{-\Delta\psi(\rho_t)}}\int_0^\Delta ds e^{-s\psi(\rho_t)}\mathbb{E}\left(X_s e^{-\rho_t X_s+s\psi(\rho_t)}\right)\\ &= \frac{1}{s_t}\frac{\psi(\rho_t)}{1-e^{-\Delta\psi(\rho_t)}}\int_0^\Delta ds e^{-s\psi(\rho_t)}\mathbb{E}\left(Y_s\right)\\ &= \frac{x_t}{s_t}\frac{\psi(\rho_t)}{1-e^{-\Delta\psi(\rho_t)}}\int_0^\Delta ds e^{-s\psi(\rho_t)}s\\ &= \frac{x_t}{s_t}\frac{\psi(\rho_t)}{1-e^{-\Delta\psi(\rho_t)}}\frac{1}{(\psi(\rho_t))^2}\left(1-e^{-\Delta\psi(\rho_t)}-\Delta\psi(\rho_t)e^{-\Delta\psi(\rho_t)}\right)\\ &\le \frac{\rho_t x_t}{\psi(\rho_t)}\frac{1}{\sqrt{t\rho_t^2\sigma^2(\rho_t)}}.\end{aligned} \tag{5.17}$$

The rightmost term in the above equation converges to zero uniformly in $x$ and $\Delta$ because

$$\frac{\rho_t x_t}{\psi(\rho_t)} = \frac{\rho_t\psi'(\rho_t)}{\psi(\rho_t)} \le 1,$$

which is in turn an easy consequence of the elementary inequality

$$\begin{aligned}\psi'(\lambda) &= b+\int_0^\infty ye^{-\lambda y}\Pi(dy) = b+\int_0^\infty da\int_a^\infty e^{-\lambda y}\Pi(dy)\\ &\le b+\int_0^\infty da e^{-\lambda a}\overline{\Pi}(a) = \frac{\psi(\lambda)}{\lambda},\end{aligned} \tag{5.18}$$

for all $\lambda > 0$. □

*Proof of the estimate (2.5).* By Lemma 3.3 we have the key identity

$$h_y^C(t,\Delta) = \mathbb{P}(T_y \in (t,t+\Delta], X_{T_y} = y) = b\int_{[0,y]}\mathbb{P}(X_t\in dz)u_\Delta(y-z).$$

Taking Laplace transform in $y$ we obtain

$$\int_0^\infty dy e^{-\lambda y}h_y^C(t,\Delta) = be^{-t\psi(\lambda)}\frac{(1-e^{-\Delta\psi(\lambda)})}{\psi(\lambda)}.$$

for any $t > 0$. Observe the identity

$$\mathbb{P}(W_\lambda \in dy) = \frac{\psi(\lambda)}{(1-e^{-\Delta\psi(\lambda)})}e^{-\lambda y}u_\Delta(y)dy,$$

with $W_\lambda$ as defined in the previous proof. We deduce therefrom the identity

$$b\mathbb{P}(Y_t^\lambda + W_\lambda \in dy) = e^{t\psi(\lambda)}\frac{\psi(\lambda)}{(1-e^{-\Delta\psi(\lambda)})}e^{-\lambda y}h_y^C(t,\Delta)dy, \qquad y\ge 0. \tag{5.19}$$

The Fourier transform of the left most term in the above equation is integrable because of the inequality

$$\left|\frac{1-e^{-\Delta\psi(\lambda-i\theta)}}{\psi(\lambda-i\theta)}\right| \le \left|\frac{1}{\psi(\lambda-i\theta)}\right| \sim \left|\frac{1}{b\theta}\right|,$$

the hypothesis (H), the Lemma 3.7 and Proposition 2 in Chapter 1 in [1]. We then deduce the identity

$$\begin{aligned} & s_t \frac{\psi(\rho_t)}{(1-e^{-\Delta\psi(\rho_t)})} e^{tH(\rho_t)} h_x^C(t,\Delta) \\ & = \frac{b}{2\pi}\int_{-\infty}^{\infty} dz \mathbb{E}\left(\exp\left\{iz\left(\left(\frac{Y_t-\mu_t}{s_t}\right)+\frac{W_{\rho_t}}{s_t}\right)\right\}\right). \end{aligned} \tag{5.20}$$

Using the arguments in the previous proofs we get that the rightmost term in the above identity equals

$$\frac{b}{\sqrt{2\pi}}(1+o(1)),$$

and the error term is uniform in $x$ and $\Delta$. This finishes the proof. □

## 6 Proof of Proposition 2.5

*Proof.* We repeat the calculation on page 9 with $\lambda = 0$ to get

$$\hat{h}_z^J(t) := \int_0^\infty e^{izy} h_y^J(t) dy = e^{-t\psi(-iz)} \frac{\psi_*(-iz)}{-iz},$$

so that

$$th_x^J(t) = \frac{t}{2\pi}\int_{-\infty}^{\infty} e^{-ixz} e^{-t\psi(-iz)} \frac{\psi_*(-iz)}{-iz} dz. \tag{6.1}$$

The integral above is well defined since the hypothesis (H) ensures the integrability in a neighbourhood of infinity, and that around zero follows from the regular variation of $\overline{\Pi}$ at infinity. Indeed, the regular variation of $\overline{\Pi}$ implies the finiteness of the integral

$$\int_1^\infty \frac{dz}{z}\overline{\Pi}(z) < \infty,$$

and some elementary calculations allow to deduce therefrom that

$$\int_{|z|<1} \left|\frac{\psi_*(-iz)}{-iz}\right| dz < \infty.$$

We write the RHS of (6.1) as $I_1 + I_2$, where

$$\begin{aligned} I_1 &= \frac{t}{2\pi}\int_{|z|\le Kc(t)} e^{-ixz} e^{-t\psi(-iz)} \frac{\psi_*(-iz)}{-iz} dz \\ &= \frac{1}{2\pi}\int_{|z|\le K} e^{-izy_t} e^{-t\psi(-iz/c(t))} \frac{t\psi_*(-iz/c(t))}{-iz} dz \\ &= \frac{1}{2\pi}\int_{|z|\le K} e^{-izy_t} e^{-\tilde{\psi}(-iz)} \frac{\tilde{\psi}(-iz)}{-iz} dz + o(1), \end{aligned}$$

where $\tilde{\psi}$ is the exponent of the limiting stable process $S$, and we use the fact that $t\psi_*(-iz/c(t)) \sim t\psi(-iz/c(t)) \to \tilde{\psi}(-iz)$ uniformly on $[-K, K]$. Clearly

$$\lim_{K\to\infty} \left|\int_{|z|>K} e^{-\tilde{\psi}(-iz)} \frac{\tilde{\psi}(-iz)}{-iz} dz\right| = 0,$$

so that

$$\lim_{K\to\infty}\lim_{t\to\infty} |I_1 - \tilde{h}_{y_t}(1)| = 0, \text{ uniformly in } y_t.$$

The result follows because, for any fixed $K$

$$\begin{aligned}&\lim_{t\to\infty}|I_2|\\&\le \lim_{t\to\infty} te^{-(t-t_0)\kappa}\int_{|z|>Kc(t)}\exp\left\{-t_0\int_0^\infty(1-\cos(zy))\Pi(dy)\right\}\frac{|\psi(-iz)|}{z}dz\\&=0,\end{aligned}$$

where $\kappa=\liminf_{|z|\to\infty}\int_0^\infty(1-\cos(zy))\Pi(dy)>0$, by the strongly non-lattice assumption.

Similarly we have the representation

$$c(t)h_x^C(t,\Delta)=\frac{bc(t)}{2\pi}\int_{-\infty}^\infty e^{-ixz}e^{-t\psi(-iz)}\frac{(1-e^{-\Delta\psi(-iz)})}{\psi(-iz)}dz,$$

the integrability following from (H) and the bound

$$\begin{aligned}\left|\frac{(1-e^{-\Delta\psi(-iz)})}{\psi(-iz)}\right|&\le\left|\frac{1}{-ibz+\psi_*(-iz)}\right|\\&\sim\frac{1}{b|z|}\text{ as }|z|\to\infty.\end{aligned}$$

Again we have the uniform estimate,

$$\begin{aligned}&\frac{bc(t)}{2\pi}\int_{|z|\le Kc(t)}e^{-ixz}e^{-t\psi(-iz)}\frac{(1-e^{-\Delta\psi(-iz)})}{\psi(-iz)}dz\\&=\frac{b}{2\pi}\int_{|z|\le K}e^{-izy_t}e^{-t\psi(-iz/c(t))}\frac{(1-e^{-\Delta\psi(-iz/c(t))})}{\psi(-iz/c(t))}dz\\&=\frac{b\Delta}{2\pi}\int_{|z|\le K}e^{-izy_t}e^{-t\tilde{\psi}(-iz)}dz+o(1),\end{aligned}$$

and the proof is concluded as before. □

**Acknowledgments.** We would like to thank two anonymous referees for their thorough revision of the paper. Their comments helped us to substantially improve the presentation of the paper.

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