Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 39, 1–39.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-3041

Skorohod and Stratonovich integration in the plane

Samy Tindel* Khalil Chouk†

Abstract

This article gives an account on various aspects of stochastic calculus in the plane. Specifically, our aim is 3-fold: (i) Derive a pathwise change of variable formula for a path $x : [0,1]^2 \to \mathbb{R}$ satisfying some Hölder regularity conditions with a Hölder exponent greater than $1/3$. (ii) Get some Skorohod change of variable formulas for a large class of Gaussian processes defined on $[0,1]^2$. (iii) Compare the bidimensional integrals obtained with those two methods, computing explicit correction terms whenever possible. As a byproduct, we also give explicit forms of corrections in the respective change of variable formulas.

Keywords: Young integrals; Rough path; Stochastic integral; Malliavin calculus.
AMS MSC 2010: Primary 60H05, Secondary 60G05 ; 60G60 ; 60H07.
Submitted to EJP on January 27, 2013, final version accepted on January 30, 2015.

1 Introduction

Stochastic calculus for processes indexed by the plane (or higher order objects) is notoriously a cumbersome topic. In order to get an idea of this fact, let us start from the simplest situation of a smooth function x indexed by $[0,1]^2$ and a regular function $\varphi \in C^2(\mathbb{R})$. Then some elementary computations show that

$$[\delta\varphi(x)]_{s_1 s_2;t_1 t_2} = \int_{[s_1,s_2]\times[t_1,t_2]} \varphi^{(1)}(x_{u;v})\, d_{uv}x_{u;v} + \int_{[s_1,s_2]\times[t_1,t_2]} \varphi^{(2)}(x_{u;v})\, d_u x_{u;v} d_v x_{u;v}, \tag{1.1}$$

for all $0 \le s_1 < s_2 \le 1$ and $0 \le t_1 < t_2 \le 1$, where we have set $[\delta y]_{s_1 s_2;t_1 t_2}$ for the planar increment of y in the rectangle $[s_1,s_2]\times[t_1,t_2]$, namely

$$[\delta y]_{s_1 s_2;t_1 t_2} \equiv y_{s_2;t_2} - y_{s_1;t_2} - y_{s_2;t_1} + y_{s_1;t_1}. \tag{1.2}$$

This simple formula already exhibits the extra term $\int \varphi^{(2)}(x_{u;v})\, d_u x\, d_v x$ with respect to integration in $\mathbb{R}$, and the mixed differential term $d_u x\, d_v x$ is one of the main source of complications when one tries to extend (1.1) to more complex situations.

*Institut Élie Cartan and INRIA Nancy-Grand Est, Université de Lorraine, France.
E-mail: samy.tindel@univ-lorraine.fr
†Ceremade, Université de Paris-Dauphine, France.
E-mail: chouk@ceremade.dauphine.fr

Moving to stochastic calculus in the plane, generalizations of (1.1) to a random process x obviously starts with change of variables formulas involving the Brownian sheet or martingales indexed by the plane. Relevant references include [3, 13, 19], and some common features of the formulas produced in these articles are the following:

- Higher order derivatives of f showing up.
- Mixed differentials involving partial derivatives of x and quadratic variation type elements.
- Huge number of terms in the formula due to boundary effects.

This non compact form of stochastic calculus in the plane has certainly been an obstacle to its development, and we shall go back to this problem later on.

Some recent advances in generalized stochastic calculus have also paved the way to change of variables formulas in the plane beyond the martingale case. One has to distinguish two type of contributions in this direction:

(a) Skorohod type formulas for the fractional Brownian sheet (abbreviated as fBs in the sequel) with Hurst parameters greater than $1/2$ have been obtained in [17] thanks to a combination of differential calculus in the plane and stochastic analysis tools inspired by [1]. A subsequent generalization to Hurst parameters smaller than $1/2$ is available in [18], invoking the notion of extended divergence introduced in [12]. Notice however that the extended divergence leads to a rather weak notion of integral, and might not be necessary when the Hurst parameters of the fBs are greater than $1/4$.

(b) The article [4] focuses on pathwise methods for stochastic calculus in the plane, and builds an analog of the rough paths theory for functions indexed by the plane. In particular, generalizations of (1.1) with Stratonovich type integrals are given for functions with Hölder regularity greater than $1/3$. The construction is deterministic and general, and only requires the existence of a stack of iterated integrals of x called rough path, denoted by $\mathbb{X}$. One can show in particular that $\mathbb{X}$ exists when x is a fBs.

The current article is a contribution to these recent advances on generalized stochastic calculus in the plane. Namely, we focus on 3 different problems: (i) A complete exposition of the Stratonovich type change of variables formula obtained through rough paths techniques. (ii) Generalization of [17] to a fairly general Gaussian process x. (iii) Comparison of Stratonovich and Skorohod formulas, analogously to the 1 dimensional situation handled in [10]. Before we further comment on these contributions, we now describe our main results more specifically.

1.1 Some general notation

Before we can turn to the description of our main results, we introduce some general notation concerning differential calculus in the plane. Let us mention first that we shall separate as much as possible the first and the second direction of integration, which will be respectively be denoted by direction 1 and direction 2. Thus the evaluation of a function $f:[0,1]^{2k}\to\mathbb{R}$ will be denoted by $f_{s_1\cdots s_k;t_1\cdots t_k}$. We also set $d_{12}x$ for the differential $d_{uv}x$ and $d_1x\,d_2x$ for the differential $d_ux\,d_vx$. In fact, since the differential element $d_1x\,d_2x$ is essential for our purposes, we further shorten it into $d_{\hat{1}\hat{2}}x$.

Another notation which will be used extensively throughout the paper is the following: we set $y=\varphi(x)$, and for all $j\geq 1$ we write y^j for the function $\varphi^{(j)}(x)$. With those first shorthands, equation (1.1) for a smooth function $x:[0,1]^2\to\mathbb{R}$ can be written as

$$\delta y = \int_1\int_2 y^1\, d_{12}x + \int_1\int_2 y^2\, d_{\hat{1}\hat{2}}x. \tag{1.3}$$

This kind of compact notation is of course useful when cumbersome computations come into the picture.

Let us anticipate a little on the notation for planar increments which will be introduced at Section 3.1: we denote by $\mathcal{P}_{k,l}$ the set of $\mathbb{R}$-valued functions involving k variables in direction 1 and l variables in direction 2, satisfying some vanishing conditions on diagonals. We mostly deal with spaces of the form $\mathcal{P}_{2,2}$ and introduce some Hölder norms. Namely, if $f \in \mathcal{P}_{2,2}(V)$, we set

$$\|f\|_{\gamma_1;\,\gamma_2} = \sup\left\{ \frac{|f_{s_1 s_2;\, t_1 t_2}|}{|s_2 - s_1|^{\gamma_1} |t_2 - t_1|^{\gamma_2}} \,;\, s_1, s_2, t_1, t_2 \in [0,1] \right\},$$

and we denote by $\mathcal{P}_{2,2}^{\gamma_1,\gamma_2}(V)$ the space of increments in $\mathcal{P}_{2,2}(V)$ whose $\|\cdot\|_{\gamma_1;\,\gamma_2}$ norm is finite.

1.2 Stratonovich type formula in the Young case

We assume here that $x : [0,1]^2 \to \mathbb{R}$ is a path such that the rectangular increments δx of x satisfy $\delta x \in \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ with $\gamma_1, \gamma_2 > 1/2$, which corresponds to the case where integration with respect to x can be handled by Young techniques in the plane. Our change of variable formula in this situation relies on the definition of 2 increments $\mathbf{x}^{1;2}, \mathbf{x}^{\hat{1};\hat{2}} \in \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ defined as follows (see also Definition 4.2 for further information):

$$\mathbf{x}^{1;2} = \int_1 \int_2 d_{12} x, \quad \text{and} \quad \mathbf{x}^{\hat{1};\hat{2}} = \int_1 \int_2 d_1 x \, d_2 x,$$

where the integrals can be understood in the Young sense.

With these notations in hand, the change of variables formula can be read as:

Theorem 1.1. *Let $x : [0,1]^2 \to \mathbb{R}$ be a path such that $x \in \mathcal{P}_{1,1}^{\gamma_1,\gamma_2}$ with $\gamma_1, \gamma_2 > 1/2$ (see equation* (3.2) *for the definition of this space). Then the planar increments (see Section 3.1 for the definition of planar increment)*

$$z^1 = \int_1 \int_2 y^1 \, d_{12} x, \quad \textit{and} \quad z^2 = \int_1 \int_2 y^2 \, d_{\hat{1}\hat{2}} x, \tag{1.4}$$

are well defined in the 2d-Young sense. Moreover:

(i) *Both z^1 and z^2 can be decomposed as:*

$$z^1 = y^1 \, \mathbf{x}^{1;2} + \rho^1, \quad \textit{and} \quad z^2 = y^2 \, \mathbf{x}^{\hat{1};\hat{2}} + \rho^2, \tag{1.5}$$

where ρ^1, ρ^2 can be decomposed as $\rho^j = \rho^{j,1} + \rho^{j,2} + \rho^{j,12}$, with $\rho^{j,1} \in \mathcal{P}_{2,2}^{2\gamma_1,\gamma_2}$, $\rho^{j,2} \in \mathcal{P}_{2,2}^{\gamma_1,2\gamma_2}$ and $\rho^{j,12} \in \mathcal{P}_{2,2}^{2\gamma_1,2\gamma_2}$.

(ii) *Provided x is a smooth path, the increments z^1 and z^2 are defined as Riemann-Stieljes integrals.*

(iii) *If x^n is a sequence of smooth functions such that the related increments $\mathbf{x}^{n;1;2}, \mathbf{x}^{n;\hat{1};\hat{2}}$ converge respectively to $\mathbf{x}^{1;2}$ and $\mathbf{x}^{\hat{1};\hat{2}}$ in $\mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$, then $z^{1,n}, z^{2,n}$ also converge respectively to z^1 and z^2. Specifically, there exist $p > 1$ and $c_\varphi > 0$ such that for $i = 1,2$ we have*

$$\|z^i - z^{i,n}\|_{\gamma_1,\gamma_2} \le c_\varphi \, [1 + \mathcal{N}_{\gamma_1,\gamma_2}(x) + \mathcal{N}_{\gamma_1,\gamma_2}(x^n)]^p \, \mathcal{N}_{\gamma_1,\gamma_2}(x - x^n)$$

with $c_\varphi = \sum_{i=1}^4 \sup_{|k| \le ||x||_\infty} \varphi(x)$ and where the norm $\mathcal{N}$ is introduced in equation (3.2).

(iv) *Some Riemann sums convergences hold true: if π_n^1 and π_n^2 are 2 partitions of $[s_1, s_2] \times [t_1, t_2]$ whose mesh goes to 0 as $n \to \infty$, then*

$$\lim_{n\to\infty} \sum_{\pi_n^1, \pi_n^2} y^1_{\sigma_i;\tau_j} \, \mathbf{x}^{1;2}_{\sigma_i \sigma_{i+1};\tau_j \tau_{j+1}} = z^1_{s_1 s_2; t_1 t_2}, \;\; \textit{and} \;\; \lim_{n\to\infty} \sum_{\pi_n^1, \pi_n^2} y^2_{\sigma_i;\tau_j} \, \mathbf{x}^{\hat{1};\hat{2}}_{\sigma_i \sigma_{i+1};\tau_j \tau_{j+1}} = z^2_{s_1 s_2; t_1 t_2}. \tag{1.6}$$

(v) *The change of variables formula* (1.3) *still holds true when integrals are understood in the Young sense.*

Observe that this theorem is not new and can be easily recovered from considerations contained in [7, 15]. However, we express it here in terms which allow easy generalizations to Skorohod type integrals and to rough situations as well.

Remark 1.2. Let us remark that the statement of this Theorem still holds true when we take a general $y^i \in \mathcal{P}_{1,1}^{\gamma_1,\gamma_2}$ (instead of $y^i = \varphi^i(x)$).

1.3 Stratonovich type formula in the rough case

Consider now a function x whose rectangular increments δx only satisfy $\delta x \in \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ with $\gamma_1, \gamma_2 > 1/3$. The definition of z^1, z^2 as in (1.4) and the equivalent of formula (1.1) require now a huge additional effort. In particular, the correct definition relies on the introduction of a collection of iterated integrals of x (called rough path above x and denoted by $\mathbb{X}$ by analogy with the 1-d case) that we proceed to describe, following [4].

The reader will soon observe that the definition of $\mathbb{X}$ involves a whole zoology of objects which are somehow tedious to describe. In this article we shall index those objects by the directions of integration, trying to separate as much as possible direction 1 and direction 2 as we already did for the first order integrals $\mathbf{x}^{1;2}$ and $\mathbf{x}^{\hat{1};\hat{2}}$. Moreover, when one tries to define iterated integrals in the plane, the following extra facts have to be taken into account:

(i) The differentials with respect to x can be in one direction only ($d_1 x$ or $d_2 x$) or bidirectional. This reflects into some indices 0 when we don't integrate in a given direction, and 1 or 2 otherwise. Furthermore, as already mentioned, our bidirectional differentials can be either of type $d_{12}x$ or $d_1 x\, d_2 x = d_{\hat{1}\hat{2}}x$. We keep our convention of indices $1;2$ for differentials of the type $d_{12}x$ and $\hat{1};\hat{2}$ for differentials of the type $d_{\hat{1}\hat{2}}x$. As an example of these conventions, we define $\mathbf{x}^{1\hat{1};0\hat{2}} \in \mathcal{P}_{2,2}$ in the following way for a smooth function x:

$$\mathbf{x}^{1\hat{1};0\hat{2}} = \int_1 d_1 x \int_2 d_{\hat{1}\hat{2}} x, \quad \text{that is} \quad \mathbf{x}^{1\hat{1};0\hat{2}}_{s_1 s_2;t_1 t_2} = \int_{s_1}^{s_2}\int_{t_1}^{t_2}\left(\int_{s_1}^{\sigma_2} d_1 x_{\sigma_1;t_1}\right) d_1 x_{\sigma_2;\tau_1} d_2 x_{\sigma_2;\tau_1}.$$

(ii) We manipulate objects which are either iterated integrals or products of iterated integrals. We indicate that one starts a new integral in one specific direction and gets a product of increments by placing a new $\int$ sign, and this is translated by a $\cdot$ in the indices of $\mathbf{x}$. For instance, modifying our previous example, we define $\mathbf{x}^{1\cdot\hat{1};0\hat{2}} \in \mathcal{P}_{3,2}$ in the following way for a smooth function x:

$$\mathbf{x}^{1\cdot\hat{1};0\hat{2}} = \int_1 d_1 x \int_1\int_2 d_{\hat{1}\hat{2}} x, \quad \text{that is} \quad \mathbf{x}^{1\cdot\hat{1};0\hat{2}}_{s_1 s_2 s_3;t_1 t_2} = \int_{s_1}^{s_2} d_1 x_{\sigma_1;t_1}\int_{s_2}^{s_3}\int_{t_1}^{t_2} d_1 x_{\sigma_2;\tau_1} d_2 x_{\sigma_2;\tau_1}.$$

Notice that those breaks in integration can occur at different steps in each direction 1 or 2. The resulting overlapping integrals are an important source of technical troubles in the pathwise computations of [4].

With these preliminary considerations in mind, our assumptions on the function x are of the following form:

Hypothesis 1.3. *The function x is such that $\delta x \in \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ with $\gamma_1, \gamma_2 > 1/3$. Moreover, the following rough path $\mathbb{X}$ can be constructed out of x:*

In the table above, all increments belong to $\mathcal{P}_{2,2}$, so that a regularity (α,β) means that the increment lyes into $\mathcal{P}_{2,2}^{\alpha,\beta}$. Furthermore, the stack $\mathbb{X}$ is a geometric rough path, insofar as there exists a regularization x^n of x such that $\lim_{n\to\infty}\|x - x^n\|_{\gamma_1,\gamma_2} = 0$ and such that all the integrals in $\mathbb{X}^n$, constructed out of x^n in the Lebesgue-Stieljes sense,

Increment	*Interpretation*	*Regularity*	*Increment*	*Interpretation*	*Regularity*
$\mathbf{x}^{1;2}$	$\int_1 \int_2 d_{12}x$	(γ_1, γ_2)	$\mathbf{x}^{\hat{1};\hat{2}}$	$\int_1 \int_2 d_{\hat{1}\hat{2}}x$	(γ_1, γ_2)
$\mathbf{x}^{11;02}$	$\int_1 d_1x \int_2 d_{12}x$	$(2\gamma_1, \gamma_2)$	$\mathbf{x}^{1\hat{1};0\hat{2}}$	$\int_1 d_1x \int_2 d_{\hat{1}\hat{2}}x$	$(2\gamma_1, \gamma_2)$
$\mathbf{x}^{01;22}$	$\int_2 d_2x \int_1 d_{12}x$	$(\gamma_1, 2\gamma_2)$	$\mathbf{x}^{0\hat{1};2\hat{2}}$	$\int_2 d_2x \int_1 d_{\hat{1}\hat{2}}x$	$(\gamma_1, 2\gamma_2)$
$\mathbf{x}^{11;22}$	$\int_1 \int_2 d_{12}x d_{12}x$	$(2\gamma_1, 2\gamma_2)$	$\mathbf{x}^{1\hat{1};2\hat{2}}$	$\int_1 \int_2 d_{12}x d_{\hat{1}\hat{2}}x$	$(2\gamma_1, 2\gamma_2)$
$\mathbf{x}^{\hat{1}1;\hat{2}2}$	$\int_1 \int_2 d_{\hat{1}\hat{2}}x d_{12}x$	$(2\gamma_1, 2\gamma_2)$	$\mathbf{x}^{\hat{1}\hat{1};\hat{2}\hat{2}}$	$\int_1 \int_2 d_{\hat{1}\hat{2}}x d_{\hat{1}\hat{2}}x$	$(2\gamma_1, 2\gamma_2)$

converge with respect to their natural respective norms in $\mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$, $\mathcal{P}_{2,2}^{2\gamma_1,\gamma_2}$, $\mathcal{P}_{2,2}^{\gamma_1,2\gamma_2}$ or $\mathcal{P}_{2,2}^{2\gamma_1,2\gamma_2}$. Note that the natural Hölder norm of a rough path is denoted by $\mathcal{N}$ in the sequel.

Remark 1.4. As we shall see at Section 4.3, Hypothesis 1.3 is not completely sufficient in order to settle a satisfying integration theory with respect to x. In fact the rough path $\mathbb{X}$ should also include higher order increments like $\mathbf{x}^{11\cdot1;022}$ or $\mathbf{x}^{1\cdot11;22\cdot2}$ (and other extra terms). We have only stated Hypothesis 1.3 here in order to keep our exposition into some reasonable bounds.

Now we can state the Stratonovich integration theorem of [4], which mimics Theorem 1.1:

Theorem 1.5. *Let $x : [0,1]^2 \to \mathbb{R}$ be a path such that $\delta x \in \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ with $\gamma_1, \gamma_2 > 1/3$ and assume the further rough path Hypothesis 1.3. Consider a function $\varphi \in C_b^8(\mathbb{R})$. Then the increments z^1 and z^2 given by* (1.4) *are well defined as continuous functions of the rough path $\mathbb{X}$. Moreover:*

(i) *The increment z^1 can be decomposed as:*

$$z^1 = y^1\,\mathbf{x}^{1;2} + y^2\,\mathbf{x}^{11;02} + y^2\,\mathbf{x}^{01;22} + y^2\,\mathbf{x}^{11;22} + y^3\,\mathbf{x}^{\hat{1}1;\hat{2}2} + \rho^1, \tag{1.7}$$

and the increment z^2 admits a decomposition of the form

$$z^2 = y^2\,\mathbf{x}^{\hat{1};\hat{2}} + y^3\,\mathbf{x}^{1\hat{1};0\hat{2}} + y^3\,\mathbf{x}^{0\hat{1};2\hat{2}} + y^3\,\mathbf{x}^{1\hat{1};2\hat{2}} + y^4\,\mathbf{x}^{\hat{1}\hat{1};\hat{2}\hat{2}} + \rho^2,$$

where ρ^1, ρ^2 are sums of increments with triple regularity $(3\gamma_1, 3\gamma_2)$ in at least one direction.

(ii) *If x^n is a sequence of smooth functions such that the related rough path $\mathbb{X}^n$ converges to $\mathbb{X}$, then $z^{1,n}, z^{2,n}$ (defined in the Lebesgue-Stieljes sense) also converge respectively to z^1 and z^2. Furthermore, there exist two constants $p \geq 1$ and $c = c_\varphi$ such that for $i = 1,2$ we have:*

$$\|z^1 - z^{1,n}\|_{\gamma_1,\gamma_2} \leq c\;[1 + \mathcal{N}(\mathbb{X}) + \mathcal{N}(\mathbb{X}^n)]^p\,\mathcal{N}(\mathbb{X} - \mathbb{X}^n),$$

with $c_\varphi = c\sum_{i=1}^8 \sup_{|k|\leq||x||_\infty} \varphi^i(k)$ for a universal constant c.

(iii) *The change of variables formula* (1.3) *still holds true when integrals are understood in the rough path sense.*

Obviously, Theorem 1.5 would be of little interest if we could not apply it to processes of interest. To this regard, our guiding example will be the fractional Brownian sheet (fBs in the sequel). Let us recall that this is a centered Gaussian process x defined on $[0,1]^2$, with a covariance function $R_{s_1s_2;t_1t_2} = \mathbb{E}[x_{s_1;t_1}x_{s_2;t_2}]$ defined by

$$R_{s_1s_2;t_1t_2} = \frac{1}{4}\left(|s_1|^{2\gamma_1} + |s_2|^{2\gamma_1} - |s_1 - s_2|^{2\gamma_1}\right)\left(|t_1|^{2\gamma_2} + |t_2|^{2\gamma_2} - |t_1 - t_2|^{2\gamma_2}\right), \tag{1.8}$$

where the Hurst parameters γ_1, γ_2 lye into $(0,1)$. Many possible representations are available for the fBs, among which we will appeal to the so-called harmonizable representation (see relation (6.1) below for further details). This allows a natural approximation of x by a sequence of smooth processes x^n thanks to a cutoff in frequency, and we recall the following convergence result established in [4]:

Proposition 1.6. *Let x a fBs with Hurst parameters $\gamma_j > 1/3$, for $j = 1, 2$. Define the regularization x^n of x given by a frequency cutoff on $B(0, n)$ in the harmonizable representation of x. Then:*

(i) *The family of iterated integral $\mathbb{X}^n$ defined in Hypothesis 1.3 associated to x^n fulfills the relation $\lim_{n,m\to\infty} \mathbb{E}[\mathcal{N}^p(\mathbb{X}^m - \mathbb{X}^n)] = 0$ for all $p \geq 1$, where the norm $\mathcal{N}$ is alluded to at Hypothesis 1.3. The limit object $\mathbb{X}$ is called rough sheet associated to x.*

(ii) *Theorem 1.5 applies to the fBs x.*

As the reader might imagine, Theorem 1.5 can also be applied to a wide range of Gaussian and non Gaussian processes. We focus here on fBs for sake of simplicity.

1.4 Skorohod integration

One of the main issue alluded to in this article is a comparison between Stratonovich and Skorohod type change of variable formulas when x is a Gaussian process exhibiting some Hölder regularity in the plane. Towards this aim, our global strategy is to use our Theorems 1.1 and 1.5 and compute corrections between Stratonovich and Skorohod type integrals.

We first focus on the Young case, assuming the same regularity conditions as in Section 1.2. We are then able to handle the case of a fairly general centered Gaussian process x whose covariance function R satisfies a factorization property of the form

$$\mathbb{E}[x_{s_1;t_1} x_{s_2;t_2}] = R_{s_1 s_2;t_1 t_2} = R^1_{s_1 s_2} R^2_{t_1 t_2}, \tag{1.9}$$

for two covariance functions R^1, R^2 on $[0, 1]$ and such that $R^1, R^2 \in \mathcal{C}^{1\text{-var}}([0, 1]^2)$ (which ensures that x is (γ_1, γ_2)-Hölder continuous with $\gamma_1, \gamma_2 > 1/2$). Notice in particular that the fBs covariance function (1.8) satisfies condition (1.9).

The standard growth assumptions on f in order to get a Skorohod formula for $f(x)$ should also be met. They will feature prominently in the sequel, and we proceed to recall them now:

Definition 1.7. *Let $k \in \mathbb{N}$, we will say that a function $f \in C^k(\mathbb{R})$ satisfies the growth condition (GC) if there exist positive constants c and λ such that*

$$\lambda < \frac{1}{4 \max_{s,t\in[0,1]} (R^1_s R^2_t)}, \quad \textit{and} \quad \max_{l=0,\ldots,k} |f^{(l)}(\xi)| \leq c\, e^{\lambda |\xi|^2} \ \textit{for all } \xi \in \mathbb{R}. \tag{1.10}$$

With these notations in hand, and denoting quite informally the Skorohod differentials by $d^\diamond$ (see Section 5.1.1 for further explanations), we can summarize our results in the following:

Theorem 1.8. *Assume x is a centered Gaussian process on $[0, 1]^2$ with a covariance function satisfying* (1.9)*, and such that the paths of x are Hölder continuous with exponent greater than $1/2$ in each direction (see* (3.2) *again for a precise definition). Consider a function $\varphi \in C^4(\mathbb{R})$ satisfying condition (GC). Then the increments*

$$z^{1,\diamond} = \int_1 \int_2 y^1\, d^\diamond_{12} x, \quad \textit{and} \quad z^{2,\diamond} = \int_1 \int_2 y^2\, d^\diamond_{\hat{1}\hat{2}} x, \tag{1.11}$$

are well defined in the Skorohod sense of Malliavin calculus. Moreover:

(i) *Some Riemann convergences hold true: if π^1_n and π^2_n are 2 partitions of $[s_1, s_2] \times [t_1, t_2]$*

whose mesh goes to 0 as $n \to \infty$, then

$$\lim_{n\to\infty} \sum_{\pi_n^1,\pi_n^2} y^1_{\sigma_i;\tau_j} \diamond \mathbf{x}^{1;2}_{\sigma_i\sigma_{i+1};\tau_j\tau_{j+1}} = z^{1,\diamond}_{s_1s_2;t_1t_2} \tag{1.12}$$

$$\lim_{n\to\infty} \sum_{\pi_n^1,\pi_n^2} y^2_{\sigma_i;\tau_j} \diamond \delta_2 x_{s_i;t_jt_{j+1}} \diamond \delta_1 x_{s_is_{i+1};t_j} = z^{2,\diamond}_{s_1s_2;t_1t_2}, \tag{1.13}$$

where $\diamond$ stands for the Wick product in the left hand side of the relations above, and where the convergence holds in both a.s and $L^2(\Omega)$ sense.

(ii) *The change of variables formula for $y = f(x)$ becomes*

$$\delta y_{s;t} = z^{1,\diamond}_{s;t} + z^{2,\diamond}_{s;t} + \frac{1}{2}\int_1\int_2 y^2_{u;v}\, d_1R^1_u\, d_2R^2_v + \frac{1}{2}\int_1\int_2 y^3_{u;v} R^1_u\, d_2R^2_v\, d^\diamond_1 x_{u;v}$$
$$+ \frac{1}{2}\int_1\int_2 y^3_{u;v} R^2_v\, d_1R^1_u\, d^\diamond_2 x_{u;v} + \frac{1}{4}\int_1\int_2 y^4_{u;v} R^1_u R^2_v\, d_1R^1_u\, d_2R^2_v. \tag{1.14}$$

(iii) *Explicit corrections between z^1, z^2 and $z^{1,\diamond}$, $z^{2,\diamond}$ can be computed (see relations* (5.15) *and* (5.22)*).*

Finally, let us move to the Skorohod change of variable in the rough situation. For simplicity of exposition, we have restricted our analysis to the fractional Brownian sheet, mainly because our computations heavily hinges on the explicit regular approximation sequence x^n given by the harmonizable representation of fBs (similarly to the construction of the rough path above x). The Skorohod change of variable (consistent with the formulas obtained in [18]) and Skorohod-Stratonovich comparison we obtain in this case are summarized as follows:

Theorem 1.9. *Assume x is a fractional Brownian sheet on $[0,1]^2$, with $\gamma_j > 1/3$ for $j = 1, 2$. Then the increments $z^{1,\diamond}, z^{2,\diamond}$ of equation* (1.4) *are well defined in the Skorohod sense of Malliavin calculus. Moreover:*

(i) *Both $z^{1,\diamond}$ and $z^{2,\diamond}$ can be seen as respective limits of $z^{n,1,\diamond}$ and $z^{n,2,\diamond}$, computed as in Theorem 1.8 for the regularized process x^n.*

(ii) *For all $f \in C^6(\mathbb{R})$, the change of variables formula* (1.14) *still holds, and can be read as:*

$$\delta y_{s;t} = z^{1,\diamond} + z^{2,\diamond} + 2\gamma_1\gamma_2 \int_1\int_2 y^2_{u;v}\, u^{2\gamma_1-1} v^{2\gamma_2-1}\, dudv + \gamma_2 \int_1\int_2 y^3_{u;v}\, u^{2\gamma_1} v^{2\gamma_2-1}\, d^\diamond_1 x_{u;v} dv$$
$$+ \gamma_1 \int_1\int_2 y^3_{u;v}\, u^{2\gamma_1-1} v^{2\gamma_2}\, d^\diamond_2 x_{u;v} du + \gamma_1\gamma_2 \int_1\int_2 y^4_{u;v}\, u^{4\gamma_1-1} v^{4\gamma_2-1}\, dudv. \tag{1.15}$$

(iii) *Explicit corrections between z^1, z^2 and $z^{1,\diamond}$, $z^{2,\diamond}$ can be computed (see relations* (6.17) *and* (6.25)*).*

1.5 Further comments

As the reader might have noticed, our paper gives a rather complete picture of pathwise Stratonovich and Itô-Skorohod integration for processes indexed by the plane. In order to put our strategy for the Itô case into perspective, notice that 2 types of methodologies are usually available for changes of variables in case of a Gaussian process x:

(a) Define a divergence type operator $\delta^\diamond$ for x and proceed by integration by parts on expressions like $\mathbb{E}[\delta f(x)\, G]$, where G is a smooth functional of x. This is the strategy invoked e.g. in [1, 10].

(b) Base the calculations on the pathwise change of variables formula of type (1.1). This formula is generally related to some converging Riemann sums like in Theorem 1.1, and one can compute corrections between Wick and ordinary products in relation (1.6). This is the method implicitly adopted in [17] and we also resort to this second strategy here, which allows to derive our Skorohod formula and its comparison with the Stratonovich formula at the same time.

Unfortunately, the Wick corrections strategy does not work for the rough case, even in the explicit situation of a fractional Brownian sheet. This mainly stems from the fact that convenient Riemann sums related to formula (1.1) are not available (so far) in the case of Theorem 1.5. This drawback led us to change our strategy again, and proceed by regularization. Indeed, as mentioned before, one can come up with an explicit regular approximation x^n of x. For this regularization, we can apply Theorem 1.8 and get some Itô-Stratonovich corrections. Invoking the fact that $\delta^\diamond$ is a closable operator, we can then take limits in our operations as $n \to \infty$. This allows to compare the changes of variables formulas (1.1) and (1.15), but the interpretation in terms of Riemann-Wick sums is obviously lost in this case. Notice that an approximation procedure (expressed in terms of the extended divergence operator) is also at the heart of [18] for irregular fBs.

Finally, let us say a few words about possible extensions of our work:

• Generalizations of Skorohod's change of variable to a Gaussian process without the factorization hypothesis (1.9) on the covariance function of x are certainly possible. However, at a technical level, one should be aware of the fact that the analysis of mixed terms like $\int_1 \int_2 y^3_{u;v} R^2_v \, d_1 R^1_u \, d^\diamond_2 x_{u;v}$ would require tools of Young integration in dimension 4. These techniques have been used e.g in [6], and the elaboration we need would certainly be cumbersome. We have thus sticked to the factorized case for R for sake of readability.

• As mentioned before, our strategy for the Skorohod formula in the rough case relies heavily on a suitable regularization of x. Instead of treating the explicit fBs example, we could have stated some general approximation assumptions satisfied in the fBs case. Once again, we have chosen to specialize our study here for sake of clarity. The general case might be handled in a subsequent paper, and we also hope to design a strategy based on Riemann-Wick sums in the next future.

Here is how our article is structured: We recall some basic notation of algebraic integration in dimension 1 at Section 2, and extend it to integration in the plane at Section 3. The Stratonovich change of variable formula is recalled at Section 4.1 for the Young case and at Section 4.3 in the rough situation. We then move to Skorohod type formulas at Sections 5 and 6, respectively for the regular and rough cases.

2 Algebraic integration in dimension 1

We recall here the minimal amount of notation concerning algebraic integration theory in $\mathbb{R}$, in order to prepare the ground for further developments in the plane. We refer to [8, 9] for a more detailed introduction.

2.1 Increments

The extended pathwise integration we will deal with is based on the notion of *increments*, together with an elementary operator δ acting on them. The algebraic structure they generate is described in [8, 9], but here we present directly the definitions of interest for us, for sake of conciseness. First of all, for a vector space V and an integer $k \geq 1$ we denote by $\mathcal{C}_k(V)$ the set of functions $g : [0,1]^k \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-*increment*, and we set $\mathcal{C}_*(V) = \cup_{k\geq 1}\mathcal{C}_k(V)$. We can now define the announced elementary operator δ on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1\cdots t_{k+1}} = \sum_{i=1}^{k+1}(-1)^{k-i} g_{t_1\cdots \hat{t}_i \cdots t_{k+1}}, \tag{2.1}$$

where $\hat{t}_i$ means that this particular argument is omitted. A fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Ker}\delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Im}\delta$.

Some simple examples of actions of δ, which will be the ones we will really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $s, u, t \in [0, 1]$, we have

$$\delta g_{st} = g_t - g_s, \quad \text{ and } \quad \delta h_{sut} = h_{st} - h_{su} - h_{ut}. \tag{2.2}$$

Furthermore, it is easily checked that $\mathcal{ZC}_{k+1}(V) = \mathcal{BC}_k(V)$ for any $k \geq 1$. In particular, the following basic property holds:

Lemma 2.1. *Let $k \geq 1$ and $h \in \mathcal{ZC}_{k+1}(V)$. Then there exists a (non unique) $f \in \mathcal{C}_k(V)$ such that $h = \delta f$.*

Lemma 2.1 can be rephrased as follows: any element $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non unique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_2(V)}$: it measures how much a given 1-increment is far from being an exact increment of a function, i.e., a finite difference.

Notice that our future discussions will mainly rely on k-increments with $k \leq 2$, for which we will make some analytical assumptions. Namely, we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_\mu = \sup_{s,t\in[0,1]} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{ and } \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V);\, \|f\|_\mu < \infty\}. \tag{2.3}$$

Obviously, the usual Hölder spaces $\mathcal{C}_1^\mu(V)$ will be determined in the following way: for a continuous function $g \in \mathcal{C}_1(V)$, we simply set

$$\|g\|_\mu = \|\delta g\|_\mu, \tag{2.4}$$

and we will say that $g \in \mathcal{C}_1^\mu(V)$ iff $\|g\|_\mu$ is finite. Notice that $\|\cdot\|_\mu$ is only a semi-norm on $\mathcal{C}_1(V)$. For $h \in \mathcal{C}_3(V)$ set in the same way

$$\begin{aligned} \|h\|_{\gamma,\rho} &= \sup_{s,u,t\in[0,1]} \frac{|h_{sut}|}{|u-s|^\gamma |t-u|^\rho} \\ \|h\|_\mu &= \inf\left\{ \sum_i \|h_i\|_{\rho_i,\mu-\rho_i};\, h = \sum_i h_i,\, 0 < \rho_i < \mu \right\}, \end{aligned} \tag{2.5}$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0, z)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V);\, \|h\|_\mu < \infty\}.$$

Eventually, let $\mathcal{C}_3^{1+}(V) = \cup_{\mu>1}\mathcal{C}_3^\mu(V)$, and notice that the same kind of norms can be considered on the spaces $\mathcal{ZC}_3(V)$, leading to the definition of some spaces $\mathcal{ZC}_3^\mu(V)$ and $\mathcal{ZC}_3^{1+}(V)$.

With these notations in mind the following proposition is a basic result, which belongs to the core of our approach to pathwise integration. Its proof may be found in a simple form in [9].

Proposition 2.2 (The Λ-map). *There exists a unique linear map $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+}(V) \to \mathcal{C}_2^{1+}(V)$ such that*

$$\delta\Lambda = \mathrm{Id}_{\mathcal{Z}\mathcal{C}_3^{1+}(V)} \quad \text{and} \quad \Lambda\delta = \mathrm{Id}_{\mathcal{C}_2^{1+}(V)}.$$

In other words, for any $h \in \mathcal{C}_3^{1+}(V)$ such that $\delta h = 0$ there exists a unique $g = \Lambda(h) \in \mathcal{C}_2^{1+}(V)$ such that $\delta g = h$. Furthermore, for any $\mu > 1$, the map Λ is continuous from $\mathcal{Z}\mathcal{C}_3^{\mu}(V)$ to $\mathcal{C}_2^{\mu}(V)$ and we have

$$\|\Lambda h\|_{\mu} \le \frac{1}{2^{\mu} - 2}\|h\|_{\mu}, \qquad h \in \mathcal{Z}\mathcal{C}_3^{\mu}(V). \tag{2.6}$$

Let us mention at this point a first link between the structures we have introduced so far and the problem of integration of irregular functions.

Corollary 2.3. *For any 1-increment $g \in \mathcal{C}_2(V)$ such that $\delta g \in \mathcal{C}_3^{1+}$, set $\ell = (\mathrm{Id} - \Lambda\delta)g$. Then there exists a unique $f \in \mathcal{C}_1(V)$, defined up to constants, such that $\ell = \delta f$ and*

$$\delta f_{st} = \lim_{|\Pi_{st}| \to 0} \sum_{i=0}^{n-1} g_{t_i\, t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \dots, t_n = t\}$ of $[s,t]$, whose mesh tends to zero. Thus, the 1-increment δf is the indefinite integral of the 1-increment g.

2.2 Products of increments

For notational sake, let us specialize now to the case $V = \mathbb{R}$, and just write $\mathcal{C}_k^{\gamma}$ for $\mathcal{C}_k^{\gamma}(\mathbb{R})$. The usual product of two increments considered on $(\mathcal{C}_*, \delta)$ is obtained by gluing one variable in each increment (see e.g [8, 9]):

Definition 2.4. *For $g \in \mathcal{C}_n$ and $h \in \mathcal{C}_m$, we denote by gh the element of $\mathcal{C}_{n+m-1}$ defined by*

$$(gh)_{t_1,\dots,t_{m+n-1}} = g_{t_1,\dots,t_n} h_{t_n,\dots,t_{m+n-1}}, \quad t_1,\dots,t_{m+n-1} \in [0,1]. \tag{2.7}$$

However, another product (defined without gluing of variables) turns out to be useful for further computations in the plane. This product is called *splitting* and is defined below:

Definition 2.5. *For $g \in \mathcal{C}_n$ and $h \in \mathcal{C}_m$, we denote by $S(g,h)$ the element of $\mathcal{C}_n \otimes \mathcal{C}_m$ defined by*

$$[S(g,h)]_{t_1,\dots,t_{m+n}} = g_{t_1,\dots,t_n} h_{t_{n+1},\dots,t_{m+n}}, \quad t_1,\dots,t_{m+n} \in [0,1]. \tag{2.8}$$

Notice that $S(g,h)$ can also be considered as an increment in $\mathcal{C}_{n+m}$, except that it is not required to vanish when $t_n = t_{n+1}$.

We now recall some elementary properties concerning products of increments:

Proposition 2.6. *Let $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then $gh \in \mathcal{C}_2$ and*

$$\delta(gh) = -\delta g\, h + g\, \delta h. \tag{2.9}$$

Furthermore, if both g and h are elements of $\mathcal{C}^1$, then $gh \in \mathcal{C}^1$ and

$$\delta(gh) = \delta g h + g \delta h$$

2.3 Iterated integrals as increments

Iterated integrals of smooth functions on $[0,1]$ are obviously particular cases of elements of $\mathcal{C}_2$, which will be of interest for us. A typical example of this kind of object is given as follows: consider $f^j \in \mathcal{C}_1^\infty$ for $j = 1, \ldots, n$ and $0 \le s_1 < s_2 \le 1$. For $n \ge 1$, we denote by $\mathcal{S}_n(s_1, s_2)$ the simplex

$$\mathcal{S}_n(s_1, s_2) = \{(\sigma_1, \ldots, \sigma_n) \in [0,1]^n;\ s_1 < \sigma_1 < \cdots < \sigma_n < s_2\} \tag{2.10}$$

and we set

$$h^{1,\ldots,n}_{s_1 s_2} \equiv \int_{\mathcal{S}_n(s_1,s_2)} df^1_{\sigma_1} \cdots df^n_{\sigma_n} = \int_{s_1}^{s_2} \int_{s_1}^{\sigma_{n-1}} \cdots \int_{s_1}^{\sigma_2} df^1_{\sigma_1} \cdots df^n_{\sigma_n}. \tag{2.11}$$

We now introduce some notation for iterated integrals which is much too complicated for integration in dimension 1, but turns out to be useful for integration in the plane. Indeed, we can alternatively denote the increment $h^{1,\cdots,n}$ defined at (2.11) by

$$h^{1,\ldots,n} = [\underbrace{d, \ldots, d}_{n\text{ times}}](f^1, \ldots, f^n), \quad \text{or} \quad h^{1,\ldots,n} = \int df^1 \cdots df^n, \tag{2.12}$$

where the integration on the n-dimensional simplex is implicit in both cases. We shall also need a small variant of these conventions: we set

$$[\underbrace{\mathrm{Id}, \ldots, \mathrm{Id}}_{j\text{ times}}, \underbrace{d, \ldots, d}_{n-j\text{ times}}](f^1, \ldots, f^n) \equiv f^1 \cdots f^j \int df^{j+1} \cdots df^n, \tag{2.13}$$

where all the products are understood as products of increments as in Definition 2.4.

3 Algebraic integration in the plane

Before going on with the two dimensional integration, let us label some notations for further use:

Notation 3.1. *We write $X \lesssim_{a,b,...} Y$ if there exist a constant c depending on $a, b, ...$ such that the quantities X, Y satisfy $X \le cY$. For a partition $\{(s_i, t_j)_{i,j}\}$ of a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$, Δ_{ij} denotes the rectangle $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$.*

section is devoted to recall the elements of algebraic integration necessary to define an integral of the form $\int_{[0,1]^2} f(x)\, dx$ for a Hölder function x in the plane with Hölder exponent greater than $1/3$. This requires a tensorization of the algebraic structures defined in the previous section, plus some extra tools that we proceed to introduce.

3.1 Planar increments

We consider here increments of a variable s (also called direction 1) and a variable t (also called direction 2), with $(s,t) \in [0,1]^2$. For a vector space V, we set

$$\mathcal{P}_{k,l}(V) = \left\{ f \in \mathcal{C}([0,1]^k \times [0,1]^l; V);\ f_{s_1 \cdots s_k; t_1 \cdots t_l} = 0 \text{ whenever } s_i = s_{i+1} \text{ or } t_j = t_{j+1} \right\}.$$

In the particular case $V = \mathbb{R}$, we simply set $\mathcal{P}_{k,l}(\mathbb{R}) \equiv \mathcal{P}_{k,l}$.

Some partial difference operators δ_1 and δ_2 with respect to the first and second direction can be defined as in the previous section. Namely, for $f \in \mathcal{P}_{k,l}(V)$ we set

$$\delta_1 : \mathcal{P}_{k,l}(V) \to \mathcal{P}_{k+1,l}(V), \qquad \delta_1 g_{s_1 \cdots s_{k+1}; t_1 \cdots t_l} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{s_1 \cdots \hat{s}_i \cdots s_{k+1}; t_1 \cdots t_l}, \tag{3.1}$$

and we define δ_2 similarly. The planar increment δ is then obtained as $\delta = \delta_1\,\delta_2$. Notice that for $f \in \mathcal{P}_{1,1}$ we have

$$\delta f_{s_1 s_2;\, t_1 t_2} = f_{s_2;\, t_2} - f_{s_2;\, t_1} - f_{s_1;\, t_2} + f_{s_1;\, t_1},$$

which is the usual rectangular increment of a function f defined on $[0,1]^2$ and is consistent with formula (1.2). Let us label the following notation for further use:

Notation 3.2. *For $j = 1, 2$, we set $\mathcal{Z}_j\mathcal{P}_{k,l} = \mathcal{P}_{k,l} \cap \ker(\delta_j)$ and $\mathcal{B}_j\mathcal{P}_{k,l} = \mathcal{P}_{k,l} \cap \mathrm{Im}(\delta_j)$. We also write $\mathcal{ZP}_{k,l}$ for $\mathcal{P}_{k,l} \cap \ker(\delta)$ and $\mathcal{BP}_{k,l}$ for $\mathcal{P}_{k,l} \cap \mathrm{Im}(\delta)$.*

As in the 1-d case, the Hölder regularity of planar increments is an essential feature of our generalized integration theory. On $\mathcal{P}_{2,2}(V)$ and $\mathcal{P}_{3,3}(V)$, it is measured by a tensorization of the Hölder norms defined at (2.3) and (2.5). Namely, if $f \in \mathcal{P}_{2,2}(V)$, we set

$$\|f\|_{\gamma_1;\,\gamma_2} = \sup\left\{ \frac{|f_{s_1 s_2;\, t_1 t_2}|}{|s_2 - s_1|^{\gamma_1} |t_2 - t_1|^{\gamma_2}}\,;\, s_1, s_2, t_1, t_2 \in [0,1] \right\},$$

and we denote by $\mathcal{P}_{2,2}^{\gamma_1,\gamma_2}(V)$ the space of increments in $\mathcal{P}_{2,2}(V)$ whose $\|\cdot\|_{\gamma_1;\,\gamma_2}$ norm is finite. Along the same lines, we say that $h \in \mathcal{P}_{3,3}^{\gamma_1,\gamma_2}(V)$ if there exist $\kappa_1, \kappa_2, \rho_1, \rho_2$ such that $\kappa_j + \rho_j = \gamma_j$, $j = 1, 2$, and

$$\sup\left\{ \frac{|h_{s_1 s_2 s_3;\, t_1 t_2 t_3}|}{|s_2 - s_1|^{\kappa_1} |s_3 - s_2|^{\rho_1} |t_2 - t_1|^{\kappa_2} |t_3 - t_2|^{\rho_2}}\,;\, s_1, s_2, s_3, t_1, t_2, t_3 \in [0,1] \right\} < \infty.$$

Similar norms, omitted here for sake of conciseness, can be defined on $\mathcal{P}_{2,3}(V)$ and $\mathcal{P}_{3,2}(V)$.

As in the 1-dimensional case, the regularities in $\mathcal{P}_{1,1}$ are measured in a slightly different way. Namely, we say that $f \in \mathcal{P}_{1,1}^{\alpha,\beta}$ if the following norm is finite:

$$\mathcal{N}_{\alpha,\beta}(f) = \|f\|_{\alpha,\beta} + \|f\|_{\alpha,1} + \|f\|_{\beta,2} + \|f\|_{\infty}, \tag{3.2}$$

where

$$\|f\|_{\alpha,1} = \sup_{(s_1,s_2,t) \in [0,1]^2 \times [0,1]} \frac{|\delta_1 f_{s_1 s_2; t}|}{|s_2 - s_1|^{\alpha}}, \quad \text{and} \quad \|f\|_{\beta,2} = \sup_{(s,t_1,t_2) \in [0,1]^2} \frac{|\delta_2 f_{s; t_1 t_2}|}{|t_2 - t_1|^{\beta}}. \tag{3.3}$$

For Hölder continuous increments with regularity greater than 1, one gets the following inversion properties, which are a direct consequence of the one dimensional Proposition 2.2:

Proposition 3.3. *Let $\gamma_1, \gamma_2 > 1$. Then:*

(1) *There exist two unique maps $\Lambda_1 : \mathcal{B}_1\mathcal{P}_{3,3}^{\gamma_1,\gamma_2} \to \mathcal{P}_{2,3}^{\gamma_1,\gamma_2}$ and $\Lambda_2 : \mathcal{B}_2\mathcal{P}_{3,3}^{\gamma_1,\gamma_2} \to \mathcal{P}_{3,2}^{\gamma_1,\gamma_2}$ such that $\delta_j\Lambda_j = \mathrm{Id}$.These maps satisfy the bound $\|\Lambda_j(h)\|_{\gamma_1,\gamma_2} \le c_{\gamma_j}\|h\|_{\gamma_1,\gamma_2}$ for $j = 1, 2$.*

(2) *There exists a unique map $\Lambda : \mathcal{BP}_{3,3}^{\gamma_1,\gamma_2} \to \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ such that $\delta\Lambda = \mathrm{Id}$. This map satisfies the bound $\|\Lambda(h)\|_{\gamma_1,\gamma_2} \le c_{\gamma_1,\gamma_2}\|h\|_{\gamma_1,\gamma_2}$.*

We do not include the proof of this proposition for sake of conciseness. Let us just mention that (as the reader might imagine) we have $\Lambda = \Lambda_1\Lambda_2$. It should also be observed that some 2-dimensional Riemann sums are related to the sewing map Λ, echoing Corollary 2.3:

Proposition 3.4. *Let $g \in \mathcal{P}_{2,2}$ satisfying the following assumptions:*

$$\delta_1 g \in \mathcal{P}_{3,2}^{\gamma_1,*}, \qquad \delta_2 g \in \mathcal{P}_{2,3}^{*,\gamma_2}, \qquad \delta g \in \mathcal{P}_{3,3}^{\gamma_1,\gamma_2},$$

for $\gamma_1, \gamma_2 > 1$, where $*$ denotes any kind of Hölder regularity. Then there exists $f \in \mathcal{P}^{1,1}$ such that

$$\delta f = [\mathrm{Id} - \Lambda_1\delta_1]\,[\mathrm{Id} - \Lambda_2\delta_2]\, g, \quad \textit{and} \quad \lim_{|\pi|\to 0} \sum_{\sigma_i,\tau_j \in \pi} g_{\sigma_i\sigma_{i+1};\tau_j\tau_{j+1}} = \delta f_{s_1 s_2; t_1 t_2},$$

where π designates a family of rectangular partitions of $[s_1, s_2] \times [t_1, t_2]$ whose mesh goes to 0.

3.2 Products of planar increments

This section is a parallel of Section 2.2, and we mainly deal here with a state space $V = \mathbb{R}$. We describe the different conventions on products of 2-d increments which will be used in the sequel, starting from the equivalent of Definition 2.4:

Definition 3.5. *For $g \in \mathcal{P}_{n_1,n_2}$ and $h \in \mathcal{P}_{m_1,m_2}$, we denote by gh the element lying in the space $\mathcal{P}_{n_1+m_1-1,n_2+m_2-1}$ defined by*

$$(gh)_{s_1,\ldots s_{n_1+m_1-1};t_1,\ldots,t_{n_2+m_2-1}} = g_{s_1,\ldots s_{n_1};t_1,\ldots,t_{n_2}} h_{s_{n_1},\ldots s_{n_1+m_1-1};t_{n_2},\ldots,t_{n_2+m_2-1}}.$$

We now define the equivalent of splitting for increments in $\mathcal{P}$:

Definition 3.6. *Let $g \in \mathcal{P}_{n_1,n_2}$ and $h \in \mathcal{P}_{m_1,m_2}$. Then:*

- *The partial splitting $S_1(g, h)$ is the element of $\mathcal{C}_{n_2+m_2-1}(\mathcal{C}_{n_1} \otimes \mathcal{C}_{m_1})$ defined by*

$$[S_1(g,h)]_{s_1,\ldots s_{n_1+m_1};t_1,\ldots,t_{n_2+m_2-1}} = g_{s_1,\ldots s_{n_1};t_1,\ldots,t_{n_2}} h_{s_{n_1+1},\ldots s_{n_1+m_1};t_{n_2},\ldots,t_{n_2+m_2-1}}.$$

- *The partial splitting $S_2(g, h)$ is the element of $\mathcal{C}_{n_1+m_1-1}(\mathcal{C}_{n_2} \otimes \mathcal{C}_{m_2})$ defined by*

$$[S_2(g,h)]_{s_1,\ldots s_{n_1+m_1-1};t_1,\ldots,t_{n_2+m_2}} = g_{s_1,\ldots s_{n_1};t_1,\ldots,t_{n_1}} h_{s_{n_1},\ldots s_{n_1+m_1-1};t_{n_2+1},\ldots,t_{n_2+m_2}}.$$

- *The splitting $S(g, h)$ is the element of $\mathcal{C}_{n_1} \otimes \mathcal{C}_{m_1}(\mathcal{C}_{n_2} \otimes \mathcal{C}_{m_2})$ defined by*

$$[S_1(g,h)]_{s_1,\ldots s_{n_1+m_1};t_1,\ldots,t_{n_2+m_2}} = g_{s_1,\ldots s_{n_1};t_1,\ldots,t_{n_2}} h_{s_{n_1+1},\ldots s_{n_1+m_1};t_{n_2+1},\ldots,t_{n_2+m_2}}.$$

We close this section by introducing a last product of increments which is labeled for further computations.

Definition 3.7. *Let $g \in \mathcal{P}_{2,1}$ and $h \in \mathcal{P}_{1,2}$. Then $g \circ h$ is the increment in $\mathcal{P}_{2,2}$ defined by $[g \circ h]_{s_1 s_2;t_1 t_2} = g_{s_1 s_2;t_1} h_{s_1;t_1 t_2}$.*

3.3 Iterated integrals as increments in the plane

The relationship between iterated integrals and increments in the plane is crucial for us. Generally speaking, an iterated integral is given as follows: consider $f^j \in \mathcal{P}_{1,1}^{\infty}$ for $j = 1, \ldots, n$ and $(s_1, s_2), (t_1, t_2) \in \mathcal{S}_2$, where we recall relation (2.10) defining simplexes. Then we set

$$\begin{aligned} h^{1,\ldots,n}_{s_1 s_2;t_1,t_2} &\equiv \int_{\mathcal{S}_n(s_1,s_2)\times\mathcal{S}_n(t_1,t_2)} d_{12} f^1_{\sigma_1;\tau_1} \cdots d_{12} f^n_{\sigma_n;\tau_n} \\ &= \int_{s_1}^{s_2}\int_{t_1}^{t_2}\int_{s_1}^{\sigma_{n-1}}\int_{t_1}^{\tau_{n-1}} \cdots \int_{s_1}^{\sigma_2}\int_{t_1}^{\tau_2} d_{12} f^1_{\sigma_1;\tau_1} \cdots d_{12} f^n_{\sigma_n;\tau_n}, \end{aligned} \tag{3.4}$$

where we recall from Section 1.1 that $d_{12} f^j_{\sigma;\tau}$ stands for $\partial^2_{\sigma\tau} f^j_{\sigma;\tau}$.

Expression (3.4) is obviously cumbersome, and it could in particular become clearer by separating the s, σ from the t, τ variables. This is where the conventions introduced in equation (2.12) turn out to be useful. Namely, one can simply tensorize (2.12) in order to write the increment $h^{1,\ldots,n}$ defined at (3.4) as

$$h^{1,\ldots,n} = [d_1, \ldots, d_1] \otimes [d_2, \ldots, d_2]\,(f^1, \ldots, f^n), \quad \text{or} \quad h^{1,\ldots,n} = \int_1\int_2 d_{12} f^1 \cdots d_{12} f^n, \tag{3.5}$$

and notice that we will mainly use the second convention throughout the paper. This notation proves to be particularly convenient when one is faced with partial integrations (as introduced in (2.13)) in both directions 1 and 2. In order to illustrate this point, let us consider the simple example

$$g \equiv [d_1, d_1] \otimes [\mathrm{Id}, d_2]\, (f^1, f^2) = \int_1 d_1 f^1 \int_2 d_{12} f^2. \tag{3.6}$$

Let us now describe the algorithm which allows to go from expression (3.6) to an integral like (3.4). It can be summarized as follows:

- For direction 1, count the number of iterated integrals starting from the left hand side (in our example this number is 2). Then interpret these integrals as integrals on the simplex in direction 1 and write the 1-variables.
- Do the same for direction 2. In our example, there is only one integral in this direction, so that variable 2 is frozen in the first differential $d_1 f^1$.

Applying this algorithm, the reader can easily check that g defined at (3.6) can be written as

$$g_{s_1 s_2; t_1 t_2} = \int_{s_1 < \sigma_1 < \sigma_2 < s_2} d_1 f^1_{\sigma_1; t_1} \int_{t_1}^{t_2} d_{12} f^2_{\sigma_2; \tau_1}.$$

For sake of conciseness, we omit generalizations of this simple example.

4 Planar integration

Before going into the computational details, let us describe the general strategy we shall follow in order to obtain our Itô-Stratonovich type change of variable formulae in case of non smooth functions x. Indeed, we start from a smooth approximation x^n to our path x and we introduce a useful notation for the remainder of the computations:

Notation 4.1. *We shall drop the index n of approximations in x^n, which means that x will stand for a generic smooth path defined on $[0,1]^2$. For a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, we also write y for the path $\varphi(x)$ and for all $j \geq 1$ we set $y^j = \varphi^{(j)}(x)$.*

With these notations in hand, for a smooth sheet x and $f \in \mathcal{C}_b^2$ it is well known that formula (1.1) holds true. Recall that we have written this relation under the following form, compatible with our convention (3.5):

$$\delta y = \int_1 \int_2 y^1 \, d_{12} x + \int_1 \int_2 y^2 \, d_{\hat{1}\hat{2}} x. \tag{4.1}$$

We shall see that this formula still holds true in the limit for x, except that the integrals involved in the right hand side of (4.1) have to be interpreted in a sense which goes beyond the Riemann-Stieltjes case. Our main task will thus be to obtain a definition of $\int_1 \int_2 y^1 \, d_{12} x$ and $\int_1 \int_2 y^2 \, d_{\hat{1}\hat{2}} x$ involving iterated integrals of x and increments of y (or y^j for $j \geq 1$) only. Though this task might overlap with some aspects of [4], we present it here because it is short enough and allows us to introduce part of our formalism.

Let us introduce what will be later interpreted as the first order elements of the planar rough path above x:

Notation 4.2. *Let $x \in \mathcal{P}_{1,1}^{\gamma_1,\gamma_2}$ with $\gamma_1, \gamma_2 > 1/2$. We set*

$$\mathbf{x}^{1;2} = \delta x, \quad \text{and} \quad \mathbf{x}^{\hat{1};\hat{2}} = [\mathrm{Id} - \Lambda_1 \delta_1][\mathrm{Id} - \Lambda_2 \delta_2]\, (\delta_1 x \, \delta_2 x)\,.$$

Notice that for smooth functions we also have

$$\mathbf{x}^{1;2} = \int_1 \int_2 d_{12} x, \quad \text{and} \quad \mathbf{x}^{\hat{1};\hat{2}} = \int_1 \int_2 d_1 x \, d_2 x.$$

We are now ready to express the integrals in (4.1) in terms of the planar sewing map $(\Lambda_i)_{i=1,2}$ and $\Lambda = \Lambda_1 \Lambda_2$ and the increments introduced in Notation 4.2.

4.1 Change of variables formula: the Young case

The following theorem gives the analog of relation (4.1) in the Young case, and is a way to recast Theorem 1.1. Notice that some extensions of these results are contained in [4].

Theorem 4.3. *Let $x \in \mathcal{P}_{1,1}^{\gamma_1,\gamma_2}$ (where $\mathcal{P}_{1,1}^{\gamma_1,\gamma_2}$ is defined by (3.2)) with $\gamma_1, \gamma_2 > 1/2$ and $\varphi \in C^4(\mathbb{R})$. With our notations 4.2 and 4.1 in mind, define two increments z^1, z^2 as*

$$z^1 = [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](y^1\,\mathbf{x}^{1;2}), \quad \textit{and} \quad z^2 = [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](y^2\,\mathbf{x}^{\hat{1};\hat{2}}). \tag{4.2}$$

Then items (i)–(v) of Theorem 1.1 hold true. Furthermore, for $j = 1, 2$ the following bound is satisfied: $\|z^j\|_{\gamma_1;\gamma_2} \le c_\varphi \mathcal{N}_{\gamma_1,\gamma_2}(x)(1 + \mathcal{N}_{\gamma_1,\gamma_2}(x))$.

Proof. We first introduce a formalism which will feature prominently in the sequel of the paper.

Step 1: Setting for our computations. Let us write the first step of our expansion in a usual integration language: for $s_1, s_2, t_1, t_2 \in [0,1]$ and a continuously differentiable function x we have

$$z^1_{s_1 s_2;t_1 t_2} = \int_{s_1}^{s_2}\int_{t_1}^{t_2} y^1_{\sigma_1;\tau_1}\, d_{12}x_{\sigma_1;\tau_1},$$

where we have written $d_{12}x_{\sigma_1;\tau_1}$ instead of $d_{\sigma\tau}x_{\sigma_1;\tau_1}$. Then according to the elementary identity $y_{\sigma_1;\tau_1} = y_{s_1;t_1} + \delta_2 y^1_{\sigma_1;t_1\tau_1}$ we obtain

$$z^1_{s_1 s_2;t_1 t_2} = \int_{s_1}^{s_2} y^1_{\sigma_1;t_1}\int_{t_1}^{t_2} d_{12}x_{\sigma_1;\tau_1} + \int_{s_1}^{s_2}\int_{t_1}^{t_2} \delta_2 y^1_{\sigma_1;t_1\tau_1}\, d_{12}x_{\sigma_1;\tau_1}.$$

Going on with this procedure, we end up with a decomposition of the form

$$\begin{aligned} z^1_{s_1 s_2;t_1 t_2} &= y^1_{s_1;t_1}\,\delta x_{s_1 s_2;t_1 t_2} + \int_{s_1}^{s_2} \delta_1 y^1_{s_1\sigma_1;t_1}\int_{t_1}^{t_2} d_{12}x_{\sigma_1;\tau_1} \\ &\quad + \int_{t_1}^{t_2}\delta_2 y^1_{s_1;t_1\tau_1}\int_{s_1}^{s_2} d_{12}x_{\sigma_1;\tau_1} + \int_{s_1}^{s_2}\int_{t_1}^{t_2}\delta y^1_{s_1\sigma_1;t_1\tau_1}\, d_{12}x_{\sigma_1;\tau_1}. \end{aligned} \tag{4.3}$$

Since further calculations with all explicit indices are cumbersome, we shall now show how to translate the above computations with the formalism of Section 3.3: we simply write

$$\begin{aligned} z^1 &= \int_1\int_2 y^1\, d_{12}x = \int_1 y^1 \int_2 d_{12}x + \int_1\int_2 d_2 y^1\, d_{12}x \\ &= y\,\delta x + \int_1 d_1 y^1 \int_2 d_{12}x + \int_2 d_2 y^1 \int_1 d_{12}x + \int_1\int_2 d_{12}y^1\, d_{12}x \\ &\equiv y^1\,\mathbf{x}^{1;2} + a^{11;02} + a^{01;22} + a^{11;22}, \end{aligned} \tag{4.4}$$

which is obviously a shorter expression than (4.3). From now on, we shall carry on our computations with this simplified formalism.

Step 2: Analysis of the integrals. Consider the term $a^{11;02}$ above. According to the definition (3.1) of δ_1 we have:

$$\begin{aligned} \delta_1 a^{11;02}_{s_1 s_2 s_3;t_1 t_2} &= a^{11;02}_{s_1 s_3;t_1 t_2} - a^{11;02}_{s_1 s_2;t_1 t_2} - a^{11;02}_{s_2 s_3;t_1 t_2} = \int_{s_2}^{s_3}\left(y^1_{s_2;t_1} - y^1_{s_1;t_1}\right)\int_{t_1}^{t_2} d_{12}x_{u;v} \\ &= \delta_1 y^1_{s_1 s_2;t_1}\,\mathbf{x}^{1;2}_{s_2 s_3;t_1 t_2}, \end{aligned}$$

which is shortened into the relation $\delta_1 a^{11;02} = \delta_1 y^1\,\mathbf{x}^{1;2}$. From this expression it is easily seen that $\delta_1 a^{11;02} \in \mathcal{P}_{3,2}^{2\gamma_1;\gamma_2}$, and since $2\gamma_1 > 1$ one can resort to Proposition 3.3 in order to get the relation $a^{11;02} = \Lambda_1(\delta_1 y\, \mathbf{x}^{1;2})$. Proceeding in the same way for $a^{01;22}$ and $a^{11;22}$ we end up with

$$a^{11;02} = \Lambda_1(\delta_1 y^1\, \mathbf{x}^{1;2}), \quad a^{01;22} = \Lambda_2(\delta_2 y^1\, \mathbf{x}^{1;2}), \quad a^{11;22} = \Lambda(\delta y^1\, \mathbf{x}^{1;2}), \tag{4.5}$$

where we observe that $\varphi \in C^3(\mathbb{R})$ is the minimal assumption in order to have $\delta y^1 \in \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ (we impose however the condition $\varphi \in C^4(\mathbb{R})$ in order to have $\delta y^2 \in \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ as well). Furthermore, according to relation (2.9) we have $\delta_1(y\, \mathbf{x}^{1;2}) = -\delta_1 y^1\, \mathbf{x}^{1;2}$, $\delta_2(y^1\, \mathbf{x}^{1;2}) = -\delta_2 y^1\, \mathbf{x}^{1;2}$ and $\delta(y^1\, \mathbf{x}^{1;2}) = \delta y^1\, \mathbf{x}^{1;2}$, so that (4.5) can be expressed as

$$a^{11;02} = -\Lambda_1\delta_1(y^1\, \mathbf{x}^{1;2}), \quad a^{01;22} = -\Lambda_2\delta_2(y^1\, \mathbf{x}^{1;2}), \quad a^{11;22} = \Lambda\delta(y^1\, \mathbf{x}^{1;2}).$$

Plugging these relations into (4.4) we get

$$z^1 = \int_1\int_2 y^1\, d_{12}x = [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](y^1\, \mathbf{x}^{1;2}),$$

for smooth functions z, which corresponds to claim (ii) in Theorem 1.1. Item (i)-(iii)-(v) are now a matter of straightforward limiting procedures on smooth sheets, and the assertions concerning z^2 are obtained exactly in the same way. Item (iv) is an easy consequence of Proposition 3.4 and expression (4.2).

□

4.2 Riemann sums decompositions

This section is meant as a preparation for Skorohod type computations. Indeed, change of variables in the Skorohod setting involve some mixed integrals with $dx\, dR$ terms, for which a suitable representation is required. It will also be convenient for us to express the integral $\int_1\int_2 y^2\, d_1x d_2x$ in different ways, so that we first recall a proposition borrowed from [4]:

Proposition 4.4. *let $x \in \mathcal{P}_{1,1}^{\gamma_1,\gamma_2}$ with $\gamma_1, \gamma_2 > 1/2$. Set $y = \varphi(x)$ for $\varphi \in C^2(\mathbb{R})$ and $z^{2,y} \equiv \int_1\int_2 y\, d_1x d_2x$, understood in the Young sense. Then the following series of identities hold true:*

$$\begin{aligned} z^{2,y} &= [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](y\, \mathbf{x}^{\hat{1};\hat{2}}) = [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](y\, \delta_1 x\, \delta_2 x) \\ &= [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](y\, \delta_2 x\, \delta_1 x) = [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](y\, \delta_1 x \circ \delta_2 x), \end{aligned}$$

where we recall that the notation $\circ$ has been introduced at Definition 3.7.

The following proposition gives different ways to express the increment $z^{2,y}$ as limit of Riemann sums.

Proposition 4.5. *Let $0 < s_1 < s_2 < 1$, $0 < t_1 < t_2 < 1$ and denote by $\pi_1 = (s_i)_i$ and $\pi_2 = (t_j)$ some partitions of the intervals $[s_1, s_2]$ and $[t_1, t_2]$ respectively. Then under the assumptions of Proposition 4.4 we have that $\int_{s_1}^{s_2}\int_{t_1}^{t_2} y_{st}\, d_{12}x_{st}$ can be written as limit of Riemann sums of the form $\lim_{|\pi_1|,|\pi_2|\to 0}\sum_{i,j} y_{s_i;t_j}\delta x_{s_is_{i+1};t_jt_{j+1}}$, and recalling that $z^{2,y} \equiv \int_1\int_2 y\, d_1x d_2x$ we also have:*

$$\begin{aligned} z^{2,y} &= \lim_{|\pi_1|,|\pi_2|\to 0}\sum_{i,j} y_{s_i;t_j}\delta_1 x_{s_is_{i+1};t_j}\delta_2 x_{s_{i+1};t_jt_{j+1}} \\ &= \lim_{|\pi_1|,|\pi_2|\to 0}\sum_{i,j} y_{s_i;t_j}\delta_2 x_{s_i;t_jt_{j+1}}\delta_1 x_{s_is_{i+1};t_{j+1}} = \lim_{|\pi_1|,|\pi_2|\to 0}\sum_{i,j} y_{s_i;t_j}\delta_1 x_{s_is_{i+1};t_j}\delta_2 x_{s_i;t_jt_{j+1}}. \end{aligned}$$

Finally we shall need an extension of the last three propositions to integrals with mixed driving noises:

Proposition 4.6. *Let $f \in \mathcal{P}_{1,1}^{\gamma_1,\gamma_2}$,$g \in \mathcal{P}_{1,1}^{\rho_1,\rho_2}$ and $h \in \mathcal{P}_{1,1}^{\beta_1,\beta_2}$ such that $\gamma_i+\rho_i > 1$, $\beta_i+\gamma_i > 1$ and $\beta_i + \rho_i > 1$ for $i = 1, 2$. Set*

$$\int_1 \int_2 f \, d_1 g \, d_2 h = [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](f \, \delta_1 g \, \delta_2 h).$$

Then we also have

$$\int_1 \int_2 f \, d_1 g d_2 h = [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](f \, \delta_2 g \, \delta_1 h) = [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](f \, \delta_1 g \circ \delta_2 h).$$

Moreover, taking up the notations of Proposition 4.5 we have the following Reimann-sum representation of our integrals:

$$\begin{aligned}\int_{s_1}^{s_2} \int_{t_1}^{t_2} f_{s;t} \, d_1 g_{s;t} d_2 h_{s;t} &= \lim_{|\pi|\to 0} \sum_{i,j} f_{s_i;t_j} \delta_1 g_{s_i s_{i+1};t_j} \delta_2 h_{s_{i+1};t_j t_{j+1}} \\ &= \lim_{|\pi|\to 0} \sum_{i,j} f_{s_i;t_j} \delta_1 g_{s_i s_{i+1};t_j} \delta_2 h_{s_i;t_j t_{j+1}} = \lim_{|\pi|\to 0} \sum_{i,j} f_{s_i;t_j} \delta_2 g_{s_i;t_j t_{j+1}} \delta_1 h_{s_i s_{i+1};t_{j+1}}.\end{aligned}$$

In addition, as a consequence of the continuity of the sewing maps we have also the following continuity estimate:

$$\left\| \int_1 \int_2 f \, d_1 g \, d_2 h \right\|_{\rho_1,\beta_2} \lesssim \mathcal{N}_{\gamma_1,\gamma_2}(f) \, \mathcal{N}_{\beta_1,\beta_2}(h) \, \mathcal{N}_{\rho_1,\rho_2}(g).$$

Proof. Let $a^1 = f \, \delta_1 g \, \delta_2 h$, $a^2 = f \, \delta_2 g \, \delta_1 h$ and $a^3 = f \, \delta_1 g \circ \delta_2 h$. Then by a simple computation we have that

$$\begin{aligned}\delta_1 a^1 &= -\delta_1 f \, \delta_1 g \, \delta_2 h - f \, \delta_1 g \, \delta h \in \mathcal{P}_{3,2}^{\min(\gamma_1+\rho_1,\rho_1+\beta_1),\beta_2} \\ \delta_2 a^1 &= -\delta_2 f \, \delta_1 g \, \delta_2 h - f \, \delta g \, \delta_2 h \in \mathcal{P}_{2,3}^{\rho_1,\min(\gamma_2+\beta_2,\rho_2+\beta_2)},\end{aligned}$$

and

$$\delta a^1 = \delta f \, \delta_1 g \, \delta_2 h + \delta_1 f \, \delta g \, \delta_2 h + \delta_2 f \, \delta_1 g \, \delta h + f \, \delta g \, \delta h \in \mathcal{P}_{3,3}^{\min(\gamma_1+\rho_1,\rho_1+\beta_1),\min(\gamma_2+\beta_2,\rho_2+\beta_2)}.$$

This means that a^1 satisfies the assumptions of Proposition 3.3, and the same is readily checked for a^2 and a^3. Thus the increments $[\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](a^j)$ are well defined for $j = 1, 2, 3$. We set $[\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](a^1) = \int_1 \int_2 f \, d_1 g d_2 h$ since both objects coincide for smooth functions f, g, h.

We now identify the increments $[\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](a^j)$ by analyzing their Riemann sums. Indeed, a straightforward application of Proposition 3.4 yields the following limits:

$$\begin{aligned}[\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](a^1) &= \lim_{|\pi|\to 0} \sum_{i,j} f_{s_i;t_j} \delta_1 g_{s_i s_{i+1};t_j} \delta_2 h_{s_{i+1};t_j t_{j+1}} \\ [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](a^2) &= \lim_{|\pi|\to 0} \sum_{i,j} f_{s_i;t_j} \delta_2 g_{s_i;t_j t_{j+1}} \delta_1 h_{s_i s_{i+1};t_{j+1}} \\ [\mathrm{Id} - \Lambda_1\delta_1][\mathrm{Id} - \Lambda_2\delta_2](a^3) &= \lim_{|\pi|\to 0} \sum_{i,j} f_{s_i;t_j} \delta_1 g_{s_i s_{i+1};t_j} \delta_2 h_{s_i;t_j t_{j+1}}.\end{aligned}$$

We now prove that the 3 increments coincide by showing that the differences between Riemann sums vanish when the mesh of the partitions go to zero. Indeed, if we call π_2 the partition in direction 2, observe for instance that

$$\begin{aligned}\lim_{|\pi_2|\to 0} \sum_{i,j} (a^1 - a^3)_{s_i s_{i+1};t_j t_{j+1}} &= \lim_{|\pi_2|\to 0} \sum_{i,j} f_{s_i;t_j} \delta_1 g_{s_i s_{i+1};t_j} \delta h_{s_i s_{i+1};t_j t_{j+1}} \\ &= \sum_i \int_{t_1}^{t_2} f_{s_i t} \delta_1 g_{s_i s_{i+1};t} \, d_2 \delta_1 h_{s_i s_{i+1};t}\end{aligned} \tag{4.6}$$

Now if we remark that $|\int_{t_1}^{t_2} f_{s_i t}\delta_1 g_{s_i s_{i+1};t}\, d_t h_{s_i s_{i+1};t}| \lesssim (s_{i+1}-s_i)^{\rho_1+\beta_1}$, we easily obtain that $\lim_{|\pi_1|\to 0}\lim_{|\pi_2|\to 0}\sum_{i,j}(a^1-a^3)_{s_i s_{i+1};t_j t_{j+1}} = 0$. The same relation holds true for a^2-a^3 by symmetry, which ends our proof.

□

4.3 Change of variable formula: the rough case

In this section, we consider a path $x \in \mathcal{P}_{1,1}^{\gamma_1,\gamma_2}$ with $\gamma_1,\gamma_2 > 1/3$ and we wish to establish the change of variables formula for $y=\varphi(x)$ (with $\varphi \in C_b^8(\mathbb{R})$) announced in Theorem 1.5. In such a general context, the integration theory with respect to x relies on the existence of a rough path $\mathbb{X}$ sitting above x. Now recall that the first elements of $\mathbb{X}$ have been introduced at Hypothesis 1.3. However, as mentioned in Remark 1.4, the rough path still has to be completed and we proceed to its description here.

Let us first introduce another indexing convention for the elements of the rough path $\mathbb{X}$, similarly to what is done at Section 1.3:

(iii) (Following (ii) at Section 1.3) In the rough case, some overlapping integrals in directions 1 and 2 difficult the regularity analysis of certain increments. This induces some splitting operations on iterated integrals, leading to some split elements of the rough path $\mathbb{X}$. We indicate this splitting procedure by a $\otimes$ in indices of $\mathbf{x}$. An example of this operation is given by the increment $\mathbf{x}^{11;2\otimes 2} \in \mathcal{C}_2(\mathcal{C}_2 \otimes \mathcal{C}_2)$:

$$\mathbf{x}^{11;2\otimes 2}_{s_1 s_2;t_1 t_2 t_3 t_4} = \int_{s_1}^{s_2}\int_{s_1}^{\sigma_2}\left(\int_{t_1}^{t_2} d_{12}x_{\sigma_1;\tau_1}\right)\left(\int_{t_3}^{t_4} d_{12}x_{\sigma_2;\tau_2}\right).$$

With this additional notation in mind, the complete description of $\mathbb{X}$ is given below:

Hypothesis 4.7. *The function x is such that $\delta x \in \mathcal{P}_{2,2}^{\gamma_1,\gamma_2}$ with $\gamma_1,\gamma_2 > 1/3$, and fulfills Hypothesis 1.3. In addition, the stack $\mathbb{X}$ of iterated integrals related to x is required to contain the elements in the table below, where we let the reader guess the natural Hölder regularities related to each increment. As in Hypothesis 1.3, the iterated integrals above*

Table 1: Further elements of $\mathbb{X}$

Increment	*Interpretation*	*Increment*	*Interpretation*
$\mathbf{x}^{1\hat{1};2\hat{2}}$	$\int_1\int_2 d_{12}x d_{\hat{1}\hat{2}}x$	$\mathbf{x}^{1\otimes 1;22}$	$\int_1\int_2 d_{12}x \otimes_1 \int_1 d_{12}x$
$\mathbf{x}^{11\cdot 1;022}$	$\int_1 d_1x \int_2 d_{12}x \int_1 d_{12}x$	$\mathbf{x}^{11\cdot 1;222}$	$\int_1\int_2 d_{12}x d_{12}x \int_1 d_{12}x$
$\mathbf{x}^{1\hat{1};0\hat{2}}$	$\int_1 d_1x \int_2 d_{\hat{1}\hat{2}}x$	$\mathbf{x}^{11\otimes 1;222}$	$\int_1\int_2 d_{12}x d_{12}x \otimes_1 \int_1 d_{12}x$
$\mathbf{x}^{11\otimes 1;022}$	$\int_1 d_1x \int_2 d_{12}x \otimes_1 \int_1 d_{12}x$	$\mathbf{x}^{11\cdot 1;2\cdot 22}$	$\int_1\int_2 d_{12}x \int_2 d_{12}x \int_1 d_{12}x$

are assumed to be limits along approximations of x by smooth functions. Furthermore, the rough path $\mathbb{X}$ should also contain all the elements $\mathbf{x}$ obtained by symmetrizing the increments of Table 1 with respect to $1\leftrightarrow 2$, as well as those for which we change the last indices $1;2$ by $\hat{1};\hat{2}$. In total, we have to assume the existence of 26 additional increments.

With the additional notation introduced above, the following theorem is one of the main contents of [4]:

Theorem 4.8. *Theorem 1.5 holds true under Hypothesis 4.7.*

5 Skorohod's calculus in the Young case

This section is devoted to relate the Young type integration theory introduced at Section 4 and the Skorohod integral in the plane handled in [17]. Specifically, we shall first generalize the Skorohod change of variables formula given in [17] for a fractional Brownian sheet with Hurst parameter greater than $1/2$ to a fairly general Gaussian process. We shall then compare this formula with Theorem 1.1 item (v).

5.1 Malliavin calculus framework

We consider in this section a centered Gaussian process $\{x_{s;t};\ (s,t)\in[0,1]^2\}$ defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, with covariance function $\mathbb{E}[x_{s_1;t_1}x_{s_2;t_2}] = R_{s_1s_2;t_1t_2}$. We now briefly define the basic elements of Malliavin calculus with respect to x and then specify a little the setting under which we shall work.

5.1.1 Malliavin calculus with respect to x

We first relate a Hilbert space $\mathcal{H}$ to our process x, defined as the closure of the linear space generated by the functions $\{\mathbf{1}_{[0,s]\times[0,t]},(s,t)\in[0,1]^2\}$ with respect to the semi define positive form $\langle \mathbf{1}_{[0,s_1]\times[0,t_1]},\mathbf{1}_{[0,s_2]\times[0,t_2]}\rangle = R_{s_1s_2;t_1t_2}$. Then the map $I_1:\mathbf{1}_{[0,s]\times[0,t]}\to x_{s;t}$ can be extended to an isometry between $\mathcal{H}$ and the first chaos generated by $\{x_{s;t};(s,t)\in[0,1]^2\}$.

Starting from the space $\mathcal{H}$, a Malliavin calculus with respect to x can now be developped in the usual way (see [10, 14] for further details). Namely, we first define a set of smooth functionals of x by

$$\mathcal{S} := \{f(I_1(\psi_1),\dots,I_1(\psi_n));\ n\in\mathbb{N}, f\in C_b^\infty(\mathbb{R}^n),\psi_1,\dots,\psi_n\in\mathcal{H}\}$$

and for $F=f(I_1(\psi_1),\dots,I_1(\psi_n))\in\mathcal{S}$ we define

$$DF=\sum_{i=1}^n \partial_i f(I_1(\psi_1),\dots,I_1(\psi_n))\,\psi_i.$$

Then D is a closable operator from $L^p(\Omega)$ into $L^p(\Omega,\mathcal{H})$. Therefore we can extend D to the closure of smooth functionals under the norm

$$\|F\|_{1,p}=\left(\mathbb{E}[|F|^p]+\mathbb{E}[\|DF\|_{\mathcal{H}}^p]\right)^{\frac{1}{p}}$$

The iteration of the operator D is defined in such a way that for a smooth random variable $F\in\mathcal{S}$ the iterated derivative D^kF is a random variable with values in $\mathcal{H}^{\otimes k}$. The domain $\mathbb{D}^{k,p}$ of D^k is the completion of the family of smooth random variables $F\in\mathcal{S}$ with respect to the semi-norm :

$$\|F\|_{k,p}=\left(\mathbb{E}[|F|^p]+\sum_{j=1}^k \mathbb{E}[\|D^jF\|_{\mathcal{H}^{\otimes j}}^p]\right)^{\frac{1}{p}}.$$

Similarly, for a given Hilbert space V we can define the space $\mathbb{D}^{k,p}(V)$ of V-valued random variables, and $\mathbb{D}^\infty(V)=\cap_{k,p\ge1}\mathbb{D}^{k,p}$.

Consider now the adjoint $\delta^\diamond$ of D. The domain of this operator is defined as the set of $u\in L^2(\Omega,\mathcal{H})$ such that $\mathbb{E}[|\langle DF,u\rangle_{\mathcal{H}}|]\lesssim\|F\|_{1,2}$, and for this kind of process $\delta^\diamond(u)$ (called Skorohod integral of u) is the unique element of $L^2(\Omega)$ such that

$$\mathbb{E}[\delta^\diamond(u)F]=\mathbb{E}[\langle DF,u\rangle_{\mathcal{H}}],\quad \text{for}\quad F\in\mathbb{D}^{1,2}.$$

Note that $\mathbb{E}[\delta^\diamond(u)]=0$ and

$$\mathbb{E}[|\delta^\diamond(u)|^2]\le\mathbb{E}[\|u\|_{\mathcal{H}}^2]+\mathbb{E}[\|Du\|_{\mathcal{H}\otimes\mathcal{H}}^2]\equiv\|u\|_{\mathbb{D}^{1,2}(\mathcal{H})}^2. \tag{5.1}$$

The following divergence type property of $\delta^\diamond$ will be useful in the sequel:

$$\delta^\diamond(Fu) = F\delta^\diamond(u) - \langle Du, F\rangle_{\mathcal{H}}, \tag{5.2}$$

and we also recall the following compatibility of $\delta^\diamond$ with limiting procedures:

Lemma 5.1. *let u_n be a sequence of elements in $Dom(\delta^\diamond)$, which converges to u in $L^2(\Omega, \mathcal{H})$. We further assume that $\delta^\diamond(u_n)$ converges in $L^2(\Omega)$ to some random variable $F \in L^2(\Omega)$. Then $u \in Dom(\delta^\diamond)$ and $\delta^\diamond(u) = F$.*

5.1.2 Wick products

Some of our results below will be expressed in terms of Rieman-Wick sums. We give a brief account on these objects, mainly borrowed from [10, 11].

Among functionals F of x such that $F \in \mathbb{D}^\infty$, the set of multiple integrals plays a special role. In order to introduce it in the context of a general process x indexed by the plane, consider an orthonormal basis $\{e_n;\, n \geq 1\}$ of $\mathcal{H}$ and let $\hat{\otimes}$ denote the symmetric tensor product. Then

$$f_n = \sum_{\text{finite}} f_{i_1,\cdots,i_n} e_{i_1} \hat{\otimes} \cdots \hat{\otimes} e_{i_n}, \quad f_{i_1,\cdots,i_n} \in \mathbb{R} \tag{5.3}$$

is an element of $\mathcal{H}^{\hat{\otimes} n}$ satisfying the relation:

$$\|f_n\|^2_{\mathcal{H}^{\hat{\otimes} n}} = \sum_{\text{finite}} |f_{i_1,\cdots,i_n}|^2 . \tag{5.4}$$

Moreover, $\mathcal{H}^{\hat{\otimes} n}$ is the completion of the set of elements like (5.3) with respect to the norm (5.4).

For an element $f_n \in \mathcal{H}^{\hat{\otimes} n}$, the multiple Itô integral of order n is well-defined. First, any element of the form given by (5.3) can be rewritten as

$$f_n = \sum_{\text{finite}} f_{j_1\cdots j_m} e_{j_1}^{\hat{\otimes} k_1} \hat{\otimes} \cdots \hat{\otimes} e_{j_m}^{\hat{\otimes} k_m}, \tag{5.5}$$

where the $j_1, \dots, j_m$ are different and $k_1 + \cdots + k_m = n$. Then, if $f_n \in \mathcal{H}^{\hat{\otimes} n}$ is given under the form (5.5), define its multiple integral as:

$$I_n(f_n) = \sum_{\text{finite}} f_{j_1,\cdots,j_m} H_{k_1}(I_1(e_{j_1})) \cdots H_{k_m}(I_1(e_{j_m})), \tag{5.6}$$

where H_k denotes the k-th normalized Hermite polynomial given by

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}} = \sum_{j \leq k/2} \frac{(-1)^j k!}{2^j\, j!\, (k-2j)!} x^{k-2j}.$$

It holds that the multiple integrals of different order are orthogonal and that

$$\mathbb{E}\left[|I_n(f_n)|^2\right] = n!\, \|f_n\|^2_{\mathcal{H}^{\hat{\otimes} n}}.$$

This last isometric property allows to extend the multiple integral for a general $f_n \in \mathcal{H}^{\hat{\otimes} n}$ by $L^2(\Omega)$ convergence. Finally, one can define the integral of $f_n \in \mathcal{H}^{\otimes n}$ by putting $I_n(f_n) := I_n(\tilde{f}_n)$, where $\tilde{f}_n \in \mathcal{H}^{\hat{\otimes} n}$ denotes the symmetrized version of f_n. Moreover, the chaos expansion theorem states that any square integrable random variable $F \in L^2(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathcal{G}$ is the σ- field generated by x, can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{with} \quad \mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{\mathcal{H}^{\hat{\otimes} n}} . \tag{5.7}$$

With these notations in mind, one way to introduce Wick products on a Wiener space is to impose the relation

$$I_n(f_n) \diamond I_m(g_m) = I_{n+m}(f_n \hat{\otimes} g_m) \tag{5.8}$$

for any $f_n \in \mathcal{H}^{\hat{\otimes} n}$ and $g_m \in \mathcal{H}^{\hat{\otimes} m}$, where the multiple integrals $I_n(f_n)$ and $I_m(g_m)$ are defined by (5.6). If $F = \sum_{n=1}^{N_1} I_n(f_n)$ and $G = \sum_{m=1}^{N_2} I_m(g_m)$, we define $F \diamond G$ by

$$F \diamond G = \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} I_{n+m}(f_n \hat{\otimes} g_m).$$

By a limit argument, we can then extend the Wick product to more general random variables (see [11] for further details). In this paper, we will take the limits in the $L^2(\Omega)$ topology.

Some corrections between ordinary and Wick products will be computed below. A simple example occurs for products of $f(x)$ by a Gaussian increment. Indeed, for a smooth function f and $g_1, g_2 \in \mathcal{H}$, it is shown in [11] that

$$f(I_1(g_1)) \diamond I_1(g_2) = f(I_1(g_1))\, I_1(g_2) - f'(I_1(g_1))\, \langle g_1, g_2 \rangle_{\mathcal{H}} . \tag{5.9}$$

We now state a result which is proven in [10, Proposition 4.1].

Proposition 5.2. *Let $F \in \mathbb{D}^{k,2}$ and $g \in \mathcal{H}^{\otimes k}$. Then*

1. *$F \diamond I_k(g)$ is well defined in $L^2(\Omega)$.*
2. *$Fg \in$ Dom $\delta^{\diamond k}$.*
3. *$F \diamond I_k(g) = \delta^{\diamond k}(Fg)$.*

5.1.3 Further assumptions and preliminary results

In order to simplify our computations, let us introduce some additional assumptions on the covariance R:

Hypothesis 5.3. *The covariance R of our centered Gaussian process x belongs to the space $\mathcal{C}^{1\text{-var}}([0,1]^4)$, and satisfies a factorization property of the form*

$$\mathbb{E}[x_{s_1;t_1} x_{s_2;t_2}] = R_{s_1 s_2; t_1 t_2} = R^1_{s_1 s_2} R^2_{t_1 t_2},$$

for two covariance functions R^1, R^2 on $[0,1]$. In addition, setting $R^i_a = R^i_{aa}$ for $a \in [0,1]$ and $i = 1,2$, we assume that $a \mapsto R^i_a$ is differentiable and we suppose that

$$|2R^i_{ab} - R^i_{aa} - R^i_{bb}| \lesssim |a-b|^{\gamma_i} \tag{5.10}$$

for all $a, b \in [0,1]$, with $\gamma_i > 1$. Finally we suppose that $(R^i)'_a = \partial_a R^i_{aa} \in L^\infty([0,1])$.

The first consequence of our Hypothesis 5.3 is that the regularity of x corresponds to the Young type regularity of Section 4. Indeed, it is readily checked that relation (5.10) yields

$$\mathbb{E}\left[(\delta x_{s_1 s_2; t_1 t_2})^2\right] \lesssim |s-s'|^{\gamma_1} |t-t'|^{\gamma_2}.$$

Since x is Gaussian, an easy application of Kolmogorov's criterion ensures that

$$x \in \mathcal{P}^{\alpha_1,\alpha_2}_{1,1}, \quad \text{with} \quad \alpha_1 = \frac{\gamma_1}{2} - \epsilon_1 > \frac{1}{2}, \quad \alpha_2 = \frac{\gamma_2}{2} - \epsilon_2 > \frac{1}{2}, \tag{5.11}$$

for arbitrarily small $\epsilon_1, \epsilon_2 > 0$. This enables us to appeal to Young's integration theory in order to define integrals of the form $\int_1 \int_2 \varphi(x)\, d_{12}x$.

Let us quote two lemmas concerning Hölder norms in the plane which will feature in our comparison between Stratonovich and Skorohod integrations. The first one deals with the composition of a Hölder process with a nonlinearity f:

Lemma 5.4. *Let $\varphi \in C^2(\mathbb{R})$, $\theta_1, \theta_2 > 0$ and a rectangle $\Delta \subset [0,1]^2$. Then on Δ we have that*

$$\|\delta_1 y\|_{\theta_1,0} \le \|y^1\|_{0;\Delta} \, \|\delta_1 x\|_{\theta_1,0}$$

and

$$\|\delta y\|_{\theta_1,\theta_2} \lesssim \left(\|y^1\|_{0;\Delta} + \|y^2\|_{0;\Delta}\right) \|\delta x\|_{\theta_1,\theta_2}(1 + \|\delta x\|_{\theta_1,\theta_2}),$$

where $\|\cdot\|_{0;\Delta}$ stands for the supremum norm on Δ and y^j still denotes $\varphi^{(j)}(x)$.

Next we also need an integral semi-norm dominating Hölder's norms in the plane. This is given by the following Garsia type result:

Lemma 5.5. *Let $p > 1$, $\theta_1, \theta_2 > 0$ and $y \in \mathcal{P}_{1,1}$. The following relation holds true: there exists a constant $M = M_{\theta_1,\theta_2} > 1$ such that*

$$\|\delta y\|^p_{\theta_1,\theta_2} \le M^p \int_{[0,1]^4} \frac{|\delta y_{u_1 u_2; v_1 v_2}|^p}{|u_2 - u_1|^{\theta_1 p + 2} |v_2 - v_1|^{\theta_2 p + 2}} du_1 du_2 dv_1 dv_2. \tag{5.12}$$

We now turn to a consequence of our additional Hypothesis 5.3 on embedding properties of the Hilbert space $\mathcal{H}$ defined above:

Lemma 5.6. *Under Hypothesis 5.3, we have $\|f\|_{\mathcal{H}} \le \|f\|_\infty \|R\|_{1\text{-var};[0,1]^4}$.*

Proof. Consider a step function $f = \sum_{ij} a_{ij} \mathbf{1}_{\Delta_{ij}}$ related to a partition $(\Delta_{ij})_{ij}$ of $[0,1]^2$. We have

$$\begin{aligned}
\|f\|^2_{\mathcal{H}} = \sum_{i,j,l,k} a_{ij} a_{lk} R^1_{s_i s_l} R^2_{t_j t_k} &= \sum_{i,j,k,l} a_{ij} a_{kl} \int_0^{s_i} \int_0^{s_l} \int_0^{t_j} \int_0^{t_k} d_{12} R^1_{s_1 s_2} d_{12} R^2_{t_1 t_2} \\
&= \int_{[0,1]^4} f_{s_1;t_1} \, f_{s_2;t_2} d_{12} R^1_{s_1 s_2} d_{12} R^2_{t_1 t_2} \le \|f\|^2_\infty \, \|R\|^2_{1\text{-var};[0,1]^4}.
\end{aligned} \tag{5.13}$$

The general case now easily follows by density of the step functions in $\mathcal{H}$.

□

Let us now recall that we work under the usual assumptions for Skorohod type change of variables formulae given at Definition 1.7 and referred to as (GC) condition in the sequel. Notice that $\max_{s,t\in[0,1]}(R^1_s R^2_t) = \max_{s,t\in[0,1]} \mathbb{E}[|x_{s;t}|^2]$. Thus condition (GC) implies that

$$\mathbb{E}\Big[\sup_{s,t\in[0,1]} |\varphi(x_{s;t})|^r\Big] < \infty, \quad \text{for all} \quad r \ge 1. \tag{5.14}$$

We now state an approximation result in $\mathcal{H}$ which proves to be useful in order to get our Itô type formula.

Proposition 5.7. *Let x be a centered Gaussian process on $[0,1]$ satisfying Hypothesis 5.3 and $\varphi \in C^1(\mathbb{R})$ such that the growth condition (GC) is fulfilled for f and $f^{(1)}$. Consider a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$ and $\pi_1 = (s_i)_i$, $\pi_2 = (t_j)_j$ two respective dissections of the intervals $[s_1, s_2]$ and $[t_1, t_2]$. Then*

$$\lim_{|\pi_1|,|\pi_2|\to 0} \mathbb{E}\left[\left\|y.\mathbf{1}_\Delta - \sum_{i,j} y_{s_i;t_j} \mathbf{1}_{\Delta_{i,j}}\right\|^2_{\mathcal{H}}\right] = 0,$$

where we have used Notation 3.1 for the rectangles $\Delta_{i,j}$.

Proof. Observe first that

$$y_{s;t} \mathbf{1}_\Delta(s,t) - \sum_{i,j} y_{s_i;t_j} \mathbf{1}_{\Delta_{ij}}(s,t) = \sum_{i,j} (y_{s;t} - y_{s_i;t_j}) \mathbf{1}_{\Delta_{i,j}}(s,t)$$

from which the following estimation is easily obtained:

$$|(y_{s;t} - y_{s_i;t_j})\mathbf{1}_{\Delta_{i,j}}(s,t)| \le \left(\sup_{(s,t)\in\Delta} |y^1_{s;t}| \max_{|s_1-s_2|\le|\pi_1|,|t_1-t_2|\le|\pi_2|} |x_{s_1;t_1} - x_{s_2;t_2}| \right) \mathbf{1}_{\Delta_{i,j}}.$$

Hence if we take expectations in this last estimation and resort to Hölder's inequality, we obtain that

$$\mathbb{E}\left[\left\| y.\mathbf{1}_\Delta - \sum_{i,j} y_{s_i;t_j}\mathbf{1}_{\Delta_{i,j}} \right\|_\infty^2 \right] \le \mathbb{E}^{1/2}\left[\sup_{(s,t)\in\Delta} |y^1_{s;t}|^4 \right] \mathbb{E}^{1/2}\left[\max_{|s-s'|\le|\pi_1|,|t-t'|\le|\pi_2|} |x_{s;t} - x_{s';t'}|^4 \right] \left\| \sum_{i,j} \mathbf{1}_{\Delta_{ij}} \right\|_\infty.$$

Now the r.h.s of this inequality goes to zero when the mesh of the partitions π_1, π_2 goes to zero by continuity properties of x (see (5.11)). Our claim thus easily stems from the embedding (5.13).

□

5.2 Itô-Skorohod type formula

We now turn to one of the main aim of this article, namely the proof of a Skorohod type change of variable formula for a general Gaussian process x defined on $[0,1]^2$, under our assumptions 5.3. Our starting point is the relation between z^1 and its Skorohod equivalent.

Proposition 5.8. *Assume x is a centered Gaussian process on $[0,1]^2$ with a covariance function satisfying* (1.9)*. Consider a function $\varphi \in C^2(\mathbb{R})$ satisfying condition (GC) and a rectangle $\Delta = [s_1,s_2]\times[t_1,t_2]$. Then we have that $y^1_\cdot \mathbf{1}_\Delta \in Dom(\delta^\diamond)$, and if we define the increment $z^{1,\diamond} \equiv \delta^\diamond(y^1_\cdot \mathbf{1}_\Delta)$ the following relation holds true:*

$$z^{1,\diamond}_{s_1s_2;t_1t_2} = z^1_{s_1s_2;t_1t_2} - \frac{1}{4}\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^2_{s;t}\, d_1R^1_s d_2R^2_t, \tag{5.15}$$

where z^1 is given by Theorem 4.3 and the second integral in the right hand side of (5.15) *is of Riemann-Stietjes type. Moreover, relation* (1.12) *holds true in the $L^2(\Omega)$ and almost sure sense.*

Remark 5.9. We have expressed all our assumptions so far in terms of Hölder type regularities, and this is why we stick to this kind of hypothesis on R here. However, generalizations to p-var type assumptions are easily conceivable, and in particular relation (5.15) is certainly verified as soon as $R \in \mathcal{C}^{1-\text{var}}([0,1]^4)$.

Proof of Proposition 5.8. Consider a sequence of partitions $\pi_n = (\pi^1_n, \pi^2_n)$ whose mesh go to 0 as $n\to\infty$. The generic elements of π_n will be denoted by (s_i,t_j). Owing to formula (5.2), we have that

$$\sum_{\pi_n} \delta^\diamond(y^1_{s_i;t_j}\mathbf{1}_{\Delta_{ij}}) = \sum_{\pi_n} y^1_{s_i;t_j}\delta x_{s_is_{i+1};t_jt_{j+1}} - \sum_{\pi_n} y^2_{s_i;t_j}\mathbb{E}[x_{s_i;t_j}\delta x_{s_is_{i+1};t_jt_{j+1}}] \tag{5.16}$$
$$= \sum_{\pi_n} y^1_{s_i;t_j}\delta x_{s_is_{i+1};t_jt_{j+1}} - \sum_{\pi_n} y^2_{s_i;t_j}(R^1_{s_is_{i+1}} - R^1_{s_is_i})(R^2_{t_jt_{j+1}} - R^2_{t_jt_j}) \equiv A^n_1 - A^n_2.$$

We now treat those two terms separately.

Step 1: Estimation of A^n_2. Recall that A^n_2 is defined by

$$A^n_2 = \sum_{\pi_n} y^2_{s_i;t_j}(R^1_{s_{i+1}s_i} - R^1_{s_is_i})(R^2_{t_{j+1}t_j} - R^2_{t_jt_j}).$$

In order to treat this term, first remark that for $k=1,2$ we have

$$R^k_{s_is_{i+1}} - R^k_{s_is_i} = \frac{1}{2}(R^k_{s_{i+1}s_{i+1}} - R^k_{s_is_i}) + \rho^k_{s_is_{i+1}},$$

where $\rho^k_{s_is_{i+1}} = \frac{1}{2}(2R^k_{s_is_{i+1}} - R^k_{s_is_i} - R^k_{s_{i+1}s_{i+1}})$. Injecting this relation in the definition of the term A^n_2 and recalling that we have set $R^k_a \equiv R^k_{aa}$, we obtain

$$\begin{aligned} A^n_2 &= 1/4 \sum_{\pi_n} y^2_{s_i;t_j}(R^1_{s_{i+1}} - R^1_{s_i})(R^2_{t_{j+1}} - R^2_{t_j}) \\ &\quad +1/2 \sum_{\pi_n} y^2_{s_i;t_j}\left[(R^2_{t_{j+1}} - R^2_{t_j})\rho^1_{s_is_{i+1}} + (R^1_{s_{i+1}} - R^1_{s_i})\rho^2_{t_jt_{j+1}} + \rho^1_{s_is_{i+1}}\rho^2_{t_jt_{j+1}}\right] \\ &\equiv A^n_{21} + A^n_{22} + A^n_{23} + A^n_{24}. \end{aligned}$$

We will now show that

$$\lim_{n\to\infty} A^n_{21} = \frac{1}{4}\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^2_{s;t}\, d_1R^1_s d_2R^2_t, \quad \text{and} \quad \lim_{n\to\infty}\sum_{j=2}^4 A^n_{2j} = 0, \tag{5.17}$$

where the limits are understood in the almost sure and L^2 sense.

Indeed, it is easily understood that the terms $A^n_{22}, A^n_{23}, A^n_{24}$ are remainder terms: according to Hypothesis 5.3 we have that $|\rho^i_{ab}| \lesssim |a-b|^{\gamma_i}$, and we get the following inequality for A^n_{22}:

$$\begin{aligned} A^n_{22} &\lesssim |\pi_1|^{\gamma_1 - 1} \sup_{(s,t)\in\Delta} |y^2_{s;t}| \sum_{\pi_n}(s_{i+1} - s_i)\,|R^2_{t_{j+1}} - R^2_{t_j}| \\ &\le |\pi_1|^{\gamma_1-1} \sup_{(s,t)\in\Delta} |y^2_{s;t}|\,(s_2 - s_1)\int_{t_1}^{t_2} |d_2R^2_t|. \end{aligned}$$

This relation, plus the condition (GC) on f, obviously entails that $\lim_{n\to\infty} A^n_{22} = 0$ in the almost sure and $L^2(\Omega)$ sense. The case of A^n_{23}, A^n_{24} follow exactly along the same lines.

We now focus on the term A^n_{21}: observe that

$$\begin{aligned} &\left| a_1 - 1/4 \int_{s_1}^{s_2}\int_{t_1}^{t_2} y^2_{s;t} d_1R^1_s d_2R^2_t \right| \\ &\qquad \lesssim \sup_{(s,t)\in\Delta} |y^2_{s;t}| \max_{|s-s'|\le|\pi_1|, |t-t'|\le|\pi_2|} |x_{s;t} - x_{s_2;t_2}| \int_{s_1}^{s_2}\int_{t_1}^{t_2} |d_1R^1_s||d_2R^2_t|. \end{aligned}$$

Invoking the same estimates as before for the Hölder norm of x and condition (GC) on f, the proof of our assertion (5.17) is now completed.

Step 2: Estimation of A^n_1. Let us set $Y^n = \sum_{\pi_n} y^1_{s_i;t_j}\mathbf{1}_{\Delta_{ij}}$ and $Y = y^1\mathbf{1}_\Delta$. Then one can recast (5.16) into $\delta^\diamond(Y^n) = A^n_1 - A^n_2$. Furthermore, we know that A^n_1 converges almost surely to z^1 as stated in Theorem 4.3. In order to show that A^n_1 also converges in L^2, we proceed as follows: (i) We show that Y^n converges to Y in $\mathbb{D}^{1,2}(\mathcal{H})$. According to standard results of Malliavin calculus (see also (5.1)), this yields the L^2 convergence of $\delta^\diamond(Y^n)$ to $\delta^\diamond(Y)$. (ii) Since we have also shown the convergence of A^n_2 in L^2, the convergence of A^n_1 in L^2 is then easily obtained from the relation $\delta^\diamond(Y^n) = A^n_1 - A^n_2$.

We are thus reduced to the convergence of Y^n to Y in $\mathbb{D}^{1,2}(\mathcal{H})$, which is obtained similarly to what is done in Proposition 5.7. Indeed, the convergence of Y^n towards Y is obtained exactly as in the latter proposition. In addition, the derivative of Y^n can be explicitly computed as:

$$D_{\sigma;\tau}Y^n_{s;t} = \sum_{\pi_n} y^2_{s_i;t_j}\mathbf{1}_{[0,s_i]\times[0,t_j]}(\sigma,\tau)\mathbf{1}_{\Delta_{ij}}(s,t).$$

The convergence of this derivative to

$$D_{\sigma;\tau}Y_{s;t}=y^2_{s;t}\mathbf{1}_{[0,s]\times[0,t]}(\sigma,\tau)\mathbf{1}_\Delta(s,t)$$

is then established again along the same lines as for Proposition 5.7.

Step 3: Conclusion. Let us summarize the results obtained in the last two steps: plugging relation (5.17) into the definition of A^n_2 and recalling the limiting behavior of A^n_1 established at Step 2, we have obtained that

$$\delta^\diamond(y^1\mathbf{1}_\Delta)=\lim_{n\to\infty}\sum_{\pi_n}\delta^\diamond(y^1_{s_i;t_j}\mathbf{1}_{\Delta_{ij}})=\int_{s_1}^{s_2}\int_{t_1}^{t_2}y^1_{s;t}d_{12}x_{s;t}-\frac{1}{4}\int_{s_1}^{s_2}\int_{t_1}^{t_2}y^2_{s;t}d_1R^1_sd_2R^2_t.$$

where the convergence is understood in both a.s and $L^2(\Omega)$ sense. This finishes our proof of relation (5.15).

As far as expression (1.12) with Wick-Riemann sums is concerned, recall that we have proved that

$$\delta^\diamond(y^1_\cdot\mathbf{1}_\Delta)=\lim_{|\pi_1|,|\pi_2|\to0}\sum_{\pi_n}\delta^\diamond(y^1_{s_i;t_j}\mathbf{1}_{\Delta_{ij}}).$$

Now invoke Proposition 5.2 for $k=1$ in order to state that

$$\delta^\diamond(y^1_{s_i;t_j}\mathbf{1}_{\Delta_{ij}})=y^1_{s_i;t_j}\diamond\delta^\diamond(\mathbf{1}_{\Delta_{ij}})=y^1_{s_i;t_j}\diamond\delta x_{s_is_{i+1};t_jt_{j+1}},$$

which ends the proof.

□

Proposition 5.8 gives a meaning to the increment $z^{1,\diamond}$ and compares them to the corresponding Stratonovich increment z^1. In order to compare change of variables formulae, we still have to define Skorohod integrals of the form $z^{2,\diamond}$, which is what we proceed to do now.

To this aim, let us start by some formal considerations: it is easily conceived that

$$\int_0^s\int_0^t y^2_{u;v}d_1^\diamond x_{u;v}d_2^\diamond x_{u;v}=\int_0^s\int_0^t\int_0^u\int_0^v y^2_{u;v}d^\diamond_{12}x_{u'v}d^\diamond_{12}x_{uv'}=\delta^{\diamond,2}(N(y^2))\tag{5.18}$$

where, similarly to [17], we set

$$N(y)_{u'u;vv'}:=y_{u,v}\,\mathbf{1}_{[0,s]\times[0,v]}(u,v')\,\mathbf{1}_{[0,u]\times[0,t]}(u',v),$$

where we integrate firstly in (u',v) and then in (u,v'), and where the notation $\delta^{\diamond,2}$ specifies that we perform double integrals in the Skorohod sense. Our objective in what follows is to give a rigorous meaning to equation (5.18).

Lemma 5.10. *Take up the notation of Proposition 5.8, and consider $f\in C^3(\mathbb{R})$ satisfying condition (GC). For a sequence of partitions $(\pi_n)_{n\geq1}$ whose mesh goes to 0 define*

$$a^{\pi_n}_{u'u;vv'}=\sum_{i,j}y^2_{s_i;t_j}\,\mathbf{1}_{[0,s_i]\times[t_j,t_{j+1}]}(u',v)\,\mathbf{1}_{[s_i,s_{i+1}]\times[0,t_j]}(u,v').\tag{5.19}$$

Then a^{π_n} converges to $N(y^2)$ in $L^2(\Omega,\mathcal{H}^{\otimes2})$ as n goes to infinity.

Proof. First notice that the tensor norm of an element $K\in\mathcal{H}^{\otimes2}$ can be bounded as:

$$\begin{aligned}\|K\|_{\mathcal{H}^{\otimes2}}&=\int_{[0,1]^8}K_{a_1a_1';b_1b_1'}K_{a_2a_2';b_2b_2'}d_{12}R^1_{a_1a_1'}d_{12}R^1_{a_2a_2'}d_{12}R^2_{b_1b_1'}d_{12}R^1_{b_2b_2'}\\&\leq\int_{[0,1]^8}|K_{a_1a_1';b_1b_1'}K_{a_2a_2';b_2b_2'}|\,|d_{12}R^1_{a_1a_1'}||d_{12}R^1_{a_2a_2'}||d_{12}R^2_{b_1b_1'}||d_{12}R^1_{b_2b_2'}|.\end{aligned}\tag{5.20}$$

Furthermore, a simple computation shows that

$$a^{\pi_n}_{u'u;vv'} - N(y^2)_{u'u;vv'} = \sum_{\pi_n} \left[y^2_{s_i;t_j} - y^2_{u;v} \right] \left[\mathbf{1}_{[0,u]\times[t_j,t_{j+1}]}(u',v) \mathbf{1}_{[s_i,s_{i+1}]\times[0,v]}(u,v') \right]$$
$$+ \sum_{\pi_n} y^2_{s_i;t_j} \left[\left(\mathbf{1}_{[0,s_i]\times[0,t_j]}(u',v') - \mathbf{1}_{[0,u]\times[0,v]}(u',v') \right) \mathbf{1}_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}(u,v) \right],$$

and thus,

$$\left| a^{\pi_n}_{u'u;vv'} - N(y^2)_{u'u;vv'} \right| \le \Big(\sup_{(a,b)\in[0,s]\times[0,t]} |y^3_{a;b}| \sup_{|a_2-a_1|\le|\pi_1|, |b_2-b_1|\le|\pi_2|} |x_{a_2;b_2} - x_{a_1;b_1}|$$
$$+ \max_{i,j} (\mathbf{1}_{[s_i,s_{i+1}]}(u') + \mathbf{1}_{[t_j,t_{j+1}]}(v')) \sup_{(a,b)\in[0,s]\times[0,t]} |y^2_{a;b}| \Big). \quad (5.21)$$

Our claims are now easily derived: on the one hand the right hand side of (5.21) converges to zero when $n \to \infty$ if $u' \neq s_i$ and $v' \neq t_j$ for all i, j. Then using inequality (5.20) and dominated convergence we obtain that a^{π_n} converges a.s to $N(y)$ in $\mathcal{H}^{\otimes 2}$. On the other hand, in order to obtain the convergence in $L^2(\Omega, \mathcal{H}^{\otimes 2})$ it suffices to use the fact that f satisfies condition (GC) and apply once again dominated convergence.

□

Now we are able to define our mixed integral in the Skorohod sense and connect it to the equivalent integral in the Young theory:

Proposition 5.11. *Assume x is a centered Gaussian process on $[0,1]^2$ with a covariance function satisfying (1.9). Consider a function $\varphi \in C^4(\mathbb{R})$ satisfying condition (GC) and a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$. Then we have that $N(y) \in Dom(\delta^{\diamond,2})$, and if we define $z^{2,\diamond} = \delta^{\diamond,2}(N(y))$ the following relation holds:*

$$z^{2,\diamond}_{s_1s_2;t_1t_2} = z^2_{s_1s_2;t_1t_2} - \frac{1}{4}\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^2_{u;v} d_1R^1_u\, d_2R^2_v - \frac{1}{2}\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^3_{u;v} R^1_u\, d_2R^2_v\, d_1x_{u;v}$$
$$- \frac{1}{2}\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^3_{u;v} R^2_v\, d_1R^1_u\, d_2x_{u;v} + \frac{1}{4}\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^4_{u;v} R^1_u R^2_v\, d_1R^1_u\, d_2R^2_v, \quad (5.22)$$

where $\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^3_{u;v} R^1_u\, d_2R^2_v\, d_1x_{u;v}$ and $\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^3_{u;v} R^2_v\, d_1R^1_u\, d_2x_{u;v}$ are defined according to Proposition 4.6. Moreover, relation (1.13) holds true in the $L^2(\Omega)$ and almost sure sense.

Proof. Like for Proposition 5.8, our strategy is as follows: consider a sequence $\pi_n = (\pi^1_n, \pi^2_n)$ whose mesh go to 0 as $n \to \infty$ and set $a^n \equiv a^{\pi_n}$ defined by (5.19). We have seen at Lemma 5.10 that $\lim_{n\to\infty} a^n = N(y)$ in $L^2(\Omega, \mathcal{H}^{\otimes 2})$. We shall now study the convergence of $\delta^\diamond(a^n)$ by means of Wick-Stratonovich corrections. Then we will conclude by invoking Proposition 5.1.

Step 1: Wick-Stratonovich corrections. According to relation (5.8) and Proposition 5.2 for $k = 2$ we obtain

$$\begin{aligned} \delta^{\diamond,2}(a^n) &= \sum_{\pi_n} \delta^{\diamond,2}(y^2_{s_i;t_j} \mathbf{1}_{[0,s_i]\times[t_j,t_{j+1}]} \otimes \mathbf{1}_{[s_i,s_{i+1}]\times[0,t_j]}) \\ &= \sum_{\pi_n} y^2_{s_i;t_j} \diamond \delta_2 x_{s_i;t_jt_{j+1}} \diamond \delta_1 x_{s_is_{i+1};t_j}. \end{aligned} \quad (5.23)$$

We now use Theorem 4.10 in [11] in order to get that $\delta^{\diamond,2}(a^n)$ can be decomposed as:

$$\begin{aligned}
&\sum_{\pi_n} y^2_{s_i;t_j}(\delta_2 x_{s_i;t_jt_{j+1}} \diamond \delta_1 x_{s_is_{i+1};t_j}) - \sum_{\pi_n} y^3_{s_i;t_j} R^1_{s_i}(R^2_{t_jt_{j+1}} - R^2_{t_jt_j})\delta_1 x_{s_is_{i+1};t_j} \\
&- \sum_{\pi_n} y^3_{s_i;t_j} R^2_{t_j}(R^1_{s_is_{i+1}} - R^1_{s_i,s_i})\delta_2 x_{s_i;t_jt_{j+1}} \\
&+ \sum_{\pi_n} y^4_{s_i;t_j} R^1_{s_i} R^2_{t_j}(R^1_{s_is_{i+1}} - R^1_{s_is_i})(R^2_{t_jt_{j+1}} - R^2_{t_jt_j}) \equiv B^n_1 - B^n_2 - B^n_3 + B^n_4.
\end{aligned} \tag{5.24}$$

Like in the proof of Proposition 5.8, we treat those 4 terms separately.

Step 2: Estimation of $B^n_1, \dots, B^n_4$. The term B^n_1 can be decomposed as

$$B^n_1 = \sum_{\pi_n} y^2_{s_i;t_j}\delta_1 x_{s_is_{i+1};t_j}\delta_2 x_{s_i;t_jt_{j+1}} - \sum_{\pi_n} y^2_{s_i;t_j}(R^1_{s_is_{i+1}} - R^1_{s_is_i})(R^2_{t_jt_{j+1}} - R^2_{t_jt_j}).$$

Moreover, the second term in the r.h.s is the same as A^n_2 in the proof of Proposition 5.8, while the convergence for $\sum_{\pi_n} y^2_{s_i;t_j}\delta_1 x_{s_is_{i+1};t_j}\delta_2 x_{s_i;t_jt_{j+1}}$ follows exactly along the same lines as A^n_1 in the same proof. We thus leave to the patient reader the task of showing that

$$\lim_{n\to\infty} B^n_1 = \int_{s_1}^{s_2}\int_{t_1}^{t_2} y^2_{s;t}\, d_1 x_{s;t} d_2 x_{s;t} - \frac{1}{4}\int_{s_1}^{s_2}\int_{t_1}^{t_2} y^2_{s;t}\, d_1 R^1_s d_2 R^2_t, \tag{5.25}$$

and we concentrate now on the other terms in (5.24).

The term $B^n_2 = \sum_{\pi_n} y^3_{s_i;t_j} R^1_{s_i}(R^2_{t_jt_{j+1}} - R^2_{t_jt_j})\delta_1 x_{s_is_{i+1};t_j}$ can be decomposed as $B^n_2 = B^n_{21} + B^n_{22}$, with

$$B^n_{21} = \frac{1}{2}\sum_{\pi_n} y^3_{s_i;t_j} R^1_{s_i}(R^2_{t_{j+1}} - R^2_{t_j})\delta_1 x_{s_is_{i+1};t_j}, \quad B^n_{22} = \sum_{\pi_n} y^3_{s_i;t_j} R^1_{s_i}\rho^2_{t_jt_{j+1}}\delta_1 x_{s_is_{i+1};t_j},$$

where we recall that we have set $\rho^k_{t_jt_{j+1}} = \frac{1}{2}(2R^k_{t_jt_{j+1}} - R^k_{t_jt_j} - R^k_{t_{j+1}t_{j+1}})$ for $k = 1, 2$.

It is now easily seen that the almost sure and L^2 convergence of B^n_2 are obtained with the same kind of considerations as for A^n_2 in the proof of Proposition 5.8. We get that

$$\lim_{n\to\infty} B^n_2 = \frac{1}{2}\int_0^s\int_0^t y^3_{u;v} R^1_u\, d_2 R^2_v\, d_1 x_{u;v},$$

and B^n_3, B^n_4 are also handled in the same way.

Step 3: Conclusion. Thanks to Step 1 and Step 2, we have obtained that $\delta^{\diamond}(a^n)$ converges to the right hand side of relation (5.22) as $n \to \infty$, in both almost sure and L^2 senses. As mentioned before, this limiting behavior plus the convergence of a^n to $N(y)$ established at Lemma 5.10 yield relation (5.22) by a direct application of Proposition 5.1. Furthermore, relation (1.13) is also a direct consequence of relation (5.23).

□

Notice that our formula (5.22) involves some mixed integrals of the form:

$$\int_0^s\int_0^t y^3_{u;v} R^2_v\, d_1 R^1_u d_2 x_{u;v},$$

which are defined as Young type integrals. The following proposition, whose proof is similar to Propositions 5.8 and 5.11 and is left to the reader for sake of conciseness, gives a meaning to the analogue integrals in the Skorohod setting.

Proposition 5.12. *Let $f \in C^4(\mathbb{R})$ be a function satisfying condition (GC). Then for every fixed $u \in [0,s]$ we have that $v \mapsto y^3_{u;v} R^2_v \in Dom(\delta^{\diamond,u})$ where $\delta^{\diamond,u}$ is the divergence operator associated to the process $(x_{u;v})_{v\in[0,t]}$. We can thus define $\int_0^s \int_0^t y^3_{u;v} R^2_v d_1 R^1_u d^{\diamond}_2 x_{u;v}$ by :*

$$\int_0^s \int_0^t y^3_{u;v} R^2_v \, d_1 R^1_u \, d^{\diamond}_2 x_{u;v} := \lim_{|\pi|\to 0} \sum_{\pi_n} \delta^{\diamond,s_i}(y^3_{s_i;t_j} \mathbf{1}_{[t_j,t_{j+1}]}) R^2_{t_j} (R^1_{s_{i+1}} - R^1_{s_i})$$

where the convergence holds in both $L^2(\Omega)$ and almost sure senses. In addition, we have the following identity:

$$\int_0^s \int_0^t y^3_{u;v} R^2_v \, d_1 R^1_u \, d^{\diamond}_2 x_{u;v} = \int_0^s \int_0^t y^3_{u;v} R^2_v \, d_1 R^1_u \, d_2 x_{u;v} - \frac{1}{2} \int_0^s \int_0^t y^4_{u;v} R^1_u R^2_v \, d_1 R^1_u \, d_2 R^2_v.$$

Finally, the integral $\int_0^s \int_0^s y^3_{u;v} R^1_u d_2 R^2_v d^{\diamond}_1 x_{u;v}$ is defined similarly.

Remark 5.13. We have defined all the integrals we needed in order to prove our Skorohod change of variable formula (1.14). Indeed, the proof of formula (1.14) is now easily deduced by injecting the identities of Propositions 5.8, 5.11 and 5.12 in the Stratonovich type formula (1.3).

6 Skorohod's calculus in the rough case

Our goal in this section is to extend the formulae given in Propositions 5.8 and 5.11 to rougher situations, namely for Gaussian processes in the plane with Hölder regularities smaller than $1/2$. This is however a harder task than in the Young case, and this is why we introduce 2 simplifications in our considerations:

(1) Instead of dealing with a general centered Gaussian process whose covariance admits the factorization property of Hypothesis 5.3, we handle here the case of a fractional Brownian sheet $(x_{s;t})_{(s,t)\in[0,1]^2}$ with Hurst parameters $\gamma_1, \gamma_2 \in (1/3, 1/2]$.

(2) The definition of our Skorohod integrals with respect to x is obtained in the following way: we first regularize x as a smooth process x^n. For this process we can still use the formulae of Propositions 5.8 and 5.11 like in the Young case. We shall then perform a limiting procedure on these formulae (this is where the specification of a concrete approximation is important), which will give our Stratonovich-Skorohod corrections. Notice however that the interpretation in terms of Riemann-Wick sums will be lost with this strategy.

As in the previous section, we start our considerations by specifying the Malliavin framework in which we are working.

6.1 Further Malliavin calculus tools

Recall that the covariance function of our fractional Brownian sheet x is given by (1.8). We can thus consider a Hilbert space $\mathcal{H}^x$ related to x exactly as in Section 5.1, where we now stress the dependence in x of $\mathcal{H}^x$ in order to differentiate it from the Hilbert space related to white noise. In particular we denote by I^x_1 the isometry between $\mathcal{H}^x$ and the first chaos generated by x.

However, the Malliavin structure related to the harmonizable representation of x will also play a prominent role in the sequel. Namely, it is well known (see e.g. [16]) that for $s,t \in [0,1]$, x can be represented as

$$x_{s;t} = c_{\gamma_1,\gamma_2} \hat{W}(Q_{s;t}) = c_{\gamma_1,\gamma_2} \int_{\mathbb{R}^2} Q_{s;t}(\xi,\eta) \, \hat{W}(d\xi, d\eta), \tag{6.1}$$

where c_{γ_1,γ_2} is a normalization constant whose exact value is irrelevant for our computations, W is the Fourier transform of the white noise on $\mathbb{R}^2$, and $Q_{s;t}$ is a kernel defined by

$$Q_{s;t}(\xi,\eta) = \frac{e^{\imath s\xi}-1}{|\xi|^{\gamma_1+\frac{1}{2}}}\,\frac{e^{\imath t\eta}-1}{|\eta|^{\gamma_2+\frac{1}{2}}}. \tag{6.2}$$

This induces us to consider the canonical Hilbert space related to $\hat{W}$, that is $\mathcal{H}^{\hat{W}} = L^2(\mathbb{R}^2)$. The relations between Malliavin calculus with respect to $\hat{W}$ and x are then summarized in the next lemma:

Lemma 6.1. *Denote by $\mathbb{D}^{x,k,p}$ (resp. $\mathbb{D}^{\hat{W},k,p}$) the Sobolev spaces related to x (resp. $\hat{W}$), and recall the notation $\mathbb{L}^{1,2} = \mathbb{D}^{\hat{W},1,2}(L^2(\mathbb{R}))$ borrowed from [14]. For $\phi : [0,1]^2 \to \mathbb{R}$, set*

$$K\phi(\xi,\eta) = \int_{[0,1]^2} \phi_{s;t}\,\partial_s\partial_t Q_{s;t}(\xi,\eta)\,dsdt, \tag{6.3}$$

where we recall that Q is defined by (6.2)*. Then the following holds true:*

(i) *We can represent the space $\mathcal{H}^x$ as the closure of the set of step functions under the norm $\|\phi\|_{\mathcal{H}^x} = \|K\phi\|_{L^2(\mathbb{R}^2)}$.*

(ii) *We have $\mathbb{D}^{x,1,2}(\mathcal{H}^x) = K^{-1}(\mathbb{L}^{1,2})$, where we recall that $\mathbb{L}^{1,2} = \mathbb{D}^{\hat{W},1,2}(L^2(\mathbb{R}))$. In addition, for any smooth function F and any $\mathcal{H}^x$-valued square integrable random variable u the following identity holds:*

$$\langle u, D^x F\rangle_{\mathcal{H}^x} = \langle Ku, D^{\hat{W}} F\rangle_{L^2(\mathbb{R}^2)}.$$

(iii) *As far as divergence operators are concerned, the relation is*

$$\text{Dom}(\delta^{x,\diamond}) = K^{-1}\text{Dom}(\delta^{\hat{W},\diamond}), \quad \text{and} \quad \delta^{x,\diamond}(u) = \delta^{\hat{W},\diamond}(Ku).$$

Proof. Let $\phi = \sum_{i,j}\phi_{i,j}\mathbf{1}_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}$ be a step function. We have that:

$$\begin{aligned} I_1^x(\phi) &= \sum_{i,j}\phi_{i,j}\delta x_{s_is_{i+1};t_jt_{j+1}} = \sum_{i,j}\phi_{i,j}\hat{W}(\delta Q_{s_is_{i+1}t_jt_{j+1}}) \\ &= \hat{W}\left(\sum_{i,j}\phi_{i,j}\delta Q_{s_is_{i+1};t_jt_{j+1}}\right) = \hat{W}(K\phi), \end{aligned} \tag{6.4}$$

which easily yields our first claim (i).

Let now F be a smooth functional of x of the form $F = f(x_{s_1;t_1},\dots,x_{s_n;t_n})$. Then

$$\begin{aligned} \mathbb{E}\left[\langle u, D^x F\rangle_{\mathcal{H}^x}\right] &= \mathbb{E}\Big[\sum_{l\in\{1,\dots,n\}}\partial_l f(x_{s_1t_1},\dots,x_{s_nt_n})\langle u, \mathbf{1}_{[0,s_l]\times[0,t_l]}\rangle_{\mathcal{H}^x}\Big] \\ &= \mathbb{E}\Big[\sum_{l\in\{1,\dots,n\}}\partial_l f(x_{s_1t_1},\dots,x_{s_nt_n})\langle Ku, K\mathbf{1}_{[0,s_l]\times[0,t_l]}\rangle_{L^2(\mathbb{R})}\Big], \end{aligned} \tag{6.5}$$

and since $K\mathbf{1}_{[0,s_l]\times[0,t_l]} = Q_{s_l,t_l}$ we end up with

$$\begin{aligned} \mathbb{E}\left[\langle u, D^x F\rangle_{\mathcal{H}^x}\right] &= \mathbb{E}\Big[\langle Ku, \sum_{l\in\{1,\dots,n\}}\partial_l f(\hat{W}(Q_{s_1t_1}),\dots,\hat{W}(Q_{s_nt_n}))Q_{s_l,t_l}\rangle_{L^2(\mathbb{R})}\Big] \\ &= \mathbb{E}\left[\langle Ku, D^{\hat{W}}F\rangle_{L^2(\mathbb{R})}\right], \end{aligned}$$

which gives our assertion (ii) by density of smooth functionals. Relation (iii) is easily derived from (ii) by duality.

□

Notice that the preceding result can be extended to second order derivatives thanks to a simple tensorization trick. We label here the result for further use:

Lemma 6.2. *Under the conditions of Lemma 6.1, set*

$$\left[K^{\otimes 2}\phi\right](\xi_1\xi_2;\eta_1\eta_2) = \int_{[0,1]^4} \phi_{s_1s_2;t_1t_2}\, \partial_{st}Q_{s_1;t_1}(\xi_1,\eta_1)\partial_{st}Q_{s_2;t_2}(\xi_2,\eta_2)\, ds_1ds_2dt_1dt_2. \tag{6.6}$$

Then for any smooth functional F and any $(\mathcal{H}^x)^{\otimes 2}$-valued square integrable random variable u we have:

$$\langle u, D^{2,x}F\rangle_{\mathcal{H}^x} = \langle K^{\otimes 2}u, D^{2,\hat{W}}F\rangle_{L^2(\mathbb{R}^4)}.$$

6.2 Embedding results

Similarly to [2], we now give an embedding result for the space $\mathcal{H}^x$ which proves to be useful for further computations.

Lemma 6.3. *Let $\gamma_1,\gamma_2 \in (0,\frac{1}{2}]$. Then the following inequality is satisfied:*

$$\begin{aligned}\|u\|_{\mathcal{H}}^2 \lesssim & \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|\delta\tilde{u}_{s_1s_2;t_1t_2}|^2}{|s_2-s_1|^{2-2\gamma_1}|t_2-t_1|^{2-2\gamma_2}}ds_1dt_1\right)ds_2dt_2 \\ &+\int_{\mathbb{R}^2}\frac{1}{|s_2-s_1|^{2-2\gamma_1}}\left(\int_0^1\left|\tilde{u}_{s_2;t}-\tilde{u}_{s_1;t}\right|^2dt\right)ds_1ds_2 \\ &+\int_{\mathbb{R}^2}\frac{1}{|t_2-t_1|^{2-2\gamma_2}}\left(\int_0^1\left|\tilde{u}_{s;t_2}-\tilde{u}_{s;t_1}\right|^2ds\right)dt_1dt_2+\int_{[0,1]^2}|\tilde{u}_{s;t}|^2dsdt,\end{aligned} \tag{6.7}$$

where we have set $\tilde{u}_{s;t} = u_{s;t}\mathbf{1}_{[0,1]^2}(s,t)$.

Proof. In this proof we only consider the case $\gamma_1,\gamma_2 < 1/2$. Indeed, if $\gamma_1 = 1/2$ or $\gamma_2 = 1/2$ then our process x is simply a Brownian motion in the first or in the second direction, and this situation is handled by L^2 norms.

For $\gamma_1,\gamma_2 < 1/2$, definitions (6.2) and (6.3) entail:

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^2}|\xi|^{1-2\gamma_1}|\eta|^{1-2\gamma_2}\left|\int_{[0,1]^2}u_{s;t}e^{\imath s\xi+\imath t\eta}dsdt\right|^2d\xi d\eta,$$

from which one deduces that $\mathcal{H}$ is isometric to $H^{1/2-\gamma_1}\otimes H^{1/2-\gamma_1}$, where $H^{1/2-\gamma_1}$ stands for the Sobolev space $W^{1/2-\gamma_1,2}$. Now we use the fact that $1/2-\gamma_1 \in (0,1/2)$, and recall that the norm defined by

$$\mathcal{N}_{1/2-\gamma_1}^2(\phi) = \int_{\mathbb{R}^2}\frac{|\phi_{s_1}-\phi_{s_2}|^2}{|s_2-s_1|^{2-2\gamma_1}}ds_1ds_2 + \int_{\mathbb{R}}|\phi_s|^2ds$$

is equivalent to the usual norm in $H^{1/2-\gamma_1}$. This yields (6.7) by tensorization.

□

The following embedding result is easily deduced from Lemma 6.3.

Corollary 6.4. *Let $\gamma_1,\gamma_2 \in (0,1/2)$ and $u \in \mathcal{P}_{1,1}^{\alpha_1,\alpha_2}$ such that $0 < \frac{1}{2}-\alpha_i < \gamma_i$. Then we have the following embedding:*

$$\|u\|_{\mathcal{H}} \lesssim \mathcal{N}_{\alpha_1,\alpha_2}(u), \tag{6.8}$$

where we recall that $\mathcal{N}_{\alpha,\beta}$ is defined by (3.2).

6.3 Strategy and preliminary results

The strategy we shall develop in order to extend Proposition 5.8 (and also Proposition 5.11) to the rough case is based on a regularization of x. Specifically, for a strictly positive integer n, set

$$x^n_{s;t} = c_{\gamma_1,\gamma_2} \int_{|\xi|,|\eta|\leq n} Q_{s;t}(\xi,\eta)\, \hat{W}(d\xi, d\eta), \tag{6.9}$$

where we recall that x and Q are respectively defined by (6.1) and (6.2). For fixed n, it is readily checked that x^n is a regular Gaussian process. Its covariance function is given by $R^n_{s_1s_2;t_1t_2} = R^{1,n}_{s_1s_2} R^{2,n}_{t_1t_2}$, where

$$R^{i,n}_{ab} = c_{\gamma_i} \int_{|\xi|\leq n} \frac{(e^{\imath a\xi}-1)(e^{-ib\xi}-1)}{|\xi|^{2\gamma_i+1}} d\xi, \quad \text{for} \quad i=1,2, \tag{6.10}$$

and hence R^n is a regular function which satisfies Hypothesis 5.3. One can thus apply Proposition 5.8 and obtains the following Skorohod-Stratonovich comparison:

$$z^{1,n,\diamond}_{s_1s_2;t_1t_2} \equiv \delta^{x^n,\diamond}(y^{n,1}_{\cdot}\mathbf{1}_\Delta) = \int_\Delta y^{n,1}_{s;t}\, d_{12}x^n_{s;t} - \frac{1}{4}\int_\Delta y^{n,2}_{s;t}\, d_1R^{1,n}_s d_2 R^{2,n}_t, \tag{6.11}$$

for $\varphi \in C^6(\mathbb{R})$ satisfying condition (GC), and where we set again $\Delta = [s_1, s_2] \times [t_1 t_2]$. Our goal is now to take limits in equation (6.11).

A first observation in this direction is that equation (6.11) involves Skorohod integrals with respect to x^n. The fact that a different integral has to be defined for each n is somehow clumsy, and this is why we have decided to express all integrals with respect to $\hat{W}$ in the remainder of our computations. Namely, the same computations as for equations (6.4) and (6.5) entail that $\delta^{x^n,\diamond}(y^n) = \delta^{\hat{W},\diamond}(K^n y^n)$, where K^n is the operator defined by

$$K^n\phi(\xi,\eta) = \mathbf{1}_{(|\xi|,|\eta|\leq n)} \int_{[0,1]^2} \phi_{s;t}\, \partial_s\partial_t Q_{s;t}(\xi,\eta) ds dt. \tag{6.12}$$

With this representation in hand, our limiting procedure can be decomposed as follows:

- Take L^2 limits in the right hand side of equation (6.11) by means of rough paths techniques.
- Show that $K^n y^n$ converges in $L^2(\Omega, L^2(\mathbb{R}))$ to Ky.

Thanks to the closability of $\delta^{\hat{W},\diamond}$, this will show the convergence of $\delta^{x^n,\diamond}(y^n\mathbf{1}_{[0,1]^2})$ to $\delta^{x,\diamond}(y\mathbf{1}_{[0,1]^2})$ and our Skorohod-Stratonovich correction formula will be obtained in this way.

We now state and prove 3 useful lemmas for our future computations. The first one deals with convergence of covariance functions:

Lemma 6.5. *For $i = 1, 2$, set $R^i_u = u^{2\gamma_i}$. Then for all $\varepsilon > 0$ we have*

$$\lim_{n\to+\infty} \|R^{i,n} - R^i\|_{2\gamma_i-\epsilon} = 0,$$

where $R^{i,n}$ is defined by (6.10).

Proof. We recall that $c_{\gamma_i}\int_{\mathbb{R}} \frac{|e^{\imath a\xi}-1|^2}{|\xi|^{2\gamma_i+1}} d\xi = a^{2\gamma_i}$. Then an elementary computation shows that

$$|\delta_i(R^{i,n} - R^i)_{ab}| = \left| c_{\gamma_i} \int_{|\xi|\geq n} \frac{cos(a\xi) - cos(b\xi)}{|\xi|^{2\gamma_i+1}} d\xi \right| \lesssim_{\gamma_i,\epsilon} |a-b|^{2\gamma_i-\epsilon} \int_{|x|\geq n} |\xi|^{-1-\epsilon} d\xi,$$

which gives $\|R^{n,i} - R^u\|_{2\gamma_i-\epsilon} \lesssim_{\gamma_i,\epsilon} \int_{|\xi|\geq n} |\xi|^{-1-\epsilon} d\xi$, and this finishes the proof.

□

Our second preliminary result ensures that x^n is an accurate approximation of x:

Proposition 6.6. *Let $p > 1$ and $0 < \varepsilon < \min(\gamma_1, \gamma_2)$. Then we have the following convergence:*

$$\lim_{n\to\infty} \mathbb{E}\left[\|x^n - x\|^p_{\gamma_1-\epsilon,\gamma_2-\epsilon}\right] = 0.$$

In addition, there exists $\lambda > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[e^{\lambda \sup_{(s,t)\in[0,1]^2} |x^n_{s;t}|^2}\right] < +\infty. \tag{6.13}$$

Proof. The definitions (6.1) of x, (6.9) of x^n plus formula (6.2) for Q allow to write, for all $n \geq 1$:

$$\begin{aligned}\mathbb{E}[|\delta(x^n - x)_{s_1s_2;t_1t_2}|^2] &= \int_{|x|,|y|\geq n} \frac{|e^{\imath s_2 x} - e^{\imath s_1 x}|^2 |e^{\imath t_2 y} - e^{\imath t_1 y}|^2}{|x|^{2\gamma_1+1}|y|^{\gamma_2+1}} dxdy \\ &\lesssim_{\gamma_1,\gamma_2} |s_1 - s_1|^{2(\gamma_1-\epsilon/2)}|t_2 - t_1|^{2(\gamma_2-\epsilon/2)} I_n\end{aligned} \tag{6.14}$$

where we have set $I_n = \int_{|x|,|y|\geq n} \frac{dx\,dy}{|x|^{1+\epsilon}|y|^{1+\epsilon}}$ (this quantity is obviously finite). Hence by Gaussian hypercontractivity and Lemma 5.5 we obtain

$$\begin{aligned}\mathbb{E}[[\|x^n - x\|^p_{\gamma_1-\epsilon,\gamma_2-\epsilon}] &\lesssim_{p,\gamma_1,\gamma_2,\epsilon} \int_{[0,1]^4} \frac{\mathbb{E}[|\delta x_{s_1s_2;t_1t_2}|^2]^{p/2}}{|s_2 - s_1|^{(\gamma_1-\epsilon)p+2}|t_2 - t_1|^{(\gamma_2-\epsilon)p+2}} dt_2 ds_2 dt_1 ds_1 \\ &\lesssim (I_n)^{p/2} \int_{[0,1]^4} |s_2 - s_1|^{p\epsilon/2-2}|t_2 - t_1|^{p\epsilon/2-1} dt_2 ds_2 dt_1 ds_1 \lesssim (I_n)^{p/2}.\end{aligned}$$

Since $\lim_{n\to\infty} I_n = 0$, we thus get $\lim_{n\to\infty} \mathbb{E}[\|x^n - x\|^p_{\gamma_1-\epsilon,\gamma_2-\epsilon}] = 0$, which is our first claim.

We now focus on the exponential integrability of $\sup x^n$. Notice that for a fixed n one can easily get those exponential estimates thanks to Fernique's lemma. However, we claim some uniformity in n here, and we thus come back to uniform estimates of moments in order to prove (6.13). Let then $r = \max(\lfloor \frac{1}{\gamma_1-\epsilon} \rfloor, \lfloor \frac{1}{\gamma_2-\epsilon} \rfloor) + 1$ and remark that $\|x^n\|_\infty \leq \|x^n\|_{\epsilon,\epsilon}$. We thus use a decomposition of the form $\mathbb{E}[e^{\lambda \sup_{(s,t)\in[0,1]^2} |x^n_{s;t}|^2}] = I^{1,n}(\lambda) + I^{2,n}(\lambda)$, where

$$I^{1,n}(\lambda) = \sum_{l=0}^{r-1} \frac{\lambda^l}{l!} \mathbb{E}[\|x^n\|^{2l}_\infty], \quad \text{and} \quad I^{2,n}(\lambda) = \sum_{l=r}^{+\infty} \frac{\lambda^l}{l!} \mathbb{E}[\|x^n\|^{2l}_\infty].$$

We now bound those 2 terms separately: one the one hand, it is readily checked that

$$I^{1,n}(\lambda) \lesssim \max_{i=0,\dots,r} \sup_{n\in\mathbb{N}} \mathbb{E}[\|x^n\|^{2i}_{\epsilon,\epsilon}] < +\infty,$$

for $\epsilon < \min(\gamma_1, \gamma_2)$. On the other hand, the bound on $I^{2,n}(\lambda)$ is obtained invoking Lemma 5.5 again. Indeed, starting from expression (5.12) and introducing a standard Gaussian random variable $\mathcal{N}$, it is easily seen that

$$\mathbb{E}[\|x^n\|^{2l}_{\epsilon,\epsilon}] \leq C_{2l,\epsilon,\epsilon} \mathbb{E}[\mathcal{N}^{2l}] \int_{[0,1]^2\times[0,1]^2} \frac{\mathbb{E}[|\delta x^n_{s_1s_2t_1t_1}|^2]^l}{|s_2 - s_1|^{2l\epsilon+2}|t_2 - t_1|^{2l\epsilon+2}} ds_1 ds_2 dt_1 dt_2$$

with $\mathcal{N}$ is a Gaussian random variable $\mathcal{N}(0,1)$. Now we have

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mathbb{E}\Big[|\delta x^n_{s_1s_2;t_1t_2}|^2\Big] &\leq \int_{\mathbb{R}^2} \frac{|e^{\imath s_2 x} - e^{\imath s_1 x}|^2 |e^{\imath t_2 y} - e^{\imath t_1 y}|^2}{|x|^{2\gamma_1+1}|y|^{2\gamma_2+1}} dxdy \\ &\leq |s_2 - s_1|^{2\gamma_1}|t_2 - t_1|^{2\gamma_2} \int_{\mathbb{R}^2} \frac{|e^{\imath x} - 1|^2 |e^{\imath y} - 1|^2}{|x|^{2\gamma_1+1}|y|^{2\gamma_2+1}} dxdy \lesssim |s_2 - s_1|^{2\gamma_1}|t_2 - t_1|^{2\gamma_2}.\end{aligned}$$

Furthermore, according to Lemma 5.5, the constant $C_{2l,\epsilon,\epsilon}$ can be taken of the form M^l for a given $M > 1$. Thus:

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\|x^n\|_{\epsilon,\epsilon}^{2l}] \lesssim M^l \, \mathbb{E}\left[\mathcal{N}^{2l}\right],$$

from which the relation $I^{2,n}(\lambda) < \infty$ is easily obtained. This finishes the proof of (6.13). □

With classical considerations concerning compositions of Hölder functions with non linearities, we finally get the following result which is labelled for further use. Its proof is omitted for sake of conciseness.

Lemma 6.7. *Let $\rho_1, \rho_2 \in (0,1)$, x^1, x^2 two increments lying in $\mathcal{P}_{1,1}^{\rho_1,\rho_2}$, and $f \in C^3(\mathbb{R})$ satisfying condition (GC). Then we have (The square below has been dropped as suggested by the referee):*

$$\mathcal{N}_{\rho_1,\rho_2}(f(x^1) - f(x^2)) \lesssim c_{x^1,x^2}\, \mathcal{N}_{\rho_1,\rho_2}(x^1 - x^2)\, \left[1 + \mathcal{N}_{\rho_1,\rho_2}(x^1) + \mathcal{N}_{\rho_1,\rho_2}(x^2)\right] \tag{6.15}$$

where we recall that $\mathcal{N}_{\alpha_1,\alpha_2}$ has been defined at equation (6.8)*. In the relation above we have also set $c_{x^1,x^2} = \exp(\theta(\sup_{(s,t)\in[0,1]^2} |x^1_{s;t}|^2 + \sup_{(s,t)\in[0,1]^2} |x^2_{s;t}|^2))$, where θ is the constant featuring in condition (GC).*

6.4 Itô-Skorohod type formula

We now turn to the limiting procedure in equation (6.11), beginning with the term involving covariances only:

Proposition 6.8. *Let $f \in C^6(\mathbb{R})$ be a function satisfying condition (GC) with a small enough parameter $\lambda > 0$ defined as in Lemma 6.7, so that the random variable c_{x^1,x^2} (also defined in Lemma 6.7) satisfies $\mathbf{E}[c^8_{x^1,x^2}] < \infty$. Consider x^n, the regularized version of x defined by* (6.9)*. Then the following convergence:*

$$\lim_{n\to+\infty} \int_{[0,1]^2} y^n_{s;t}\, d_1 R^{1,n}_s d_2 R^{2,n}_t = \gamma_1\gamma_2 \int_{[0,1]^2} y_{s;t} s^{2\gamma_1 - 1} t^{2\gamma_2 - 1} ds dt \tag{6.16}$$

holds in $L^2(\Omega)$.

Proof. The integrals involved in (6.16) are all of Young type. Owing to Proposition 4.6, we thus have:

$$\int_{[0,1]^2} y^n_{s;t} d_1 R^{1,n}_s d_2 R^{2,n}_t = [(\mathrm{Id} - \Lambda_1\delta_1)(\mathrm{Id} - \Lambda_2\delta_2)]\,(y^n \delta_1 R^{1,n} \delta_2 R^{2,n}).$$

By continuity of the sewing maps Λ and Λ_i, the desired convergence will thus stem from the relations $\lim_{n\to 0} A^{1,n} = 0$ and $\lim_{n\to 0} A^{2,n} = 0$, where for $\epsilon > 0$ we set:

$$A^{1,n} := \sum_{i=1}^{2} \|\delta_i R^{i,n} - \delta_i R^i\|_{2\gamma_i - \epsilon}, \quad \text{and} \quad A^{2,n} := \mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y^n - y).$$

Now the relation $\lim_{n\to 0} A^{1,n} = 0$ is obviously a direct consequence of Lemma 6.5. As far as $A^{2,n}$ is concerned, we start from relation (6.15) and apply Hölder's inequality. This yields

$$\begin{aligned}&\mathbb{E}[(\mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y^n - y))^2] \\ &\quad\lesssim \mathbb{E}^{1/4}[(c_{x^1,x^2})^8]\, \mathbb{E}^{1/2}[\|x^n - x\|^4_{\gamma_1-\epsilon,\gamma_2-\epsilon}]\, \mathbb{E}^{1/4}[(1 + N_{\gamma_1-\epsilon,\gamma_2-\epsilon}(x^n) + N_{\rho_1,\rho_2}(x))^8].\end{aligned}$$

Then according to Proposition 6.6 we see that the r.h.s of this last equation vanishes when n goes to infinity, which proves our claim. □

We now compute the correction terms in z^1, that is the equivalent of Proposition 5.8.

Proposition 6.9. *Let x be a fBs with Hurst parameters $\gamma_1, \gamma_2 > 1/3$. Consider a function $\varphi \in C^8(\mathbb{R})$ satisfying condition (GC) and a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$. Then we have that $y^1_\cdot \mathbf{1}_\Delta \in Dom(\delta^\diamond)$, and if we define the increment $z^{1,\diamond} \equiv \delta^\diamond(y^1_\cdot \mathbf{1}_\Delta)$ the following relation holds true:*

$$z^{1,\diamond}_{s_1 s_2; t_1 t_2} = z^1_{s_1 s_2; t_1 t_2} - \gamma_1\gamma_2 \int_\Delta y^2_{s;t} s^{2\gamma_1 - 1} t^{2\gamma_2 - 1} ds dt, \tag{6.17}$$

where z^1 is the rough integral given by Theorem 1.5.

Proof. Let us start from the corrections for the regularized process x^n, for which we can appeal to Proposition 5.8. We obtain relation (6.11), written here again for convenience:

$$z^{1,n,\diamond}_{s_1 s_2; t_1 t_2} \equiv \delta^{x^n,\diamond}(y^{n,1}_\cdot \mathbf{1}_\Delta) = \int_\Delta y^{n,1}_{s;t} d_{12} x^n_{s;t} - \frac{1}{4}\int_\Delta y^{n,2}_{s;t} d_1 R^{1,n}_s d_2 R^{2,n}_t. \tag{6.18}$$

Now putting together Proposition 6.8 and the continuity of the rough path integral, we get convergence of the r.h.s of (6.18) in probability. Thus one can write:

$$\lim_{n\to+\infty} \delta^{x^n,\diamond}(y^n) = \int_\Delta y^1_{s;t} d_{12} x_{s;t} - \gamma_1\gamma_2 \int_\Delta y^2_{s;t} s^{\gamma_1 - 1} t^{\gamma_2 - 1}\, dt ds,$$

where the integral with respect to x is interpreted in the sense of Theorem 1.5.

Let us further analyze the convergence of $\delta^{x^n,\diamond}(y^n)$: recall that this quantity can be written as $\delta^{\hat{W},\diamond}(K^n y^n)$, where K^n is defined by (6.12) or specifically as

$$(K^n y^n)(\xi,\eta) = \frac{\imath\xi\, \imath\eta}{|\xi|^{\gamma_1+1/2}|\eta|^{\gamma_2+1/2}} \left(\int_\Delta y^n_{uv} e^{\imath\xi u + \imath\eta v} du dv\right) \mathbf{1}_{(|\xi|,|\eta|\le n)} \tag{6.19}$$

Hence, owing to closability of the operator $\delta^{\hat{W},\diamond}$, the proof of (6.17) is reduced to show that $K^n y^{n,1}$ converges in $L^2(\Omega; L^2(\Delta))$ to Ky^1. Now expression (6.19) easily entails that

$$\begin{aligned}\left|\left|K^n y^{n,1} - Ky\right|\right|_{L^2(\mathbb{R})} &\le \|y^{n,1} - y^1\|_{\mathcal{H}} \\ &\quad + \int_{|\xi|,|\eta|\ge n} |\xi|^{1-2\gamma_1}|\eta|^{1-2\gamma_2} \left|\int_\Delta y^1_{u;v} e^{\imath\xi u + \imath\eta v} du dv\right|^2 d\xi d\eta,\end{aligned}$$

and we shall bound the 2 terms on the r.h.s of this inequality.

Indeed, on the one hand we consider $\gamma_1, \gamma_2 > 1/4$ and $\epsilon > 0$ small enough. This gives

$$\begin{aligned}&\mathbb{E}\left[\int_{\|(\xi,\eta)\|_\infty \ge n} |\xi|^{1-2\gamma_1}|\eta|^{1-2\gamma_2} \left|\int_\Delta y^1_{u;v} e^{\imath\xi u + \imath\eta v} du dv\right|^2 d\xi d\eta\right] \\ &\qquad\lesssim n^{-\epsilon}\, \mathbb{E}\left[(\mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y))^2\right] \overset{n\to+\infty}{\longrightarrow} 0.\end{aligned}$$

On the other hand, Corollary 6.4 asserts that $\|y - y^n\|_{\mathcal{H}} \lesssim \mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y^n - y)$, and the r.h.s of this relation vanishes as $n \to \infty$ thanks to Proposition 6.6. This concludes our proof.

□

In order to complete our comparison between Itô and Stratonovich formulae, we still have to compare the Skorohod type increment $z^{2,\diamond}$ and the rough integral z^2. As a previous step, let us give an intermediate result concerning some mixed integrals in R, x:

Proposition 6.10. *Let* $\varphi \in C^6(\mathbb{R})$ *satisfying condition (GC) for a small enough parameter* λ *as in Proposition 6.8, and recall that for the fractional Brownian sheet* x *we have* $R^i_u = u^{2\gamma_i}$ *for* $i = 1, 2$. *Then the integral*

$$\int y^1 R^1\, d_1 x\, d_2 R^2 = [(\mathrm{Id} - \Lambda_1 \delta_1)(\mathrm{Id} - \Lambda_2 \delta_2)] \left(y^1 R^1 \delta_1 x \delta_2 R^2 + \frac{1}{2} y^2 R^1 (\delta_1 x)^2 \delta_2 R^2 \right) \quad (6.20)$$

is well defined a.s, in the sense of Proposition 3.4. Moreover the following convergence takes place in $L^2(\Omega)$*:*

$$\lim_{n\to+\infty} \int_\Delta y^{1,n}_{uv} R^{1,n}_u d_1 x^n_{uv} d_2 R^{2,n}_v = \int_\Delta y^1_{u;v} R^1_u d_1 x_{u;v} d_2 R^2_v. \quad (6.21)$$

Finally, the same kind of result is still verified when one interchanges directions 1 and 2 in relation (6.20)*.*

Proof. Let us first check that the integral in (6.20) is well-defined in the sense of Proposition 4.6. To this aim, set $A = y^1 R^1 \delta_1 x\, \delta_2 R^2 + \frac{1}{2} y^2 R^1 (\delta_1 x)^2 \delta_2 R^2$. Then a simple application of Proposition 2.6 yields $\delta_1 A = z\, \delta_2 R^2$, with

$$z = \left(y^2 \delta_1 x - \delta_1 y^1 \right) R^1 \delta_1 x - \frac{1}{2} \delta_1 (y^2 R^1)(\delta_1 x)^2 - y^1 \delta_1 R^1 \delta_1 x.$$

It is thus easily seen that $\delta_1 A \in \mathcal{P}^{3\gamma_1-\epsilon, 2\gamma_2}_{3,2}$ for an arbitrary small ϵ, thanks to the fact that $x \in \mathcal{P}^{\gamma_1-\epsilon,\gamma_2-\epsilon}_{1,1}$ and $\delta_1 y^1 - y^2 \delta_1 x \in \mathcal{P}^{2\gamma_1-\epsilon,\gamma_2}_{2,1}$ almost surely. It is worth noting at this point that this regularity property is not valid without the term $1/2 y^2 R^1 (\delta_1 x)^2 \delta_2 R^2$ in equation (6.20). Also observe that with the same kind of considerations we also have that $\delta_2 A \in \mathcal{P}^{\gamma_1, 3\gamma_2-\epsilon}_{2,3}$.

Let us now compute δA: we have

$$\delta A = -\delta_2 z\, \delta_2 R^2 = -\left(A_1 + A_2 \right) \delta_2 R^2,$$

where

$$A_1 = g\, R^1 \delta_1 x, \quad \text{with} \quad g = \delta y^1 - \delta_2 y^2 \delta_1 x - y^2 \delta x,$$

and where setting $(\delta_1 x \circ_1 \delta x)_{s_1 s_2; t_1 t_2} = \delta_1 x_{s_1 s_2 t_1} \delta x_{s_1 s_2; t_1 t_2}$ similarly to Definition 3.7, we have

$$A_2 = \delta_2 y^1 \delta_1 R^1 \delta_1 x - \{\delta_1 y^1 - y^2 \delta_1 x\} R^1 \delta x - \frac{1}{2} \delta y^2 (\delta_1 x)^2 - \frac{1}{2} \delta_1 y^2 \{\delta x \circ_1 \delta_1 x + \delta_1 x \circ_1 \delta x\}.$$

The reader can now easily check that $A_2 \in \mathcal{P}^{3\gamma_1-\epsilon, \gamma_2-\epsilon}_{2,2}$. In order to check the regularity of A_1, observe that g is of the form $g = \delta_2 h$, with

$$\begin{aligned} h_{s_1 s_2; t} &:= (\delta_1 y^1_{s_1 s_2; t} - y^2_{s_1; t}) \delta_1 x_{s_2 s_2; t} \\ &= \left(\int_0^1 d\theta\, \theta \int_0^1 d\theta' y^2 (x_{s_1; t} + \theta\theta' \delta_1 x_{s_1 s_2; t}) \right) (\delta_1 x_{s_1 s_2; t})^2. \end{aligned} \quad (6.22)$$

Computing $\delta_2 h$ with formula (6.22), one obtains that $A^1 = \delta_2 h R^1 \delta_1 x \in \mathcal{P}^{3\gamma_1-\epsilon,\gamma_2-\epsilon}_{2,2}$.

Let us summarize our last considerations: we have seen that both A_1 and A_2 lye into $\mathcal{P}^{3\gamma_1-\epsilon,\gamma_2-\epsilon}_{2,2}$, and recalling that $\delta A = -(A_1 + A_2)\delta_2 R^2$, we obtain $\delta A \in \mathcal{P}^{3\gamma_1-\epsilon, 3\gamma_2-\epsilon}_{3,3}$. We have also checked that $\delta_1 A \in \mathcal{P}^{3\gamma_1-\epsilon, 2\gamma_2}_{3,2}$ and $\delta_2 A \in \mathcal{P}^{2\gamma_1, 3\gamma_2-\epsilon}_{2,3}$. Gathering all this information, we have checked the assumptions of Proposition 3.4 for the increment A, which justifies expression (6.20).

Now we focus on the convergence formula (6.21). We start by observing that for all $n \geq 1$ the following representation holds true:

$$\int_\Delta y^{n,1}_{uv} R^{1,n}_u d_2 x^n_{uv} d_2 R^{2,n}_v = [(\mathrm{Id} - \Lambda_1 \delta_1)(\mathrm{Id} - \Lambda_2 \delta_2)] A^n$$

with $A^n = y^{n,1} R^{1,n} \delta_2 R^{2,n} + 1/2 y^{n,1} R^{1,n} (\delta_1 x^n)^2 \delta_2 R^{2,n}$. Hence, owing to the continuity of the planar sewing-maps $(\Lambda_i)_{i=1,2}$ and Λ, our claim (6.21) is reduced to prove that the sequences $\|A^n - A\|_{\gamma_1-\epsilon, 2\gamma_2-\epsilon}, \|\delta_1(A^n - A)\|_{3\gamma_1-\epsilon, 2\gamma_2-\epsilon}, \|\delta_2(A^n - A)\|_{\gamma_1-\epsilon, 3\gamma_2-\epsilon}$ and $\|\delta(A^n - A)\|_{3\gamma_1-\epsilon, 3\gamma_2-\epsilon}$ converge in $L^2(\Omega)$ and almost surely to 0. Furthermore, it is readily checked that those convergences all stem from the relations

$$\lim_{n\to 0} \|R^{i,n} - R\|_{2\gamma_i-\epsilon} + \sum_{i=0}^{5} \mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y^{n,i} - y^i) + \|(\delta_1 x^n)^2 - (\delta_1 x)^2\|_{2\gamma_1-\epsilon,1} = 0 \quad (6.23)$$

and

$$\lim_{n\to 0} \|\delta_2 (h - h^n)\|_{2\gamma_1-\epsilon,\gamma_2-\epsilon} = 0, \quad \text{with} \quad h^n = \delta_1 y^{n,1} - y^{n,2} \delta_1 x^n, \quad (6.24)$$

where the limits take place in some $L^p(\Omega)$ with a sufficiently large p, and where we recall that h is defined by (6.22). We now turn to the proof of those two relations.

To begin with, note that the convergence (6.23) easily stems from Lemma 6.5 for the terms R, Proposition 6.6 for the terms x and Lemma 6.7 for the terms y. In order to prove (6.24), we invoke again the integral representation (6.22) for both h^n and h. Then some elementary considerations (omitted here for sake of conciseness) allow to reduce the problem to the following relation:

$$L^p(\Omega) - \lim_{n\to\infty} \left(\mathbb{E}[\mathcal{N}_{\gamma_1-\epsilon,\gamma_1-\epsilon}(x^n - x)^p] + \sum_{i=0}^{3} \mathbb{E}[\mathcal{N}^p_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y^{n,i} - y^i)^p] \right) = 0.$$

This last relation is a direct consequence of Proposition 6.6 and composition with non linearities, whenever φ satisfies the growth condition (GC) with a small parameter $\lambda > 0$ chosen as in Proposition 6.8. The proof is now finished.

□

We can now state our result concerning the Itô-Stratonovich correction for the mixed stochastic integral $\int y d_1 x d_2 x$:

Theorem 6.11. *Let x be a fBs with Hurst parameters $\gamma_1, \gamma_2 > 1/3$. Consider a function $\varphi \in C^8(\mathbb{R})$ satisfying condition (GC) and a rectangle $\Delta = [s_1, s_2] \times [t_1, t_2]$. Then we have that $N(y^2) \in Dom(\delta^{\diamond,2})$, and if we define the Skorohod integral $z^{2,\diamond}$ as $\delta^{\diamond,2}(N(y^2))$, the following particular case of relation* (5.22) *holds:*

$$z^{2,\diamond}_{s_1 s_2; t_1 t_2} = z^2_{s_1 s_2; t_1 t_2} - \gamma_1\gamma_2 \int_\Delta y^2_{u;v} u^{2\gamma_1 - 1} v^{2\gamma_2 - 1} du dv - \gamma_2 \int_\Delta y^3_{u;v} u^{2\gamma_1} v^{2\gamma_2 - 1} dv d_1 x_{u;v}$$
$$- \gamma_1 \int_\Delta y^3_{u;v} u^{2\gamma_1-1} v^{2\gamma_2} du d_2 x_{u;v} + \gamma_1\gamma_2 \int_\Delta y^4_{u;v} u^{4\gamma_1-1} v^{4\gamma_2-1} du dv. \quad (6.25)$$

Proof. We follow the same strategy as for Theorem 6.9: apply first Proposition 5.11 for the regularized process x^n, which yields:

$$\int_\Delta y^{n,2}_{u;v} d_1^\diamond x^n_{uv} d_2^\diamond x^n_{uv} = \int_\Delta y^{n,2}_{u;v} d_1 x^n_{u;v} d_2 x^n_{uv} - 1/4 \int_\Delta y^{n,2}_{u;v} d_1 R^{1,n}_u d_2 R^{2,n}_v$$
$$- \frac{1}{2} \int_\Delta y^{n,3}_{u;v} R^{1,n}_u d_2 R^{2,n}_v d_1 x_{u;v} - \frac{1}{2} \int_\Delta y^{n,3}_{u;v} R^{2,n}_v d_1 R^{1,n}_u d_2 x^n_{uv}$$
$$+ 1/4 \int_\Delta y^{n,4}_{u;v} R^{1,n}_u R^{2,n}_v d_1 R^{1,n}_u d_2 R^{n,2}_v. \quad (6.26)$$

Now our preliminary results allow to take limits in relation (6.26). Indeed, owing to Propositions 6.8 and 6.10 plus the continuity of the rough increment z^2 given at Theorem 1.5, we obtain the convergence in probability for the four first terms in the r.h.s of equation (6.26). Moreover the last term also converges in $L^2(\Omega)$, thanks to the same arguments as in the proof of the Proposition 6.8. We thus get the convergence of the r.h.s of equation (6.26) to the r.h.s of equation (6.25), and also the fact that $z^{2,\diamond}$ converges in $L^2(\Omega)$. Like in the proof of Theorem 6.9, the proof of (6.25) is thus reduced to show the L^2 convergence of the integrand defining $z^{2,\diamond}$.

However, mimicking again the proof of Theorem 6.9, it is easily seen that

$$\int_0^1\int_0^1 y^{n,2}_{u;v} d_1^\diamond x^n_{uv} d_2^\diamond x^n_{uv} := \delta^{x^n,\diamond,2}\left(N(y^{n,2})\right) = \delta^{\hat{W},\diamond,2}\left(K^{n,\otimes 2}N(y^{n,2})\right),$$

where we recall that $K^{\otimes 2}$ is defined by (6.6) and

$$\left[K^{n,\otimes 2}\phi\right](x_1x_2;y_1y_2) = \mathbf{1}_{\{|x_1|,|x_2|,|y_1|,|y_2|\le n\}}\ \left[K^{\otimes 2}\phi\right](x_1x_2;y_1y_2).$$

It thus remains to show that $K^{n,\otimes 2}(N(y^{n,2}))$ converges to $K^{\otimes 2}(N(y^2)$ in $L^2(\Omega, L^2(\mathbb{R}^4))$. Towards this aim, we introduce the further notation $u_{s;t}(\xi,\eta) = y^2_{s;t}(e^{\imath s\xi}-1)(e^{\imath t\eta}-1)$, $u^n_{s;t}(\xi,\eta) = y^{n,2}_{s;t}(e^{\imath s\xi}-1)(e^{\imath t\eta}-1)$ and

$$\begin{aligned}
\hat{u}(\xi_1,\xi_2;\eta_1,\eta_2) &= \int_\Delta u_{s;t}(\xi_1,\eta_1)e^{\imath s\xi_2+\imath t\eta_2}dsdt \\
\hat{u}^n(\xi_1,\xi_2;\eta_1,\eta_2) &= \int_\Delta u^n_{s;t}(\xi_1,\eta_1)e^{\imath s\xi_2+\imath t\eta_2}dsdt.
\end{aligned}$$

Then note that

$$\|(K^n)^{\otimes 2}(N(y^n)) - K^{\otimes 2}(N(y)\|^2_{L^2(\mathbb{R}^4)} \le I^n_1 + I^n_2 + I^n_3,$$

where

$$\begin{aligned}
I^n_1 &= \int_{|\xi_1|,|\eta_1|\ge n}\left(\int_{\mathbb{R}^2}|\xi_2|^{1-2\gamma_1}|\eta_2|^{1-2\gamma_2}\left|\hat{u}(\xi_1,\xi_2;\eta_1,\eta_2)\right|^2 d\xi_2 d\eta_2\right)\frac{d\xi_1 d\eta_1}{|\xi_1|^{2\gamma_1+1}|\eta_1|^{2\gamma_2+1}} \\
I^n_2 &= \int_{\mathbb{R}^2}\left(\int_{|\xi_2|,|\eta_2|\ge n}|\xi_2|^{1-2\gamma_1}|\eta_2|^{1-2\gamma_2}\left|\hat{u}(\xi_1,\xi_2;\eta_1,\eta_2)\right|^2 d\xi_2 d\eta_2\right)\frac{d\xi_1 d\eta_1}{|\xi_1|^{2\gamma_1+1}|\eta_1|^{2\gamma_2+1}}
\end{aligned}$$

and

$$\begin{aligned}
I^n_3 = \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|\xi_2|^{1-2\gamma_1}|\eta_2|^{1-2\gamma_2}\left|\hat{u}(\xi_1,\xi_2;\eta_1,\eta_2)-\hat{u}^n(\xi_1,\xi_2;\eta_1,\eta_2)\right|^2 d\xi_2 d\eta_2\right)& \\
\times\frac{d\xi_1 d\eta_1}{|\xi_1|^{2\gamma_1+1}|\eta_1|^{2\gamma_2+1}}&.
\end{aligned}$$

In order to bound those 3 terms, observe that

$$\mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(u(\xi,\eta)) \lesssim \mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y)(1+|\xi|^{\gamma_1-\epsilon}+|\eta|^{\gamma_2-\epsilon}+|\xi|^{\gamma_1-\epsilon}|\eta|^{\gamma_2-\epsilon})$$

and thus Corollary 6.4 entails that

$$\mathbb{E}[I^n_1] \lesssim \mathbb{E}\left[\left(\mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y^2)\right)^2\right]\int_{|\xi|,|\eta|\ge n}\frac{1}{|\xi|^{\epsilon+1}|\eta|^{\epsilon+1}}d\xi d\eta \overset{n\to+\infty}{\longrightarrow} 0,$$

and

$$\mathbb{E}[I^n_2] \lesssim n^{-\epsilon}\,\mathbb{E}\left[\left(\mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y^2)\right)^2\right] \overset{n\to+\infty}{\longrightarrow} 0.$$

As far as I_3^n is concerned, we remark that

$$\mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(u(\xi,\eta)-u^n(\xi,\eta)) \lesssim \mathcal{N}_{\gamma_1-\epsilon,\gamma_2-\epsilon}(y^2-y^{n,2})(1+|\xi|^{\gamma_1-\epsilon}+|\eta|^{\gamma_2-\epsilon}+|\xi|^{\gamma_1-\epsilon}|\eta|^{\gamma_2-\epsilon})$$

and then we can conclude along the same lines as in Theorem 6.11 that $\mathbb{E}[I_3^n]$ vanishes as n goes to infinity. This finishes the proof.

□

The last step in order to go from Theorem 6.11 to Theorem 1.9 is to convert Stratonovich into Skorohod type integrals in the right hand side of relation (6.25). To this aim, let us first recall the following one-parameter result:

Proposition 6.12. *Let B a fractional brownian motion with hurst parameters $1/2 \geq \gamma > 1/3$ then we have that $u \mapsto \varphi(B_u)u^{2\gamma} \in Dom(\delta^{\diamond,B})$ and we have that*

$$\int_{[0,1]} \varphi(B_u)u^{2\gamma} d^{\diamond} B_u = \int_{[0,1]} \varphi(B_u)u^{2\gamma} dB_u - \gamma \int_{[0,1]} \varphi'(B_u)u^{4\gamma-1}du$$

Proof. Use exactly the same arguments of the Proposition (6.9) for the one parameters setting. □

Now the Corollary below is the key to the conversion of Theorem 6.11 into Theorem 1.9:

Corollary 6.13. *For $\gamma_i > 1/3$ and $\varphi \in C^6(\mathbb{R})$ then for every $v \in [0,1]$ $u \mapsto y^3_{u;v}u^{2\gamma_1} \in Dom(\delta^{\diamond,x_{\cdot;v}})$ and the following formula hold true*

$$\int_{\Delta} y^3_{u;v}u^{2\gamma_1}v^{2\gamma_2-1}d_1^{\diamond}x_{u;v}dv = \int_{\delta} y^3_{u;v}u^{2\gamma_1}v^{2\gamma_2-1}d_1x_{u;v}d_2v - \gamma_1 \int_{\Delta} y^4_{u;v}u^{4\gamma_1-1}v^{4\gamma_2-1}dudv$$

Proof. we recall that $x_{u,v} \overset{(\text{law})}{=} v^{\gamma_2}B_u$ with B is a fBm with hurst parameter γ_1 and then it suffice to use the proposition (6.4). □

References

[1] E. Alòs, O. Mazet, D. Nualart: Stochastic calculus with respect to Gaussian processes. *Ann. Prob.* **29** (2001), no. 2, 766–801. MR-1849177

[2] X. Bardina, M. Jolis: Multiple fractional integral with Hurst parameter less than $\frac{1}{2}$. *Stoch. Proc. Appl.* **116** (2006), 463–479. MR-2199559

[3] Cairoli, J. Walsh: Stochastic integrals in the plane. *Acta Math.* **134** (1975), 111–183. MR-0420845

[4] K.Chouk, M. Gubinelli: Rough sheets. Work in progress (2013).

[5] G. Da Prato, P. Malliavin, Paul, D. Nualart: Compact families of Wiener functionals. *C. R. Acad. Sci. Paris Sér. I Math.* **315** (1992), no. 12, 1287–1291.

[6] P. Friz, S. Riedel: Convergence rates for the full Gaussian rough paths. *Arxiv* preprint (2011). MR-3161527

[7] P. Friz, N. Victoir: *Multidimensional dimensional processes seen as rough paths.* Cambridge University Press, 2010. MR-2604669

[8] M. Gubinelli: Controlling rough paths. *J. Funct. Anal.* **216**, 86-140 (2004). MR-2091358

[9] M. Gubinelli, S. Tindel: Rough evolution equations. *Ann. Probab.* **38** (2010), no. 1, 1–75. MR-2599193

[10] Y. Hu, M. Jolis, S. Tindel: On Stratonovich and Skorohod stochastic calculus for Gaussian processes. *Ann. Probab.* **41** (2013), no. 3, 1656–1693. MR-3098687

[11] Y. Hu, J-A. Yan: Wick calculus for nonlinear Gaussian functionals. Acta Math. Appl. Sin. Engl. Ser. 25 (2009), no. 3, 399-414. MR-2506982

[12] J. León, D. Nualart: An extension of the divergence operator for Gaussian processes. *Stochastic Process. Appl.* **115** (2005), no. 3, 481–492. MR-2118289

[13] D. Nualart: Une formule d'Itô pour les martingales continues à deux indices et quelques applications. *Ann. Inst. H. Poincaré Probab. Statist.* **20** (1984), no. 3, 251–275.

[14] D. Nualart: *The Malliavin Calculus and Related Topics.* Probability and its Applications. Springer-Verlag, 2nd Edition, (2006). MR-2200233

[15] L. Quer-Sardanyons, S. Tindel: The 1-d stochastic wave equation driven by a fractional Brownian sheet. *Stochastic Process. Appl.* **117** (2007), no. 10, 1448–1472. MR-2353035

[16] Samorodnitsky, G., Taqqu, M. S.: *Stable non-Gaussian random processes.* Chapman and Hall, (1994). MR-1280932

[17] C.A. Tudor, F. Viens: Itô formula and local time for the fractional Brownian sheet. *Electron. J. Probab.* **8** (2003), no. 14, 31 pp. MR-1998763

[18] C.A. Tudor, F. Viens: Itô formula for the two-parameter fractional Brownian motion using the extended divergence operator. *Stochastics* **78** (2006), no. 6, 443–462. MR-2281680

[19] E. Wong, M. Zakai: Differentiation formulas for stochastic integrals in the plane. *Stochastic Processes Appl.* **6** (1977/78), no. 3, 339–349. MR-0651571

Acknowledgments. This article was finalized while S. Tindel was visiting the University of Kansas. He wishes to express his gratitude to this institution for its warm hospitality.

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)

Economical model of EJP-ECP

- Low cost, based on free software (OJS[1])
- Non profit, sponsored by IMS[2], BS[3], PKP[4]
- Purely electronic and secure (LOCKSS[5])

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

[1] OJS: Open Journal Systems http://pkp.sfu.ca/ojs/
[2] IMS: Institute of Mathematical Statistics http://www.imstat.org/
[3] BS: Bernoulli Society http://www.bernoulli-society.org/
[4] PK: Public Knowledge Project http://pkp.sfu.ca/
[5] LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[6] IMS Open Access Fund: http://www.imstat.org/publications/open.htm