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Large deviations for the empirical distribution in the branching random walk*

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Abstract

We consider the branching random walk $(Z_n)_{n\geq 0}$ on $\mathbb{R}$ where the underlying motion is of a simple random walk and branching is at least binary and at most decaying exponentially in law. For a large class of measurable sets $A \subseteq \mathbb{R}$, it is well known that $\bar{Z}_n(\sqrt{n}A) \to \nu(A)$ almost surely as $n \to \infty$, where $\bar{Z}_n$ is the particles empirical distribution at generation n and ν is the standard Gaussian measure on $\mathbb{R}$. We therefore analyze the rate at which $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) > \nu(A) + \epsilon)$ and $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) < \nu(A) - \epsilon)$ go to zero for any $\epsilon > 0$. We show that the decay is doubly exponential in either n or $\sqrt{n}$, depending on A and ϵ and find the leading coefficient in the top exponent. To the best of our knowledge, this is the first time such large deviation probabilities are treated in this model.

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1 Introduction and Results

In this work we analyze the decay of probabilities of certain unlikely deviation events involving the Branching Random Walk (henceforth BRW). As far as we know, very little has been done in this direction, although, after optimal law of large numbers and central limit theorem type results have been fully obtained, both the question and the events we consider seem to us natural and fundamental. To fix notation and context, we begin by briefly describing the model (1.1) and giving a short account of some of the relevant results in its analysis (1.2). Precise statements for the results in this paper then follow (1.3), along with the idea of the proof and accompanying remarks. Complete proofs for all statements are given in Section 2.

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1.1 Setup

The BRW model traces the evolution by means of reproduction and motion of a population of particles on the real line, carried out synchronously in discrete steps or *generations*. We denote by Z_n (henceforth the *particles measure*) the population at time $n = 0, 1, \dots$, which we describe as a point measure on $\mathbb{R}$ with a mass 1 per particle. The process is formally defined as follows. Initially there is a single particle at the origin $Z_0 = \delta_0$. It evolves in one generation to a random point measure Z_1. Although one may consider any law for Z_1, often and in this paper as well, attention is restricted to evolution by means of independent reproduction and motion. That is, Z_1 is realized by the particle giving birth to a random number of descendants, which are then moving independently (of each other and of their number) according to some common spatial distribution F, while the original particle is dying.

At any further generation $n \geq 2$ we have (conditioned on Z_{n-1}),

$$Z_n = \sum_{x \in Z_{n-1}} \widetilde{Z}_1^x \,, \tag{1.1}$$

where $\widetilde{Z}_1^x(\cdot)$ has the same distribution as $Z_1(\cdot - x)$ and $\{\widetilde{Z}_1^x :\, x \in Z_{n-1}\}$ are independent. Here and later, for a point measure ζ with integer masses, we write $x \in \zeta$ iff x is an atom of ζ, that is if $\zeta(x) = \zeta(\{x\}) > 0$. We use $(x :\, x \in \zeta)$ for the multi-set of atoms of ζ, where each atom x is repeated $\zeta(x)$ times. Moreover, if this multi-set is used as an index set (as above), different copies of the same atom are considered different indices.

Despite the old age of this model it is still quite central in pure and applied probability. It remains a popular model for describing and analyzing phenomena in various applied disciplines, such as biology, population dynamics and computer science. At the same time, due to the fundamentality of the stochastic dynamics it captures, it is frequently found in various seemingly unrelated mathematical models (e.g. the Gaussian Free Field [12], Interacting Particle System [20]). Finally, there are aspects of the model which are still not understood or only beginning to be understood now (e.g. its extremal process [2]). For the classical theory of BRW, we direct the reader to the survey by Ney [22] and the books by Révész [25], Harris [15], and Asmussen and Hering [3].

1.2 Known Results

Since the population-size process $(|Z_n|)_{n\geq 0} = (Z_n(\mathbb{R}))_{n \geq 0}$ is a standard Galton Watson process, it is well known that once reproduction is super-critical

$$\beta = \mathbb{E}|Z_1| > 1 \tag{1.2}$$

and assuming

$$\mathbb{E}|Z_1| \log |Z_1| < \infty \tag{1.3}$$

then for the *normalized particles measure* $\widehat{Z}_n = \beta^{-n} Z_n$ we have almost surely

$$\lim_{n\to\infty} |\widehat{Z}_n| = |\widehat{Z}| \,, \tag{1.4}$$

where $|\widehat{Z}|$ is some non-negative random variable with $\mathbb{E}|\widehat{Z}| = 1$ and almost-surely,

$$|\widehat{Z}| > 0 \quad \Longleftrightarrow \quad |Z_n| > 0 \,,\ \forall n \geq 1 \,. \tag{1.5}$$

The optimal version of this theorem is due to Kesten and Stigum [19]. If $\beta \leq 1$ and $\mathbb{P}(|Z_1| = 1) \neq 1$ the population dies out with probability 1. From now on, we shall assume both (1.2) and

$$\mathbb{P}(|Z_1| = 0) = 0 \;, \tag{1.6}$$

This will guarantee the non-trivial survival of the process almost-surely.

When displacement is considered as well, an analogous result to the above, conjectured by Harris [15], first proved by Stan [26], and then proved under optimal conditions by Kaplan [18] is

$$\lim_{n\to\infty} \widehat{Z}_n(\sqrt{n}A) = |\widehat{Z}|\nu(A) \qquad \mathbb{P}\text{-a.s.}\,, \tag{1.7}$$

Here $A \in \mathcal{A}_0$ where

$$\mathcal{A}_0 = \{(-\infty, x] :\ x \in \mathbb{R}\}\,, \tag{1.8}$$

ν is the standard Gaussian measure on $\mathbb{R}$, and the assumptions are (1.2), (1.3) for branching, and zero mean and unit variance for the motion, that is

$$\int x\,\mathrm{dF}(x) = 0 \quad ; \qquad \int x^2\,\mathrm{dF}(x) = 1\,. \tag{1.9}$$

Writing $\bar{\zeta} = \zeta/|\zeta|$ for a finite measure ζ, if follows from (1.4) and (1.7) that the *particles empirical distribution* $\bar{Z}_n$ satisfies

$$\lim_{n\to\infty} \bar{Z}_n(\sqrt{n}A) = \nu(A) \tag{1.10}$$

(recall that we assume also (1.6)).

Once leading order asymptotics (1.4), (1.7) have been obtained, second-order terms, or the question of the rate of the convergence, can be approached. For the population size, Heyde [16] has shown that under $\mathbb{E}|Z_1|^2 < \infty$, for some (explicit) $\alpha_0 > 0$, as $n \to \infty$

$$\alpha_0|\widehat{Z}|^{-1/2}\beta^{n/2}(|\widehat{Z}_n| - |\widehat{Z}|) \Rightarrow N(0,1)\,. \tag{1.11}$$

For the particles measures, more recently Chen [13] has proved that for all $A \in \mathcal{A}_0$,

$$\sqrt{n}\big(\widehat{Z}_n(\sqrt{n}A) - |\widehat{Z}|\nu(A)\big) = \varphi_1(n)|\widehat{Z}| + \alpha_1\widehat{M} + o(1)\,, \tag{1.12}$$

as $n \to \infty$, where $\alpha_1 > 0$, $\varphi_1(\cdot)$ is a bounded function, and $\widehat{M}$ is some random variable - all explicitly defined. In this case, motion is of a simple random walk and branching admits the same assumptions as in Heyde's.

Having settled the main questions in the "typical deviations" regime, it is natural to turn to the regime of atypical or large deviations. Results here are not as abundant. For $|Z_n|$, Athreya [4] has considered the following probabilities:

$$\mathbb{P}\big(\big||Z_{n+1}|/|Z_n| - \beta\big| > \Delta\big) \quad \text{and} \qquad \mathbb{P}\big(\big||\widehat{Z}_n| - |\widehat{Z}|\big| > \Delta\big)\,, \tag{1.13}$$

for $\Delta > 0$ and under the assumptions of exponential moments. If $p = \mathbb{P}(|Z_1| = 1) > 0$, he showed that the probability on the left is

$$\lambda_0(\Delta)p^n(1 + o(1)) \tag{1.14}$$

for some explicitly defined $\lambda_0(\Delta) > 0$ and otherwise, it is at most

$$\alpha_1(\Delta)\exp(-\lambda_1(\Delta)b^n)\,, \tag{1.15}$$

where b is the first integer for which $\mathbb{P}(|Z_1| = b) > 0$ and $\lambda_1(\Delta), \alpha_1(\Delta) > 0$. For the probability on the right, he obtained the bound

$$\mathbb{P}\big(\big||\widehat{Z}_n| - |\widehat{Z}|\big| > \Delta\big) \le C\exp\big(-C'\Delta^{2/3}(\beta^{1/3})^n\big)\,. \tag{1.16}$$

Above $C, C' > 0$ are some universal constants. See also [23, 14]. Different atypicality is treated by Jones [17] and Biggins and Bingham [8] who investigate the left and right tail of $|\widehat{Z}|$.

For the BRW, much effort has been directed into estimating the number of particles which deviate linearly away from the mean displacement in the underlying motion. It is a classical result by Biggins [7] that for any $A \in \mathcal{A}_0$,

$$\lim_{n\to\infty} n^{-1} \log Z_n(nA) = -\inf_{x\in A} \Lambda^*(x) \qquad \mathbb{P}\text{-a.s.}\,, \tag{1.17}$$

if the r.h.s. is positive and otherwise $Z_n(nA) \to 0$ a.s. Here Λ^* is the Legendre-Fenchel transform of $\Lambda(\theta) = \log \mathbb{E} \int e^{\theta x}\, \mathrm{d}Z_1(x)$, which is assumed to be finite. This can be also used to obtain the speed of the left (or right) most particle as $\inf\{x : \Lambda^*(x) < 0\}$, although to obtain sharper results, different methods have been used (c.f. Bramson [10, 11], and Addario-Berry and Reed [1]).

Perhaps closest to the type of large-deviation analysis we do here is the result by Athreya and Kang in [5], where instead of a motion in $\mathbb{R}$, particles move according to some positive-recurrent Markov chain with invariant measure π. Along with a local version of (1.10), they find that the probability that at time n the fraction of particles at state s is at least $\Delta > 0$ away from $\pi(s)$ decays exponentially as $\lambda(\Delta)p^n$ for some explicit $\lambda(\Delta) > 0$ and with p as in (1.14), which is assumed to be positive. Nevertheless, this is still quite far from what we do here. First, a random walk on $\mathbb{R}$ is typically null recurrent or transient (unless degenerate). Second, we in fact assume $p_1 = 0$ and thus obtain very different decay scales.

1.3 New Results

In this work we analyze large deviation probabilities of the form:

$$\mathbb{P}(|\bar{Z}_n(\sqrt{n}A) - \nu(A)| > \Delta)\,, \tag{1.18}$$

for a large class of measurable sets $A \subseteq \mathbb{R}$ and $\Delta > 0$. In light of (1.10), the above clearly decays in n and we aim to understand how fast.

1.3.1 Assumptions

We make the following assumptions. For branching, we shall assume that $|Z_1|$ is non-deterministic, that $\mathbb{E}e^{\theta|Z_1|} < \infty$ for θ in some neighborhood of 0 and that $\mathbb{P}(|Z_1| \geq 2) = 1$. The last condition guarantees that exponential growth of the population size is unavoidable. Although the case of $\mathbb{P}(|Z_1| \geq 2) < 1$ is an interesting problem, it is of a different nature as it permits using strategies which suppress the branching in order to realize large deviation events. This will result in a different scale for the decay in (1.18). For the underlying motion, we shall assume simple random walk steps. The precise step distribution will not change the result, as long as it has mean zero and bounded or rapidly decaying (at least doubly exponential) tails. Again, allowing for steps with fatter tails would have given rise to strategies which exploit these tails to achieve the unlikely events, resulting in a problem of a different nature and a different scale for the decay of (1.18). For the same reason, branching at fixed times is crucial for our results. Branching at random times (e.g. exponentially distributed, as in the branching Brownian motion model), would also result in a different scale of decay.

1.3.2 Main Theorems

We are now ready to state our main result. Let $\mathcal{A}$ be the algebra generated by $\mathcal{A}_0$ (defined in (1.8)) and set $b = \min\{k : \mathbb{P}(|Z_1| = k) > 0\} \geq 2$. For $A \in \mathcal{A}$ non-empty define

$\tilde{I}_A, \tilde{J}_A, I_A, J_A : (0,1) \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\begin{aligned} \tilde{I}_A(p) &= \inf\{|x| : \nu(A-x) \geq p\,,\ x \in \mathbb{R}\}\,, & (1.19)\\ \tilde{J}_A(p) &= \inf\big\{r : \sup_{x\in\mathbb{R}} \nu\big((A-x)/\sqrt{1-r}\big) \geq p\,,\ r \in [0,1)\big\} & (1.20)\end{aligned}$$

and

$$\begin{aligned} I_A(p) &= (\log b)\tilde{I}_A(p)\,, & (1.21)\\ J_A(p) &= (\log b)\tilde{J}_A(p)\,. & (1.22)\end{aligned}$$

Then,

Theorem 1.1. *Let $A \in \mathcal{A} \setminus \{\emptyset\}$ and $p \in \big(\nu(A), 1\big)$. If $I_A(p) < \infty$ then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log\big[-\log \mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \geq p\big)\big] = I_A(p)\,, \tag{1.23}$$

whenever p is a continuity point of I_A. If $I_A(p) = \infty$ then

$$\lim_{n\to\infty} \frac{1}{n} \log\big[-\log \mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \geq p\big)\big] = J_A(p)\,, \tag{1.24}$$

whenever p is a continuity point of J_A.

Replacing A by A^c and p by $1-p$ in Theorem 1.1, one has
Theorem 1′. *Let $A \in \mathcal{A} \setminus \{\mathbb{R}\}$ and $p \in \big(0, \nu(A)\big)$. If $I_{A^c}(1-p) < \infty$ then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log\big[-\log \mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \leq p\big)\big] = I_{A^c}(1-p)\,, \tag{1.25}$$

whenever $1-p$ is a continuity point of I_{A^c}. If $I_{A^c}(1-p) = \infty$ then

$$\lim_{n\to\infty} \frac{1}{n} \log\big[-\log \mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \leq p\big)\big] = J_{A^c}(1-p)\,, \tag{1.26}$$

whenever $1-p$ is a continuity point of J_{A^c}.

1.3.3 Remarks.

As follows from Lemma 2.1 below, for $A \in \mathcal{A} \setminus \{\emptyset\}$ and $p \in \big(\nu(A), 1\big)$, either $I_A(p) \in (0,\infty)$ or $I_A(p) = \infty$ and $J_A(p) \in (0, \log b)$. Thus on a double-exponential scale, Theorem 1.1 (and Theorem 1′) capture the right first-order asymptotics for the decay of the probability of a large deviation in the empirical distribution for such A's and p's. A double-exponential scale of large deviations appears in a handful of earlier works (eg. [21]), but here it is a natural scale, as the number of particles is exponential in n.

The restriction to continuity points of the rate function is commensurate with the usual large deviation formulation applied to sets of the form $[p,\infty)$ (only that in our case, the scale of decay is double exponential). Indeed, in an LDP formulation the upper bound on the $\limsup$ for the decay rate of the measure of $[p,\infty)$ is given as the infimum of the rate function over the interior of this set, namely (p,∞). The lower bound on the $\liminf$ is the infimum over the closure of the set, namely $[p,\infty)$. Only at continuity points of the rate function, is it guaranteed that these infima match and a limit for the rate of decay exists. In our case, there are indeed choices of sets A and discontinuity points p such that the limits in Theorem 1.1 (and Theorem 1′) do not exist, as the next proposition exemplifies. Nevertheless for any $A \in \mathcal{A}$, it is easy to see that the functions I_A and J_A possess only finitely many discontinuity points.

Proposition 1.2. *There exists* $a, x > 0$ *such that with* $A = x + [-a, a]$ *and* $p = \nu([-a, +a])$ *we have* $I_A(p) \in (0, \infty)$, $I_A(p+) = \infty$ *and* (1.23) *does not hold.*

The statement in the theorem still holds if we replace the weak inequality in (1.23), (1.24), (1.25), (1.26) by a strong one. Our proof for the lower bound on $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p)$ works verbatim for $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) > p)$.

The restriction to intervals of the form $(-\infty, x]$, $(y, x]$ and (y, ∞) in $\mathcal{A}$ is quite arbitrary and the theorem still holds if $\mathcal{A}$ is the algebra generated by sets of the form $(-\infty, x)$ or more generally, the set of all finite unions of disjoint intervals which either contain their endpoints or do not, or contain only one of them and can be finite or infinite, but as long as their interior is non-empty.

On the other hand, Theorem 1.1 cannot be expected to hold for all Borel sets, nor even all continuity sets of ν. Indeed, the following shows that there are simple enough sets for which the decay in (1.18) has neither linear nor radical rate on a double exponential scale.

Proposition 1.3. *For all* $\alpha \in (1/2, 1)$ *and* $p \in (0, 1)$, *there exists a set* A, *which is a countable union of disjoint finite intervals, such that as* $n \to \infty$

$$\log\big[-\log \mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \geq p\big)\big] \sim (\log b) n^{\alpha} \tag{1.27}$$

1.3.4 Idea of Proof

It is usually the case in the realm of large deviations that obtaining decay asymptotics for probabilities of unlikely events amounts to finding (and proving that it is such) an optimal (that is least "costly" in terms of probability) "strategy" for realizing the unlikely event. Consider therefore $A \in \mathcal{A}$ and $p \in (\nu(A), 1)$ as in the conditions of Theorem 1.1. What is the optimal strategy for having at least p fraction of the population in the set $\sqrt{n}A$ at time n instead of the likely $\nu(A)$?

As it turns out, among all possible strategies one needs to consider only two: a *shift* strategy and a *dilation* strategy. Given $x \in \mathbb{R}$, in the shift strategy all particles move together and constantly in either the left or the right direction for $w = |x|\sqrt{n}$ generations, so that in the end they are all at $x\sqrt{n}$ (up to rounding). This can be done with probability $\exp(-b^{|x|\sqrt{n}(1+o(1))})$ by keeping the number of particles at its minimum. Relative to the position of the particles at generation w, the target set has now "shifted" by $-x\sqrt{n}$. Therefore after dividing by the CLT scaling of $\sqrt{n}$, each particle at generation w will typically have (asymptotically) a fraction of $\nu(A - x)$ of its descendants in $\sqrt{n}A$, and this will also be the fraction for the entire population. Consequently, if there exists x for which $\nu(A - x) \geq p$, this strategy will realize the event $\{\bar{Z}_n(\sqrt{n}A) \geq p\}$ at the sole cost of "steering" the population for the first w generations. This can be done with probability at least $\exp(-e^{I_A(p)\sqrt{n}(1+o(1))})$ if x is chosen closest to 0.

If there is no x for which $\nu(A - x) \geq p$, a dilation strategy is employed, whereby all particles move together for $w' = r'n + |x'|\sqrt{n}$ generations ($x' \in \mathbb{R}$, $r' \in (0, 1)$) such that at generation w' they are all at position $x'\sqrt{n}$ (up to rounding effects, this can be achieved, for instance, by all particles moving in a constant direction $|x'|\sqrt{n}$ steps and then alternate between $+1$ and -1 steps for $r'n$ steps more). If r', x' are chosen such that $\nu((A - x')/\sqrt{1 - r'}) \geq p$ then, as in the shift case, the typical fraction of decedents in $\sqrt{n}A$ at time n, coming from each particle at time w' will be at least p and hence also the fraction among the total population. The probabilistic cost of this strategy is therefore incurred just in the first w' generations, and by keeping reproduction at its minimum, it can be $\exp(-b^{r'n(1+o(1))})$. Choosing the smallest r' possible, $\{\bar{Z}_n(\sqrt{n}A) \geq p\}$ can be achieved by a strategy which has probability $\exp(-e^{J_A(p)n(1+o(1))})$.

Of course these strategies only give lower bounds for the probability in question.

One therefore must also show that other strategies would not cost less, namely a corresponding upper bound. This is indeed done in the proof of the theorem. In fact, the optimal strategy is, in some sense, unique. To state a formal version of this assertion, let $B_A(p)$ for $A \in \mathcal{A} \setminus \{\emptyset\}$ and $p \in (0,1)$ be the set

$$B_A(p) = \{x :\ \nu(A - x) \geq p\}\,. \tag{1.28}$$

Then,

Proposition 1.4. *Let $A \in \mathcal{A} \setminus \{\emptyset\}$ and $p \in \big(\nu(A), 1\big)$. If $I_A(p) < \infty$ and p is a continuity point of $I_A(p)$, then for any small enough $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\Big(\bar{Z}_{w_\epsilon(n)}\big(\sqrt{n}B_A(p-\epsilon)\big) \geq 1-\epsilon \;\Big|\; \bar{Z}_n(\sqrt{n}A) \geq p\Big) = 1\,, \tag{1.29}$$

where

$$w_\epsilon(n) = (\tilde{I}_A(p) + \epsilon^3)\sqrt{n}\,. \tag{1.30}$$

If $I_A(p) = \infty$ and p is a continuity point of $J_A(p)$ then for any small enough $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\bar{Z}_{w_\epsilon(n)}\Big(\sqrt{n(1-\tilde{J}_A(p)}\, B_{A/\sqrt{1-\tilde{J}_A(p)}}(p-\epsilon)\Big) \geq 1-\epsilon \;\Big|\; \bar{Z}_n(\sqrt{n}A) \geq p\Big) = 0\,, \tag{1.31}$$

where

$$w_\epsilon(n) = (\tilde{J}_A(p) + \epsilon^3)n\,. \tag{1.32}$$

1.3.5 Extension to higher dimensions

The results presented here carry over to any dimension $d \geq 2$ (for the underlying motion) with only straight-forward modifications. On a qualitative level, both the phenomenon and the proofs are the same. More explicitly, we now have a simple d-dimensional random walk. The collection of subsets to consider $\mathcal{A}$ now contains all unions of disjoint multi-dimensional rectangles (bounded or unbounded) in $\mathbb{R}^d$. The definitions of $\tilde{I}_A(p)$ and $\tilde{J}_A(p)$ (and subsequently $I_A(p)$ and $J_A(p)$) are similar, only that $\mathbb{R}^d$ replaces $\mathbb{R}$ and the l^1 norm replaces the absolute value. With these changes, Theorem 1 holds without change. Similarly, the proofs only necessitate simple modifications.

2 Proofs

In this section we provide proofs for the statements in (1.3). We first introduce further notation (2.1) and prove some preliminary lemmas (2.2). These are then used to prove Theorem 1.1 and Theorem 1′ (2.3) and Propositions 1.2, 1.3 and 1.4 (2.4 - 2.6).

2.1 A bit more notation

The space of all particles measures, that is, finite point measures on $\mathbb{R}$ with integer masses, will be denoted by $\mathcal{Z}$. For $\zeta \in \mathcal{Z}$, we denote by $(Z_n^\zeta)_{n\geq 0}$ a BRW process with a similar evolution as $(Z_n)_{n\geq 0}$, only that initially $Z_0 = \zeta$. We will write Z_n^x in place of $Z_n^{\delta_x}$ for short. ν_n is the distribution of the position of a simple random walk after n steps. For $u \in \mathbb{R}$, as usual, $u^+ = \max(0,u)$ and $u^- = -(-u)^+$. We will use C, C', C'' to denote positive constants whose value is immaterial and changes from one use to the other. Constant values which are used more than once are denoted $C_0, C_1, ..$, and their values become fixed the first time they appear in the text.

2.2 Preliminaries

Lemma 2.1. *Let $A \in \mathcal{A}$ be non-empty and $p \in (0,1)$.*

1. $(\rho, \xi) \mapsto \nu(\rho A + \xi) \in \mathrm{C}^\infty(\mathbb{R}^2)$.
2. *If $\tilde{I}_A(p) \in [0,\infty)$ then there exists $x \in \mathbb{R}$ with $|x| = \tilde{I}_A(p)$ such that*
$$\nu(A - x) \geq p\,. \tag{2.1}$$
3. $\tilde{J}_A(p) \in [0,1)$ *and there exists $x \in \mathbb{R}$ such that with $r = \tilde{J}_A(p)$*
$$\nu\big((A-x)/\sqrt{1-r}\big) \geq p\,. \tag{2.2}$$
4. *If $p > \nu(A)$ then either $\tilde{I}_A(p) \in (0,\infty)$ or $\tilde{I}_A(p) = \infty$, $\tilde{J}_A(p) \in (0,1)$.*

Proof. Part 1 follows from the dominated convergence theorem and standard arguments once we write
$$\nu(\rho A + \xi) = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{(\rho t + \xi)^2}{2}} \rho \,\mathrm{d}t \tag{2.3}$$
since the integrand is in $\mathrm{C}^\infty(\mathbb{R}^2)$.

For part 2 and 3, if $A = \mathbb{R}$, then $\tilde{I}_A(p) = \tilde{J}_A(p) = 0$, and there is nothing to prove. Otherwise, define
$$\varphi_A(r,x) = \nu((A-x)/\sqrt{1-r}) \tag{2.4}$$
which is in $\mathrm{C}^\infty([0,1) \times \mathbb{R})$ by part 1. Therefore $\{x \in \mathbb{R} :\ \varphi_A(0,x) \geq p\}$ is a closed set, which, if non-empty, must contain a minimizer of $|\cdot|$. This shows part 2.

For part 3, if A contains a half-infinite interval, then since $\varphi_A(0,x) \to 1 > p$ if $x \to +\infty$ or $x \to -\infty$, we must have $\tilde{I}_A(p) < \infty$. Therefore $\tilde{J}_A(p) = 0$, and (2.2) is satisfied with $r = 0$ and x from part 2. Otherwise, A is a finite union of finite intervals, and so there must exist $R < 1$, $M < \infty$ such that

- $\varphi_A(r,x) \geq p$ for some $0 \leq r \leq R$ and x with $|x| \leq M$.
- $\varphi_A(r,x) < p/2$ for all $0 \leq r \leq R$ and x with $|x| > M$.

Thus, $\tilde{J}_A(p)$ is the infimum of the continuous function r over the non-empty compact set
$$\{(r,x) : \varphi_A(r,x) \geq p,\ 0 \leq r \leq R,\ |x| \leq M\}\,, \tag{2.5}$$
which gives part 3.

Finally, if $p > \nu(A)$, then $\tilde{I}_A(p) > 0$ by part 2. At the same time, if $\tilde{J}_A(p) = 0$, then $\tilde{I}_A(p) < \infty$ by part 3. This takes care of part 4. $\square$

Below is a standard result concerning the uniformity of the convergence to the Normal distribution under the CLT.

Lemma 2.2. *Let $A \subseteq \mathbb{R}$ be a continuity set of $\nu(A)$, i.e. $\nu(\partial A) = 0$ and $R > 0$. Then,*
$$\lim_{n\to\infty} \sup_{\rho \in [R^{-1}, R]} \sup_{\xi \in \mathbb{R}} |\nu_n(\sqrt{n}(\rho A + \xi)) - \nu(\rho A + \xi)| = 0\,. \tag{2.6}$$

Proof. By Theorem 2 in [9], it is enough to check that
$$\lim_{\delta \to 0} \sup_{\xi,\rho} \nu\big((\partial(\rho A + \xi))^\delta\big) = 0, \tag{2.7}$$
where for a set $D \subset \mathbb{R}$, we set $D^\delta = \{x \in \mathbb{R} :\ \inf_{y \in D} |x-y| < \delta\}$ and the supremum is over ρ and ξ as in the statement in the lemma. Since ν is equivalent to λ, Lebesgue measure on $\mathbb{R}$, we may show (2.7) with λ in place of ν. But,
$$\lambda\big((\partial(\rho A + \xi))^\delta\big) = \lambda\big(\rho(\partial A)^{\delta/\rho} + \xi\big) \leq R\lambda\big((\partial A)^{R\delta}\big)\,, \tag{2.8}$$
The last term goes to 0 as $\delta \to 0$, since $\lambda(\partial A) = 0$. $\square$

We shall need the following uniform Chernoff-Cramér-type upper bound.

Lemma 2.3. *Let $\mathbb{X}$ be a family of random variables on $\mathbb{R}$ with zero mean such that for some $\theta_0 > 0$*

$$\sup_{X\in\mathbb{X}} \mathbb{E}e^{\theta_0 X} < \infty \quad \text{and} \quad \sup_{X\in\mathbb{X}} \mathbb{E}(X^-)^2 < \infty \,. \tag{2.9}$$

Then there exists $C > 0$ such that for any $\Delta > 0$ small enough, any $m \geq 1$ and $X_1, \dots, X_m$ independent copies of random variables in $\mathbb{X}$

$$\mathbb{P}\big(\tfrac{1}{m}\sum_{i=1}^{m} X_i \,>\, \Delta\big) \leq e^{-C\Delta^2 m} \tag{2.10}$$

Proof. Using the exponential Chebyshef's inequality we may bound the l.h.s. in (2.10) for any $0 < \theta \leq \theta_1 < \theta_0$ by

$$\exp\Big\{ -m\big(\Delta\theta - m^{-1}\sum_{i=1}^{m} L_{X_i}(\theta)\big)\Big\} \,, \tag{2.11}$$

where we use $L_X(\theta) = \log \mathbb{E}e^{\theta X}$ for the log moment generating function of X. Since $L_X(\theta)$ is in $\mathrm{C}^\infty([0,\theta_0))$ due to (2.9), we may use Taylor expansion to write (note that the first two terms are 0)

$$L_X(\theta) = \tfrac{1}{2}L''_X(\widehat{\theta})\theta^2 \,, \tag{2.12}$$

for some $\widehat{\theta} \in (0,\theta)$. Now if we denote by $M_X(\widehat{\theta}) = \mathbb{E}e^{\widehat{\theta} X}$ the moment generating function of X then

$$L''_X(\widehat{\theta}) = \frac{M''_X(\widehat{\theta})M_X(\widehat{\theta}) - (M'_X(\widehat{\theta}))^2}{M_X^2(\widehat{\theta})} < CM_X(\theta_0) + \mathbb{E}(X^-)^2 \,. \tag{2.13}$$

This follows since $M_X(\widehat{\theta}) \geq 1$ via Jensen's inequality and since

$$M''_X(\widehat{\theta}) = \mathbb{E}X^2 e^{\widehat{\theta} X} \leq \mathbb{E}X^2 1_{X<0} + C\mathbb{E}e^{\theta_0 X} 1_{X\geq 0} \,, \tag{2.14}$$

for some $C > 0$ independent of $X \in \mathbb{X}$. Therefore (2.9) implies that there exists $K > 0$ for which

$$\sup_{X\in\mathbb{X}} \sup_{\widehat{\theta}\in(0,\theta_1)} L''_X(\widehat{\theta}) < K \tag{2.15}$$

and thus

$$\Delta\theta - m^{-1}\sum_{i=1}^{m} L_{X_i}(\theta) \geq \Delta\theta - \tfrac{1}{2}K\theta^2 \,. \tag{2.16}$$

Using this bound with $\theta = \Delta/K$ in (2.11) and assuming Δ is small enough, the result follows with $C = (2K)^{-1}$ in (2.10). □

The last lemma can be used to prove the following.

Lemma 2.4. *There exists $C, C' > 0$ such that for all $\Delta > 0$ sufficiently small, $A \subset \mathbb{R}$, $\zeta \in \mathcal{Z}$ and $n \geq 1$,*

$$\mathbb{P}\Big(\bar{Z}_n^\zeta(A) > \tfrac{1}{|\zeta|}\sum_{x\in\zeta} \nu_n(A-x) + \Delta\Big) \leq Ce^{-C'\Delta^2|\zeta|} \,. \tag{2.17}$$

The same holds if we replace $>$ with $<$ and $+\Delta$ with $-\Delta$.

Proof. Using

$$\bar{Z}_n^{\zeta}(A) = \frac{\frac{1}{|\zeta|}\sum_{x\in\zeta}\widehat{Z}_n^x(A)}{\frac{1}{|\zeta|}\sum_{x\in\zeta}|\widehat{Z}_n^x|}\,, \tag{2.18}$$

the l.h.s. of (2.17) is bounded above by

$$\mathbb{P}\big(\tfrac{1}{|\zeta|}\sum_{x\in\zeta}\widehat{Z}_n^x(\mathbb{R}) < 1-\tfrac{\Delta}{2}\big) + \mathbb{P}\big(\tfrac{1}{|\zeta|}\sum_{x\in\zeta}\widehat{Z}_n^x(A) > \tfrac{1}{|\zeta|}\sum_{x\in\zeta}\nu_n(A-x)+\tfrac{\Delta}{3}\big) \tag{2.19}$$

as long as Δ is small enough. Now Theorem 4 in [4] gives a uniform bound on the moment generating function $e^{\theta\widehat{Z}_n(\mathbb{R})}$ for all $n\geq 1$ and $\theta\in[0,\theta_0]$, for some $\theta_0>0$. This uniform bound can be extended to include also the moment generating functions of (the stochastically smaller) $\widehat{Z}_n^x(A)$ for all $A\subseteq\mathbb{R}$ and $x\in\mathbb{R}$ in the same range of θ. The non-negativity of all these random variables imply that we may extend the bound also to all $\theta<0$. Thus, it is not difficult to see that the family of random variables

$$\mathbb{X} = \{\pm(\widehat{Z}_n(A)-\nu_n(A)) :\ n\geq 1,\ A\subseteq\mathbb{R}\} \tag{2.20}$$

satisfies the conditions in Lemma 2.3, whence (2.19) is bounded above by $Ce^{-C'\Delta^2|\zeta|}$ for some $C,C'>0$ as desired.

Replacing A with A^{c} , we obtain (2.17) with $<$, $-\Delta$ in place of $>$, $+\Delta$. □

2.3 Proof of Theorem 1.1

Fix A and p as in the conditions of the theorem. There are two cases to consider, according to whether $I_A(p)$ is finite or not.

2.3.1 The Case $I_A(p)<\infty$.

We shall prove the lower and upper bounds separately.

Lower bound. Let $\epsilon>0$ be arbitrarily small. From the assumed continuity of I_A (and therefore $\tilde{I}_A$) at p and Lemma 2.1 part 2, it follows that there exists $x\in\mathbb{R}$, $\delta>0$ such that

$$\nu(A-x)\geq p+\delta \quad\text{and}\quad |x|<\tilde{I}_A(p)+\epsilon\,. \tag{2.21}$$

Set

$$w=\lfloor |x|\sqrt{n}\rfloor\mathrm{sgn}(x) \quad;\qquad m=n-|w| \quad;\qquad \zeta=b^{|w|}\delta_w \tag{2.22}$$

and write

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\geq p)\geq \mathbb{P}(Z_{|w|}=\zeta)\ \mathbb{P}(\bar{Z}_m^{\zeta}(\sqrt{n}A)\geq p) \tag{2.23}$$

The first factor can be lower bounded by $\exp\{-Cb^{|w|}\}$ as the event $\{Z_{|w|}=\zeta\}$ is equivalent to having all particles in the first $|w|$ generations give birth to b children, all of whom take either a $+1$ step or a -1 step, depending on the sign of x. This requires that at most $C'b^{|w|}$ independent particles make certain branching/walking choices, all of which have a uniformly positive probability.

The second factor in (2.23) can be bounded below by

$$\big(\mathbb{P}(\bar{Z}_m(\sqrt{n}A-w)\geq p)\big)^{|\zeta|}\,. \tag{2.24}$$

By (2.21), the probability in the above expression is further bounded below by

$$\mathbb{P}\left(\bar{Z}_m\left(\sqrt{m}\left(\sqrt{\frac{n}{m}}(A-x)+\frac{x\sqrt{n}-w}{\sqrt{m}}\right)\right)\geq \nu\big(A-x\big)-\delta\right)\,. \tag{2.25}$$

Now, set $\rho = \sqrt{\frac{n}{m}}$ and $\xi = \frac{x\sqrt{n}-w}{\sqrt{m}}$. Since $\rho = 1 + O(1/\sqrt{n})$ and $\xi = O(1/\sqrt{n})$, by Lemma 2.1 part 1 and Lemma 2.2 we have

$$\Big|\nu(A-x) - \nu_m\big(\sqrt{m}(\rho(A-x)+\xi)\big)\Big| = o(1)\,. \tag{2.26}$$

Therefore, for n large enough (2.25) is bounded below by

$$\mathbb{P}\Big(\bar{Z}_m\big(\sqrt{m}(\rho(A-x)+\xi)\big) \geq \nu_m\big(\sqrt{m}(\rho(A-x)+\xi)\big) - \delta/2\Big)\,. \tag{2.27}$$

Applying Lemma 2.4 we see that the latter goes to 1 when $n \to \infty$.

Plugging this back into (2.24), recalling that $|\zeta| = b^{|w|}$, the second factor in (2.23) is bounded below by $\exp\{-C' b^{|w|}\}$. Combining the bounds on both factors in (2.23) we arrive at

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \geq \exp\{-Cb^{|x|\sqrt{n}}\} = \exp\big\{-e^{(\log b)(\tilde{I}_A(p)+\epsilon)\sqrt{n}+C'}\big\} \tag{2.28}$$

for n large enough. Since ϵ was arbitrary, the lower bound follows.

Upper bound. Let $\epsilon > 0$ be arbitrarily small and set

$$|w| = \lfloor(\tilde{I}_A(p)-\epsilon)\sqrt{n}\rfloor \quad ; \qquad m = n - |w|\,. \tag{2.29}$$

Conditioning on the particles measure ζ at generation $|w|$, we have

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) = \sum_{\zeta} \mathbb{P}(\bar{Z}_m^{\zeta}(\sqrt{n}A) \geq p)\mathbb{P}(Z_{|w|} = \zeta)\,. \tag{2.30}$$

Any such ζ must satisfy $\mathrm{supp}(\zeta) \subseteq [-|w|, +|w|]$. Therefore there exists $\delta > 0$, such that for all such ζ and $z \in \zeta$,

$$\nu(A - z/\sqrt{n}) \leq \max_{|y| \leq \tilde{I}_A(p)-\epsilon} \nu(A-y) = p - \delta\,. \tag{2.31}$$

This follows from the definition of $\tilde{I}_A(p)$.

Using Lemma 2.1 and Lemma 2.2, we further obtain for n large,

$$\frac{1}{|\zeta|}\sum_{z\in\zeta}\nu_m(\sqrt{n}A - z) \leq \frac{1}{|\zeta|}\sum_{z\in\zeta}\nu\big(\sqrt{\tfrac{n}{m}}A - \tfrac{z}{\sqrt{m}}\big) + \frac{\delta}{2} < p - \frac{\delta}{3}\,. \tag{2.32}$$

Then Lemma 2.4 implies that $\mathbb{P}(\bar{Z}_m^{\zeta}(\sqrt{n}A) \geq p)$ is bounded above by

$$\mathbb{P}\Big(\bar{Z}_m^{\zeta}(\sqrt{n}A) \geq \tfrac{1}{|\zeta|}\sum_{z\in\zeta}\nu_m(\sqrt{n}A - z) + \tfrac{\delta}{3}\Big) \leq Ce^{-C'|\zeta|}\,. \tag{2.33}$$

As $|\zeta| \geq b^{|w|}$ we have from (2.30) for n large enough,

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \leq \exp\big\{-e^{(\log b)(\tilde{I}_A(p)-\epsilon)\sqrt{n}-C}\big\}\,, \tag{2.34}$$

and this concludes the upper bound as ϵ was arbitrary.

2.3.2 The Case $I_A(p) = \infty$.

The proof in this case is technically similar to the proof in the previous case, although the "optimal" strategy for achieving the desired deviation is different.

Lower bound. Let $\epsilon > 0$ be arbitrarily small. By continuity of J_A (and therefore $\tilde{J}_A$) at p and Lemma 2.1 part 3, we may find $r \in (0,1)$, $x \in \mathbb{R}$ and $\delta > 0$, such that

$$r < \big(\tilde{J}_A(p) + \epsilon\big) \quad \text{and} \quad \nu\big((A-x)/\sqrt{1-r}\big) \geq p + \delta \tag{2.35}$$

Set

$$q = 2\lfloor rn/2 \rfloor \quad ; \quad w = \lfloor |x|\sqrt{n} \rfloor \mathrm{sgn}(x) \quad ; \quad s = q + |w| \quad ; \quad \zeta = b^s \delta_w \tag{2.36}$$

and write

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \geq \mathbb{P}(Z_s = \zeta)\mathbb{P}(\bar{Z}^{\zeta}_{n-s}(\sqrt{n}A) \geq p)\,. \tag{2.37}$$

The first factor on the r.h.s. is at least $\exp\{-Cb^s\}$ since the event there can be achieved by having all particles give birth to b children in the first s generations, make only $+1$ or -1 steps in the first $|w|$ generations (depending on the sign of x), and then alternate between $+1$ and -1 steps in the succeeding q generations. This requires that $C'b^s$ independent particles make certain branching/walking choices, all of which have a uniformly positive probability.

The second factor is bounded below by

$$\big(\mathbb{P}(\bar{Z}_{n-s}(\sqrt{n}A - w) \geq p)\big)^{|\zeta|} \tag{2.38}$$

Setting

$$m = n - s \quad , \quad \rho = \sqrt{\frac{n}{m}(1-r)} \quad , \quad \xi = \frac{x\sqrt{n} - w}{\sqrt{m}}\,, \tag{2.39}$$

and using (2.35), we may bound below the probability in (2.38) by

$$\mathbb{P}\Big(\bar{Z}_m\big(\sqrt{m}\big(\rho\tfrac{A-x}{\sqrt{1-r}} + \xi\big)\big) \geq \nu\big(\rho\tfrac{A-x}{\sqrt{1-r}} + \xi\big) + \nu\big(\tfrac{A-x}{\sqrt{1-r}}\big) - \nu\big(\rho\tfrac{A-x}{\sqrt{1-r}} + \xi\big) - \delta\Big)\,. \tag{2.40}$$

Now $\rho = 1 + O(1/\sqrt{n})$ and $\xi = O(1/\sqrt{n})$. Hence by Lemma 2.1 and Lemma 2.2 for n large enough, the latter is at least

$$\mathbb{P}\Big(\bar{Z}_m\big(\sqrt{m}\big(\rho\tfrac{A-x}{\sqrt{1-r}} + \xi\big)\big) \geq \nu_m\big(\sqrt{m}\big(\rho\tfrac{A-x}{\sqrt{1-r}} + \xi\big) - \delta/2\Big)\,. \tag{2.41}$$

In light of Lemma 2.4, this goes to 1 as $n \to \infty$ and therefore for n large the second factor in (2.37) is bounded below by $e^{-C|\zeta|} \geq \exp\{-C'b^s\}$.

Plugging the two lower bounds in (2.37) we obtain for n large enough

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \geq \exp\{-e^{(\log b)s + C}\} \geq \exp\big\{-e^{(\log b)(\tilde{J}_A(p) + 2\epsilon)n}\big\}\,. \tag{2.42}$$

The lower bound then follows from the arbitrariness of ϵ.

Upper bound. As in the previous case, let $\epsilon > 0$ be small enough and set

$$q_\epsilon = \big\lfloor \big(\tilde{J}_A(p) - \epsilon\big)n \big\rfloor \quad ; \qquad m = n - q_\epsilon\,. \tag{2.43}$$

This time we condition on the particles measure ζ at generation q_ϵ:

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) = \sum_{\zeta} \mathbb{P}(\bar{Z}^{\zeta}_m(\sqrt{n}A) \geq p)\mathbb{P}(Z_{q_\epsilon} = \zeta)\,. \tag{2.44}$$

Now, from the definition of $\tilde{J}_A(p)$ it follows that there exists $\delta > 0$ such that for all $\epsilon' \in [\epsilon, 2\epsilon]$ and $z \in \mathbb{R}$,

$$\nu\Big(\tfrac{A-z}{\sqrt{1-\tilde{J}_A(p)+\epsilon'}}\Big) \leq p - \delta\,. \tag{2.45}$$

Therefore, for any measure ζ and n large enough by Lemma 2.1 and 2.2

$$\frac{1}{|\zeta|}\sum_{z \in \zeta} \nu_m(\sqrt{n}A - z) \leq \frac{1}{|\zeta|}\sum_{z \in \zeta} \nu\big(\sqrt{\tfrac{n}{m}}A - \tfrac{z}{\sqrt{m}}\big) + \frac{\delta}{2} \leq p - \frac{\delta}{2}\,. \tag{2.46}$$

Using Lemma 2.4 we have that $\mathbb{P}(\bar{Z}_m^\zeta(\sqrt{n}A) \geq p)$ is bounded above by

$$\mathbb{P}\left(\mathbb{P}(\bar{Z}_m^\zeta(\sqrt{n}A) \geq \frac{1}{|\zeta|}\sum_{z\in\zeta}\nu_m(\sqrt{n}A - z) + \frac{\delta}{2}\right) \leq Ce^{-C'|\zeta|}\,. \tag{2.47}$$

But if ζ is a possible particle measure at generation q_ϵ, then $|\zeta| \geq b^{q_\epsilon}$. Hence from (2.44) we obtain for n large enough,

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \leq e^{-Cb^{q_\epsilon}} \leq \exp\left\{ - e^{(\log b)(\tilde{J}_A(p)-\epsilon)n - C'}\right\}, \tag{2.48}$$

and since ϵ is arbitrary the upper bound follows. □

2.4 Proof of Proposition 1.2

Fix $a = \frac{3}{8}M^{-1}$ and $x = \frac{1}{2}M^{-1}$ for some positive integer M whose value will be determined later. Set $A = x + [-a,a]$ and $p = \nu([-a,+a])$. Since $x \mapsto \nu(x + [-a,a])$ has a unique maximizer $x = 0$, it is clear that $I_A(p) = (\log b)x$ and $I_A(p+) = \infty$. We shall now show that for M large enough we also have

$$\begin{aligned}
\limsup_{n\to\infty} \tfrac{1}{\sqrt{n}} \log\big(- \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p)\big) \quad &\geq \quad 2(\log b)x\,, && (2.49)\\
\liminf_{n\to\infty} \tfrac{1}{\sqrt{n}} \log\big(- \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p)\big) \quad &\leq \quad (\log b)x\,. && (2.50)
\end{aligned}$$

which shows that a limit indeed fails to exist.

In what follows, we shall make use of the following three facts. The first is a sharp estimate on the error term in CLT, given by the Edgeworth Expansion (see Theorem VI.6 in [24]). Recall that the latter says that if $t \in \mathbb{N}$ tends to ∞ then uniformly in $z \in \mathbb{R}$

$$\nu_t\big((-\infty,z)\big) = \nu\big(\big(-\infty, \tfrac{z}{\sqrt{t}}\big)\big) + t^{-1/2}S(z)f(z/\sqrt{t}) + o\big(t^{-1/2}\big)\,, \tag{2.51}$$

where f is the standard Normal density, S is the 1-periodic odd function on $\mathbb{R}$ given by

$$S(u) = \begin{cases} \frac{1}{2} - (u \bmod 1) & u \bmod 1 \in (0,1)\,, \\ 0 & u \bmod 1 = 0 \end{cases} \tag{2.52}$$

and $u \bmod 1$ means $u - \lfloor u \rfloor$.

The second is the Taylor expansion of $(\rho,\xi) \mapsto \nu\big(\rho[-z,z]+\xi\big)$ around $(\rho,\xi) = (1,0)$ for some $z > 0$:

$$\nu\big(\rho[-z,z]+\xi\big) = \nu([-z,z]) + C_z(\rho-1) - C'_z\xi^2 + O_z\big((\rho-1)^2 + \xi^3\big)\,, \tag{2.53}$$

where $C_z = 2zf(z) > 0$ and $C'_z = -f'(z) > 0$. The third is the easily verified assertion that for any $z > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\nu([-z,z]+\xi) > 0 \ \text{ if } \ \xi < 0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\nu([-z,z]+\xi) < 0 \ \text{ if } \ \xi > 0\,. \tag{2.54}$$

Starting with (2.49), let $w = \lfloor 2x\sqrt{n}\rfloor$, $m = n - w$ and note that

$$\sqrt{\tfrac{n}{m}} = 1 + xn^{-1/2} + O(n^{-1})\,. \tag{2.55}$$

We would like to bound from above $\nu_m\big(\sqrt{n}A - y\big)$ for any $y \in \mathbb{Z}$ and n large enough. Writing:

$$\begin{aligned}
\nu_m\big(\sqrt{n}A - y\big) &= \nu\big(\sqrt{\tfrac{n}{m}}A - \tfrac{y}{\sqrt{m}}\big) \\
&\quad + m^{-1/2}\Big(S\big(\sqrt{n}(a+x) - y\big)f\big(\sqrt{\tfrac{n}{m}}(a+x) - \tfrac{y}{\sqrt{m}}\big) \\
&\qquad - S\big(\sqrt{n}(-a+x) - y\big)f\big(\sqrt{\tfrac{n}{m}}(-a+x) - \tfrac{y}{\sqrt{m}}\big)\Big) + o\big(m^{-1/2}\big)
\end{aligned} \tag{2.56}$$

and using (2.53), (2.54) and (2.55), we see that

$$\nu_m\big(\sqrt{n}A - y\big) \le p - C\,, \tag{2.57}$$

if $|y - x\sqrt{n}| \ge n^{1/2}$ and

$$\nu_m\big(\sqrt{n}A - y\big) \le p - C'_a n^{-1/3} + O\big(n^{-1/2}\big)\,. \tag{2.58}$$

if $|y - x\sqrt{n}| \in \big(n^{1/3}, n^{1/2}\big)$.

If, on the other hand, $|y - x\sqrt{n}| \le n^{1/3}$ then the r.h.s. of (2.56) is bounded above by

$$p + C_a x n^{-1/2} + n^{-1/2} f(a)\Big(S\big(\sqrt{n}(a+x)\big) - S\big(\sqrt{n}(-a+x)\big)\Big) + o\big(n^{-1/2}\big)\,. \tag{2.59}$$

This is because $y \in \mathbb{Z}$ and S is 1-periodic. Therefore for $n = \big((8k+2)M\big)^2$ where k is a positive integer, we have $(\sqrt{n}x) \bmod 1 = 0$ and $(\sqrt{n}a) \bmod 1 = \frac{3}{4}$, which imply together with (2.59) that

$$\nu_m(\sqrt{n}A - y) \le p + n^{-1/2}\big(2af(a)x + 2f(a)S(3/4)\big) + o\big(n^{-1/2}\big)\,. \tag{2.60}$$

We may therefore choose M large enough such that the coefficient of $n^{-1/2}$ above is smaller than $-\delta$ for some $\delta > 0$.

All together, with the above choice of M, we have

$$\nu_m\big(\sqrt{n}A - y\big) \le p - \delta n^{-1/2} \tag{2.61}$$

for all $y \in \mathbb{Z}$, $n = \big((8k+2)M\big)^2$ and $k \in \mathbb{N}$ large enough. Therefore for any measure ζ for the particles at time w, we have

$$\frac{1}{|\zeta|}\sum_{y\in\zeta} \nu_m\big(\sqrt{n}A - y\big) \le \sup_{y\in\mathbb{Z}} \nu_m\big(\sqrt{n}A - y\big) \le p - \delta n^{-1/2}\,. \tag{2.62}$$

It follows then from Lemma 2.4 and $|\zeta| \ge b^w$ that

$$\mathbb{P}\big(\bar{Z}^\zeta_m(\sqrt{n}A) \ge p\big) \le Ce^{-C'|\zeta|\delta^2 n^{-1}} \le e^{-e^{2(\log b)x\sqrt{n}(1-o(1))}}\,. \tag{2.63}$$

By conditioning on the particle measure in generation w, the same bound holds for $\mathbb{P}\big(\bar{Z}^\zeta_n(\sqrt{n}A) \ge p\big)$ for all n as above. This gives (2.49).

Turning to (2.50), let this time $n = (8kM)^2$ for k integer and large enough, $w = x\sqrt{n}$ and $m = n - w$. Observe that

$$\sqrt{\frac{n}{m}} = 1 + \tfrac{1}{2}xn^{-1/2} + O(n^{-1})\,, \quad w \bmod 1 = (x\sqrt{n}) \bmod 1 = (a\sqrt{n}) \bmod 1 = 0\,. \tag{2.64}$$

Therefore, by (2.51) and (2.53) for some $\delta > 0$ and k large enough,

$$\nu_m\big(\sqrt{n}A - w\big) = \nu\Big(\sqrt{\tfrac{n}{m}}[-a,a]\Big) + o\big(n^{-1/2}\big) \ge p + \tfrac{1}{2}C_a x n^{-1/2} + o\big(n^{-1/2}\big) \ge p + \delta n^{-1/2}\,. \tag{2.65}$$

We may now set $\zeta = b^w \delta_w$ and use Lemma 2.4 to write

$$\begin{aligned}\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) &\ge \mathbb{P}(Z_w = \zeta)\,\mathbb{P}\big(\bar{Z}^\zeta_m(\sqrt{n}A) \ge p\big) \\ &\ge e^{-Cb^w}\Big(1 - \mathbb{P}\big(\bar{Z}^\zeta_m(\sqrt{n}A) < -\delta n^{-1/2} + \tfrac{1}{|\zeta|}\sum_{y\in\zeta}\nu_m(\sqrt{n}A - y)\big)\Big) \\ &\ge e^{-e^{(\log b)x\sqrt{n}(1+o(1))}}\big(1 - C'e^{-C''b^w\delta^2 n^{-1}}\big) \ge e^{-e^{(\log b)x\sqrt{n}(1+o(1))}}\,.\end{aligned} \tag{2.66}$$

The latter holds for all n as above and this shows (2.50). □

2.5 Proof of Proposition 1.3

Let $\alpha \in (1/2, 1)$ and $p \in (0,1)$ be given. Fix some $\delta > 0$ (small enough for the arguments below to hold), $\gamma \in (1-\alpha, 1/2)$ and $a > 0$ such that $\nu(A_0) = p$ where $A_0 = [-a, +a]$. For any integer $k \geq 1$ set $A_k = x_k + r_k \cdot A_0$ where

$$x_k = k^{1+\delta}\,, \qquad r_k = \sqrt{1 - k^{-\frac{(1-\alpha)(1+\delta)}{\alpha-1/2}} + k^{-\frac{\gamma(1+\delta)}{\alpha-1/2}}}\,. \tag{2.67}$$

Finally, for some $k_0 > 0$ to be chosen later, set

$$A = \bigcup_{k=k_0}^{\infty} A_k\,. \tag{2.68}$$

We shall now argue that (1.27) is satisfied with the above A, α and p.

Lower bound. For any n large enough, set $k = \lceil n^{(\alpha-1/2)/(1+\delta)} \rceil$, $w = \lfloor x_k \sqrt{n} \rfloor$, $m = n - w$, $\zeta = b^w \delta_w$ and write

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \geq \mathbb{P}(Z_w = \zeta)\; \mathbb{P}(\bar{Z}^{\zeta}_m(\sqrt{n}A) \geq p)\,. \tag{2.69}$$

The first factor in the r.h.s. is at least $\exp\{-Cb^w\} \geq \exp\{-b^{n^{\alpha}(1+o(1)}\}$, as the event can be achieved by all particles multiplying at rate b and having their descendants take a $+1$ step for w generations.

For the second factor in the r.h.s. of (2.69), observe that

$$\begin{aligned} \nu_m\big(\sqrt{n}A - w\big) &= \nu_m\big(\sqrt{n}(A - x_k) + O(1)\big) \geq \nu_m\big(\sqrt{n} r_k A_0 + O(1)\big) \\ &= \nu_m\Big(\sqrt{m}\Big(\sqrt{\tfrac{n}{m}} r_k A_0 + O\big(m^{-1/2}\big)\Big)\Big) = \nu\Big(\sqrt{\tfrac{n}{m}} r_k A_0\Big) + O\big(m^{-1/2}\big)\,, \end{aligned} \tag{2.70}$$

where the last inequality follows from Lemma 2.1 part 1 and the fact that $|\nu_m(\rho A_0 + \xi) - \nu(\rho A_0 + \xi)| = O(m^{-1/2})$ uniformly in $(\rho, \xi) \in \mathbb{R}_+ \times \mathbb{R}$, as shown in [6] (see (2.5) and the paragraph that follows). Now since $\rho \mapsto \nu(\rho A_0)$ is smooth and strictly increasing and since

$$r_k \geq \big(1 - n^{-(1-\alpha)}\big)^{1/2} + n^{-\gamma}\big(1 + o(1)\big)\,, \quad \sqrt{\tfrac{n}{m}} \geq \big(1 - n^{-(1-\alpha)}\big)^{-1/2}\big(1 + O\big(n^{-1}\big)\big) \tag{2.71}$$

the right most term in (2.70) is at least

$$\nu\big(A_0\big(1 + n^{-\gamma}(1 + o(1))\big)\big) \geq p + Cn^{-\gamma}\,. \tag{2.72}$$

for some $C > 0$. Setting $\Delta = Cn^{-\gamma}$, with C as above, the second factor in the r.h.s. of (2.69) is at least

$$\big(\mathbb{P}\big(\bar{Z}_m(\sqrt{n}A - w) \geq \nu_m(\sqrt{n}A - w) - \Delta\big)\big)^{|\zeta|}\,. \tag{2.73}$$

Since $\Delta^2 |\zeta| \to \infty$, Lemma 2.4 then shows that the latter goes to 1 with n.

Upper bound. Let $\epsilon > 0$ be arbitrarily small and set $w = \lfloor (1-\epsilon) n^{\alpha} \rfloor$ and $m = n - w$. By conditioning on the particles measure in generation w, it is clear that

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \leq \max_{\zeta}\; \mathbb{P}(\bar{Z}^{\zeta}_m(\sqrt{n}A) \geq p) \tag{2.74}$$

where the maximum is taken over all feasible particles measures ζ for generation w. For such ζ, we may write

$$\frac{1}{|\zeta|} \sum_{z \in \zeta} \nu_m(\sqrt{n}A - z) \leq \max_{z \in \zeta} \nu_m(\sqrt{n}A - z) \tag{2.75}$$

$$\leq \max_{z \in \zeta} \nu\big(\sqrt{\tfrac{n}{m}} A - \tfrac{z}{\sqrt{m}}\big) + O(n^{-1/2}) \tag{2.76}$$

$$\leq \max_{|y| \leq (1-\epsilon) n^{\alpha-1/2}} \nu\big(\sqrt{\tfrac{n}{m}}(A - y)\big) + O(n^{-1/2})\,, \tag{2.77}$$

where for the second inequality, we have used

$$\limsup_{m\to\infty} \sup_{\rho\in[1/2,2]} \sup_{\xi\in\mathbb{R}} m^{1/2} |\nu_m(\sqrt{m}(\rho A+\xi)) - \nu(\rho A+\xi)| < \infty \,, \tag{2.78}$$

which holds for the set A in light of (2.5) of [6].

Consider now some y in the range of the maximum in (2.77) and find the index k of the closest point to y among $(x_k)_{k\geq k_0}$. We can then write

$$\sqrt{\tfrac{n}{m}}(A-y) = \sqrt{\tfrac{n}{m}}\big(A_k - y\big) \cup \sqrt{\tfrac{n}{m}}\big((A\setminus A_k) - y\big) \tag{2.79}$$

and bound the Gaussian measure of each set separately.

The measure of the first set is upper bounded by (using Lemma 2.1)

$$\begin{aligned} \nu\big(\sqrt{\tfrac{n}{m}}(A_k - x_k)\big) &= \nu\big(\sqrt{\tfrac{n}{m}} r_k \cdot A_0\big) && (2.80)\\ &\leq \nu(A_0) - C\big(1-\sqrt{\tfrac{n}{m}} r_k\big) && (2.81)\\ &\leq p - C\left(\big(1-r_k\big) - \big(\sqrt{\tfrac{n}{m}}-1\big)\right) . && (2.82)\end{aligned}$$

For the second set in (2.79), note that from the definition of A it follows that

$$(A\setminus A_k) - y \subseteq (-Ck^\delta, +Ck^\delta)^{\mathrm{c}} \,, \tag{2.83}$$

for large enough k. Then, using a standard bound on the tails of ν, we obtain

$$\nu\Big(\sqrt{\tfrac{n}{m}}\big((A\setminus A_k)-y\big)\Big) \leq C' e^{-Ck^{2\delta}} \,. \tag{2.84}$$

Combining the two bounds, we have

$$\nu\big(\sqrt{\tfrac{n}{m}}(A-y)\big) \leq p - C''\left(\big(1-r_k\big) - C'e^{-Ck^{2\delta}} - \big(\sqrt{\tfrac{n}{m}}-1\big)\right) \tag{2.85}$$

Now if k_0 is chosen large enough, the r.h.s. above is maximized when k is the largest possible. At the same time, the choices of k and y imply

$$(k-1)^{1+\delta} < y \leq (1-\epsilon)n^{\alpha-1/2} \tag{2.86}$$

which gives an upper bound on k. Using this in (2.85) we infer that the r.h.s. of (2.77) is bounded above by

$$p - C\left(n^{-(1-\alpha)}/2 - (1-\epsilon)n^{-(1-\alpha)}/2\right)(1-o(1)) \leq p - C'\epsilon n^{-(1-\alpha)} \,. \tag{2.87}$$

We may now use Lemma 2.4 and the fact that $|\zeta| \geq b^w$ to conclude that

$$\mathbb{P}(\bar{Z}_m^\zeta(\sqrt{n}A) \geq p) \leq C\exp(-C'\epsilon^2 n^{-2(1-\alpha)} b^{(1-\epsilon)n^\alpha}) \,. \tag{2.88}$$

This finishes the proof as ϵ was arbitrary. $\square$

2.6 Proof of Proposition 1.4

By Theorem 1.1 and Bayes rule, it is enough to show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log\Big[-\log \mathbb{P}\Big(\bar{Z}_n(\sqrt{n}A) \geq p \;\Big|\; \bar{Z}_{w_\epsilon(n)}\big(\sqrt{n}B_A^{\mathrm{c}}(p-\epsilon)\big) > \epsilon\Big) > I_A(p)\Big] \,. \tag{2.89}$$

Since $|Z_{w_\epsilon(n)}| > e^{(I_A(p)+\epsilon^3)\sqrt{n}}$, this will follow by conditioning on $Z_{w_\epsilon(n)}$ and using Lemma 2.4, provided that we can show that on

$$\big\{\bar{Z}_{w_\epsilon(n)}\big(\sqrt{n}B_A^{\mathrm{c}}(p-\epsilon)\big) > \epsilon\big\} \tag{2.90}$$

we have

$$\frac{1}{|Z_{w_\epsilon(n)}|}\sum_{x\in Z_{w_\epsilon(n)}} \nu_{n-w_\epsilon(n)}\big(\sqrt{n}A-x\big) < p-\Delta_\epsilon \tag{2.91}$$

for all n large enough and some $\Delta_\epsilon > 0$.

To check this, let ζ be any realization of $Z_{w_\epsilon(n)}$ satisfying (2.90) and use Lemma 2.1 and 2.2 to write

$$\begin{aligned}
\frac{1}{|\zeta|}\sum_{x\in\zeta}\nu_{n-w_\epsilon(n)}\big(\sqrt{n}A-x\big) &\le \frac{1}{|\zeta|}\sum_{x\in\zeta}\nu\big(A-x/\sqrt{n}\big)+o(1)\\
&= \frac{1}{|\zeta|}\sum_{x\in\zeta\cap\sqrt{n}B_A^{\mathrm{c}}(p-\epsilon)}\nu\big(A-x/\sqrt{n}\big)+\frac{1}{|\zeta|}\sum_{x\in\zeta\cap\sqrt{n}B_A(p-\epsilon)}\nu\big(A-x/\sqrt{n}\big)+o(1)\\
&\le (p-\epsilon)\bar\zeta\big(\sqrt{n}B_A^{\mathrm{c}}(p-\epsilon)\big) \;+\; (p+C\epsilon^3)\bar\zeta\big(\sqrt{n}B_A(p-\epsilon)\big)+o(1)\,,
\end{aligned} \tag{2.92}$$

where the last inequality comes from the definition of $B_A(p-\epsilon)$ and the bound

$$\begin{aligned}
\sup_{x\in\zeta}\nu\big(A-x/\sqrt{n}\big) &\le \sup_{|x|\le w_\epsilon(n)}\nu\big(A-x/\sqrt{n}\big)=\\
&= \sup_{|x|\le \tilde I_A(p)+\epsilon^3}\nu\big(A-x\big)\le p+\epsilon^3
\end{aligned} \tag{2.93}$$

which is valid in light of the definition of $\tilde I_A(p)$ and Lemmas 2.1 and 2.2.

Bounding the last expression in (2.92) above by $p-\epsilon\bar\zeta\big(\sqrt{n}B_A^{\mathrm{c}}(p-\epsilon)\big)+\epsilon^3+o(1)$ and using the fact that ζ is a realization of $Z_{w_\epsilon(n)}$ on (2.90) we infer that (2.91) can be made to hold with $\Delta_\epsilon = \frac{1}{2}\epsilon^2$ for ϵ small enough and any large n, as desired.

Turning to the case $I_A(p)=\infty$, since in this case $w_\epsilon(n)$ is chosen such that $|Z_{w_\epsilon(n)}| > e^{(J_A(p)+\epsilon^3)n}$, here also the desired result will follow once we show that (2.91) holds on

$$\Big\{\bar Z_{w_\epsilon(n)}\Big(\sqrt{n(1-\tilde J_A(p)}\,B^{\mathrm{c}}_{A/\sqrt{1-\tilde J_A(p)}}(p-\epsilon)\Big)>\epsilon\Big\} \tag{2.94}$$

Proceeding as before, we take ζ which is a possible realization of $Z_{w_\epsilon(n)}$ on the event (2.94) and use Lemmas 2.1 and 2.2 to write

$$\begin{aligned}
\frac{1}{|\zeta|}\sum_{x\in\zeta}\nu_{n-w_\epsilon(n)}\big(\sqrt{n}A-x\big) \;&\le\; \frac{1}{|\zeta|}\sum_{x\in\zeta}\nu\Big(\tfrac{A}{\sqrt{1-\tilde J_A(p)}}-\tfrac{x}{\sqrt{n(1-\tilde J_A(p))}}\Big)+\epsilon^3+o(1)\\
&\le (p-\epsilon)\bar\zeta\Big(\sqrt{n(1-\tilde J_A(p)}\,B^{\mathrm{c}}_{A/\sqrt{1-\tilde J_A(p)}}(p-\epsilon)\Big)\\
&\quad + p\bar\zeta\Big(\sqrt{n(1-\tilde J_A(p)}\,B_{A/\sqrt{1-\tilde J_A(p)}}(p-\epsilon)\Big)+\epsilon^3+o(1)\,,
\end{aligned} \tag{2.95}$$

where the last inequality follows by the definition of the set $B_A(p)$ and from

$$\sup_{x\in\zeta}\nu\Big(\tfrac{A}{\sqrt{1-\tilde J_A(p)}}-\tfrac{x}{\sqrt{n(1-\tilde J_A(p))}}\Big)=p\,, \tag{2.96}$$

which holds in light of the definition of $\tilde J_A(p)$ and Lemma 2.1.

Bounding the last expression in (2.95) above by

$$p-\epsilon\bar\zeta\Big(\sqrt{n(1-\tilde J_A(p)}\,B^{\mathrm{c}}_{A/\sqrt{1-\tilde J_A(p)}}(p-\epsilon)\Big)+\epsilon^3+o(1)\,, \tag{2.97}$$

we may again choose $\Delta_\epsilon=\frac{1}{2}\epsilon^2$ for (2.91) to hold on the event (2.94) for small $\epsilon>0$ and all n large enough, as desired. □

References

[1] L. Addario-Berry and B. Reed. Minima in branching random walks. *Ann. Probab.*, 37(3):1044–1079, 2009. MR-2537549

[2] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching brownian motion. *Probability Theory and Related Fields*, 157(3-4):535–574, 2013. MR-3129797

[3] S. Asmussen and H. Hering. *Branching processes.* Springer, 1983. MR-0701538

[4] K. Athreya. Large deviation rates for branching processes–i. single type case. *The Annals of Applied Probability*, 4(3):779–790, 1994. MR-1284985

[5] K. B. Athreya and H.-J. Kang. Some limit theorems for positive recurrent branching Markov chains. I, II. *Adv. in Appl. Probab.*, 30(3):693–710, 711–722, 1998. MR-1663545

[6] R. Bhattacharya. On errors of normal approximation. *The Annals of Probability*, pages 815–828, 1975. MR-0467879

[7] J. Biggins. Chernoff's theorem in the branching random walk. *Journal of Applied Probability*, pages 630–636, 1977. MR-0464415

[8] J. D. Biggins and N. H. Bingham. Large deviations in the supercritical branching process. *Adv. in Appl. Probab.*, 25(4):757–772, 1993. MR-1241927

[9] P. Billingsley and F. Topsoe. Uniformity in weak convergence. *Probability Theory and Related Fields*, 7:1–16, 1967. 10.1007/BF00532093. MR-0209428

[10] M. Bramson. Maximal displacement of branching brownian motion. *Communications on Pure and Applied Mathematics*, 31(5):531–581, 1978. MR-0494541

[11] M. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.*, 44(285):iv+190, 1983. MR-0705746

[12] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 65(1):1–20, 2012. MR-2846636

[13] X. Chen. Exact convergence rates for the distribution of particles in branching random walks. *Annals of Applied Probability*, pages 1242–1262, 2001. MR-1878297

[14] K. Fleischmann and V. Wachtel. Lower deviation probabilities for supercritical Galton–Watson processes. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, 43(2):233–255, 2007. MR-2303121

[15] T. E. Harris. *The theory of branching processes.* Die Grundlehren der Mathematischen Wissenschaften, Bd. 119. Springer-Verlag, Berlin, 1963. MR-0163361

[16] C. C. Heyde. Some central limit analogues for supercritical Galton-Watson processes. *J. Appl. Probability*, 8:52–59, 1971. MR-0282422

[17] O. D. Jones. Large deviations for supercritical multitype branching processes. *J. Appl. Probab.*, 41(3):703–720, 2004. MR-2074818

[18] N. Kaplan. A note on the branching random walk. *Journal of Applied Probability*, 19(2):pp. 421–424, 1982. MR-0649980

[19] H. Kesten and B. P. Stigum. A limit theorem for multidimensional Galton-Watson processes. *Ann. Math. Statist.*, 37:1211–1223, 1966. MR-0198552

[20] T. M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999. MR-1717346

[21] P. Mörters and N. Sidorova. A class of weakly self-avoiding walks. *Journal of Statistical Physics*, 133(2):255–269, 2008. MR-2448533

[22] P. Ney. Branching random walk. In *Spatial stochastic processes*, volume 19 of *Progr. Probab.*, pages 3–22. Birkhäuser Boston, Boston, MA, 1991. MR-1144089

[23] P. E. Ney and A. N. Vidyashankar. Local limit theory and large deviations for supercritical branching processes. *Ann. Appl. Probab.*, 14(3):1135–1166, 2004. MR-2071418

[24] V. V. Petrov. *Sums of independent random variables.* Berlin, 1975. MR-0388499

[25] P. Révész. *Random walks of infinitely many particles.* World Scientific Publishing Co. Inc., River Edge, NJ, 1994. MR-1645302

[26] A. Stam. On a conjecture by Harris. *Probability Theory and Related Fields*, 5(3):202–206, 1966.

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