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Quenched large deviations for multiscale diffusion processes in random environments*

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Abstract

We consider multiple time scales systems of stochastic differential equations with small noise in random environments. We prove a quenched large deviations principle with explicit characterization of the action functional. The random medium is assumed to be stationary and ergodic. In the course of the proof we also prove related quenched ergodic theorems for controlled diffusion processes in random environments that are of independent interest. The proof relies entirely on probabilistic arguments, allowing to obtain detailed information on how the rare event occurs. We derive a control, equivalently a change of measure, that leads to the large deviations lower bound. This information on the change of measure can motivate the design of asymptotically efficient Monte Carlo importance sampling schemes for multiscale systems in random environments.

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1 Introduction

Let $0 < \varepsilon, \delta \ll 1$ and consider the process $(X^\epsilon, Y^\epsilon) = \{(X^\epsilon_t, Y^\epsilon_t), t \in [0,T]\}$ taking values in the space $\mathbb{R}^m \times \mathbb{R}^{d-m}$ that satisfies the system of stochastic differential equation (SDE's)

$$\begin{aligned} dX^\epsilon_t &= \left[\frac{\epsilon}{\delta} b\left(Y^\epsilon_t, \gamma\right) + c\left(X^\epsilon_t, Y^\epsilon_t, \gamma\right)\right] dt + \sqrt{\epsilon}\sigma\left(X^\epsilon_t, Y^\epsilon_t, \gamma\right) dW_t, \\ dY^\epsilon_t &= \frac{1}{\delta}\left[\frac{\epsilon}{\delta} f\left(Y^\epsilon_t, \gamma\right) + g\left(X^\epsilon_t, Y^\epsilon_t, \gamma\right)\right] dt + \frac{\sqrt{\epsilon}}{\delta}\left[\tau_1\left(Y^\epsilon_t, \gamma\right) dW_t + \tau_2\left(Y^\epsilon_t, \gamma\right) dB_t\right], \\ X^\epsilon_0 &= x_0, \quad Y^\epsilon_0 = y_0 \end{aligned} \tag{1.1}$$

where $\delta = \delta(\epsilon) \downarrow 0$ such that $\epsilon/\delta \uparrow \infty$ as $\epsilon \downarrow 0$. Here, (W_t, B_t) is a $2\kappa-$dimensional standard Wiener process. We assume that for each fixed $x \in \mathbb{R}^m$, $b(\cdot,\gamma), c(x,\cdot,\gamma), \sigma(x,\cdot,\gamma), f(\cdot,\gamma)$, $g(x,\cdot,\gamma), \tau_1(\cdot,\gamma)$ and $\tau_2(\cdot,\gamma)$ are stationary and ergodic random fields. We denote by $\gamma \in \Gamma$ the element of the related probability space. If we want to emphasize the dependence on the initial point and on the random medium, we shall write $\left(X^{\epsilon,(x_0,y_0),\gamma}, Y^{\epsilon,(x_0,y_0),\gamma}\right)$ for the solution to (1.1).

The system (1.1) can be interpreted as a small-noise perturbation of dynamical systems with multiple scales. The slow component is X and the fast component is Y. We study the regime where

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the homogenization parameter goes faster to zero than the strength of the noise does. The goal of this paper is to obtain the quenched large deviations principle associated to the component X, that is associated with the slow motion. The case of large deviations for such systems in periodic media for all possible interactions between ϵ and δ, i.e., $\epsilon/\delta \to 0, c \in (0,\infty)$ or ∞, was studied in [33], see also [1, 6, 11]. In [33] (see also [7]), it was assumed that the coefficients are periodic with respect to the $y-$variable and based on the derived large deviations principle, asymptotically efficient importance sampling Monte Carlo methods for estimating rare event probabilities were obtained. In the current paper, we focus on quenched (i.e. almost sure with respect to the random environment) large deviations for the case $\epsilon/\delta \uparrow \infty$ and the situation is more complex when compared to the periodic case since the coefficients are now random fields themselves and the fast motion does not take values in a compact space.

We treat the large deviations problem via the lens of the weak convergence framework, [5], using entirely probabilistic arguments. This framework transforms the large deviations problem to convergence of a stochastic control problem. The current work is certainly related to the literature in random homogenization, see [15, 16, 17, 18, 20, 22, 23, 24, 25, 26, 27, 29]. Our work is most closely related to [16, 20], where stochastic homogenization for Hamilton-Jacobi-Bellman (HJB) equations was studied. The authors in [16, 20] consider the case $\delta = \epsilon$ with the fast motion being $Y = X/\delta$ and with the coefficients $b = f = 0$ in a general Hamiltonian setting. In both papers the authors briefly discuss large deviations for diffusions (i.e., when the Hamiltonian is quadratic) and the action functional is given as the Legendre-Fenchel transform of the effective Hamiltonian and the case studied there is $\delta = \epsilon$. Moreover, in [19, 36] the large deviations principle for systems like (1.1) is considered in the case $\epsilon = \delta$ with the coefficients $b = f = 0$. In [19, 36] the coefficients are deterministic (i.e., not random fields as in our case) and stability type conditions for the fast process Y are assumed in order to guarantee ergodicity. Lastly, related annealed homogenization results (i.e. on average and not almost sure with respect to the medium) for uncontrolled multiscale diffusions as in (1.1) in the case $\epsilon = 1$, $\delta \downarrow 0$ and $Y = X/\delta$ have been recently obtained in [29]. Under different assumptions on the structure of the coefficients, the opposite case to ours where $\epsilon/\delta \downarrow 0$ has been partially considered in [6, 11, 32, 33].

In contrast to most of the aforementioned literature, in this paper, we study the case $\epsilon/\delta \uparrow \infty$ and we use entirely probabilistic arguments. Because $\epsilon/\delta \uparrow \infty$, we also need to consider the additional effect of the macroscopic problem (i.e., what is called cell problem in the periodic homogenization literature) due to the highly oscillating term $\frac{\epsilon}{\delta}\int_0^T b\left(Y_t^\epsilon, \gamma\right) dt$. We use entirely probabilistic arguments and because the homogenization parameter goes faster to zero that the strength of the noise does, we are able to derive an explicit characterization of the quenched large deviations principle and detailed information on the change of measure leading to its proof, Theorem 3.5. Due to the presence of the highly oscillatory term $\frac{\epsilon}{\delta}\int_0^T b\left(Y_t^\epsilon, \gamma\right) dt$, the change of measure in question depends on the macroscopic problem and we determine this dependence explicitly. Additionally, in the course of the proof, we obtain quenched (i.e., almost sure with respect to the random environment) ergodic theorems for uncontrolled and controlled random diffusion processes that may be of independent interest, Theorem 3.3 and Appendix A. It is of interest to note that for the purposes of proving the Laplace principle, which is equivalent to the large deviations principle, one can constrain the variational problem associated with the stochastic control representation of exponential functionals to a class of L^2 controls with specific dependence on δ, ϵ, Lemma 5.1.

Partial motivation for this work comes from chemical physics, molecular dynamics and climate modeling, e.g., [35, 8, 30, 37], where one is often interested in simplified models that preserve the large deviation properties of the system in the case where $\delta \ll \epsilon$, i.e., in the case where δ is orders of magnitude smaller than ϵ. Other related models where the regime of interest is $\epsilon/\delta \uparrow \infty$ have been considered in [1, 6, 7, 10, 11, 13, 33]. When rare events are of interest, then large deviations theory comes into play. As mentioned before, we are able to derive an explicit characterization of the quenched large deviations principle, Theorem 3.5. The explicit form of the derived large deviations action functional and of the control achieving the large deviations bound give useful information which can be used to design provably efficient importance sampling schemes for estimation of related rare event probabilities. In the case of a periodic fast motion, the design of large deviations inspired efficient Monte Carlo importance sampling schemes was investigated in [7, 8, 33]. The paper [7] also includes importance sampling numerical simulations in the case of diffusion moving in a random multiscale environment in dimension one. In the present paper, we focus on rigorously developing the large deviations theory and the design of asymptotically efficient importance sampling schemes in random environments is addressed in [34].

The rest of the paper is organized as follows. In Section 2 we set-up notation, state our assumptions and review known results from the literature on random homogenization that will be useful for our purposes. In Section 3 we state our main results. Sections 4, 5 and 6 contain the proofs of the main results of the paper, i.e., quenched homogenization results for pairs of controlled diffusions and occupation measures in random environments and the large deviations principle with the explicit characterization of the action functional. The Appendix A contains the proofs of the necessary quenched ergodic theorems for controlled diffusion processes in random environments.

2 Assumptions, notation and review of useful known results

In this section we setup notation and pose the main assumptions of the paper. In this section, and for the convenience of the reader, we also review well known results from the literature on random homogenization that will be useful for our purposes. The content of this section is classical.

We start by describing the properties of the random medium. Let $(\Gamma,\mathcal{G},\nu)$ be the probability space of the random medium and as in [14], a group of measure-preserving transformations $\{\tau_y, y\in\mathbb{R}^d\}$ acting ergodically on Γ.

Definition 2.1. *We assume that the following hold.*

i. *τ_y preserves the measure, namely $\forall y\in\mathbb{R}^{d-m}$ and $\forall A\in\mathcal{G}$ we have $\nu(\tau_y A)=\nu(A)$.*

ii. *The action of $\{\tau_y : y\in\mathbb{R}^{d-m}\}$ is ergodic, that is if $A=\tau_y A$ for every $y\in\mathbb{R}^d$ then $\nu(A)=0$ or 1.*

iii. *For every measurable function f on $(\Gamma,\mathcal{G},\nu)$, the function $(y,\gamma)\mapsto f(\tau_y\gamma)$ is measurable on $\big(\mathbb{R}^{d-m}\times\Gamma,\mathbb{B}(\mathbb{R}^{d-m})\otimes\mathcal{G}\big)$.*

For $\tilde\phi\in L^2(\Gamma)$ (i.e., a square integrable function in Γ), we define the operator $T_y\tilde\phi(\gamma)=\tilde\phi(\tau_y\gamma)$. It is known, e.g. [22], that T_y forms a strongly continuous group of unitary maps in $L^2(\Gamma)$. Moreover, if the limit exists, the infinitesimal generator D_i of T_y in the direction i is defined by

$$D_i\tilde\phi=\lim_{h\downarrow 0}\frac{T_{he_i}\tilde\phi-\tilde\phi}{h}. \tag{2.1}$$

and is a closed and densely defined generator.

Next, for $\tilde\phi\in L^2(\Gamma)$, we define $\phi(y,\gamma)=\tilde\phi(\tau_y\gamma)$. This definition guarantees that ϕ will be a stationary and ergodic random field on $\mathbb{R}^{d-m}$. Similarly, for a measurable function $\tilde\phi:\mathbb{R}^m\times\Gamma\mapsto\mathbb{R}^m$ we consider the (locally) stationary random field $(x,y)\mapsto\tilde\phi(x,\tau_y\gamma)=\phi(x,y,\gamma)$.

We follow this procedure to define the random fields $b,c,\sigma,f,g,\tau_1,\tau_2$ that play the role of the coefficients of (1.1), which then guarantees that they are ergodic and stationary random fields. In particular, we start with $L^2(\Gamma)$ functions $\tilde b(\gamma),\tilde c(x,\gamma),\tilde\sigma(x,\gamma),\tilde f(\gamma),\tilde g(x,\gamma)$, $\tilde\tau_1(\gamma),\tilde\tau_2(\gamma)$ and we define the coefficients of (1.1) via the relations $b(y,\gamma)=\tilde b(\tau_y\gamma), c(x,y,\gamma)=\tilde c(x,\tau_y\gamma),\sigma(x,y,\gamma)=\tilde\sigma(x,\tau_y\gamma), f(y,\gamma)=\tilde f(\tau_y\gamma), g(x,y,\gamma)=\tilde g(x,\tau_y\gamma),\tau_1(y,\gamma)=\tilde\tau_1(\tau_y\gamma)$ and $\tau_2(y,\gamma)=\tilde\tau_2(\tau_y\gamma)$.

The main assumption for the coefficients of (1.1) is as follows.

Condition 2.2. *i. The functions $b(y,\gamma),c(x,y,\gamma),\sigma(x,y,\gamma),f(y,\gamma),g(x,y,\gamma),\tau_1(y,\gamma)$ and $\tau_2(y,\gamma)$ are $C^1(\mathbb{R}^{d-m})$ in y and $C^1(\mathbb{R}^m)$ in x with all partial derivatives continuous and globally bounded in x and y.*

ii. For every fixed $\gamma\in\Gamma$, the diffusion matrices $\sigma\sigma^T$ and $\tau_1\tau_1^T+\tau_2\tau_2^T$ are uniformly nondegenerate.

It is known that under Condition 2.2, there exists a filtered probability space $(\Omega,\mathcal{F},\mathfrak{F}_t,\mathbb{P})$ such that for every given initial point $(x_0,y_0)\in\mathbb{R}^m\times\mathbb{R}^{d-m}$, for every $\gamma\in\Gamma$ and for every $\epsilon,\delta>0$ there exists a strong Markov process $(X^\epsilon_t,Y^\epsilon_t,t\geq 0)$ satisfying (1.1). However, if we define a probability measure $\mathcal{P}=\nu\otimes\mathbb{P}$ on the product space $\Gamma\times\Omega$, then when considered on the probability space $(\Gamma\times\Omega,\mathcal{G}\otimes\mathcal{F},\mathcal{P})$, $\{(X^\epsilon_t,Y^\epsilon_t),t\geq 0\}$ is not a Markov process.

From the previous discussion it is easy to see that the periodic case is a special case of the previous setup. Indeed, we can consider the periodic case with period 1, Γ to be the unit torus and ν to be Lebesgue measure on Γ. For every $\gamma \in \Gamma$, the shift operators $\tau_y\gamma = (y+\gamma) \mod 1$ and we have $\phi(y,\gamma) = \tilde{\phi}(y+\gamma)$ for a periodic function $\tilde{\phi}$ with period 1.

For every $\gamma \in \Gamma$, we define next the operator

$$\mathcal{L}^{\gamma} = f(y,\gamma)\nabla_y \cdot + tr \left[\left(\tau_1(y,\gamma)\tau_1^T(y,\gamma) + \tau_2(y,\gamma)\tau_2^T(y,\gamma) \right) \nabla_y \nabla_y \cdot \right]$$

and we let Y_t^{γ} to be the corresponding Markov process. It follows from [26, 24, 22], that we can associate the canonical process on Γ defined by the environment γ, which is a Markov process on Γ with continuous transition probability densities with respect to d-dimensional Lebesgue measure, e.g., [22]. In particular, we let

$$\begin{aligned} \gamma_t &= \tau_{Y_t^{\gamma}}\gamma \\ \gamma_0 &= \tau_{y_0}\gamma \end{aligned} \tag{2.2}$$

Definition 2.3. *We denote the infinitesimal generator of the Markov process γ_t by*

$$\tilde{L} = \tilde{f}(\gamma)D \cdot + \mathrm{tr} \left[\left(\tilde{\tau}_1(\gamma)\tilde{\tau}_1^T(\gamma) + \tilde{\tau}_2(\gamma)\tilde{\tau}_2^T(\gamma) \right) D^2 \cdot \right],$$

where D was defined in (2.1).

Following [24], we assume the following condition on the structure of the operator defined in Definition 2.3. This condition allows to have a closed form for the unique ergodic invariant measure for the environment process $\{\gamma_t\}_{t\geq 0}$, Proposition 2.6.

Condition 2.4. *We can write the operator $\tilde{L}$ in the following generalized divergence form*

$$\tilde{L} = \frac{1}{\tilde{m}(\gamma)} \left[\sum_{i,j} D_i \left(\tilde{a}\tilde{a}_{i,j}^T(\gamma) D_j \cdot \right) + \sum_j \tilde{\beta}_j(\gamma) D_j \cdot \right]$$

where $\tilde{\beta}_j = \tilde{m}\tilde{f}_j - \sum_i D_i \left(\left(\tilde{\tau}_1\tilde{\tau}_1^T + \tilde{\tau}_2\tilde{\tau}_2^T \right)_{i,j} \tilde{m} \right)$ and $\tilde{a}\tilde{a}_{i,j}^T = \left(\tilde{\tau}_1\tilde{\tau}_1^T + \tilde{\tau}_2\tilde{\tau}_2^T \right)_{i,j} \tilde{m}$. We assume that $\tilde{m}(\gamma)$ is bounded from below and from above with probability 1, that there exist smooth $\tilde{d}_{i,j}(\gamma)$ such that $\tilde{\beta}_j = \sum_j D_j \tilde{d}_{i,j}$ with $|\tilde{d}_{i,j}| \leq M$ for some $M < \infty$ almost surely and

$$\textit{div } \tilde{\beta} = 0 \textit{ in distribution}, \quad \textit{i.e.,} \int_{\Gamma} \sum_{j=1}^{d} \tilde{\beta}_j(\gamma) D_j \tilde{\phi}(\gamma)\nu(d\gamma) = 0, \quad \forall \tilde{\phi} \in \mathcal{H}^1,$$

where the Sobolev space $\mathcal{H}^1 = \mathcal{H}^1(\nu)$ is the Hilbert space equipped with the inner product

$$(\tilde{f}, \tilde{g})_1 = \sum_{i=1}^{d} (D_i\tilde{f}, D_i\tilde{g}).$$

Example 2.5. A trivial example that satisfies Condition 2.4 is the gradient case. Let $\tilde{f}(\gamma) = -D\tilde{Q}(\gamma)$ and $\tilde{\tau}_1(\gamma) = \sqrt{2D} =$ constant and $\tilde{\tau}_2(\gamma) = 0$. Then, we have that $\tilde{m}(\gamma) = \exp[-\tilde{Q}(\gamma)/D]$ and $\tilde{\beta}_j = 0$ for all $1 \leq j \leq d$. Moreover, if $\tilde{m} = 1$ and $\tilde{d}_{i,j}$ are constants then the operator is of divergence form.

Next, we recall some classical results from random homogenization.

Proposition 2.6 ([24] and Theorem 2.1 in [22])**.** *Assume Conditions 2.2 and 2.4. Define a measure on $(\Gamma, \mathcal{G})$ by*

$$\pi(d\gamma) \doteq \frac{\tilde{m}(\gamma)}{\mathrm{E}^{\nu}\tilde{m}(\cdot)}\nu(d\gamma).$$

Then π is the unique ergodic invariant measure for the environment process $\{\gamma_t\}_{t\geq 0}$.

We will denote by E^{ν} and by E^{π} the expectation operator with respect to the measures ν and π respectively. We remark here that since $\tilde{m}$ is bounded from above and from below, $\mathcal{H}^1(\nu)$ and $\mathcal{H}^1(\pi)$ are equivalent. We also need to introduce the macroscopic problem, known as cell problem in the periodic homogenization literature or corrector in the homogenization literature in general. This is needed in order to address the situation $\tilde{b} \neq 0$. For every $\rho > 0$, we consider the solution to the auxiliary problem on Γ.

$$\rho\tilde{\chi}_{\rho} - \tilde{L}\tilde{\chi}_{\rho} = \tilde{b}. \tag{2.3}$$

Let us review some well known facts related to the solution to this auxiliary problem, e.g., see [22, 15]. By Lax-Milgram lemma, equation (2.3) has a unique weak solution in the abstract Sobolev space $\mathcal{H}^1$. Moreover, letting $\mathcal{R}_{\rho}\tilde{h}(\gamma) = \int_0^{\infty} e^{-\rho t}\mathrm{E}_{\gamma}\tilde{h}(\gamma_t)dt$, for every $\tilde{h} \in L^2(\Gamma)$, we have

$$\tilde{\chi}_{\rho}(\cdot) = \mathcal{R}_{\rho}\tilde{b}(\cdot),$$

As in [24, 26], there is a constant K that is independent of ρ such that

$$\rho\mathrm{E}^{\pi}\left[\tilde{\chi}_{\rho}(\cdot)\right]^2 + \mathrm{E}^{\pi}\left[D\tilde{\chi}_{\rho}(\cdot)\right]^2 \leq K$$

By Proposition 2.6 in [22] we then get that $\tilde{\chi}_{\rho}$ has an $\mathcal{H}^1$ strong limit, i.e., there exists a $\tilde{\chi}_0 \in \mathcal{H}^1(\pi)$ such that

$$\lim_{\rho\downarrow 0}\left\|\tilde{\chi}_{\rho}(\cdot) - \tilde{\chi}_0(\cdot)\right\|_1 = 0$$

and that

$$\lim_{\rho\downarrow 0}\rho\mathrm{E}^{\pi}\left[\tilde{\chi}_{\rho}(\cdot)\right]^2 = 0.$$

This implies that $D\tilde{\chi}_{\rho} \in L^2(\pi)$ and that it has a $L^2(\pi)$ strong limit, i.e., there exists a $\tilde{\xi} \in L^2(\pi)$ such that

$$\lim_{\rho\downarrow 0}\left\|D\tilde{\chi}_{\rho} - \tilde{\xi}\right\|_{L^2}^2 = 0$$

In addition, since $\tilde{b}$ is bounded under Condition 2.2, $\tilde{\chi}_{\rho}$ is also bounded. This follows because the resolvent operator $\mathcal{R}_{\rho}$ corresponding to the operator $\rho I - \mathcal{L}$ is associated to a $L^{\infty}(\Gamma)$ contraction semigroup, see Section 2.2 of [22].

Moreover, as in Proposition 3.2. of [24], we have that for almost all $\gamma \in \Gamma$

$$\delta\chi_0(y/\delta, \gamma) \to 0, \text{ as } \delta \downarrow 0, \text{ a.s. } y \in \mathcal{Y}.$$

3 Main results

In this section we present the statement of the main results of the paper. In preparation for stating the large deviations theorem, we first recall the concept of a Laplace principle.

Definition 3.1. *Let $\{X^{\epsilon}, \epsilon > 0\}$ be a family of random variables taking values in a Polish space $\mathcal{S}$ and let I be a rate function on $\mathcal{S}$. We say that $\{X^{\epsilon}, \epsilon > 0\}$ satisfies the Laplace principle with rate function I if for every bounded and continuous function $h : \mathcal{S} \to \mathbb{R}$*

$$\lim_{\epsilon\downarrow 0} -\epsilon \ln \mathbb{E}\left[\exp\left\{-\frac{h(X^{\epsilon})}{\epsilon}\right\}\right] = \inf_{x\in\mathcal{S}}\left[I(x) + h(x)\right]. \tag{3.1}$$

If the level sets of the rate function (equivalently action functional) are compact, then the Laplace principle is equivalent to the corresponding large deviations principle with the same rate function (Theorems 2.2.1 and 2.2.3 in [5]).

In order to establish the quenched Laplace principle, we make use of the representation theorem for functionals of the form $\mathbb{E}\left[e^{-\frac{1}{\epsilon}h(X^{\epsilon,\gamma})}\right]$ in terms of a stochastic control problem. Such representations were first derived in [4].

Let $\mathcal{A}$ be the set of all $\mathfrak{F}_s-$progressively measurable n-dimensional processes $u \doteq \{u(s), 0 \leq s \leq T\}$ satisfying

$$\mathbb{E}\int_0^T \|u(s)\|^2 ds < \infty,$$

In the present case, let $Z(\cdot) = (W(\cdot), B(\cdot))$ and $n = 2k$. Then, for the given $\gamma \in \Gamma$ we have the representation

$$-\epsilon \ln \mathbb{E}_{x_0,y_0}\left[\exp\left\{-\frac{h(X^\epsilon)}{\epsilon}\right\}\right] = \inf_{u\in\mathcal{A}} \mathbb{E}_{x_0,y_0}\left[\frac{1}{2}\int_0^T \left[\|u_1(s)\|^2 + \|u_2(s)\|^2\right] ds + h(\bar{X}^\epsilon)\right] \tag{3.2}$$

where the pair $(\bar{X}^\epsilon, \bar{Y}^\epsilon)$ is the unique strong solution to

$$\begin{aligned}
d\bar{X}_t^\epsilon &= \left[\frac{\epsilon}{\delta} b\left(\bar{Y}_t^\epsilon, \gamma\right) + c\left(\bar{X}_t^\epsilon, \bar{Y}_t^\epsilon, \gamma\right) + \sigma\left(\bar{X}_t^\epsilon, \bar{Y}_t^\epsilon, \gamma\right) u_1(t)\right] dt + \sqrt{\epsilon}\sigma\left(\bar{X}_t^\epsilon, \bar{Y}_t^\epsilon, \gamma\right) dW_t, \\
d\bar{Y}_t^\epsilon &= \frac{1}{\delta}\left[\frac{\epsilon}{\delta} f\left(\bar{Y}_t^\epsilon, \gamma\right) + g\left(\bar{X}_t^\epsilon, \bar{Y}_t^\epsilon, \gamma\right) + \tau_1\left(\bar{Y}_t^\epsilon, \gamma\right) u_1(t) + \tau_2\left(\bar{Y}_t^\epsilon, \gamma\right) u_2(t)\right] dt \\
&\quad + \frac{\sqrt{\epsilon}}{\delta}\left[\tau_1\left(\bar{Y}_t^\epsilon, \gamma\right) dW_t + \tau_2\left(\bar{Y}_t^\epsilon, \gamma\right) dB_t\right], \qquad (3.3) \\
\bar{X}_0^\epsilon &= x_0, \;\; \bar{Y}_0^\epsilon = y_0
\end{aligned}$$

This representation implies that in order to derive the Laplace principle for $\{X^\epsilon\}$, it is enough to study the limit of the right hand side of the variational representation (3.2). The first step in doing so is to consider the weak limit of the slow motion $\bar{X}^\epsilon$ of the controlled couple (3.3).

Fix $\gamma \in \Gamma$ and let us define for notational convenience $\mathcal{Z} = \mathbb{R}^\kappa$ and $\mathcal{Y} = \mathbb{R}^{d-m}$. Due to the involved controls, it is convenient to introduce the following occupation measure. Let $\Delta = \Delta(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$ that will be chosen later on and is used to exploit a time-scale separation. Let A_1, A_2, B, Θ be Borel sets of $\mathcal{Z}, \mathcal{Z}, \Gamma, [0,T]$ respectively. Let $u_i^\epsilon \in A_i, i = 1, 2$ and let $(\bar{X}^\epsilon, \bar{Y}^\epsilon)$ solve (3.3) with u_i^ϵ in place of u_i. We associate with $(\bar{X}^\epsilon, \bar{Y}^\epsilon)$ and u_i^ϵ a family of occupation measures $\mathrm{P}^{\epsilon,\Delta,\gamma}$ defined by

$$\mathrm{P}^{\epsilon,\Delta,\gamma}(A_1 \times A_2 \times B \times \Theta) = \int_\Theta \left[\frac{1}{\Delta}\int_t^{t+\Delta} 1_{A_1}(u_1^\epsilon(s)) 1_{A_2}(u_2^\epsilon(s)) 1_B\left(\tau_{\bar{Y}_s^\epsilon}\gamma\right) ds\right] dt,$$

assuming that $u_i^\epsilon(t) = 0$ for $i = 1, 2$ if $t > T$. Next, we introduce the notion of a viable pair, see also [6]. Such a notion will allow us to characterize the limiting behavior of the pair $\left(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma}\right)$.

Definition 3.2. *Define the function in $L^2(\Gamma)$*

$$\tilde{\lambda}(x, \gamma, z_1, z_2) = \tilde{c}(x, \gamma) + \tilde{\xi}(\gamma)\tilde{g}(x, \gamma) + \tilde{\sigma}(x, \gamma) z_1 + \tilde{\xi}(\gamma)\left(\tilde{\tau}_1(\gamma) z_1 + \tilde{\tau}_2(\gamma) z_2\right)$$

where $\tilde{\xi}$ is the L^2 limit of $D\tilde{\chi}_\rho$ as $\rho \downarrow 0$ that is defined in Section 2. Consider the operator $\tilde{L}$ defined in Definition 2.3. We say that a pair $(\psi, \mathrm{P}) \in \mathcal{C}([0,T]; \mathbb{R}^m) \times \mathcal{P}(\mathcal{Z} \times \mathcal{Z} \times \Gamma \times [0,T])$ is viable with respect to $(\tilde{\lambda}, \tilde{L})$ and we write $(\psi, \mathrm{P}) \in \mathcal{V}$, if the following hold.

- *The function ψ is absolutely continuous and P is square integrable in the sense that*

$$\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma\times[0,T]} |z|^2 \mathrm{P}(dz_1 dz_2 d\gamma dt) < \infty.$$

- *For all $t \in [0,T]$, $\mathrm{P}(\mathcal{Z} \times \mathcal{Z} \times \Gamma \times [0,t]) = t$. Thus, P can be decomposed as $\mathrm{P}(dz_1 dz_2 d\gamma dt) = \mathrm{P}_t(dz_1 dz_2 d\gamma) dt$ such that $\mathrm{P}_t(\mathcal{Z} \times \mathcal{Z} \times \Gamma) = 1$.*
- *For all $t \in [0,T]$, (ψ, P) satisfy the ODE*

$$\psi_t = x_0 + \int_0^t \left[\iint_{\mathcal{Z}\times\mathcal{Z}\times\Gamma} \tilde{\lambda}(\psi_s, \gamma, z_1, z_2) \mathrm{P}_s(dz_1 dz_2 d\gamma)\right] ds. \tag{3.4}$$

  and for a given P, there is a unique well defined ψ satisfying (3.4).
- *For a.e. $t \in [0,T]$,*

$$\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma} \tilde{L}\tilde{f}(\gamma) \mathrm{P}_t(dz_1 dz_2 d\gamma) = 0 \tag{3.5}$$

  for all $\tilde{f} \in \mathcal{D}(\tilde{L})$.

For notational convenience later on, let us also define

$$\tilde{\lambda}_{\rho}(x,\gamma,z_1,z_2)=\tilde{c}(x,\gamma)+D\tilde{\chi}_{\rho}(\gamma)\tilde{g}(x,\gamma)+\tilde{\sigma}(x,\gamma)z_1+D\tilde{\chi}_{\rho}(\gamma)\left(\tilde{\tau}_1(\gamma)z_1+\tilde{\tau}_2(\gamma)z_2\right)$$

Now, that we have defined the notion of a viable pair we are ready to present the law of large numbers results for controlled pairs $\left(\bar{X}^{\epsilon,\gamma},\mathrm{P}^{\epsilon,\Delta,\gamma}\right)$.

Theorem 3.3. *Assume Conditions 2.2 and 2.4. Fix the initial point $(x_0,y_0)\in\mathbb{R}^m\times\mathcal{Y}$ and consider a family $\{u^\epsilon=(u_1^\epsilon,u_2^\epsilon),\epsilon>0\}$ of controls (that may depend on γ) in $\mathcal{A}$ satisfying a.s. with respect to $\gamma\in\Gamma$, the bound A.11 and*

$$\sup_{\epsilon>0}\mathbb{E}\int_0^T\left[\left\|u_1^\epsilon(s)\right\|^2+\left\|u_2^\epsilon(s)\right\|^2\right]ds<\infty \tag{3.6}$$

Then the family $\{(\bar{X}^{\epsilon,\gamma},\mathrm{P}^{\epsilon,\Delta,\gamma}),\epsilon>0\}$ is tight almost surely with respect to $\gamma\in\Gamma$. Given any subsequence of $\{(\bar{X}^{\epsilon},\mathrm{P}^{\epsilon,\Delta}),\epsilon>0\}$, there exists a subsubsequence that converges in distribution with limit $(\bar{X},\mathrm{P})$ almost surely with respect to $\gamma\in\Gamma$. With probability 1, the limit point $(\bar{X},\mathrm{P})\in\mathcal{V}$, according to Definition 3.2.

Next, we are ready to state the quenched Laplace principle for $\{X^\epsilon,\epsilon>0\}$.

Theorem 3.4. *Let $\{(X^\epsilon,Y^\epsilon),\epsilon>0\}$ be, for fixed $\gamma\in\Gamma$, the unique strong solution to (1.1) and assume that $\epsilon/\delta\uparrow\infty$. We assume that Conditions 2.2 and 2.4 hold. Define*

$$S(\phi)=\inf_{(\phi,\mathrm{P})\in\mathcal{V}}\left[\frac{1}{2}\int_{\mathcal{Z}\times\mathcal{Z}\times\mathcal{Y}\times[0,T]}\left[\left\|z_1\right\|^2+\left\|z_2\right\|^2\right]\mathrm{P}(dz_1dz_2dydt)\right], \tag{3.7}$$

with the convention that the infimum over the empty set is ∞. Then, we have

i. The level sets of S are compact. In particular, for each $s<\infty$, the set

$$\Phi_s=\{\phi\in\mathcal{C}([0,T];\mathbb{R}^m):S(\phi)\leq s\}$$

is a compact subset of $\mathcal{C}([0,T];\mathbb{R}^m)$.

ii. For every bounded and continuous function h mapping $\mathcal{C}([0,T];\mathbb{R}^m)$ into $\mathbb{R}$

$$\lim_{\epsilon\downarrow0}-\epsilon\ln\mathbb{E}_{x_0,y_0}\left[\exp\left\{-\frac{h(X^{\epsilon,\gamma})}{\epsilon}\right\}\right]=\inf_{\phi\in\mathcal{C}([0,T];\mathbb{R}^m),\phi_0=x_0}\left[S(\phi)+h(\phi)\right].$$

almost surely with respect to $\gamma\in\Gamma$.

In other words, under the imposed assumptions, $\{X^{\epsilon,\gamma},\epsilon>0\}$ satisfies the quenched large deviations principle with action functional S.

Actually, it turns out that in this case we can compute the quenched action functional in closed form.

Theorem 3.5. *Let $\{(X^{\epsilon,\gamma},Y^{\epsilon,\gamma}),\epsilon>0\}$ be, for fixed $\gamma\in\Gamma$, the unique strong solution to (1.1). Under Conditions 2.2 and 2.4, $\{X^{\epsilon,\gamma},\epsilon>0\}$ satisfies, almost surely with respect to $\gamma\in\Gamma$, the large deviations principle with rate function*

$$S(\phi)=\begin{cases}\frac{1}{2}\int_0^T(\dot{\phi}(s)-r(\phi(s)))^Tq^{-1}(\phi(s))(\dot{\phi}(s)-r(\phi(s)))ds & \text{if } \phi\in\mathcal{AC}([0,T];\mathbb{R}^m) \text{ and } \phi(0)=x_0\\ +\infty & \text{otherwise.}\end{cases}$$

where

$$r(x)=\lim_{\rho\downarrow0}\mathrm{E}^{\pi}\left[\tilde{c}(x,\cdot)+D\tilde{\chi}_{\rho}(\cdot)\tilde{g}(x,\cdot)\right]=\mathrm{E}^{\pi}[\tilde{c}(x,\cdot)+\tilde{\xi}(\cdot)\tilde{g}(x,\cdot)]$$

$$\begin{aligned}q(x)&=\lim_{\rho\downarrow0}\mathrm{E}^{\pi}\left[(\tilde{\sigma}(x,\cdot)+D\tilde{\chi}_{\rho}(\cdot)\tilde{\tau}_1(\cdot))(\tilde{\sigma}(x,\cdot)+D\tilde{\chi}_{\rho}(\cdot)\tilde{\tau}_1(\cdot))^T+(D\tilde{\chi}_{\rho}(\cdot)\tilde{\tau}_2(\cdot))\left(D\tilde{\chi}_{\rho}(\cdot)\tilde{\tau}_2(\cdot)\right)^T\right]\\&=\mathrm{E}^{\pi}\left[(\tilde{\sigma}(x,\cdot)+\tilde{\xi}(\cdot)\tilde{\tau}_1(\cdot))(\tilde{\sigma}(x,\cdot)+\tilde{\xi}(\cdot)\tilde{\tau}_1(\cdot))^T+\left(\tilde{\xi}(\cdot)\tilde{\tau}_2(\cdot)\right)\left(\tilde{\xi}(\cdot)\tilde{\tau}_2(\cdot)\right)^T\right]\end{aligned}$$

Notice that the coefficients $r(x)$ and $q(x)$ that enter into the action functional are those obtained if we had first taken to (1.1) $\delta \downarrow 0$ with ϵ fixed and then consider the large deviations for the homogenized system. This is in accordance to intuition since in the case $\epsilon/\delta \uparrow \infty$, δ goes to zero faster than ϵ. This implies that homogenization should occur first as it indeed does and then large deviations start playing a role.

4 Proof of Theorem 3.3

In this section we prove Theorem 3.3. Tightness is established in Subsection 4.1, whereas the identification of the limit point is done in Subsection 4.2.

4.1 Tightness of the controlled pair $\left\{(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma}), \epsilon, \Delta > 0\right\}$.

In this section we prove that the family $\{(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma}), \epsilon > 0\}$, is almost surely tight with respect to $\gamma \in \Gamma$ where $\Delta = \Delta(\epsilon) \downarrow 0$. The following proposition takes care of tightness and uniform integrability of $\{\mathrm{P}^{\epsilon,\Delta,\gamma}, \epsilon > 0\}$.

Lemma 4.1. *Assume Conditions 2.2 and 2.4. Let $\{u^{\epsilon,\gamma}, \epsilon > 0, \gamma \in \Gamma\}$ be a family of controls in $\mathcal{A}$ such that Conditions A.10 and A.11 of Lemma A.6 hold. The following hold*

i. For every $\eta > 0$, there is a set N_η (the same N_η identified in Lemma A.6) with $\pi(N_\eta) \geq 1 - \eta$ such that for every $\gamma \in N_\eta$ and for every bounded sequence $\Delta \in \mathcal{H}_1^{N_\eta}$ (i.e. a sequence that satisfies Condition A.2), the family $\{\mathrm{P}^{\epsilon,\Delta,\gamma}, \epsilon > 0\}$ is tight as $\epsilon \downarrow 0$.

ii. The family $\{\mathrm{P}^{\epsilon,\Delta,\gamma}, \epsilon > 0\}$ is uniformly integrable, in the sense that

$$\lim_{M\to\infty} \sup_{\epsilon>0,\gamma\in\Gamma} \mathbb{E} \int_{\{(z_1,z_1)\in\mathcal{Z}^2:[\|z_1\|+\|z_2\|]\geq M\}\times\Gamma\times[0,T]} [\|z_1\| + \|z_2\|]\, \mathrm{P}^{\epsilon,\Delta,\gamma}(dz_1 dz_1 d\tilde{\gamma} dt) = 0$$

Proof. (i). Let us first prove the first part of the Lemma. It is clear that we can write

$$\mathrm{P}^{\epsilon,\Delta,\gamma}(A_1 \times A_2 \times B \times \Theta) = \int_\Theta \mathrm{P}_t^{\epsilon,\Delta,\gamma}(A_1 \times A_2 \times B)dt$$

where

$$\mathrm{P}_t^{\epsilon,\Delta,\gamma}(A_1 \times A_2 \times B) = \left[\frac{1}{\Delta}\int_t^{t+\Delta} 1_{A_1}(u_1^{\epsilon,\gamma}(s)) 1_{A_2}(u_2^{\epsilon,\gamma}(s)) 1_B\left(\tau_{\bar{Y}_s^\epsilon}\gamma\right) ds\right] dt,$$

Let us denote by $\mathrm{P}_{1,t}^{\epsilon,\Delta,\gamma}(A_1 \times A_2)$ and by $\mathrm{P}_{2,t}^{\epsilon,\Delta,\gamma}(B)$ the first and second marginals of $\mathrm{P}_t^{\epsilon,\Delta,\gamma}(A_1 \times A_2 \times B)$ respectively. Namely,

$$\mathrm{P}_{1,t}^{\epsilon,\Delta,\gamma}(A_1 \times A_2) = \mathrm{P}_t^{\epsilon,\Delta,\gamma}(A_1 \times A_2 \times \Gamma), \text{ and } \mathrm{P}_{2,t}^{\epsilon,\Delta,\gamma}(B) = \mathrm{P}_t^{\epsilon,\Delta,\gamma}(\mathcal{Z} \times \mathcal{Z} \times B)$$

It is clear that tightness of $\{\mathrm{P}^{\epsilon,\Delta,\gamma}, \epsilon > 0\}$ is a consequence of tightness of $\{\mathrm{P}_{1,t}^{\epsilon,\Delta,\gamma}, \epsilon > 0\}$ and of $\{\mathrm{P}_{2,t}^{\epsilon,\Delta,\gamma}, \epsilon > 0\}$.

Let us first consider tightness of $\{\mathrm{P}_{1,t}^{\epsilon,\Delta,\gamma}, \epsilon > 0\}$. For this purpose, we claim that the function

$$g(r) = \int_{\mathcal{Z}\times\mathcal{Z}\times[0,T]} \left[\|z_1\|^2 + \|z_2\|^2\right] r(dz_1 dz_2 dt), \quad r \in \mathcal{P}(\mathcal{Z} \times \mathcal{Z} \times [0,T])$$

is a tightness function, i.e., it is bounded from below and its level sets $R_k = \{r \in \mathcal{P}(\mathbb{R}^{2k} \times [0,T]) : g(r) \leq k\}$ are relatively compact for each $k < \infty$. Notice that the second marginal of every $r \in \mathcal{P}(\mathcal{Z} \times \mathcal{Z} \times [0,T])$ is the Lebesgue measure.

Chebyshev's inequality implies

$$\sup_{r\in R_k} r\left(\{(z_1,z_2) \in \mathcal{Z} \times \mathcal{Z} : [\|z_1\| + \|z_2\|] > M\} \times [0,T]\right) \leq \sup_{r\in R_k} \frac{g(r)}{M^2} \leq \frac{k}{M^2}.$$

Hence, R_k is tight and thus relatively compact as a subset of $\mathcal{P}$.

Since g is a tightness function, by Theorem A.3.17 of [5] tightness of $\{\mathrm{P}^{\epsilon,\Delta,\gamma}_{1,t}, \epsilon > 0\}$ will follow if we prove that

$$\sup_{\epsilon\in(0,1]} \mathbb{E}\left[g(\mathrm{P}^{\epsilon,\Delta,\gamma}_{1,t} \otimes \mathrm{Leb}_{[0,T]})\right] < \infty,$$

where $\mathrm{Leb}_{[0,T]}$ denotes Lebesgue measure in $[0,T]$. However, by (3.6)

$$\begin{aligned}\sup_{\epsilon\in(0,1]} \mathbb{E}\left[g(\mathrm{P}^{\epsilon,\Delta,\gamma}_{1,t} \otimes \mathrm{Leb}_{[0,T]})\right] &= \sup_{\epsilon\in(0,1]} \mathbb{E}\left[\int_0^T \int_{\mathcal{Z}\times\mathcal{Z}} \left[\|z_1\|^2 + \|z_2\|^2\right] \mathrm{P}^{\epsilon,\Delta,\gamma}_{1,t}(dz_1 dz_2) dt\right] \\ &= \sup_{\epsilon\in(0,1]} \mathbb{E} \int_0^T \frac{1}{\Delta} \int_t^{t+\Delta} \left[\|u^\epsilon_1(s)\|^2 + \|u^\epsilon_2(s)\|^2\right] ds dt \\ &< \infty,\end{aligned}$$

uniformly in $\gamma \in \Gamma$, which concludes the tightness proof for $\{\mathrm{P}^{\epsilon,\Delta,\gamma}_{1,t}, \epsilon > 0\}$.

Let us now consider tightness of $\{\mathrm{P}^{\epsilon,\Delta,\gamma}_{2,t}, \epsilon > 0\}$. For this purpose we notice that for every $\gamma \in \Gamma$ and every $\tilde{\phi} \in L^2(\Gamma) \cap L^1(\pi)$ we have

$$\int_\Gamma \tilde{\phi}(\tilde{\gamma}) \mathrm{P}^{\epsilon,\Delta,\gamma}_{2,t}(d\tilde{\gamma}) = \frac{1}{\Delta} \int_t^{t+\Delta} \tilde{\phi}\left(\tau_{\bar{Y}^\epsilon_s}\gamma\right) ds = \frac{1}{\Delta}\int_t^{t+\Delta} \phi\left(\bar{Y}^\epsilon_s, \gamma\right) ds.$$

Let us fix $\eta > 0$. Then, by Lemma A.6 we know that there exists $N_\eta \subset \Gamma$ with $\pi(N_\eta) \geq 1 - \eta$ such that for every bounded sequence $\Delta \in \mathcal{H}^{N_\eta}_1$ we have

$$\lim_{\epsilon\downarrow 0} \sup_{\gamma\in N_\eta} \sup_{0\leq t\leq T} \mathbb{E}\left|\frac{1}{\Delta}\int_t^{t+\Delta} \phi\left(\bar{Y}^\epsilon_s, \gamma\right) ds - \bar{\phi}\right| = 0$$

or equivalently

$$\lim_{\epsilon\downarrow 0} \sup_{\gamma\in N_\eta} \sup_{0\leq t\leq T} \mathbb{E}\left|\int_\Gamma \tilde{\phi}(\tilde{\gamma}) \mathrm{P}^{\epsilon,\Delta,\gamma}_{2,t}(d\tilde{\gamma}) - \bar{\phi}\right| = 0 \tag{4.1}$$

Now, as a probability measure in a Polish space π is itself tight. So, there exists a compact subset of Γ, say K_η, such that

$$\pi(K_\eta) \geq 1 - \eta/2.$$

Therefore, using (4.1) and the latter bound, we get that for ϵ sufficiently small, say $\epsilon < \epsilon_0(\eta)$ and for every $\gamma \in N_\eta$ and $t \in [0,T]$, we have

$$\inf_{\epsilon\in(0,\epsilon_0(\eta))} \mathbb{E}\left[\mathrm{P}^{\epsilon,\Delta,\gamma}_{2,t}(K_\eta)\right] \geq 1 - \eta$$

which implies that, uniformly in $\gamma \in N_\eta$, the measure valued random variables $\{\mathrm{P}^{\epsilon,\Delta,\gamma}_{2,t}(\cdot), \epsilon \in (0, \epsilon_0(\eta))\}$ are tight.

(ii). Uniform integrability of the family $\{\mathrm{P}^{\epsilon,\Delta,\gamma}, \epsilon > 0\}$ follows by

$$\begin{aligned}&\mathbb{E}\left[\int_{\{(z_1,z_2)\in\mathcal{Z}\times\mathcal{Z}:[\|z_1\|+\|z_2\|]>M\}\times\Gamma\times[0,T]} [\|z_1\| + \|z_2\|]\, \mathrm{P}^{\epsilon,\Delta}(dz_1 dz_2 d\tilde{\gamma} dt)\right] \\ &\leq \frac{2}{M}\mathbb{E}\left[\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma\times[0,T]} \left[\|z_1\|^2 + \|z_2\|^2\right] \mathrm{P}^{\epsilon,\Delta}(dz_1 dz_2 d\tilde{\gamma} dt)\right] \\ &= \frac{2}{M}\mathbb{E}\int_0^T \frac{1}{\Delta}\int_t^{t+\Delta} \left[\|u^\epsilon_1(s)\|^2 + \|u^\epsilon_2(s)\|^2\right] ds dt\end{aligned}$$

and the fact that

$$\sup_{\epsilon>0,\gamma\in\Gamma} \mathbb{E}\int_0^T \frac{1}{\Delta}\int_t^{t+\Delta} \left[\|u^\epsilon_1(s)\|^2 + \|u^\epsilon_2(s)\|^2\right] ds dt < \infty.$$

This concludes the proof of the lemma. □

Lemma 4.2. *Assume Conditions 2.2 and 2.4. Let $\{u^{\epsilon,\gamma}, \epsilon > 0, \gamma \in \Gamma\}$ be a family of controls in $\mathcal{A}$ as in Lemma 4.1. Moreover, fix $\eta > 0$, and consider the set N_η with $\pi(N_\eta) \geq 1 - \eta$ from Lemma A.6. Then, for every $\gamma \in N_\eta$, the family $\{\bar{X}^{\epsilon,\gamma}, \epsilon > 0\}$ is relatively compact as $\epsilon \downarrow 0$.*

Proof. It suffices to prove that for every $\eta > 0$

$$\lim_{\theta \downarrow 0} \limsup_{\epsilon \downarrow 0} \mathbb{P}\left[\sup_{t_1, t_2 < T, |t_1 - t_2| < \theta} \left\| \bar{X}^{\epsilon,\gamma}_{t_1} - \bar{X}^{\epsilon,\gamma}_{t_2} \right\| > \eta \right] = 0$$

Recalling the auxiliary problem (2.3) and the discussion succeeding it, we apply Itô formula (see also [24]), to rewrite $\bar{X}^{\epsilon,\gamma}_{t_1} - \bar{X}^{\epsilon,\gamma}_{t_2}$ as

$$\begin{aligned}
\bar{X}^{\epsilon,\gamma}_{t_1} - \bar{X}^{\epsilon,\gamma}_{t_2} \quad = \quad & \int_{t_1}^{t_2} \lambda\left(\bar{X}^{\epsilon,\gamma}_s, \bar{Y}^{\epsilon,\gamma}_s, u_1(s), u_2(s)\right) ds \\
& -\delta\left[\chi_0\left(\bar{Y}^{\epsilon,\gamma}_{t_2}\right) - \chi_0\left(\bar{Y}^{\epsilon,\gamma}_{t_1}\right)\right] \\
& +\sqrt{\epsilon}\int_{t_1}^{t_2} (\sigma + \xi\tau_1)\left(\bar{X}^{\epsilon,\gamma}_s, \bar{Y}^{\epsilon,\gamma}_s\right) dW_s + \sqrt{\epsilon}\int_{t_1}^{t_2} \xi\tau_2\left(\bar{Y}^{\epsilon,\gamma}_s\right) dB_s \\
& = B^{\epsilon,\gamma}_1 + B^{\epsilon,\gamma}_2 + B^{\epsilon,\gamma}_3
\end{aligned}$$

where $B^{\epsilon,\gamma}_i$ is the i^{th} line of the right hand side of the last display.

First we treat the term $B^{\epsilon,\gamma}_3$. It suffices to discuss one of the two stochastic integrals, let's say the first one. In particular, by Itô isometry, Lemma A.6, we have, that there is a set N_η with $\pi(N_\eta) \geq 1 - \eta$ such that for every $\gamma \in N_\eta$,

$$\lim_{\epsilon \downarrow 0} \left| \mathbb{E}\left\| \int_{t_1}^{t_2} \left(\tilde{\sigma}\left(\bar{X}^{\epsilon,\gamma}_s, \cdot\right) + \tilde{\xi}\tilde{\tau}_1(\cdot)\right) dW_s \right\|^2 - \int_{t_1}^{t_2} \mathrm{E}^{\pi}\left[\left\| \left(\tilde{\sigma}\left(\bar{X}^{\epsilon,\gamma}_s, \cdot\right) + \tilde{\xi}\tilde{\tau}_1(\cdot)\right) \right\|^2\right] ds \right| \to 0$$

as $\epsilon \downarrow 0$. In a similar fashion we can also treat the stochastic integral with respect to the Brownian motion B. Hence, for every $\gamma \in N_\eta$

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left\| B^{\epsilon,\gamma}_3 \right\|^2 = 0$$

Next, we treat $B^{\epsilon,\gamma}_1$. Lemma A.6 and the uniform bound (A.10), implies that for every $\gamma \in N_\eta$

$$\lim_{|t_2 - t_1| \to 0} \lim_{\epsilon \downarrow 0} \mathbb{E}\left\| B^{\epsilon,\gamma,t_2-t_1}_1 \right\|^2 = 0$$

Similarly, one can show that $\lim_{\epsilon \downarrow 0} \mathbb{E}\left\| B^{\epsilon,\gamma}_2 \right\| = 0$. Therefore, tightness of $\{\bar{X}^{\epsilon,\gamma}, \epsilon > 0\}$ follows for $\gamma \in N_\eta$. □

4.2 Identification of the limit points.

In this section we prove that any weak limit point of the tight sequence $\left\{(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma}), \epsilon > 0\right\}$ is a viable pair, i.e., it satisfies Definition 3.2. Let $(\bar{X}, \mathrm{P})$ be an accumulation point (in distribution) of $(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma})$ as $\epsilon, \Delta \downarrow 0$. Due to the Skorokhod representation, we may assume that there is a probability space, where this convergence holds with probability 1. The constraint (3.6) and Fatou's lemma guarantee that with probability 1,

$$\int_{\mathcal{Z} \times \mathcal{Z} \times \Gamma \times [0,T]} \left[\|z_1\|^2 + \|z_2\|^2\right] \bar{\mathrm{P}}(dz_1 dz_2 d\gamma dt) < \infty.$$

Moreover, since $\mathrm{P}^{\epsilon,\Delta,\gamma}(\mathcal{Z} \times \mathcal{Z} \times \Gamma \times [0,t]) = t$ for every $t \in [0,T]$ and using the fact that $\bar{\mathrm{P}}(\mathcal{Z} \times \mathcal{Z} \times \Gamma \times [0,t])$ is continuous as a function of $t \in [0,T]$ and that $\bar{\mathrm{P}}(\mathcal{Z} \times \mathcal{Z} \times \Gamma \times \{t\}) = 0$ we obtain $\bar{\mathrm{P}}(\mathcal{Z} \times \mathcal{Z} \times \Gamma \times [0,t]) = t$ and that $\bar{\mathrm{P}}$ can be decomposed as $\mathrm{P}(dz_1 dz_2 d\gamma dt) = \mathrm{P}_t(dz_1 dz_2 d\gamma) dt$ with $\mathrm{P}_t(\mathcal{Z} \times \mathcal{Z} \times \Gamma) = 1$.

Let us next prove that $(\bar{X}, \bar{\mathrm{P}})$ satisfy (3.4). We will use the martingale problem. In particular, let ζ be a smooth bounded function, $\phi \in \mathcal{C}^2(\mathbb{R}^m)$ compactly supported, $\{\tilde{z}_j\}_{j=1}^q$ be a family of bounded, smooth and compactly supported functions and for $r \in \mathcal{P}\left(\mathcal{Z} \times \mathcal{Z} \times \Gamma \times [0,T]\right)$, $t \in [0,T]$ define

$$(r, \tilde{z}_j)_t = \int_{\mathcal{Z} \times \mathcal{Z} \times \Gamma \times [0,t]} \tilde{z}_j(z_1, z_2, \gamma, s) r(dz_1 dz_2 d\gamma ds)$$

Then, in order to show (3.4), it is enough to show that for any $0 < t_1 < t_2 < \cdots < t_m < t < t+r \leq T$, the following limit holds almost surely with respect to $\gamma \in \Gamma$ as $\epsilon \downarrow 0$

$$\begin{aligned}\mathbb{E}\Big\{\zeta\left(\bar{X}^{\epsilon,\gamma}_{t_i}, (\mathrm{P}^{\epsilon,\Delta,\gamma}, z_j)_{t_i}, i \leq m, j \leq q\right) \Big[\phi(\bar{X}^{\epsilon,\gamma}_{t+r}) - \phi(\bar{X}^{\epsilon,\gamma}_t) \\ - \int_t^{t+r} \left[\lim_{\rho \to 0} \int_{\mathcal{Z} \times \mathcal{Z} \times \Gamma} \tilde{\lambda}_\rho(\bar{X}^{\epsilon,\gamma}_s, \gamma, z_1, z_2) \mathrm{P}_s(dz_1 dz_2 d\gamma)\right] \nabla \bar{\phi}(\bar{X}^{\epsilon,\gamma}_s) ds\Big]\Big\} \to 0 \end{aligned} \tag{4.2}$$

Let us define

$$\mathcal{L}^{\epsilon,\Delta,\rho}_s \phi(x) = \int_{\mathcal{Z} \times \mathcal{Z} \times \Gamma} \tilde{\lambda}_\rho(x, \gamma, z_1, z_2) \mathrm{P}^{\epsilon,\Delta,\gamma}_s(dz_1 dz_2 d\gamma) \nabla \phi(x)$$

where

$$\mathrm{P}^{\epsilon,\Delta,\gamma}_s(dz_1 dz_2 d\gamma) = \frac{1}{\Delta} \int_s^{s+\Delta} 1_{z_1}(u^\epsilon_1(\theta)) 1_{z_2}(u^\epsilon_2(\theta)) 1_B\left(\tau_{\bar{Y}^{\epsilon,\gamma}_\theta} \gamma\right) d\theta$$

Then, weak convergence of the pair $(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma})$ and uniform integrability of $\mathrm{P}^{\epsilon,\Delta,\gamma}$ as indicated by Lemma 4.1, shows that almost surely with respect to $\gamma \in \Gamma$

$$\mathbb{E}\left[\int_t^{t+r} \mathcal{L}^{\epsilon,\Delta,\rho}_s \phi(\bar{X}^{\epsilon,\gamma}_s) ds - \int_t^{t+r} \left[\lim_{\rho \to 0} \int_{\mathcal{Z} \times \mathcal{Z} \times \Gamma} \tilde{\lambda}_\rho(\bar{X}^{\epsilon,\gamma}_s, \gamma, z_1, z_2) \mathrm{P}_s(dz_1 dz_2 d\gamma)\right] \nabla \phi(\bar{X}^{\epsilon,\gamma}_s) ds\right] \to 0$$

as $\epsilon \downarrow 0$ and $\rho = \rho(\epsilon) \downarrow 0$. Hence, in order to prove (4.2), it is sufficient to prove that almost surely with respect to $\gamma \in \Gamma$

$$\mathbb{E}\left\{\zeta\left(\bar{X}^{\epsilon,\gamma}_{t_i}, (\mathrm{P}^{\epsilon,\Delta,\gamma}, z_j)_{t_i}, i \leq m, j \leq q\right) \left[\phi(\bar{X}^{\epsilon,\gamma}_{t+r}) - \phi(\bar{X}^{\epsilon,\gamma}_t) - \int_t^{t+r} \mathcal{L}^{\epsilon,\Delta,\rho}_s \phi(\bar{X}^{\epsilon,\gamma}_s) ds\right]\right\} \to 0$$

Recall the auxiliary problem (2.3) and consider a function $\phi \in \mathcal{C}^2(\mathbb{R}^m)$ with compact support. Let us write $\chi_\rho = (\chi_{1,\rho}, \ldots, \chi_{m,\rho})$ for the components of the vector solution to (2.3), and consider $\psi_{\ell,\rho}(x, y, \gamma) = \chi_{\ell,\rho}(y, \gamma) \partial_{x_\ell} \phi(x)$ for $\ell \in \{1, \ldots, m\}$. Set $\psi_\rho(x, y, \gamma) = (\psi_{1,\rho}, \ldots, \psi_{m,\rho})$. It is easy to see that $\tilde{\psi}_\rho(x, \gamma)$ satisfies the resolvent equation

$$\rho \tilde{\psi}_{\ell,\rho}(x, \cdot) - \tilde{L} \tilde{\psi}_{\ell,\rho}(x, \cdot) = \tilde{h}_\ell(x, \cdot) \tag{4.3}$$

where we have defined $\tilde{h}_\ell(x, \cdot) = \tilde{b}_\ell(\cdot) \partial_{x_\ell} \phi(x)$. By Itô formula and making use of (4.3), we obtain

$$
\begin{aligned}
&\mathbb{E}\left\{\zeta\left(\bar{X}^{\epsilon,\gamma}_{t_i},(\mathrm{P}^{\epsilon,\Delta,\gamma},z_j)_{t_i},i\le m,j\le q\right)\left[\phi(\bar{X}^{\epsilon,\gamma}_{t+r})-\phi(\bar{X}^{\epsilon,\gamma}_{t})-\int_t^{t+r}\mathcal{L}^{\epsilon,\Delta,\rho}_s\phi(\bar{X}^{\epsilon,\gamma}_s)ds\right]\right\}\\
&=\mathbb{E}\left\{\zeta\left(\cdots\right)\left[\int_t^{t+r}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma,u^\epsilon_1(s),u^\epsilon_2(s)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)ds-\int_t^{t+r}\mathcal{L}^{\epsilon,\Delta,\rho}_s\phi(\bar{X}^{\epsilon,\gamma}_s)ds\right]\right\}\\
&\quad+\delta\mathbb{E}\left\{\zeta\left(\cdots\right)\int_t^{t+r}\sum_{\ell=1}^m\left((c+\sigma u^\epsilon_1(s))\partial_x\psi_{\ell,\rho}+\epsilon\frac{1}{2}\mathrm{tr}\left[\partial^2_x\psi_{\ell,\rho}\right]\right)\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s\right)\right\}ds\\
&\quad+\epsilon\mathbb{E}\left\{\zeta\left(\cdots\right)\int_t^{t+r}\sum_{\ell=1}^m\mathrm{tr}\left[\sigma\tau_1^TD\partial_x\psi_{\ell,\rho}\right]\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s\right)\right\}ds\\
&\quad+\epsilon 2\mathbb{E}\left\{\zeta\left(\cdots\right)\int_t^{t+r}\mathrm{tr}\left[\sigma\sigma^T\left(\bar{Y}^{\epsilon,\gamma}_s\right)\nabla^2\phi\left(\bar{X}^{\epsilon,\gamma}_s\right)\right]\right\}ds\\
&\quad+\frac{\epsilon}{\delta}\rho\mathbb{E}\left\{\zeta\left(\cdots\right)\int_t^{t+r}\chi_\rho\left(\bar{Y}^{\epsilon,\gamma}_s\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)ds\right\}\\
&\quad-\delta\sum_{\ell=1}^m\mathbb{E}\left\{\zeta\left(\cdots\right)\left(\psi_{\ell,\rho}\left(\bar{X}^{\epsilon,\gamma}_{t+r},\bar{Y}^{\epsilon,\gamma}_{t+r}\right)-\psi_{\ell,\rho}\left(\bar{X}^{\epsilon,\gamma}_{t},\bar{Y}^{\epsilon,\gamma}_{t}\right)\right)\right\}\\
&=\sum_{i=1}^6\mathbb{E}B^{\epsilon,\gamma}_i
\end{aligned}
\tag{4.4}
$$

where $\mathbb{E}B^{\epsilon,\gamma}_i$ is the i^{th} line on the right hand side of (4.4). We want to show that each of those terms goes to zero almost surely with respect to $\gamma\in\Gamma$.

Condition 2.2 and the bound (3.6) give us that

$$\mathbb{E}\left|B^{\epsilon,\gamma}_2\right|+\mathbb{E}\left|B^{\epsilon,\gamma}_3\right|\to 0,\quad \text{as } \epsilon\downarrow 0$$

Due to the boundedness and compact support of functions ζ and ϕ, we also get that almost surely in $\gamma\in\Gamma$

$$\mathbb{E}\left|B^{\epsilon,\gamma}_4\right|\to 0,\quad \text{as } \epsilon\downarrow 0$$

By choosing $\rho=\rho(\epsilon)=\frac{\delta^2}{\epsilon}$, we also have that almost surely in $\gamma\in\Gamma$

$$\mathbb{E}\left|B^{\epsilon,\gamma}_5\right|+\mathbb{E}\left|B^{\epsilon,\gamma}_6\right|\to 0,\quad \text{as } \epsilon\downarrow 0$$

Let us next consider $B^{\epsilon,\gamma}_1$. We have

$$
\begin{aligned}
\mathbb{E}B^{\epsilon,\gamma}_1&=\mathbb{E}\left\{\zeta\left(\cdots\right)\left[\int_t^{t+r}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma,u^\epsilon_1(s),u^\epsilon_2(s)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)ds-\int_t^{t+r}\mathcal{L}^{\epsilon,\Delta,\rho}_s\phi(X^{\epsilon,\gamma}_s)ds\right]\right\}\\
&=\mathbb{E}\left\{\zeta\left(\cdots\right)\left[\int_t^{t+r}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma,u^\epsilon_1(s),u^\epsilon_2(s)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)ds-\right.\right.\\
&\qquad\left.\left.-\int_t^{t+r}\frac{1}{\Delta}\int_s^{s+\Delta}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_\theta,\gamma,u^\epsilon_1(\theta),u^\epsilon_2(\theta)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)d\theta ds\right]\right\}\\
&=\mathbb{E}\left\{\zeta\left(\cdots\right)\left[\int_t^{t+r}\frac{1}{\Delta}\int_s^{s+\Delta}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_\theta,\bar{Y}^{\epsilon,\gamma}_\theta,\gamma,u^\epsilon_1(\theta),u^\epsilon_2(\theta)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_\theta)d\theta ds-\right.\right.\\
&\qquad\left.\left.-\int_t^{t+r}\frac{1}{\Delta}\int_s^{s+\Delta}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_\theta,\gamma,u^\epsilon_1(\theta),u^\epsilon_2(\theta)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)d\theta ds\right]\right\}\\
&\quad+\mathbb{E}\left\{\zeta\left(\cdots\right)\left[\int_t^{t+r}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma,u^\epsilon_1(s),u^\epsilon_2(s)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)ds-\right.\right.\\
&\qquad\left.\left.-\int_t^{t+r}\frac{1}{\Delta}\int_s^{s+\Delta}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_\theta,\bar{Y}^{\epsilon,\gamma}_\theta,\gamma,u^\epsilon_1(\theta),u^\epsilon_2(\theta)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_\theta)d\theta ds\right]\right\}\\
&=\mathbb{E}B^{\epsilon,\gamma}_{1,1}+\mathbb{E}B^{\epsilon,\gamma}_{1,2}
\end{aligned}
$$

Let us first treat $\mathbb{E}B^{\epsilon,\gamma}_{1,1}$.

$$\begin{aligned}
&\mathbb{E}B^{\epsilon,\gamma}_{1,1} = \\
&= \mathbb{E}\left\{\zeta(\cdots)\left[\int_t^{t+r}\frac{1}{\Delta}\int_s^{s+\Delta}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_\theta,\bar{Y}^{\epsilon,\gamma}_\theta,\gamma,u^\epsilon_1(\theta),u^\epsilon_2(\theta)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_\theta)d\theta ds-\right.\right.\\
&\qquad\left.\left.-\int_t^{t+r}\frac{1}{\Delta}\int_s^{s+\Delta}\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_\theta,\gamma,u^\epsilon_1(\theta),u^\epsilon_2(\theta)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)d\theta ds\right]\right\}\\
&= \mathbb{E}\left\{\zeta(\cdots)\left[\int_t^{t+r}\frac{1}{\Delta}\int_s^{s+\Delta}\tilde{\lambda}_\rho\left(\bar{X}^{\epsilon,\gamma}_\theta,\tau_{\bar{Y}^{\epsilon,\gamma}_\theta}\gamma,u^\epsilon_1(\theta),u^\epsilon_2(\theta)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_\theta)d\theta ds-\right.\right.\\
&\qquad\left.\left.-\int_t^{t+r}\frac{1}{\Delta}\int_s^{s+\Delta}\tilde{\lambda}_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\tau_{\bar{Y}^{\epsilon,\gamma}_\theta}\gamma,u^\epsilon_1(\theta),u^\epsilon_2(\theta)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)d\theta ds\right]\right\}\\
&\to 0,\quad \text{as } \epsilon\downarrow 0,
\end{aligned}$$

by continuity of $\tilde{\lambda}_\rho$ on the first argument, stationarity and the uniform integrability obtained in Lemma 4.1.

Next we treat $\mathbb{E}B^{\epsilon,\gamma}_{1,2}$. We have

$$\begin{aligned}
\mathbb{E}\left|B^{\epsilon,\gamma}_{1,2}\right| \leq C_0\Bigg\{&\mathbb{E}\int_0^\Delta\left|\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma,u^\epsilon_1(s),u^\epsilon_2(s)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)\right|ds\\
&+\mathbb{E}\int_t^{t+\Delta}\left|\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma,u^\epsilon_1(s),u^\epsilon_2(s)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)\right|ds\Bigg\}
\end{aligned}$$

where C_0 is a finite constant. Choose $\Delta\downarrow 0$ such that $\Delta/\frac{\delta^2}{\varepsilon}\uparrow\infty$. Then, we have

$$\begin{aligned}
&\mathbb{E}\int_0^\Delta\left|\lambda_\rho\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma,u^\epsilon_1(s),u^\epsilon_2(s)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)\right|ds\\
&\leq \mathbb{E}\int_0^\Delta\left|\left(c\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma\right)+D\chi_\rho\left(\bar{Y}^{\epsilon,\gamma}_s,\gamma\right)g\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma\right)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)\right|ds\\
&+\mathbb{E}\int_0^\Delta\left|\left(\sigma\left(\bar{X}^{\epsilon,\gamma}_s,\bar{Y}^{\epsilon,\gamma}_s,\gamma\right)u^\epsilon_1(s)+D\chi_\rho\left(\bar{Y}^{\epsilon,\gamma}_s,\gamma\right)\left[\tau_1\left(\bar{Y}^{\epsilon,\gamma}_s,\gamma\right)u^\epsilon_1(s)+\tau_2\left(\bar{Y}^{\epsilon,\gamma}_s,\gamma\right)u^\epsilon_2(s)\right]\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_s)\right|ds\\
&\leq \Delta\frac{\frac{\delta^2}{\epsilon}}{\Delta}\mathbb{E}\int_0^{\Delta/\frac{\delta^2}{\epsilon}}\left|\left(c\left(\bar{X}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\gamma\right)+D\chi_\rho\left(\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\gamma\right)g\left(\bar{X}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\gamma\right)\right)\nabla\phi(\bar{X}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s})\right|ds\\
&+\sqrt{\Delta}\sqrt{\frac{\frac{\delta^2}{\epsilon}}{\Delta}\mathbb{E}\int_0^{\Delta/\frac{\delta^2}{\epsilon}}\left\|\left(\sigma\left(\bar{X}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\gamma\right)+D\chi_\rho\left(\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\gamma\right)\tau_1\left(\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\gamma\right)\right)\nabla\phi\right\|^2ds\,\mathbb{E}\int_0^\Delta\left\|u^\epsilon_1(s)\right\|^2ds}\\
&+\sqrt{\Delta}\sqrt{\frac{\frac{\delta^2}{\epsilon}}{\Delta}\mathbb{E}\int_0^{\Delta/\frac{\delta^2}{\epsilon}}\left\|\left(D\chi_\rho\left(\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\gamma\right)\tau_2\left(\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\gamma\right)\right)\nabla\phi\right\|^2ds\,\mathbb{E}\int_0^\Delta\left\|u^\epsilon_2(s)\right\|^2ds}\\
&\leq \Delta\frac{\frac{\delta^2}{\epsilon}}{\Delta}\mathbb{E}\int_0^{\Delta/\frac{\delta^2}{\epsilon}}\left\|\left(\tilde{c}\left(\bar{X}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\tau_{\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s}}\gamma\right)+D\tilde{\chi}_\rho\left(\tau_{\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s}}\gamma\right)\tilde{g}\left(\bar{X}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\tau_{\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s}}\gamma\right)\right)\nabla\phi\right\|ds\\
&+\sqrt{\Delta}\sqrt{\frac{\frac{\delta^2}{\epsilon}}{\Delta}\mathbb{E}\int_0^{\Delta/\frac{\delta^2}{\epsilon}}\left\|\left(\tilde{\sigma}\left(\bar{X}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s},\tau_{\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s}}\gamma\right)+D\tilde{\chi}_\rho\left(\tau_{\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s}}\gamma\right)\tilde{\tau}_1\left(\tau_{\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s}}\gamma\right)\right)\nabla\phi\right\|^2ds\,\mathbb{E}\int_0^\Delta\left\|u^\epsilon_1(s)\right\|^2ds}\\
&+\sqrt{\Delta}\sqrt{\frac{\frac{\delta^2}{\epsilon}}{\Delta}\mathbb{E}\int_0^{\Delta/\frac{\delta^2}{\epsilon}}\left\|\left(D\tilde{\chi}_\rho\left(\tau_{\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s}}\gamma\right)\tilde{\tau}_2\left(\tau_{\bar{Y}^{\epsilon,\gamma}_{(\delta^2/\epsilon)s}}\gamma\right)\right)\nabla\phi\right\|^2ds\,\mathbb{E}\int_0^\Delta\left\|u^\epsilon_2(s)\right\|^2ds}\\
&\to 0,\quad \text{as } \epsilon,\Delta\downarrow 0,\Delta/\frac{\delta^2}{\varepsilon}\uparrow\infty,
\end{aligned}$$

by Lemma A.6, Condition 2.2 and the uniform bound (3.6). Hence, we obtain that almost surely with respect to $\gamma\in\Gamma$,

$$\mathbb{E}\left|B_{1,2}^{\epsilon,\gamma}\right|\to 0.$$

This concludes the proof of (3.4). Next, we treat (3.5). Consider $\tilde{\phi}\in L^2(\Gamma)$ stationary, ergodic random field on $\mathbb{R}^{d-m}$. Let $\phi(y,\gamma)=\tilde{\phi}(\tau_y\gamma)$ and assume that $\phi(\cdot,\gamma)\in C_b^2(\mathbb{R}^{d-m})$. Define the formal operators

$$\mathcal{G}^{0,\gamma}_{x,y,\gamma,z_1,z_2}\phi(y,\gamma)=\left[g(x,y,\gamma)+\tau_1(y,\gamma)z_1+\tau_2(y,\gamma)z_2\right]D\phi(y,\gamma)$$

and

$$\mathcal{G}^{1,\epsilon,\gamma}_{x,y,\gamma,z_1,z_2}\phi(y,\gamma)=\frac{\epsilon}{\delta^2}\mathcal{L}^{\gamma}\phi(y,\gamma)+\frac{1}{\delta}\mathcal{G}^{0,\gamma}_{x,y,z_1,z_2}\phi(y,\gamma)$$

Following the customary notation we write $\tilde{\mathcal{G}}^{0,\gamma}_{x,\gamma,z_1,z_2}\tilde{\phi}(\gamma)=\left[\tilde{g}(x,\gamma)+\tilde{\tau}_1(\gamma)z_1+\tilde{\tau}_2(\gamma)z_2\right]D\tilde{\phi}(\gamma)$ and analogously for $\tilde{\mathcal{G}}^{1,\epsilon,\gamma}_{x,\gamma,z_1,z_2}\tilde{\phi}(\gamma)$.

For each fixed $\gamma\in\Gamma$, the process

$$\begin{aligned}
M_t^{\epsilon,\gamma}&=\phi(\bar{Y}_t^{\epsilon,\gamma})-\phi(\bar{Y}_0^{\epsilon,\gamma})-\int_0^t\mathcal{G}^{1,\epsilon,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\bar{Y}_s^{\epsilon,\gamma},u_1^\epsilon(s),u_2^\epsilon(s)}\phi(\bar{Y}_s^{\epsilon,\gamma})ds\\
&=\frac{\sqrt{\epsilon}}{\delta}\int_0^t\left\langle D\phi(\bar{Y}_s^{\epsilon,\gamma}),\tau_1(\bar{Y}_s^{\epsilon,\gamma})dW_s\right\rangle+\frac{\sqrt{\epsilon}}{\delta}\int_0^t\left\langle D\phi(\bar{Y}_s^{\epsilon,\gamma}),\tau_2(\bar{Y}_s^{\epsilon,\gamma})dB_s\right\rangle
\end{aligned}$$

is an $\mathfrak{F}_t-$martingale. Set $h(\epsilon)=\frac{\delta^2}{\epsilon}$ and write

$$\begin{aligned}
&h(\epsilon)M_t^{\epsilon,\gamma}-h(\epsilon)\left[\phi(\bar{Y}_t^{\epsilon,\gamma})-\phi(\bar{Y}_0^{\epsilon,\gamma})\right]\\
&\quad+h(\epsilon)\left[\int_0^t\frac{1}{\Delta}\left(\int_s^{s+\Delta}\mathcal{G}^{1,\epsilon,\gamma}_{\bar{X}_\theta^{\epsilon,\gamma},\bar{Y}_\theta^{\epsilon,\gamma},u_1^\epsilon(\theta),u_2^\epsilon(\theta)}\phi(\bar{Y}_\theta^{\epsilon,\gamma})d\theta\right)ds-\int_0^t\mathcal{G}^{1,\epsilon,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\bar{Y}_s^{\epsilon,\gamma},u_1^\epsilon(s),u_1^\epsilon(s)}\phi(\bar{Y}_s^{\epsilon,\gamma})ds\right]\\
&\quad=-\frac{\delta}{\epsilon}\int_0^t\frac{1}{\Delta}\left[\int_s^{s+\Delta}\left(\mathcal{G}^{0,\gamma}_{\bar{X}_\theta^{\epsilon,\gamma},\bar{Y}_\theta^{\epsilon,\gamma},u_1^\epsilon(\theta),u_2^\epsilon(\theta)}\phi(\bar{Y}_\theta^{\epsilon,\gamma})-\mathcal{G}^{0,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\bar{Y}_\theta^{\epsilon,\gamma},u_1^\epsilon(\theta),u_2^\epsilon(\theta)}\phi(\bar{Y}_\theta^{\epsilon,\gamma})\right)d\theta\right]ds\\
&\quad-\frac{\delta}{\epsilon}\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma\times[0,t]}\tilde{\mathcal{G}}^{0,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\gamma,z_1,z_2}\tilde{\phi}(\gamma)\bar{\mathrm{P}}^{\epsilon,\Delta,\gamma}(dz_1dz_2d\gamma ds)\\
&\quad-\int_0^t\frac{1}{\Delta}\int_s^{s+\Delta}\mathcal{L}^{\gamma}\phi(\bar{Y}_\theta^{\epsilon,\gamma},\gamma)d\theta ds
\end{aligned}\tag{4.5}$$

The boundedness of ϕ and of its derivatives imply that almost surely in $\gamma\in\Gamma$

$$\mathbb{E}\left[\left|h(\epsilon)M_t^{\epsilon,\gamma}\right|^2+\left|h(\epsilon)\left[\phi(\bar{Y}_t^{\epsilon,\gamma})-\phi(\bar{Y}_0^{\epsilon,\gamma})\right]\right|\right]\to 0,\quad\text{as }\epsilon\downarrow 0$$

Moreover, we have almost surely in $\gamma\in\Gamma$

$$\begin{aligned}
&h(\epsilon)\mathbb{E}\left|\int_0^t\frac{1}{\Delta}\left(\int_s^{s+\Delta}\mathcal{G}^{1,\epsilon,\gamma}_{\bar{X}_\theta^{\epsilon,\gamma},\bar{Y}_\theta^{\epsilon,\gamma},u_1^\epsilon(\theta),u_2^\epsilon(\theta)}\phi(\bar{Y}_\theta^{\epsilon,\gamma})d\theta\right)ds-\int_0^t\mathcal{G}^{1,\epsilon,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\bar{Y}_s^{\epsilon,\gamma},u_1^\epsilon(s),u_2^\epsilon(s)}\phi(\bar{Y}_s^{\epsilon,\gamma})ds\right|\\
&\leq h(\epsilon)\mathbb{E}\int_0^\Delta\left|\mathcal{G}^{1,\epsilon,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\bar{Y}_s^{\epsilon,\gamma},u_1^\epsilon(s),u_2^\epsilon(s)}\phi(\bar{Y}_s^{\epsilon,\gamma})\right|ds+h(\epsilon)\mathbb{E}\int_t^{t+\Delta}\left|\mathcal{G}^{1,\epsilon,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\bar{Y}_s^{\epsilon,\gamma},u_1^\epsilon(s),u_2^\epsilon(s)}\phi(\bar{Y}_s^{\epsilon,\gamma})\right|ds\\
&\leq\mathbb{E}\int_0^\Delta\left|\mathcal{L}\phi(\bar{Y}_s^{\epsilon,\gamma})\right|ds+\frac{\delta}{\epsilon}\mathbb{E}^\gamma\int_0^\Delta\left|\mathcal{G}^{0,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\bar{Y}_s^{\epsilon,\gamma},u_1^\epsilon(s),u_2^\epsilon(s)}\phi(\bar{Y}_s^{\epsilon,\gamma})\right|ds\\
&+\mathbb{E}\int_t^{t+\Delta}\left|\mathcal{L}\phi(\bar{Y}_s^{\epsilon,\gamma})\right|ds+\frac{\delta}{\epsilon}\mathbb{E}^\gamma\int_t^{t+\Delta}\left|\mathcal{G}^{0,\gamma}_{\bar{X}_s^{\epsilon,\gamma},\bar{Y}_s^{\epsilon,\gamma},u_1^\epsilon(s),u_2^\epsilon(s)}\phi(\bar{Y}_s^{\epsilon,\gamma})\right|ds\\
&\leq\Delta C_0\left[1+\frac{\delta}{\epsilon}\mathbb{E}\int_0^T\left\|u^\epsilon(s)\right\|^2ds\right]\\
&\to 0,\quad\text{as }\epsilon\downarrow 0,
\end{aligned}$$

due to (3.6) and $\Delta = \Delta(\epsilon) \downarrow 0$. The constant C_0 depends on the upper bound of the coefficients and on β, T.

The first term on the right hand side of (4.5) goes to zero in probability, almost surely with respect to $\gamma \in \Gamma$, due to continuous dependence of $\mathcal{G}^{0,\gamma}_{x,y,z_1,z_2}\phi(y,\gamma)$ on $x \in \mathbb{R}^m$, tightness of $\bar{X}^{\epsilon,\gamma}$, stationarity and $\delta/\epsilon \downarrow 0$.

The second term on the right hand side of (4.5) also goes to zero in probability, almost surely with respect to $\gamma \in \Gamma$, due to continuous dependence of $\mathcal{G}^{0,\gamma}_{x,y,z_1,z_2}\phi(y,\gamma)$ on $x \in \mathbb{R}^m$, tightness of $(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma})$, uniform integrability of $\mathrm{P}^{\epsilon,\Delta,\gamma}$ (Lemma 4.1) and the fact that $\delta/\epsilon \downarrow 0$.

Lastly, we consider the third term on the right hand side of (4.5). We have

$$\begin{aligned}\int_0^t \frac{1}{\Delta}\int_s^{s+\Delta} \mathcal{L}^{\gamma}\phi(\bar{Y}^{\epsilon,\gamma}_{\theta},\gamma)d\theta ds &= \int_0^t \frac{1}{\Delta}\int_s^{s+\Delta} \tilde{\mathcal{L}}\tilde{\phi}(\tau_{\bar{Y}^{\epsilon,\gamma}_{\theta}}\gamma)d\theta ds\\ &= \int_0^t \int_{\mathcal{Z}\times\Gamma} \tilde{\mathcal{L}}\tilde{\phi}(\gamma)\mathrm{P}^{\epsilon,\Delta,\gamma}(dzd\gamma ds)\end{aligned}$$

Due to weak convergence of $(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma})$, the last term converges, almost surely with respect to $\phi \in \Gamma$ to $\int_0^t \int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma} \tilde{\mathcal{L}}\tilde{\phi}(\gamma)\bar{\mathrm{P}}(dz_1dz_2d\gamma ds)$. Hence, since the rest of the terms converge to 0, as $\epsilon \downarrow 0$, we obtain in probability, almost surely in $\gamma \in \Gamma$

$$\int_0^t \int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma} \tilde{\mathcal{L}}\tilde{\phi}(\gamma)\bar{\mathrm{P}}(dz_1dz_2d\gamma ds) = 0$$

for almost all $t \in [0,T]$, which together with continuity in $t \in [0,T]$ conclude the proof of (3.5).

5 Compactness of level sets and quenched lower and upper bounds

Compactness of level sets of the rate function is standard and will not be repeated here (e.g., Subsection 4.2. of [6] or [12]).

Let us now prove the quenched lower bound. First we remark that we can restrict attention to controls that satisfy Conditions A.10 and A.11, which are required in order for Lemma A.6 to be true. For this purpose we have the following lemma, whose proof is deferred to the end of this section.

Lemma 5.1. *Let $(\bar{X}^{\epsilon,\gamma}_s, \bar{Y}^{\epsilon,\gamma}_s)$ be the strong solution to (3.3) and assume Conditions 2.2 and 2.4. Then, the infimum of the representation in (3.2) can be taken over all controls that satisfy Conditions A.10 and A.11.*

Based on Lemma 5.1, we can restrict attention to controls satisfying Conditions A.10 and A.11. Given such controls, we construct the controlled pair $(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma})$ based on such a family of controls. Then, Theorem 3.3 implies tightness of the pair $\left\{(\bar{X}^{\epsilon,\gamma}, \mathrm{P}^{\epsilon,\Delta,\gamma}), \epsilon, \Delta > 0\right\}$. Let us denote by $(\bar{X}, \bar{\mathrm{P}}) \in \mathcal{V}$ an accumulation point of the controlled pair in distribution, almost surely with respect to $\gamma \in \Gamma$. Then, by Fatou's lemma we conclude the proof of the lower bound. Indeed

$$\begin{aligned}\liminf_{\epsilon\downarrow 0} -\epsilon\log\mathbb{E}\left[e^{-\frac{1}{\epsilon}h(X^{\epsilon})}\right] &\geq \liminf_{\epsilon\downarrow 0}\mathbb{E}\left[\frac{1}{2}\int_0^T\left[\|u_1^{\epsilon,\gamma}(s)\|^2 + \|u_2^{\epsilon,\gamma}(s)\|^2\right]ds + h(\bar{X}^{\epsilon,\gamma})\right]\\ &\geq \liminf_{\epsilon\downarrow 0}\mathbb{E}\left[\frac{1}{2}\int_0^T\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma}\left[\|z_1\|^2 + \|z_2\|^2\right]\mathrm{P}^{\epsilon,\Delta,\gamma}(dz_1dz_2d\gamma ds) + h(\bar{X}^{\epsilon,\gamma})\right]\\ &\geq \inf_{(\phi,\mathrm{P})\in\mathcal{V}}\left[\frac{1}{2}\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma\times[0,T]}\left[\|z_1\|^2 + \|z_2\|^2\right]\mathrm{P}(dz_1dz_2d\gamma ds) + h(\phi)\right]\end{aligned}$$

which concludes the proof of the Laplace principle lower bound.

It remains to prove the quenched upper bound for the Laplace principle. To do so, we fix a bounded and continuous function $h : \mathcal{C}([0,T];\mathbb{R}^m) \mapsto \mathbb{R}$, and we show that

$$\limsup_{\epsilon\downarrow 0} -\epsilon\log\mathbb{E}\left[e^{-\frac{1}{\epsilon}h(X^{\epsilon})}\right] \leq \inf_{\phi\in\mathcal{C}([0,T];\mathbb{R}^m)}\left\{S(\phi) + h(\phi)\right\}$$

The idea is to fix a nearly optimizer of the right hand side of the last display and construct the control which attains the given upper bound. Fix $\eta > 0$ and consider $\psi \in \mathcal{C}([0,T];\mathbb{R}^m)$ with $\psi_0 = x_0$ such that

$$S(\psi) + h(\psi) \leq \inf_{\phi \in \mathcal{C}([0,T];\mathbb{R}^m)} \{S(\phi) + h(\phi)\} + \eta < \infty$$

Boundedness of h implies that $S(\psi) < \infty$ which means that ψ is absolutely continuous. Since the local rate function $L^o(x,v)$, defined in (6.2), is continuous and bounded as a function of $(x,v) \in \mathbb{R}^m$, standard mollification arguments (Lemmas 6.5.3 and 6.5.5 in [5]) allow to assume that $\dot{\psi}$ is piecewise constant. Next, we define the elements of $L^2(\Gamma)$

$$\tilde{u}_{1,\rho}(t,x,\gamma) = \left(\tilde{\sigma}(x,\gamma) + D\tilde{\chi}_\rho(\gamma)\tilde{\tau}_1(\gamma)\right)^T q^{-1}(x)(\dot{\psi}_t - r(x))$$

and

$$\tilde{u}_{2,\rho}(t,x,\gamma) = \left(D\tilde{\chi}_\rho(\gamma)\tilde{\tau}_2(\gamma)\right)^T q^{-1}(x)(\dot{\psi}_t - r(x))$$

and the associated stationary fields $u_{1,\rho}(t,x,y,\gamma) = \tilde{u}_{1,\rho}(t,x,\tau_y\gamma)$ and $u_{2,\rho}(t,x,y,\gamma) = \tilde{u}_{2,\rho}(t,x,\tau_y\gamma)$. We recall that $\tilde{\chi}_\rho$ satisfies the auxiliary problem in (2.3). Let us consider now the solution $\left(\bar{X}^\epsilon_t, \bar{Y}^\epsilon_t\right)$ of (3.3) with the control $u(t) = (u_1(t), u_2(t))$ being

$$u^{\epsilon,\rho,\gamma}_t = \left(u_{1,\rho}\left(t, \bar{X}^\epsilon_t, \bar{Y}^\epsilon_t, \gamma\right), u_{2,\rho}\left(t, \bar{X}^\epsilon_t, \bar{Y}^\epsilon_t, \gamma\right)\right).$$

Then, replacing $c(x,y,\gamma)$ by $c(t,x,y,\gamma) = c(x,y,\gamma) + \sigma(y,\gamma)u_{1,\rho}(t,x,y,\gamma)$, and $g(x,y,\gamma)$ by $g(t,x,y,\gamma) = g(x,y,\gamma) + \tau_1(y,\gamma)u_{1,\rho}(t,x,y,\gamma) + \tau_2(y,\gamma)u_{2,\rho}(t,x,y,\gamma)$ Theorem A.6 implies that

$$\bar{X}^\epsilon \to \bar{X} \quad \text{in law, almost surely with respect to } \gamma \in \Gamma,$$

as $\epsilon \downarrow 0$ where we have that w.p. 1 the limit is

$$\begin{aligned}
\bar{X}_t &= x_0 + \int_0^t \lim_{\rho\downarrow 0} \mathrm{E}^\pi\left[\tilde{c}(\bar{X}_s,\cdot) + D\tilde{\chi}_\rho(\cdot)\tilde{g}(\bar{X}_s,\cdot) + \left(\tilde{\sigma}(\bar{X}_s,\cdot) + D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot)\right)\tilde{u}_{1,\rho}(s,\bar{X}_s,\cdot)\right.\\
&\qquad\left. + \left(D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\right)\tilde{u}_{2,\rho}(s,\bar{X}_s,\cdot)\right] ds\\
&= x_0 + \int_0^t \lim_{\rho\downarrow 0} \mathrm{E}^\pi\left[\tilde{c}(\bar{X}_s,\cdot) + D\tilde{\chi}_\rho(\cdot)\tilde{g}(\bar{X}_s,\cdot)\right] ds\\
&\qquad + \int_0^t \lim_{\rho\downarrow 0} \mathrm{E}^\pi\left[\left(\tilde{\sigma}(\bar{X}_s,\cdot) + D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot)\right)\tilde{u}_{1,\rho}(s,\bar{X}_s,\cdot) + \left(D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\right)\tilde{u}_{2,\rho}(s,\bar{X}_s,\cdot)\right] ds\\
&= x_0 + \int_0^t r(\bar{X}_s)ds + \int_0^t \lim_{\rho\downarrow 0} \mathrm{E}^\pi\left\{\left[\left(\tilde{\sigma}(\bar{X}_s,\cdot) + D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot)\right)\left(\tilde{\sigma}(\bar{X}_s,\cdot) + D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot)\right)^T\right.\right.\\
&\qquad\left.\left. + \left(D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\right)\left(D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\right)^T\right] q^{-1}(\bar{X}_s)(\dot{\psi}_s - r(\bar{X}_s))\right\} ds\\
&= x_0 + \int_0^t r(\bar{X}_s)ds + \int_0^t \mathrm{E}^\pi\left\{\left[\left(\tilde{\sigma}(\bar{X}_s,\cdot) + D\tilde{\xi}(\cdot)\tilde{\tau}_1(\cdot)\right)\left(\tilde{\sigma}(\bar{X}_s,\cdot) + D\tilde{\xi}(\cdot)\tilde{\tau}_1(\cdot)\right)^T\right.\right.\\
&\qquad\left.\left. + \left(D\tilde{\xi}(\cdot)\tilde{\tau}_2(\cdot)\right)\left(D\tilde{\xi}(\cdot)\tilde{\tau}_2(\cdot)\right)^T\right] q^{-1}(\bar{X}_s)(\dot{\psi}_s - r(\bar{X}_s))\right\} ds\\
&= x_0 + \int_0^t r(\bar{X}_s)ds + \int_0^t q(\bar{X}_s)q^{-1}(\bar{X}_s)(\dot{\psi}_s - r(\bar{X}_s))ds\\
&= x_0 + \psi_t - \psi_0\\
&= \psi_t.
\end{aligned}$$

Moreover, by Theorem A.6 we have that for any $\eta > 0$, there exists a N_η with $\nu[N_\eta] > 1 - \eta$ such that

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \sup_{\gamma\in N_\eta} \sup_{0\leq t\leq T} \mathbb{E}\Bigg[\frac{1}{2}\int_0^T &\left[\left\|u^{\epsilon,\gamma}_{1,\rho}(s)\right\|^2 + \left\|u^{\epsilon,\gamma}_{2,\rho}(s)\right\|^2\right] ds\\
&- \frac{1}{2}\int_0^T \lim_{\rho\downarrow 0} \mathrm{E}^\pi\left[\left\|u^{\epsilon,\gamma}_{1,\rho}(s, X^\epsilon_s,\cdot)\right\|^2 + \left\|u^{\epsilon,\gamma}_{2,\rho}(s, X^\epsilon_s,\cdot)\right\|^2\right] ds\Bigg| = 0
\end{aligned}$$

Therefore, noticing that for each fixed $x \in \mathbb{R}^m$ and almost every $t \in [0,T]$

$$\begin{aligned}\lim_{\rho\downarrow 0} \mathrm{E}^{\pi}\left[\left\|u^{\epsilon,\gamma}_{1,\rho}(s,x,\cdot)\right\|^2 + \left\|u^{\epsilon,\gamma}_{2,\rho}(s,x,\cdot)\right\|^2\right] ds &= (\dot{\psi}_s - r(x))^T q^{-1}(x) q(x) q^{-1}(x)(\dot{\psi}_s - r(x)) \\ &= L^0(x, \dot{\psi}_s),\end{aligned}$$

we finally obtain

$$\begin{aligned}\limsup_{\epsilon\downarrow 0} -\epsilon \log \mathbb{E}\left[e^{-\frac{1}{\epsilon}h(X^\epsilon)}\right] &= \limsup_{\epsilon\downarrow 0} \inf_{u\in\mathcal{A}} \mathbb{E}\left[\frac{1}{2}\int_0^T \left[\|u_1(s)\|^2 + \|u_2(s)\|^2\right] ds + h(\bar{X}^\epsilon)\right] \\ &\leq \limsup_{\epsilon\downarrow 0} \mathbb{E}\left[\frac{1}{2}\int_0^T \left[\left\|u^{\epsilon,\gamma}_{1,\rho}(s)\right\|^2 + \left\|u^{\epsilon,\gamma}_{2,\rho}(s)\right\|^2\right] ds + h(\bar{X}^\epsilon)\right] \\ &\leq [S(\psi) + h(\psi)] \\ &\leq \inf_{\phi\in\mathcal{C}([0,T];\mathbb{R}^m)} \{S(\phi) + h(\phi)\} + \eta.\end{aligned}$$

The first line follows from the representation (3.2) and the second line from the choice of the particular control. The third line follows from he convergence of the X^ϵ and of the cost functional using the continuity of h. Then, the fourth line follows from the fact $\bar{X}_t = \psi_t$. Since the last statement is true for every $\eta > 0$ the proof of the upper bound is done.

We conclude this section with the proof of Lemma 5.1.

Proof of Lemma 5.1. First, we explain why Condition A.10 can be assumed without loss of generality. Without loss of generality, we can consider a function $h(x)$ that is bounded and uniformly Lipschitz continuous in $\mathbb{R}^m$. Namely, there exists a constant L_h such that

$$|h(x) - h(y)| \leq L_h \|x - y\|$$

and $\|h\|_\infty = \sup_{x\in\mathbb{R}^m} |h(x)| < \infty$. We recall that the representation

$$-\epsilon \log \mathbb{E}\left[e^{-\frac{1}{\epsilon}h(X^\epsilon_T)}\right] = \inf_{u\in\mathcal{A}} \mathbb{E}\left[\frac{1}{2}\int_0^T \|u(s)\|^2 ds + h(\bar{X}^\epsilon_T)\right] \tag{5.1}$$

is valid in a γ by γ basis.

Fix $a > 0$. Then for every $\epsilon > 0$, there exists a control $u^\epsilon \in \mathcal{A}$ such that

$$-\epsilon \log \mathbb{E}\left[e^{-\frac{1}{\epsilon}h(X^\epsilon_T)}\right] \geq \mathbb{E}\left[\frac{1}{2}\int_0^T \|u^\epsilon(s)\|^2 ds + h(\bar{X}^\epsilon_T)\right] - a. \tag{5.2}$$

So, letting $M_0 = \|h\|_\infty = \sup_{x\in\mathbb{R}^m} |h(x)|$ we easily see that such a control u^ϵ should satisfy

$$\sup_{\epsilon>0,\gamma\in\Gamma} \mathbb{E}\left[\frac{1}{2}\int_0^T \|u^\epsilon(s)\|^2 ds\right] \leq M_1 = 2M_0 + a.$$

Given that the latter bound has been established, the claim that in proving the Laplace principle lower bound one can assume Condition A.10 without loss of generality, follows by the last display and the representation (5.1) as in the proof of Theorem 4.4 of [3]. In particular, it follows by the arguments in [3] that if the last display holds, then it is enough to assume that for given $a > 0$ the controls satisfy the bound

$$\int_0^T \|u^\epsilon(s)\|^2 ds < N,$$

with

$$N \geq \frac{4M_0(4M_0 + a)}{a}$$

which proves that in proving the Laplace principle lower bound one can assume Condition A.10 without loss of generality.

Second, we explain why Condition A.11 can be assumed without loss of generality. It is clear by the representation (5.1) that the trivial bound holds

$$-\epsilon \log \mathbb{E}\left[e^{-\frac{1}{\epsilon}h(X^\epsilon_T)}\right] \leq \mathbb{E}h(X^\epsilon_T),$$

where the control $u^\epsilon(\cdot)=0$ is used to evaluate the right hand side. Thus, we only need to consider controls that satisfy

$$\mathbb{E}\left[\frac{1}{2}\int_0^T \left\|u^{\epsilon,\gamma}(s)\right\|^2 ds + h(\bar{X}^\epsilon_T)\right] \leq \mathbb{E}h(X^\epsilon_T)$$

which by the Lipschitz assumption on h, implies that

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left\|u^{\epsilon,\gamma}(s)\right\|^2 ds\right] &\leq \mathbb{E}\left|h(X^\epsilon_T)-h(\bar{X}^\epsilon_T)\right| \\ &\leq L_h \mathbb{E}\left\|X^\epsilon_T-\bar{X}^\epsilon_T\right\|.\end{aligned}$$

Let us next define the processes $\left(\hat{\hat{X}}^\epsilon_t,\hat{\hat{Y}}^\epsilon_t\right)=\left(\bar{X}^\epsilon_{\frac{\delta^2 t}{\epsilon}},\bar{Y}^\epsilon_{\frac{\delta^2 t}{\epsilon}}\right)$ and $\left(\hat{X}^\epsilon_t,\hat{Y}^\epsilon_t\right)=\left(X^\epsilon_{\frac{\delta^2 t}{\epsilon}},Y^\epsilon_{\frac{\delta^2 t}{\epsilon}}\right)$. It is easy to see that $\left(\hat{\hat{X}}^\epsilon_t,\hat{\hat{Y}}^\epsilon_t\right)$ satisfies the SDE

$$\begin{aligned}
d\hat{\hat{X}}^\epsilon_t &= \delta b\left(\hat{\hat{Y}}^\epsilon_t,\gamma\right)dt+\frac{\delta^2}{\epsilon}\left[c\left(\hat{\hat{X}}^\epsilon_t,\hat{\hat{Y}}^\epsilon_t,\gamma\right)+\sigma\left(\hat{\hat{X}}^\epsilon_t,\hat{\hat{Y}}^\epsilon_t,\gamma\right)u_1(\delta^2 t/\epsilon)\right]dt+\delta\sigma\left(\hat{\hat{X}}^\epsilon_t,\hat{\hat{Y}}^\epsilon_t,\gamma\right)dW_t,\\
d\hat{\hat{Y}}^\epsilon_t &= f\left(\hat{\hat{Y}}^\epsilon_t,\gamma\right)dt+\frac{\delta}{\epsilon}\left[g\left(\hat{\hat{X}}^\epsilon_t,\hat{\hat{Y}}^\epsilon_t,\gamma\right)+\tau_1\left(\hat{\hat{Y}}^\epsilon_t,\gamma\right)u_1(\delta^2 t/\epsilon)+\tau_2\left(\hat{\hat{Y}}^\epsilon_t,\gamma\right)u_2(\delta^2 t/\epsilon)\right]dt\\
&\qquad +\left[\tau_1\left(\hat{\hat{Y}}^\epsilon_t,\gamma\right)dW_t+\tau_2\left(\hat{\hat{Y}}^\epsilon_t,\gamma\right)dB_t\right],\\
\hat{\hat{X}}^\epsilon_0 &= x_0,\ \ \hat{\hat{Y}}^\epsilon_0=y_0,
\end{aligned}$$

and $\left(\hat{X}^\epsilon_t,\hat{Y}^\epsilon_t\right)$ satisfies the same SDE with the control $u^\epsilon_1(\cdot)=u^\epsilon_2(\cdot)=0$.

So, we basically have that

$$\begin{aligned}\frac{1}{\epsilon}\mathbb{E}\left[\int_0^{\frac{\delta^2 T}{\epsilon}} \left\|u^{\epsilon,\gamma}(s)\right\|^2 ds\right] &\leq L_h\frac{1}{\epsilon}\mathbb{E}\left\|X^\epsilon_{\frac{\delta^2 T}{\epsilon}}-\bar{X}^\epsilon_{\frac{\delta^2 T}{\epsilon}}\right\| \\ &= L_h\frac{\delta}{\epsilon}\mathbb{E}\left\|\frac{1}{\delta}\hat{X}^\epsilon_T-\frac{1}{\delta}\hat{\hat{X}}^\epsilon_T\right\|.\end{aligned}$$

For notational convenience, we define

$$\nu^\epsilon_T \doteq \frac{1}{\epsilon}\mathbb{E}\left[\int_0^{\frac{\delta^2 T}{\epsilon}} \left\|u^{\epsilon,\gamma}(s)\right\|^2 ds\right]$$

and

$$m^\epsilon_T \doteq \mathbb{E}\left\|\frac{1}{\delta}\hat{X}^\epsilon_T-\frac{1}{\delta}\hat{\hat{X}}^\epsilon_T\right\|^2+\mathbb{E}\left\|\hat{Y}^\epsilon_T-\hat{\hat{Y}}^\epsilon_T\right\|^2.$$

Since for $x>0$, the function x^2 is increasing, the latter inequality, followed by Jensen's inequality give us

$$\left|\nu^\epsilon_T\right|^2 \leq \left|L_h\frac{\delta}{\epsilon}\right|^2 m^\epsilon_T.$$

The next step is to derive an upper bound of m^ϵ_T in terms of $|\nu^\epsilon_T|^2$. Writing down the differences of $\hat{X}^\epsilon_T - \hat{\bar{X}}^\epsilon_T$ and $\hat{Y}^\epsilon_T - \hat{\bar{Y}}^\epsilon_T$, squaring, taking expectation and using Lipschitz continuity of the functions $b, c, f, g, \sigma, \tau_1, \tau_2$ and boundedness of σ, τ_1, τ_2 we obtain the inequality

$$m^\epsilon_T \le C_0 \int_0^T m^\epsilon_s ds + C_1 \left\{ \left| \frac{\delta}{\epsilon} \mathbb{E}\left[\int_0^T \left\| u_1^{\epsilon,\gamma}\left(\frac{\delta^2 s}{\epsilon}\right) \right\| ds \right] \right|^2 + \left| \frac{\delta}{\epsilon} \mathbb{E}\left[\int_0^T \left\| u_2^{\epsilon,\gamma}\left(\frac{\delta^2 s}{\epsilon}\right) \right\| ds \right] \right|^2 \right\},$$

where the constants C_0, C_1 depends only on the Lipschitz constants of $b, c, f, g, \sigma, \tau_1, \tau_2$ and on the sup norm of σ, τ_1, τ_2. Defining for notational convenience

$$|a^\epsilon_T|^2 \doteq \frac{\delta}{\epsilon} \mathbb{E}\left[\int_0^T \left\| u_1^{\epsilon,\gamma}\left(\frac{\delta^2 s}{\epsilon}\right) \right\| ds \right] + \frac{\delta}{\epsilon} \mathbb{E}\left[\int_0^T \left\| u_2^{\epsilon,\gamma}\left(\frac{\delta^2 s}{\epsilon}\right) \right\| ds \right].$$

Gronwall lemma, gives us

$$m^\epsilon_T \le C_1 \left|a^\epsilon_T\right|^2 + C_0 C_1 \int_0^T \left|a^\epsilon_s\right|^2 e^{C_0(T-s)} ds.$$

Let us now rewrite and upper bound $|a^\epsilon_T|^2$. We notice that, Hölder inequality followed by Young's inequality give us

$$\begin{aligned}
|a^\epsilon_T|^2 &= \left| \frac{\epsilon}{\delta} \frac{1}{\epsilon} \mathbb{E}\left[\int_0^{\frac{\delta^2 T}{\epsilon}} \left\| u_1^{\epsilon,\gamma}(s) \right\| ds \right] \right|^2 + \left| \frac{\epsilon}{\delta} \frac{1}{\epsilon} \mathbb{E}\left[\int_0^{\frac{\delta^2 T}{\epsilon}} \left\| u_2^{\epsilon,\gamma}(s) \right\| ds \right] \right|^2 \\
&\le \frac{1}{\delta^2} \frac{\delta^2 T}{\epsilon} \mathbb{E}\left[\int_0^{\frac{\delta^2 T}{\epsilon}} \left\| u^{\epsilon,\gamma}(s) \right\|^2 ds \right] \\
&= T \frac{1}{\epsilon} \mathbb{E}\left[\int_0^{\frac{\delta^2 T}{\epsilon}} \left\| u^{\epsilon,\gamma}(s) \right\|^2 ds \right] \\
&= T \nu^\epsilon_T \\
&\le \frac{T^2}{2} + \frac{|\nu^\epsilon_T|^2}{2}.
\end{aligned}$$

Putting these estimates together, we obtain

$$\begin{aligned}
|\nu^\epsilon_T|^2 &\le L_h^2 \left| \frac{\delta}{\epsilon} \right|^2 m^\epsilon_T \\
&\le L_h^2 C_1 \left| \frac{\delta}{\epsilon} \right|^2 \left[|a^\epsilon_T|^2 + C_0 \int_0^T |a^\epsilon_s|^2 e^{C_0(T-s)} ds \right] \\
&\le L_h^2 C_1 \left| \frac{\delta}{\epsilon} \right|^2 \left[\left(\frac{T^2}{2} + \frac{|\nu^\epsilon_T|^2}{2} \right) + C_0 \int_0^T \left(\frac{s^2}{2} + \frac{|\nu^\epsilon_s|^2}{2} \right) e^{C_0(T-s)} ds \right].
\end{aligned}$$

Therefore, by choosing δ/ϵ sufficiently small such that $L_h^2 C_1 \left|\frac{\delta}{\epsilon}\right|^2 \le 1$, we have

$$\begin{aligned}
\frac{|\nu^\epsilon_T|^2}{2} &\le L_h^2 C_1 \left| \frac{\delta}{\epsilon} \right|^2 \left[\frac{T^2}{2} + C_0 \int_0^T \left(\frac{s^2}{2} + \frac{|\nu^\epsilon_s|^2}{2} \right) e^{C_0(T-s)} ds \right] \\
&\le L_h^2 C_1 \left| \frac{\delta}{\epsilon} \right|^2 \left[\frac{T^2}{2} + \frac{T^2}{2}(e^{C_0 T} - 1) + C_0 \int_0^T \frac{|\nu^\epsilon_s|^2}{2} e^{C_0(T-s)} ds \right] \\
&= L_h^2 C_1 \left| \frac{\delta}{\epsilon} \right|^2 \left[\frac{T^2}{2} e^{C_0 T} + C_0 \int_0^T \frac{|\nu^\epsilon_s|^2}{2} e^{C_0(T-s)} ds \right].
\end{aligned}$$

Thus, we have

$$e^{-C_0 T}\frac{|\nu^\epsilon_T|^2}{2} \leq L_h^2 C_1 \left|\frac{\delta}{\epsilon}\right|^2 \left[\frac{T^2}{2} + C_0\int_0^T e^{-C_0 s}\frac{|\nu^\epsilon_s|^2}{2}ds\right].$$

So, letting $\beta^\epsilon_T = L_h^2 C_1 \frac{T^2}{2}\left|\frac{\delta}{\epsilon}\right|^2$ and $\theta^\epsilon = L_h^2 C_1 C_0\left|\frac{\delta}{\epsilon}\right|^2$, Gronwall lemma guarantees that

$$e^{-C_0 T}\frac{|\nu^\epsilon_T|^2}{2} \leq \beta^\epsilon_T + \theta^\epsilon\int_0^T \beta^\epsilon_s e^{\theta^\epsilon(T-s)}ds.$$

Since β^ϵ_T and θ^ϵ go uniformly in $\gamma\in\Gamma$ to zero at the speed $O((\frac{\delta}{\epsilon})^2)$ as $\epsilon\downarrow 0$, we get that

$$|\nu^\epsilon_T|^2 \leq C(\delta/\epsilon)^2,$$

where the constant C, depends on T, but not on ϵ,δ or γ. This concludes the argument of why Condition A.11 can be assumed without loss of generality. □

6 Proof of Theorem 3.5

In this section we prove that the explicit expression of the large deviation's action functional is given by Theorem 3.5.

Due to Theorem 3.4, we only need to prove that the rate function given in (3.7) can be written in the form of Theorem 3.5. First, we notice that one can write (3.7) in terms of a local rate function, in the form

$$S(\phi) = \int_0^T L^r(\phi_s,\dot\phi_s)ds$$

where we have defined

$$L^r(x,v) = \inf_{\mathrm{P}\in\mathcal{A}^r_{x,v}} \frac{1}{2}\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma}\left[\|z_1\|^2+\|z_2\|^2\right]\mathrm{P}(dz_1dz_2d\gamma)$$

and

$$\mathcal{A}^r_{x,v} = \Bigg\{\mathrm{P}\in\mathcal{P}\left(\mathcal{Z}\times\mathcal{Z}\times\Gamma\right) : \int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma}\tilde L\tilde f(\gamma)\mathrm{P}(dz_1dz_2d\gamma) = 0, \quad \forall \quad \tilde f\in\mathcal{D}(\tilde L)$$
$$\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma}\left[\|z_1\|^2+\|z_2\|^2\right]\mathrm{P}(dz_1dz_2d\gamma)<\infty, \text{ and } v = \lim_{\rho\to 0}\int_{\mathcal{Z}\times\mathcal{Z}\times\Gamma}\tilde\lambda_\rho(x,\gamma,z_1,z_2)\mathrm{P}(dz_1dz_2d\gamma)\Bigg\}$$

This follows directly by the definition of a viable pair (Definition 3.2). We call this representation the "relaxed" formulation since the control is characterized as a distribution on $\mathcal{Z}\times\mathcal{Z}$ rather than an element of $\mathcal{Z}\times\mathcal{Z}$. However, as we shall demonstrate below, the structure of the problem allows us to rewrite the relaxed formulation of the local rate function in terms of an ordinary formulation of an equivalent local rate function, where the control is indeed given as an element of $\mathcal{Z}\times\mathcal{Z}$. In preparation for this representation, we notice that any element $\mathrm{P}\in\mathcal{P}\left(\mathcal{Z}\times\mathcal{Z}\times\Gamma\right)$ can be written of a stochastic kernel on $\mathcal{Z}\times\mathcal{Z}$ given Γ and a probability measure on Γ, namely

$$\mathrm{P}(dz_1dz_2d\gamma) = \eta(dz_1dz_2|\gamma)\pi(d\gamma).$$

Hence, by the definition of viability, we obtain for every $\tilde f\in\mathcal{D}(\tilde L)$ that

$$\int_\Gamma \tilde L\tilde f(\gamma)\pi(d\gamma) = 0$$

where we used the independence of $\tilde L$ on z to eliminate the stochastic kernel η. Then Proposition 2.6 guarantees that π takes the form

$$\pi(d\gamma) \doteq \frac{\tilde m(\gamma)}{\mathrm{E}^\nu\tilde m(\cdot)}\nu(d\gamma)$$

and is actually an invariant, ergodic and reversible probability measure for the process associated with the operator $\tilde{L}$, or equivalently for the environment process γ_t as given by (2.2). Next, since the cost $\|z\|^2$ is convex in $z=(z_1,z_2)$ and $\tilde{\lambda}_\rho$ is affine in z, the relaxed control formulation can be easily written in terms of the ordinary control formulation

$$L^o(x,v)=\inf_{\tilde{u}\in\mathcal{A}^o_{x,v}}\frac{1}{2}\mathrm{E}^\pi\left[\|\tilde{u}_1(\cdot)\|^2+\|\tilde{u}_2(\cdot)\|^2\right] \tag{6.1}$$

and

$$\mathcal{A}^o_{x,v}=\left\{\tilde{u}=(\tilde{u}_1,\tilde{u}_2):\Gamma\mapsto\mathbb{R}^d:\mathrm{E}^\pi\left[\|\tilde{u}_1(\cdot)\|^2+\|\tilde{u}_2(\cdot)\|^2\right]<\infty,\text{ and } v=\lim_{\rho\to 0}\mathrm{E}^\pi\left[\tilde{\lambda}_\rho(x,\cdot,\tilde{u}_1(\cdot),\tilde{u}_2(\cdot))\right]\right\}.$$

Jensen's inequality and the fact that $\tilde{\lambda}_\rho(x,\gamma,z_1,z_2)$ is affine in z imply $L^r(x,v)\geq L^o(x,v)$. For the reverse inequality, for given $\tilde{u}=(\tilde{u}_1,\tilde{u}_2)$ one can define a corresponding relaxed control by $\mathrm{P}(dz_1dz_2d\gamma)=\delta_{(\tilde{u}_1(\gamma),\tilde{u}_2(\gamma))}(dz_1dz_2)\pi(d\gamma)$.

The next step is to prove that the infimization problem in (6.1) can be solved explicitly and in particular that

$$L^o(x,v)=\frac{1}{2}(v-r(x))^Tq^{-1}(x)(v-r(x)) \tag{6.2}$$

where

$$r(x)=\lim_{\rho\downarrow 0}\mathrm{E}^\pi\left[\tilde{c}(x,\cdot)+D\tilde{\chi}_\rho(\cdot)\tilde{g}(x,\cdot)\right]=\mathrm{E}^\pi[\tilde{c}(x,\cdot)+\tilde{\xi}(\cdot)\tilde{g}(x,\cdot)]$$

$$\begin{aligned}q(x)&=\lim_{\rho\downarrow 0}\mathrm{E}^\pi\left[(\tilde{\sigma}(x,\cdot)+D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot))(\tilde{\sigma}(x,\cdot)+D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot))^T+\left(D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\right)\left(D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\right)^T\right]\\&=\mathrm{E}^\pi\left[(\tilde{\sigma}(x,\cdot)+\tilde{\xi}(\cdot)\tilde{\tau}_1(\cdot))(\tilde{\sigma}(x,\cdot)+\tilde{\xi}(\cdot)\tilde{\tau}_1(\cdot))^T+\left(\tilde{\xi}(\cdot)\tilde{\tau}_2(\cdot)\right)\left(\tilde{\xi}(\cdot)\tilde{\tau}_2(\cdot)\right)^T\right]\end{aligned}$$

Let us first prove that for every $\tilde{u}=(\tilde{u}_1,\tilde{u}_2)\in\mathcal{A}^o_{x,v}$

$$\mathrm{E}^\pi\|\tilde{u}(x,\cdot)\|^2\geq(v-r(x))^Tq^{-1}(x)(v-r(x)). \tag{6.3}$$

By definition, any $\tilde{u}=(\tilde{u}_1,\tilde{u}_2)\in\mathcal{A}^o_{x,v}$ satisfies

$$\begin{aligned}v&=\lim_{\rho\to 0}\mathrm{E}^\pi\left[\tilde{\lambda}_\rho(x,\cdot,\tilde{u}_1(\cdot),\tilde{u}_1(\cdot))\right]\\&=r(x)+\lim_{\rho\to 0}\mathrm{E}^\pi\left[(\tilde{\sigma}(x,\cdot)+D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot))\,\tilde{u}_1(\cdot)+D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\tilde{u}_2(\cdot)\right].\end{aligned}$$

Treating x as a parameter, define

$$\hat{v}=v-r(x)=\lim_{\rho\to 0}\mathrm{E}^\pi\left[(\tilde{\sigma}(x,\cdot)+D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot))\,\tilde{u}_1(\cdot)+D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\tilde{u}_2(\cdot)\right],$$

and for notational convenience set

$$\tilde{\kappa}_{1,\rho}(x,\gamma)=\tilde{\sigma}(x,\gamma)+D\tilde{\chi}_\rho(\gamma)\tilde{\tau}_1(\gamma)\quad\text{ and }\tilde{\kappa}_{2,\rho}(x,\gamma)=D\tilde{\chi}_\rho(\gamma)\tilde{\tau}_2(\gamma)$$

Next, we drop writing explicitly the dependence on the parameter x and we write $q^{-1}=W^TW$, where W is an invertible matrix, so that $\hat{v}^Tq^{-1}\hat{v}=\|W\hat{v}\|^2$. Without loss of generality, we assume that $\tilde{u}\in L^2(\Gamma)$ is such that $\mathrm{E}^\pi\|\tilde{u}(\cdot)\|^2=1$. By Cauchy-Schwartz inequality in $\mathbb{R}^m$ we have

$$\begin{aligned}\|W\hat{v}\|^2&=\left\langle W\hat{v},W\lim_{\rho\downarrow 0}\mathrm{E}^\pi\left[\tilde{\kappa}_{1,\rho}(\cdot)\tilde{u}_1(\cdot)+\tilde{\kappa}_{2,\rho}(\cdot)\tilde{u}_2(\cdot)\right]\right\rangle\\&=\lim_{\rho\downarrow 0}\mathrm{E}^\pi\left[\left\langle\tilde{u}_1(\cdot),\tilde{\kappa}^T_{1,\rho}(\cdot)W^TW\hat{v}\right\rangle+\left\langle\tilde{u}_2(\cdot),\tilde{\kappa}^T_{2,\rho}(\cdot)W^TW\hat{v}\right\rangle\right]\\&\leq\lim_{\rho\downarrow 0}\left(\mathrm{E}^\pi\left[\left\|\tilde{\kappa}^T_{1,\rho}(\cdot)W^TW\hat{v}\right\|^2+\left\|\tilde{\kappa}^T_{2,\rho}(\cdot)W^TW\hat{v}\right\|^2\right]\right)^{1/2}\\&=\lim_{\rho\downarrow 0}\left(\hat{v}^TW^TW\mathrm{E}^\pi\left[\tilde{\kappa}_{1,\rho}(\cdot)\tilde{\kappa}^T_{1,\rho}(\cdot)+\tilde{\kappa}_{2,\rho}(\cdot)\tilde{\kappa}^T_{2,\rho}(\cdot)\right]W^TW\hat{v}\right)^{1/2}\\&=\left(\hat{v}^TW^TWqW^TW\hat{v}\right)^{1/2}\\&=\|W\hat{v}\|.\end{aligned}$$

If $\|W\hat{v}\| = 0$, then (6.3) holds automatically. If $\|W\hat{v}\| \neq 0$, then the last display implies $\|W\hat{v}\| \leq 1$, which directly proves that

$$\mathrm{E}^{\pi} \|\tilde{u}(x,\cdot)\|^2 = 1 \geq \|W\hat{v}\|^2 = (v - r(x))^T q^{-1}(x)(v - r(x)).$$

To prove that the inequality becomes an equality when taking the infimum over all $\tilde{u} \in \mathcal{A}^o_{x,v}$, we need to find a $\tilde{u} \in L^2(\Gamma)$ which attains the infimum. Define the elements of $L^2(\Gamma)$

$$\tilde{u}_{1,\rho}(x,\gamma;v) = \left(\tilde{\sigma}(\gamma) + D\tilde{\chi}_\rho(\gamma)\tilde{\tau}_1(\gamma)\right)^T q^{-1}(x)(v - r(x))$$

and

$$\tilde{u}_{2,\rho}(x,\gamma;v) = \left(D\tilde{\chi}_\rho(\gamma)\tilde{\tau}_2(\gamma)\right)^T q^{-1}(x)(v - r(x))$$

and set $\tilde{u}_\rho(x,\cdot;v) = (\tilde{u}_{1,\rho}(x,\cdot;v), \tilde{u}_{2,\rho}(x,\cdot;v))$. A straightforward computation yields

$$\begin{aligned}\mathrm{E}^{\pi} \|\tilde{u}_\rho(x,\cdot;v)\|^2 = (v - r(x))^T q^{-1}(x) \mathrm{E}^{\pi} \Big[&(\tilde{\sigma}(x,\cdot) + D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot))(\tilde{\sigma}(x,\cdot) + D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_1(\cdot))^T + \\ &+ \left(D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\right)\left(D\tilde{\chi}_\rho(\cdot)\tilde{\tau}_2(\cdot)\right)^T \Big] q(x)(v - r(x))\end{aligned}$$

Thus, letting $\rho \downarrow 0$, we obtain

$$\lim_{\rho \downarrow 0} \mathrm{E}^{\pi} \|\tilde{u}_\rho(x,\cdot;v)\|^2 = (v - r(x))^T q^{-1}(x)(v - r(x))$$

Hence, the element $\tilde{u} \in L^2(\Gamma)$ that we are looking for is the $L^2(\pi)$ limit of $\tilde{u}_\rho$ as defined above. This is well defined, since by Proposition 2.6 in [22] $D\tilde{\chi}_\rho$ has a well defined $L^2(\pi)$ strong limit. Therefore, we have proven that

$$L^o(x,v) = \frac{1}{2}(v - r(x))^T q^{-1}(x)(v - r(x))$$

which concludes the proof of Theorem 3.5.

A Quenched ergodic theorems

In this appendix we prove quenched ergodic theorems that are required for the proof of Theorem 3.3. For notational convenience and without loss of generality, we mostly consider a process Y driven by a single Brownian motion with diffusion coefficient $\kappa(y,\gamma)$ such that $\kappa\kappa^T = \tau_1\tau_1^T + \tau_2\tau_2^T$.

We prove the required ergodic result, Lemma A.6 in a progressive way. First, in Lemma A.1 we recall the classical ergodic theorem. This is strengthened in Lemma A.3 to cover cases of time shifts, uniformly with respect to $t \in [0,T]$. Then, in Lemmas A.4-A.5 we consider the case of perturbing the drift of the process by small perturbations (uncontrolled and controlled case). The latter result together with the standard technique of freezing the slow component yield the proof of the ergodic statement in Lemma A.6.

A.1 No time shifts, i.e. $t = 0$

Lemma A.1. *Consider the process $Y^{\epsilon,y_0,\gamma}_t$ satisfying the SDE*

$$Y^{\epsilon,y_0,\gamma}_t = y_0 + \frac{\epsilon}{\delta^2}\int_0^t f(Y^{\epsilon,y_0,\gamma}_s,\gamma)ds + \frac{\sqrt{\epsilon}}{\delta}\int_0^t \kappa(Y^{\epsilon,y_0,\gamma}_s,\gamma)dW_s. \tag{A.1}$$

Consider also a function $\tilde{\Psi} \in L^2(\Gamma) \cap L^1(\pi)$ and define $\Psi(y,\gamma) = \tilde{\Psi}(\tau_y\gamma)$. Assume that $\Psi : \mathbb{R}^{d-m} \times \Gamma \mapsto \mathbb{R}$ is measurable.

Denote $\bar{\Psi} \doteq \int_\Gamma \tilde{\Psi}(\gamma)\pi(d\gamma)$. Then for any sequence $h(\epsilon)$ that is bounded from above and such that $\delta^2/[\epsilon h(\epsilon)] \downarrow 0$ (note that in particular $h(\epsilon)$ could be a constant), there is a set N of full $\pi-$measure such that for every $\gamma \in N$

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left|\frac{1}{h(\epsilon)}\int_0^{h(\epsilon)} \Psi(Y^{\epsilon,y_0,\gamma}_s,\gamma)ds - \bar{\Psi}\right| = 0.$$

Proof of Lemma A.1. Let $\hat{Y}_t^{y_0,\gamma} = Y^{\epsilon,y_0,\gamma}_{\delta^2 t/\epsilon}$. Note that $\hat{Y}_t^{y_0,\gamma}$ satisfies

$$\hat{Y}_t^{y_0,\gamma} = y_0 + \int_0^t f(\hat{Y}_s^{y_0,\gamma},\gamma)ds + \int_0^t \kappa(\hat{Y}_s^{y_0,\gamma},\gamma)dW_s, \tag{A.2}$$

and also that $\pi(d\gamma)$ is the invariant ergodic probability measure for the environment process $\gamma_t = \tau_{\hat{Y}_t^{y_0,\gamma}}\gamma$ (Proposition 2.6).

Suppose that $\delta^2/[\epsilon h(\epsilon)] \downarrow 0$. By the ergodic theorem, there is a set N of full $\pi-$measure such that for any $\gamma \in N$

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\mathbb{E}\left[\frac{1}{h(\epsilon)}\int_0^{h(\epsilon)}\Psi(Y_s^{\epsilon,y_0,\gamma},\gamma)ds\right] &= \lim_{\epsilon\downarrow 0}\mathbb{E}\left[\frac{\delta^2}{\epsilon h(\epsilon)}\int_0^{\frac{\epsilon h(\epsilon)}{\delta^2}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds\right]\\
&= \lim_{\epsilon\downarrow 0}\mathbb{E}\left[\frac{\delta^2}{\epsilon h(\epsilon)}\int_0^{\frac{\epsilon h(\epsilon)}{\delta^2}}\tilde{\Psi}(\tau_{\hat{Y}_s^{y_0,\gamma}}\gamma)ds\right]\\
&= \lim_{\epsilon\downarrow 0}\mathbb{E}\left[\frac{\delta^2}{\epsilon h(\epsilon)}\int_0^{\frac{\epsilon h(\epsilon)}{\delta^2}}\tilde{\Psi}(\gamma_s)ds\right]\\
&= \bar{\Psi}.
\end{aligned}$$

□

It follows from Egoroff's theorem that for every $\eta > 0$ there is a set N_η with $\pi\left[N_\eta\right] > 1-\eta$, such that

$$\lim_{\epsilon\downarrow 0}\sup_{\gamma\in N_\eta}\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_0^{h(\epsilon)}\Psi(Y_s^{\epsilon,y_0,\gamma})ds - \bar{\Psi}\right| = 0.$$

A.2 Time shifts and uniformity

For notational purposes we will write that $h(\epsilon) \in \mathcal{H}_1^{N_\eta}$, if the pair $(h(\epsilon), N_\eta)$ satisfies Condition A.2.

Condition A.2. *Let $\tilde{\Psi} \in L^2(\Gamma)\cap L^1(\pi)$ and define the measurable function $\Psi(y,\gamma) = \tilde{\Psi}(\tau_y\gamma)$. For $\gamma \in \Gamma$ define*

$$\theta^\gamma(u) = \sup_{r>u}\mathbb{E}\left|\frac{1}{r}\int_0^r \Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds - \bar{\Psi}\right|.$$

For any $\eta \in (0,1)$, there exists a set N_η with $\pi(N_\eta) \geq 1-\eta$ and a sequence $\{h(\epsilon), \epsilon > 0\}$ such that the following are satisfied:

i. $\frac{\delta^2/\epsilon}{h(\epsilon)} \to 0$ *as* $\epsilon \downarrow 0$,

ii. there exists $\beta \in (0,1)$ *such that* $\frac{\sup_{\gamma\in N_\eta}\theta^\gamma\left(\frac{1}{(\delta^2/\epsilon)^\beta}\right)}{h(\epsilon)} \to 0$, *as* $\epsilon \downarrow 0$, *and*

iii. $\frac{1}{h(\epsilon)}\sup_{\gamma\in N_\eta}\sup_{t\in[0,T]}\mathbb{E}\left|\left(\delta^2/\epsilon\right)\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds - t\bar{\Psi}\right| \to 0$ *as* $\epsilon \downarrow 0$

Lemma A.3 shows that one in fact can find a pair $(h(\epsilon), N_\eta)$ satisfies Condition A.2 in order to prove a uniform in time $t \in [0,T]$, ergodic theorem.

Lemma A.3. *Consider the setup and notations of Lemma A.1. Fix $\eta > 0$. Then there exists a set N_η such that $\pi(N_\eta) \geq 1-\eta$ and $h(\epsilon) \in \mathcal{H}_1^{N_\eta}$ such that*

$$\lim_{\epsilon\downarrow 0}\sup_{\gamma\in N_\eta}\sup_{t\in[0,T]}\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)}\Psi(Y_s^{\epsilon,y_0,\gamma},\gamma)ds - \bar{\Psi}\right| = 0. \tag{A.3}$$

Proof of Lemma A.3. We start with the following decomposition

$$
\begin{aligned}
&\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)}\Psi(Y_s^{\epsilon,y_0,\gamma},\gamma)ds-\bar{\Psi}\right| = \mathbb{E}\left|\frac{\delta^2/\epsilon}{h(\epsilon)}\int_{\frac{t}{\delta^2/\epsilon}}^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-\bar{\Psi}\right| \\
&= \mathbb{E}\left|\frac{\delta^2/\epsilon}{h(\epsilon)}\int_0^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-\frac{\delta^2/\epsilon}{h(\epsilon)}\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-\bar{\Psi}\right| \\
&= \mathbb{E}\left|\frac{t+h(\epsilon)}{h(\epsilon)}\left(\frac{\delta^2/\epsilon}{t+h(\epsilon)}\int_0^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-\bar{\Psi}\right)-\frac{t}{h(\epsilon)}\left(\frac{\delta^2/\epsilon}{t}\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-\bar{\Psi}\right)\right| \\
&\leq \frac{T+h(\epsilon)}{h(\epsilon)}\mathbb{E}\left|\frac{\delta^2/\epsilon}{t+h(\epsilon)}\int_0^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-\bar{\Psi}\right|+\frac{1}{h(\epsilon)}\mathbb{E}\left|\delta^2/\epsilon\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-t\bar{\Psi}\right| \\
&\leq \frac{T+h(\epsilon)}{h(\epsilon)}\sup_{r>\frac{t+h(\epsilon)}{\delta^2/\epsilon}}\mathbb{E}\left|\frac{1}{r}\int_0^r\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-\bar{\Psi}\right|+\frac{1}{h(\epsilon)}\mathbb{E}\left|\delta^2/\epsilon\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-t\bar{\Psi}\right| \\
&\leq \frac{T+1}{h(\epsilon)}\theta^\gamma\left(\frac{t+h(\epsilon)}{\delta^2/\epsilon}\right)+\frac{1}{h(\epsilon)}\mathbb{E}^\gamma\left|\left(\delta^2/\epsilon\right)\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-t\bar{\Psi}\right|
\end{aligned}
$$

by choosing $h(\epsilon)<1$ and defining

$$
\theta^\gamma(u)=\sup_{r>u}\mathbb{E}\left|\frac{1}{r}\int_0^r\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-\bar{\Psi}\right|.
$$

Thus, we have proven that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)}\Psi(Y_s^{\epsilon,y_0,\gamma},\gamma)ds-\bar{\Psi}\right|\leq \\
&\quad\leq\frac{T+1}{h(\epsilon)}\sup_{t\in[0,T]}\theta^\gamma\left(\frac{t+h(\epsilon)}{\delta^2/\epsilon}\right)+\frac{1}{h(\epsilon)}\sup_{t\in[0,T]}\mathbb{E}\left|\left(\delta^2/\epsilon\right)\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-t\bar{\Psi}\right| \qquad \text{(A.4)}
\end{aligned}
$$

Let us first treat the second term on the right hand side of (A.4). By the ergodic theorem, Lemma A.1, and Egoroff's theorem we know that there exists a set N_η with $\pi(N_\eta)\geq 1-\eta$ such that

$$
\lim_{\epsilon\downarrow 0}\sup_{\gamma\in N_\eta}\sup_{t\in[0,T]}\mathbb{E}\left|\left(\delta^2/\epsilon\right)\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-t\bar{\Psi}\right|=0
$$

So, if we choose $h(\epsilon)\downarrow 0$ such that

$$
\lim_{\epsilon\downarrow 0}\frac{1}{h(\epsilon)}\sup_{\gamma\in N_\eta}\sup_{t\in[0,T]}\mathbb{E}^\gamma\left|\left(\delta^2/\epsilon\right)\int_0^{\frac{t}{\delta^2/\epsilon}}\Psi(\hat{Y}_s^{y_0,\gamma},\gamma)ds-t\bar{\Psi}\right|=0
$$

we have that the second term on the right hand side of (A.4) goes to zero. Next, we treat the first term on the right hand side of (A.4). Since, the function $\theta^\gamma(u)$ is decreasing, we get that

$$
\theta^\gamma\left(\frac{t+h(\epsilon)}{\delta^2/\epsilon}\right)\leq\theta^\gamma\left(\frac{h(\epsilon)}{\delta^2/\epsilon}\right)
$$

Thus, we have obtained that for every $\gamma\in\Gamma$

$$
\sup_{t\in[0,T]}\theta^\gamma\left(\frac{t+h(\epsilon)}{\delta^2/\epsilon}\right)\leq\sup_{t\in[0,T]}\theta^\gamma\left(\frac{h(\epsilon)}{\delta^2/\epsilon}\right)=\theta^\gamma\left(\frac{h(\epsilon)}{\delta^2/\epsilon}\right)
$$

Notice that because $h(\epsilon)$ is chosen such that $\frac{\delta^2/\epsilon}{h(\epsilon)}\downarrow 0$, Lemma A.1 and Egoroff's theorem, imply that

$$
\lim_{\epsilon\downarrow 0}\sup_{\gamma\in N_\eta}\theta^\gamma\left(\frac{h(\epsilon)}{\delta^2/\epsilon}\right)=0
$$

Therefore, the first term on the right hand side of (A.4) goes to zero, if we can choose $h(\epsilon)$, such that $\sup_{\gamma\in N_\eta}\theta^\gamma\left(\frac{h(\epsilon)}{\delta^2/\epsilon}\right)/h(\epsilon)\downarrow 0$. This is a little bit tricky here because the argument of θ depends on $h(\epsilon)$. However, this can be done as follows. Fix $\beta\in(0,1)$ (e.g., $\beta=1/2$) and choose $h(\epsilon)\geq\left(\delta^2/\epsilon\right)^{1-\beta}$. Then, the monotonicity of f, implies that

$$\theta^\gamma\left(\frac{h(\epsilon)}{\delta^2/\epsilon}\right)\leq\theta^\gamma\left(\frac{1}{(\delta^2/\epsilon)^\beta}\right)\downarrow 0$$

This proves that we can choose $h(\epsilon)$ such that the first term of the right hand of (A.4) goes to zero. The claim follows, by noticing that the previous computations imply that we can choose $h(\epsilon)$ that may go to zero, but slowly enough, such that both the first and the second term on the right hand side of (A.4) go to zero. □

A.3 Ergodic theorems with perturbation by small drift-Uncontrolled case

Lemma A.4. *Consider the process $Y_t^{\epsilon,y_0,\gamma}$ satisfying the SDE*

$$Y_t^{\epsilon,x,y_0,\gamma}=y_0+\frac{\epsilon}{\delta^2}\int_0^t f(Y_s^{\epsilon,x,y_0,\gamma},\gamma)ds+\frac{1}{\delta}\int_0^t g(s,x,Y_s^{\epsilon,x,y_0,\gamma},\gamma)ds+\frac{\sqrt{\epsilon}}{\delta}\int_0^t\kappa(Y_s^{\epsilon,x,y_0,\gamma},\gamma)dW_s$$

Let us consider a function $\tilde{\Psi}:[0,T]\times\mathbb{R}^m\times\Gamma$ such that $\tilde{\Psi}(t,x,\cdot)\in L^2(\Gamma)\cap L^1(\pi)$ and define $\Psi(t,x,y,\gamma)=\tilde{\Psi}(t,x,\tau_y\gamma)$. We assume that the function $\Psi:[0,T]\times\mathbb{R}^m\times\mathbb{R}^{d-m}\times\Gamma\mapsto\mathbb{R}$ is measurable, piecewise constant in t and uniformly continuous in x with respect to (t,y).

Denote $\bar{\Psi}(t,x)\doteq\int_\Gamma\tilde{\Psi}(t,x,\gamma)\pi(d\gamma)$ for all $(t,x)\in[0,T]\times\mathbb{R}^m$. Fix $\eta>0$. Then there exists a set N_η such that $\pi(N_\eta)\geq 1-\eta$ and $h(\epsilon)\in\mathcal{H}_1^{N_\eta}$ such that

$$\lim_{\epsilon\downarrow 0}\sup_{\gamma\in N_\eta}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)}\Psi(s,x,Y_s^{\epsilon,x,y_0,\gamma},\gamma)ds-\bar{\Psi}(t,x)\right|=0$$

locally uniformly with respect to the parameter $x\in\mathbb{R}^m$.

Proof of Lemma A.4. Let us set $\hat{Y}_t^{\epsilon,x,y_0,\gamma}=Y_{\delta^2t/\epsilon}^{\epsilon,x,y_0,\gamma}$. Notice that $\hat{Y}_t^{\epsilon,x,y_0,\gamma}$ satisfies

$$\hat{Y}_t^{\epsilon,x,y_0,\gamma}=y_0+\int_0^t f(\hat{Y}_s^{\epsilon,x,y_0,\gamma},\gamma)ds+\frac{\delta}{\epsilon}\int_0^t g\left(\frac{\delta^2}{\epsilon}s,x,\hat{Y}_s^{\epsilon,x,y_0,\gamma},\gamma\right)ds+\int_0^t\kappa(\hat{Y}_s^{\epsilon,x,y_0,\gamma},\gamma)dW_s$$

Slightly abusing notation, we denote by $Y_t^{\epsilon,y_0,\gamma}$ and $\hat{Y}_t^{y_0,\gamma}$ the processes corresponding to $Y_t^{\epsilon,x,y_0,\gamma}$ and $\hat{Y}_t^{\epsilon,x,y_0,\gamma}$ with $c(t,x,y)=0$.

Lemma A.3 guarantees that the statement of the Lemma is true for $Y_t^{\epsilon,y_0,\gamma}$, namely that there exists a set N_η such that $\pi(N_\eta)\geq 1-\eta$ and $h(\epsilon)\in\mathcal{H}_1^{N_\eta}$ such that

$$\lim_{\epsilon\downarrow 0}\sup_{\gamma\in N_\eta}\sup_{t\in[0,T]}\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)}\Psi(s,x,Y_s^{\epsilon,y_0,\gamma},\gamma)ds-\bar{\Psi}(t,x)\right|=0. \tag{A.5}$$

The fact that the convergence is also locally uniform with respect to the parameter $x\in\mathbb{R}^m$ follows by the uniform continuity of Ψ in x. This implies that in Lemma A.3, we can choose the sequence $h(\epsilon)$ so that the convergence holds uniformly with respect to x in each bounded region, see for example Theorem II.3.11 in [31].

To translate this statement to what we need we use Girsanov's theorem on the absolutely continuous change of measures on the space of trajectories in $C([0,T];\mathbb{R}^{d-m})$. Let

$$\phi(s,x,y,\gamma)=-\kappa^{-1}(y,\gamma)g(s,x,y,\gamma)$$

and define the quantity

$$M_T^{\epsilon,\gamma}=e^{\frac{\delta}{\epsilon}\frac{1}{\sqrt{2}}\int_0^T\phi(\delta^2s/\epsilon,x,\hat{Y}_s^{y_0,\gamma},\gamma)dW_s-\frac{1}{2}\left(\frac{\delta}{\epsilon}\right)^2\frac{1}{2}\int_0^T\left\|\phi(\delta^2s/\epsilon,x,\hat{Y}_s^{y_0,\gamma},\gamma)\right\|^2ds}$$

Then, by the aforementioned Girsanov's theorem, for each$\gamma \in \Gamma$, $M_T^{\epsilon,\gamma}$ is a $\mathbb{P}^\gamma$ martingale. Therefore, we obtain

$$
\begin{aligned}
\mathbb{E}\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)} \Psi(s,x,Y_s^{\epsilon,x,y_0,\gamma},\gamma)ds &= \mathbb{E}\frac{\delta^2/\epsilon}{h(\epsilon)}\int_{\frac{t}{\delta^2/\epsilon}}^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}} \Psi(s,x,\hat{Y}_s^{\epsilon,x,y_0,\gamma},\gamma)ds \\
&= \mathbb{E}\left[\left(\frac{\delta^2/\epsilon}{h(\epsilon)}\int_{\frac{t}{\delta^2/\epsilon}}^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}} \Psi(s,x,\hat{Y}_s^{y_0,\gamma},\gamma)ds\right)M_T^{\epsilon,\gamma}\right]
\end{aligned}
$$

Next, we prove that, for every $\gamma \in \Gamma$, $M_T^{\epsilon,\gamma}$ converges to 1 in probability as $\epsilon \downarrow 0$. For this purpose, let us write $M_T^{\epsilon,\gamma} = e^{\mathcal{E}_T^{\epsilon,\gamma}}$, where

$$
\mathcal{E}_T^{\epsilon,\gamma} = \frac{\delta}{\epsilon}\frac{1}{\sqrt{2}}\int_0^T \phi(\delta^2 s/\epsilon, x, \hat{Y}_s^{y_0,\gamma},\gamma)dW_s - \frac{1}{2}\left(\frac{\delta}{\epsilon}\right)^2\frac{1}{2}\int_0^T \left\|\phi(\delta^2 s/\epsilon, x, \hat{Y}_s^{y_0,\gamma},\gamma)\right\|^2 ds
$$

Notice that

$$
\mathcal{E}_T^{\epsilon,\gamma} = N_T^{\epsilon,\gamma} - \frac{1}{2}\left\langle N^{\epsilon,\gamma}\right\rangle_T
$$

where

$$
N_T^{\epsilon,\gamma} = \frac{\delta}{\epsilon}\frac{1}{\sqrt{2}}\int_0^T \phi(\delta^2 s/\epsilon, x, \hat{Y}_s^{y_0,\gamma},\gamma)dW_s
$$

Since, ϕ is by assumption bounded, we obtain that $N_T^{\epsilon,\gamma}$ is a continuous martingale and $\langle N^{\epsilon,\gamma}\rangle_T$ is its quadratic variation. Boundedness of ϕ and the assumption $\delta/\epsilon \downarrow 0$ as $\epsilon \downarrow 0$, implies that

$$
\begin{aligned}
\lim_{\epsilon\downarrow 0}\sup_{\gamma\in\Gamma}\mathbb{E}\left\langle N^{\epsilon,\gamma}\right\rangle_T &= \lim_{\epsilon\downarrow 0}\sup_{\gamma\in\Gamma}\frac{1}{2}\left(\frac{\delta}{\epsilon}\right)^2\mathbb{E}\int_0^T \left\|\phi(\delta^2 s/\epsilon, x, \hat{Y}_s^{y_0,\gamma},\gamma)\right\|^2 ds \\
&= 0.
\end{aligned} \tag{A.6}
$$

Hence, uniformly in $\gamma \in \Gamma$, $\langle N^{\epsilon,\gamma}\rangle_T$ converges to 0 in probability and by Problem 1.9.2 in [21], the same convergence holds for the martingale $N_T^{\epsilon,\gamma}$ as well. Thus, we have obtained that uniformly in $\gamma \in \Gamma$

$$
M_T^{\epsilon,\gamma} = e^{\mathcal{E}_t^{\epsilon,\gamma}} \text{ converges to } 1 \text{ in probability, as } \epsilon \downarrow 0. \tag{A.7}
$$

Moreover, (A.7) together with Scheffé's theorem (Theorem 16.12 in [2]) imply that

$$
\sup_{\gamma\in\Gamma}\mathbb{E}\left|M_T^{\epsilon,\gamma} - 1\right| \to 0, \text{ as } \epsilon \downarrow 0. \tag{A.8}
$$

In fact, boundedness of ϕ implies that for every $\epsilon \in (0,1)$ and $\gamma \in \Gamma$, $M_T^{\epsilon,\gamma}$ is a square integrable martingale. The latter statement and convergence in probability (A.7), imply that

$$
\sup_{\gamma\in\Gamma}\mathbb{E}\left|M_T^{\epsilon,\gamma} - 1\right|^2 \to 0, \text{ as } \epsilon \downarrow 0. \tag{A.9}
$$

Now that (A.9) has been established, we continue with the proof of the lemma. Choose $h(\epsilon)$, such that (A.5) holds, we obtain

$$
\begin{aligned}
&\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)} \Psi(s,x,Y_s^{\epsilon,x,y_0,\gamma},\gamma)ds - \bar{\Psi}(t,x)\right| \\
&\quad = \mathbb{E}\left|\left(\frac{\delta^2/\epsilon}{h(\epsilon)}\int_{\frac{t}{\delta^2/\epsilon}}^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}} \Psi(s,x,\hat{Y}_s^{y_0,\gamma},\gamma)ds\right)M_T^{\epsilon,\gamma} - \bar{\Psi}(t,x)\right| \\
&\quad \le \mathbb{E}\left|\frac{\delta^2/\epsilon}{h(\epsilon)}\int_{\frac{t}{\delta^2/\epsilon}}^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}} \Psi(s,x,\hat{Y}_s^{y_0,\gamma},\gamma)ds - \bar{\Psi}(t,x)\right| \\
&\qquad + \mathbb{E}\left|\left(\frac{\delta^2/\epsilon}{h(\epsilon)}\int_{\frac{t}{\delta^2/\epsilon}}^{\frac{t+h(\epsilon)}{\delta^2/\epsilon}} \Psi(s,x,\hat{Y}_s^{y_0,\gamma},\gamma)ds\right)\left(M_T^{\epsilon,\gamma} - 1\right)\right|
\end{aligned}
$$

Clearly, the first term converges to zero by (A.5). The second term also converges to zero by Hölder's inequality, Lemma A.3 applied to Ψ^2 and (A.9).

The claim that the convergence is locally uniformly with respect to the parameter $x \in \mathbb{R}^m$ follows by the fact that this is true for (A.5). $\square$

A.4 Ergodic theorems with perturbation by small drift-Controlled case

Lemma A.5. *Fix $T < \infty$ and consider $\mathcal{A}$ to be the set of progressively measurable controls such that*

$$\int_0^T \|u^\epsilon(s)\|^2 \, ds < N, \tag{A.10}$$

where the constant N does not depend on ϵ, δ, T or γ and additionally such that for $\delta/\epsilon \ll 1$

$$\frac{1}{\epsilon}\mathbb{E}\int_0^{\frac{\delta^2 T}{\epsilon}} \|u^\epsilon(s)\|^2 \, ds \leq C\delta/\epsilon, \tag{A.11}$$

where the constant C depends on T, but not on ϵ, δ or γ. Consider the process $\bar{Y}_t^{\epsilon,x,y_0,\gamma}$ satisfying the SDE

$$\begin{aligned}\bar{Y}_t^{\epsilon,x,y_0,\gamma} &= y_0 + \frac{\epsilon}{\delta^2}\int_0^t f(\bar{Y}_s^{\epsilon,x,y_0,\gamma},\gamma)ds + \frac{1}{\delta}\int_0^t \left[g(s,x,\bar{Y}_s^{\epsilon,x,y_0,\gamma},\gamma) + \kappa(\bar{Y}_s^{\epsilon,x,y_0,\gamma},\gamma)u^\epsilon(s)\right] ds\\ &\quad + \frac{\sqrt{\epsilon}}{\delta}\int_0^t \kappa(\bar{Y}_s^{\epsilon,x,y_0,\gamma},\gamma)dW_s\end{aligned}$$

Let us consider a function $\tilde{\Psi} : [0,T] \times \mathbb{R}^m \times \Gamma$ such that $\tilde{\Psi}(t,x,\cdot) \in L^2(\Gamma) \cap L^1(\pi)$ and define $\Psi(t,x,y,\gamma) = \tilde{\Psi}(t,x,\tau_y\gamma)$. We assume that the function $\Psi : [0,T] \times \mathbb{R}^m \times \mathbb{R}^{d-m} \times \Gamma \mapsto \mathbb{R}$ is measurable, piecewise constant in t and uniformly continuous in x with respect to (t,y).

Denote $\bar{\Psi}(t,x) \doteq \int_\Gamma \tilde{\Psi}(t,x,\gamma)\pi(d\gamma)$ for all $(t,x) \in [0,T] \times \mathbb{R}^m$. Fix $\eta > 0$. Then there exists a set N_η such that $\pi(N_\eta) \geq 1-\eta$ and $h(\epsilon) \in \mathcal{H}_1^{N_\eta}$ such that

$$\lim_{\epsilon\downarrow 0}\sup_{\gamma\in N_\eta}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)}\Psi(s,x,\bar{Y}_s^{\epsilon,x,y_0,\gamma},\gamma)ds - \bar{\Psi}(t,x)\right| = 0$$

locally uniformly with respect to the parameter $x \in \mathbb{R}^m$.

Proof of Lemma A.5. Let us set $\hat{\bar{Y}}_t^{\epsilon,x,y_0,\gamma} = \bar{Y}_{\delta^2 t/\epsilon}^{\epsilon,x,y_0,\gamma}$. Notice that $\hat{\bar{Y}}_t^{\epsilon,x,y_0,\gamma}$ satisfies

$$\begin{aligned}\hat{\bar{Y}}_t^{\epsilon,x,y_0,\gamma} &= y_0 + \int_0^t f(\hat{\bar{Y}}_s^{\epsilon,x,y_0,\gamma},\gamma)ds + \frac{\delta}{\epsilon}\int_0^t \left[g\left(\frac{\delta^2}{\epsilon}s,x,\hat{\bar{Y}}_s^{\epsilon,x,y_0,\gamma},\gamma\right) + \kappa(\hat{\bar{Y}}_s^{\epsilon,x,y_0,\gamma},\gamma)u^\epsilon\left(\delta^2 s/\epsilon\right)\right] ds\\ &\quad + \int_0^t \kappa(\hat{\bar{Y}}_s^{x,y_0,\gamma,\epsilon},\gamma)dW_s\end{aligned}$$

Essentially, based on the condition of the allowable controls (A.10), the arguments of the uncontrolled case, Lemma A.4, go through verbatim. The only place that needs some discussion is in regards to the proof of the statement corresponding to (A.6). Let us show now how this term can be treated. In the controlled case we have that

$$\phi(s,x,y,\gamma) = -\kappa^{-1}(y,\gamma)g(s,x,y,\gamma) - u^\epsilon(s)$$

and we want to prove that for every $\gamma \in \Gamma$

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left\langle N^{\epsilon,\gamma}\right\rangle_T = \lim_{\epsilon\downarrow 0}\frac{1}{2}\left(\frac{\delta}{\epsilon}\right)^2\mathbb{E}\int_0^T \left\|\phi(\delta^2 s/\epsilon,x,\hat{Y}_s^{y_0,\gamma},\gamma)\right\|^2 ds = 0. \tag{A.12}$$

It is clear that

$$\mathbb{E}\left\langle N^{\epsilon,\gamma}\right\rangle_T = \frac{1}{2}\left(\frac{\delta}{\epsilon}\right)^2 \mathbb{E}\int_0^T \left\|\phi(\delta^2 s/\epsilon, x, \hat{Y}_s^{y_0}, \gamma)\right\|^2 ds$$
$$\leq \left(\frac{\delta}{\epsilon}\right)^2 \mathbb{E}\int_0^T \left\|\kappa^{-1}(\hat{Y}_s^{y_0},\gamma) g(\delta^2 s/\epsilon, x, \hat{Y}_s^{y_0}, \gamma)\right\|^2 ds + \left(\frac{\delta}{\epsilon}\right)^2 \mathbb{E}\int_0^T \left\|u^{\epsilon}\left(\delta^2 s/\epsilon\right)\right\|^2 ds$$

The first term of the right hand side of the last display goes to zero by the boundedness of $\left\|\kappa^{-1}g\right\|^2$ (as in Lemma A.4). So we only need to consider the second term. Here we use Condition A.11. In particular, we notice that Condition A.11 gives

$$\lim_{\epsilon\downarrow 0}\left(\frac{\delta}{\epsilon}\right)^2 \mathbb{E}\int_0^T \left\|u^{\epsilon}\left(\delta^2 s/\epsilon\right)\right\|^2 ds = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{E}\int_0^{\delta^2 T/\epsilon}\left\|u^{\epsilon}(s)\right\|^2 ds = 0$$

uniformly in $\gamma\in\Gamma$. Thus we have completed the proof of (A.12). This concludes the proof of the lemma. □

A.5 Ergodic theorem with explicit dependence on the slow process.

In this subsection we consider the pair $(\bar{X}_s^{\epsilon,\gamma}, \bar{Y}_s^{\epsilon,\gamma})$ satisfying (3.3) and the purpose is to prove Lemma A.6.

Lemma A.6. *Consider the set-up, assumptions and notations of Lemma A.5. Fix $\eta > 0$. Then there exists a set N_η such that $\pi(N_\eta)\geq 1-\eta$ and $h(\epsilon)\in\mathcal{H}_1^{N_\eta}$ such that*

$$\lim_{\epsilon\downarrow 0}\sup_{\gamma\in N_\eta}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{1}{h(\epsilon)}\int_t^{t+h(\epsilon)}\Psi\left(s,\bar{X}_s^{\epsilon,\gamma},\bar{Y}_s^{\epsilon,\gamma},\gamma\right)ds - \bar{\Psi}(t,\bar{X}_t^{\epsilon,\gamma})\right| = 0.$$

Sketch of proof of Lemma A.6. Due to Lemma A.5, the statement follows by using the standard argument of freezing the slow component, see for example Chapter 7.9 of [12] or [28]. Details are omitted. □

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