The shape of large balls
in highly supercritical percolation

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Abstract

We exploit a connection between distances in the infinite percolation cluster, when the parameter is close to one, and the discrete-time TASEP on $\mathbb{Z}$. This shows that when the parameter goes to one, large balls in the cluster are asymptotically shaped near the axes like arcs of parabola.

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1 Introduction

Our main issue in this paper is to describe the shape of large balls in the infinite two-dimensional percolation cluster, when the percolation parameter is close to one. This problem is closely related to first-passage percolation, a model introduced in the 60's by Hammersley and Welsh [7] in which one estimates the minimal distance $D(\mathbf{0}, x)$ between the origin $\mathbf{0}$ and a given point x of $\mathbb{Z}^2$, when edges have i.i.d. positive finite lengths. Distances in the cluster correspond in this framework to the extreme case where edges have lengths 1 with probability $p \in (0, 1)$ and infinite length with probability $1 - p$. We refer to [3] for a recent survey on first passage percolation and shape theorems.

In first passage percolation, Kingman's subadditive ergodic theorem is the crucial tool to study the asymptotics of distances between distant points. Garet and Marchand ([5], Th.3.2) adapted this argument to the case where edges may have infinite length. They proved the existence, for all z in $\mathbb{R}^2$, of a constant $\mu(z)$ such that, if we denote by $[nz]$ one of the closest lattice points to nz, on the event[1] $\{\mathbf{0} \leftrightarrow \infty\}$ and along the subsequence $\{\mathbf{0} \leftrightarrow [nz]\}$, we have a.s.

$$\lim_{\substack{n \to \infty \\ \mathbf{0} \leftrightarrow [nz]}} \frac{D(\mathbf{0}, [nz])}{n} = \mu(z). \tag{1.1}$$

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[1]We write $v \leftrightarrow w$ if $v, w \in \mathbb{Z}^2$ belong to the same connected component, and $v \leftrightarrow \infty$ if v is in the infinite cluster.

Very little is known about μ, except when z belongs to the *oriented percolation cone*. This cone is defined as the set of points z such that there exists with probability 1 an infinite open path in the direction of z taking only east/north edges. For all z in this cone, $\mu(z)$ is obviously equal to $|z|_1$. Marchand [11] showed that μ differs from $|.|_1$ outside this cone and previously Durrett [4] had shown that this cone is delimited near the x-axis by a line $y = t_p x$ where $t_p = 1 - p + \mathrm{o}(1 - p)$. Hence, we are interested in this paper in the remaining region $|y| \leq (1 - p)x$.

Theorem 1.1. *For all $0 \leq \lambda \leq 1$, on the event $\{\mathbf{0} \leftrightarrow \infty\}$, almost surely,*

$$\mu_p(\lambda) := \lim_{\substack{n \to \infty \\ \mathbf{0} \leftrightarrow [n(1,\lambda(1-p))]}} \frac{D\left(\mathbf{0}, [n(1, \lambda(1 - p))]\right)}{n} = 1 + (1 - p)\frac{1 + \lambda^2}{2} + \mathcal{O}\left((1 - p)^2\right),$$

where the $\mathcal{O}\left((1 - p)^2\right)$ term is uniform in λ.

Let us also note that we actually obtain for all p the following non-asymptotic lower bound for μ_p, which is sharp when p goes to one. It is a consequence of Corollary 2.6 in Section 2.

Theorem 1.2. *For all $p > 1/2$ and $0 \leq \lambda \leq 1$,*

$$\mu_p(\lambda) \geq 2 - \sqrt{1 - (1 - p)(1 + \lambda^2) + \lambda^2(1 - p)^2}.$$

Outside the cone, the exact limiting shape of large balls remains unknown. Very recently, Auffinger and Damron [1] showed that the corresponding limiting shape in first passage percolation is differentiable at the edge of the cone, thereby excluding the possibility of a polygon. It is believed that the limiting shape is strictly convex near axes, our result roughly says that, when p is close to one, the four corners of L^1 balls are replaced by curves looking like arcs of parabola, as in the (schematic) figure below.
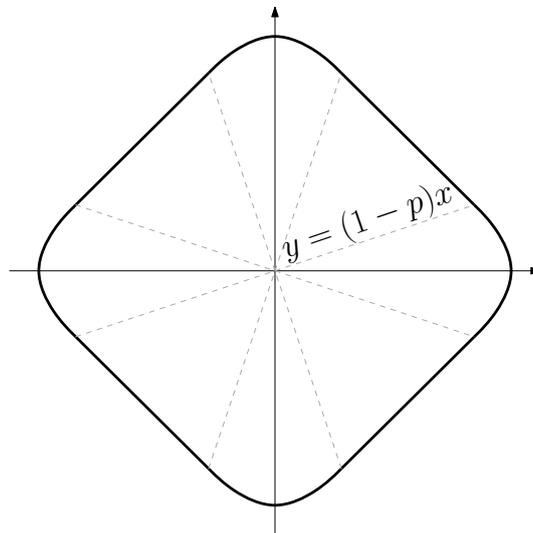
The general strategy of the proof is based on a connection introduced in [2] between the discrete-time *totally asymmetric simple exclusion process* (TASEP) and distances on the percolation cluster. The TASEP is used by physicists as a simple model for nonequilibrium phenomena, it is known to be connected to a large class of combinatorial models : the corner-growth model, last passage percolation, random matrices (see [10] for a survey). It seems that this connection with distances in the infinite percolation cluster appeared for the first time in [2].

The exact correspondence with TASEP holds with a simplified model of percolation, which is described in Section 2. The lower bound for μ_p follows easily. For the upper bound, we first need a careful analysis of geodesics in the simplified model in order to show that they can be modified into an open path in the percolation cluster.

A novelty of the present paper is that we connect the limiting shape of large balls in percolation to the parabolic limiting shape of out-of-equilibrium TASEP on $\mathbb{Z}$. This strategy differs notably from that of [2] where we worked in a large box around the x-axis, and thus percolation was in correspondence with the stationary TASEP on a finite interval. This restricted the analysis to points whose height was sublinear in n and therefore gave results only for $\lambda = 0$. This correspondence with TASEP is now more transparent and it allows us to use more efficiently some known results on TASEP.

As already mentioned, we introduce in the proof a simplified model, called *Cross Model* in the sequel. A nice feature of this model is that we are able to give an explicit description of geodesics in terms of the motion of particles in the TASEP (see Lemma 4.1 below). Note also that, since fluctuations of TASEP are known to be of order $n^{1/3}$ ([9], Theorem 1.2), fluctuations of distances in the Cross Model are of the same order, as believed for first passage percolation on $\mathbb{Z}^2$.

2 The connection with TASEP

2.1 Percolation in the Cross Model

As a first step, we study a two-dimensional random graph in which distances to the origin behave much like distances in the infinite percolation cluster and are strongly connected to the TASEP.

Here is the context we will deal with in the whole section. Let $\mathbb{Z}^\times$ be the graph on the vertices of $\mathbb{Z}^2$, with three kinds of edges:

- Vertical edges $\{(i, j) \to (i, j+1), i \in \mathbb{Z}, j \in \mathbb{Z}\}$;
- Horizontal edges $\{(i, j) \to (i+1, j), i \in \mathbb{Z}, j \in \mathbb{Z}\}$;
- Diagonal edges $\{(i, j) \to (i+1, j+1) \text{ and } (i, j) \to (i+1, j-1)\}$.

We assign length 1 to each vertical and horizontal edge, and length 2 to each diagonal edge. We now set $\varepsilon = 1 - p > 0$ and call *Cross Model* the percolation on $\mathbb{Z}^\times$ defined by:

(i) Diagonal and vertical edges are all open,

(ii) Each horizontal edge is open (resp. closed) independently with probability $1 - \varepsilon$ (resp. ε).

Remark 2.1. *Let us first motivate this simplified model.*

1. *In classical percolation on $\mathbb{Z}^2$ with ε close to zero, a very large proportion (greater than $1 - 6\varepsilon^2$) of unit squares of $\mathbb{Z}^2$ have at most one closed edge. In such squares, the addition of two diagonal edges of length 2 does not change the time needed to cross the square from one corner to the other.*

2. *The opening of vertical edges should not be significant at first order since, as we will see later, a typical geodesic between $\mathbf{0}$ and $(n, n\lambda\varepsilon)$ in $\mathbb{Z}^\times$ takes less than $2n\varepsilon$ vertical edges, a proportion ε only of them being closed in the original model of percolation.*

For $(i, j) \in \mathbb{Z}^2$, let $D^\times(i, j)$ be the distance between $(0, 0)$ and (i, j) in the Cross Model (see an example in Fig. 1). Since vertical and diagonal edges are open, every point in $\mathbb{Z}^\times$ is connected in the Cross Model to $\mathbf{0}$, hence $D^\times(i, j)$ is finite for every (i, j). Let us write down some obvious consequences of the construction: for $i \geq 0$,

- All the geodesics joining $\mathbf{0}$ to (i,j) only make N,NE,E,SE,S steps.
- Along each vertical edge $|D^\times(i,j) - D^\times(i,j+1)| = 1$.
- Along each open horizontal edge, $D^\times(i+1,j) = D^\times(i,j) + 1$.
- Along each closed horizontal edge, $D^\times(i+1,j) = D^\times(i,j) + 1$ or $+3$.
- Along each diagonal edge, $D^\times(i+1,j\pm 1) = D^\times(i,j) + 0$ or $+2$.

We also set $\mathbf{D}_i^\times$ for the (infinite) i-th column of distances $\{D^\times(i,j), j \in \mathbb{Z}\}$. Note that obviously $\mathbf{D}_0^\times = (\dots, 2, 1, 0, 1, 2, \dots)$. The aim of the present section is to identify the law of the Markov chain $\left(\mathbf{D}_i^\times\right)_{i \geq 0}$.
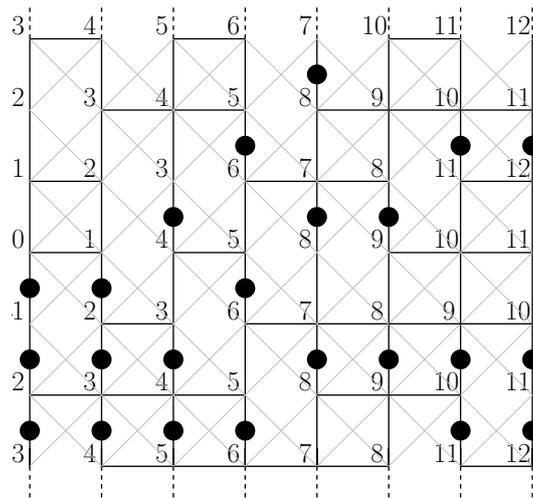

Figure 1: A configuration of percolation in $\mathbb{Z}^\times$ with the associated distances $\mathbf{D}_0^\times, \dots \mathbf{D}_7^\times$, together with particles. Note the importance of diagonal edges: $D^\times(2,1) = 3$ thanks to the diagonal edge $(1,0) \to (2,1)$.

To do so, we introduce a particle system closely related to the process $(\mathbf{D}_i^\times)_{i \geq 0}$. Let us consider the state space $\{\bullet, \circ\}^{\mathbb{Z}}$ (identified to $\{1, 0\}^{\mathbb{Z}}$), and denote its elements in the form

$$(\dots, y^{-2}, y^{-1}, y^0, y^1, y^2, \dots).$$

Let $(\mathbf{Y}_i)_{i \geq 0}$ be the process with values in $\{\bullet, \circ\}^{\mathbb{Z}}$ defined as follows :

$$\forall j \in \mathbb{Z}, \qquad Y_i^j = \begin{cases} \bullet = 1 \text{ if } D^\times(i,j) = D^\times(i,j-1) - 1. \\ \circ = 0 \text{ if } D^\times(i,j) = D^\times(i,j-1) + 1. \end{cases}$$

Let say that the site j is *occupied* by a particle at time i if $Y_i^j = \bullet$ and *empty* otherwise. We think about a particle at site j at time i as being actually located on the edge between $(i, j-1)$ and (i,j) as drawn on our pictures.

The main observation is that if we see time going from left to right, then the displacement of particles follows a discrete-time TASEP on $\mathbb{Z}$, that we define now:

Definition 2.2. *The discrete-time* Totally Asymmetric Simple Exclusion Process *(TASEP) on $\mathbb{Z}$ with parameter α is the Markov chain with state space $\{\bullet, \circ\}^{\mathbb{Z}}$ with initial condition y_0 defined by*

$$\dots = y_0^{-3} = y_0^{-2} = y_0^{-1} = y_0^0 = \bullet, \qquad \circ = y_0^1 = y_0^2 = y_0^3 = \dots$$

and whose evolution is as follows: at time $t + 1$, for each j, a particle at position j (if any) moves one step forward if the site $j + 1$ is empty at time t, with probability α and independently from the other particles.
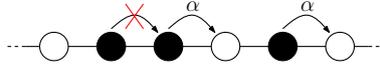
Proposition 2.3. *The process $(\mathbf{Y}_i)_{i \geq 0}$ has the law of TASEP on $\mathbb{Z}$ with parameter ε.*

Proposition 3 in [2] states the same result in the framework of a finite interval of $\mathbb{Z}$, and the proof is identical. Let us say however some words about it. The main point is that modifications in the particle configuration might only occur when a site (i, j) at some distance ℓ lies between two sites of the line $\{i\} \times \mathbb{Z}$ which are at a distance $\ell + 1$. In this case there is, at time i, a particle below j and no particle above j. The particle below j moves one step forward if and only if the horizontal edge $(i, j) \to (i + 1, j)$ is closed (which occurs with probability ε).

Let $J_{n,j}$ be the *current* of the TASEP at time n in j, that is the number of particles which have passed through position j before time n:

$$J_{n,j} := \operatorname{card}\left\{\ell > j, y_n^\ell = \bullet\right\}.$$

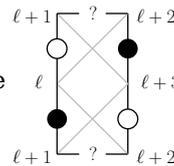**Lemma 2.4.** *For each $n, j \geq 0$,*

$$D^\times(n, j) = n + j + 2J_{n,j},$$

where $J_{n,j}$ is the current of the TASEP with parameter ε.

Proof. Let us first prove this assertion for $j = 0$. As already noticed, distances along each horizontal edge $(i, 0) \to (i + 1, 0)$ differ by 1 or 3. This implies that

$$D^\times(n, 0) = n + 2 \operatorname{card}\left\{0 \leq i \leq n - 1 \mid D^\times(i + 1, 0) = D^\times(i, 0) + 3\right\}.$$

But $D^\times(i+1, 0) = D^\times(i, 0) + 3$ occurs only in the case , *i.e.* when a particle jumps across the x-axis. Then $D^\times(n, 0) \overset{\text{law}}{=} n + 2J_{n,0}$.

To prove the Lemma for any $j \geq 0$, we have to compute $D^\times(n, j) - D^\times(n, 0)$. But by construction of the particles

$$D^\times(n, j) - D^\times(n, 0) = j - 2 \operatorname{card}\left\{\text{particles between } 0 \text{ and } j \text{ at time } n\right\},$$

and this finishes the proof. □

2.2 Asymptotics in the Cross Model

Proposition 2.5. *For any $x > 0$ and $-x\varepsilon \leq y \leq x\varepsilon$, we have almost surely and in L^1*

$$\frac{J_{\lfloor Nx \rfloor, \lfloor Ny \rfloor}}{N} \overset{N \to \infty}{\to} j(x, y) := \tfrac{1}{2}(x - y) - \tfrac{1}{2}\sqrt{(1 - \varepsilon)(x^2 - y^2/\varepsilon)}.$$

Note that if on the contrary $y > x\varepsilon$, it is clear that $J_{\lfloor Nx \rfloor, \lfloor Ny \rfloor}/N \to 0$, because the right-most particle in the TASEP is at time N at position $N\varepsilon + \mathrm{o}(N)$.

Proof of Proposition 2.5. With extra work this can be seen as a consequence of the work by Jockusch, Propp and Shor on the discrete-time TASEP ([8], Theorem 2). Here we deduce it from the results by Johansson [9] on last passage percolation (LPP) with geometric passage times with parameter ε (we refer to [12] for the connection between discrete-time TASEP and last passage percolation).

For the reader's convenience, we detail the computations. To define the model of LPP with geometric weights, let $(g_{i,j})_{i,j\geq 1}$ be i.i.d. geometric variables with parameter ε. For a point (i,j) in the quadrant $\{i,j \geq 1\}$, we write $G^\star(x,y)$ for the *last passage time* at (x,y), *i.e.*

$$G^\star(x,y) := \max_{\gamma:(1,1)\to(x,y)} \sum_{(i,j)\in\gamma} g_{i,j}$$

where the $\max$ is taken over the $\binom{x+y-2}{x-1}$ paths with North/East steps going from $(1,1)$ to (x,y). Johansson ([9], Theorem 1.1, see also [12], Theorem 2.2) has shown that for all $a, b > 0$

$$\frac{G^\star(\lfloor Na \rfloor, \lfloor Nb \rfloor)}{N} \to \Psi(a,b) := \frac{a+b+2\sqrt{(1-\varepsilon)ab}}{\varepsilon},$$

where the convergence holds a.s. and in L^1 (note that the p in Johansson's article, corresponds to $\varepsilon = 1-p$ with our notations). Thanks to a plain correspondence between TASEP and LPP (see [12] Proposition 1.2) there is coupling between LPP and discrete-time TASEP with parameter ε such that for any integers $A \geq B \geq 1$

$$J_{G^\star(A,B),A-B} = B. \tag{2.1}$$

For $a \geq b > 0$, let us write

$$\frac{J_{\lfloor N\Psi(a,b)\rfloor,\lfloor N(a-b)\rfloor}}{N} = \frac{J_{G^\star(\lfloor Na\rfloor,\lfloor Nb\rfloor),\lfloor N(a-b)\rfloor}}{N} + \left(\frac{J_{\lfloor N\Psi(a,b)\rfloor,\lfloor N(a-b)\rfloor}}{N} - \frac{J_{G^\star(\lfloor Na\rfloor,\lfloor Nb\rfloor),\lfloor N(a-b)\rfloor}}{N}\right).$$

Using (2.1), the first term in the right-hand side goes to b. The second term goes to zero almost surely since for any n, n', j we have $|J_{n,j} - J_{n',j}| \leq |n - n'|$.

We are looking for $a = a(x,y)$ and $b = b(x,y)$ such that $\Psi(a,b) = x$ and $a - b = y$. This is possible if $y \leq x\varepsilon$ and in this case we obtain

$$\frac{J_{\lfloor Nx\rfloor,\lfloor Ny\rfloor}}{N} \overset{N\to\infty}{\to} b(x,y) = \tfrac{1}{2}(x-y) - \tfrac{1}{2}\sqrt{(1-\varepsilon)(x^2 - y^2/\varepsilon)}.$$

$\square$

Taking $x = 1, y = \varepsilon\lambda$ in the Proposition, we obtain with Lemma 2.4 the following asymptotics for the distances in the Cross Model. Note that from now on, we skip the integer parts $\lfloor . \rfloor$ in order to lighten notations.

Corollary 2.6. *In the Cross Model, for any $0 < \varepsilon, \lambda \leq 1$,*

$$\frac{D^\times(n, n\varepsilon\lambda)}{n} \overset{n\to\infty}{\to} f(\lambda,\varepsilon) := 2 - \sqrt{1 - \varepsilon(1+\lambda^2) + \lambda^2\varepsilon^2} = 1 + \frac{\varepsilon}{2}(1+\lambda^2) + \mathcal{O}(\varepsilon^2),$$

where the convergence is almost sure and in L^1.

3 The lower bound

With the asymptotics found in the Cross Model, we are now able to obtain the lower bound for the distances in standard percolation in $\mathbb{Z}^2$. Adding diagonal edges to $\mathbb{Z}^2$

decreases distances, so by an obvious coupling between percolation in $\mathbb{Z}^2$ and in the Cross Model we have

$$\frac{D(\mathbf{0}, (n, n\varepsilon\lambda))}{n} \geq \frac{D^\times(n, n\varepsilon\lambda)}{n}.$$

Letting n go to infinity, we get

$$\liminf_{n\to\infty} \frac{D(\mathbf{0}, (n, n\varepsilon\lambda))}{n} \geq f(\lambda, \varepsilon) = 1 + \frac{\varepsilon}{2}(1 + \lambda^2) + \mathcal{O}(\varepsilon^2).$$

This result holds almost surely so in particular on the event $\{\mathbf{0} \leftrightarrow \infty\}$.

4 The upper bound

The proof of the upper bound is more delicate. We first present a canonical construction of a geodesic in the Cross Model, and then show how to modify it to obtain an almost optimal path between $\mathbf{0}$ and $(n, n\varepsilon\lambda)$ in the original model.

4.1 The construction of a canonical geodesic

Starting from the end $E := (n, n\varepsilon\lambda)$, we construct backwards a geodesic $\pi^\times$, in the Cross Model, joining $\mathbf{0}$ to E. An important feature of this construction is that it only depends on the trajectory of the particles.

The reader is invited to follow the construction on the following example (here $E = (7, 1)$ and $\pi^\times$ is drawn in red, $(\sigma_1, \sigma_2, \dots)$ stands for the sequence of particles ranked according to their height):
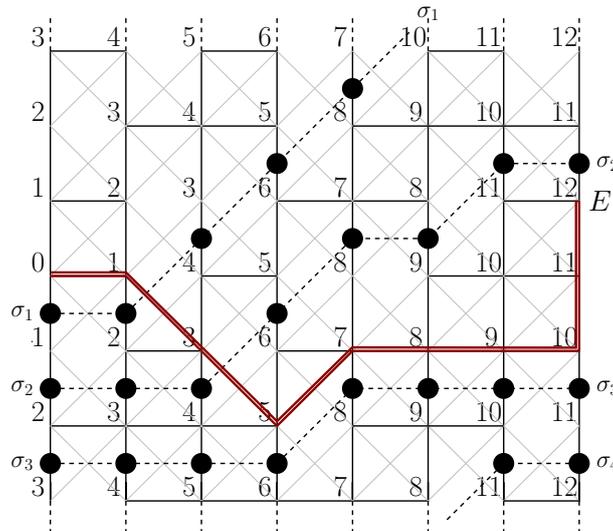
The path starts (backwards) from E by taking some vertical edges in the following way:

- if there is no particle on the vertical edge just below E (as in the example), we go down until finding the first vertex that is just below an empty edge and just above an edge with a particle (in the example, until being at $(7, -1)$ just above particle σ_3);

- if, on the contrary, there is a particle on the edge just below E, we go up until finding the first vertex that is below an empty edge and above an edge with a particle.

We now proceed from right to left by taking n horizontal or diagonal edges going to zero, so that each site of the path is just below an empty edge and just above a particle.

Let us write it more formally. After the first vertical edges, we are at a site with a certain particle σ_p just below; let us denote by (i, j) this site, and ℓ its distance to the origin.

Then, three cases may occur:

Case A. Particle σ_p had jumped at time $i - 1$. Then the path follows the diagonal edge $(i, j) \to (i - 1, j - 1)$. Note that there is still an empty edge just above, if not σ_p would not have moved.
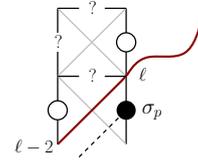
Case B. Particle σ_{p-1} was just above σ_p at time $i - 1$. Then necessarily it moved (since edge $(i, j) \to (i, j + 1)$ is now empty). The path follows the diagonal edge $(i, j) \to (i - 1, j + 1)$.
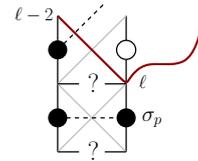
Case C. At time $i - 1$, there is no particle above σ_p. This implies that $(i, j) \to (i - 1, j)$ is open (if not, σ_p would have moved). The path follows this edge, and doing so it stays just above σ_p.
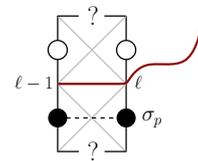
Let us record two features of this path $\pi^\times$:

- it takes only E, NE, SE edges until reaching $\{x = n\}$ (it takes exactly n such steps), and then possibly taking some additional vertical edges in the form $(n, j) \to (n, j \pm 1)$ to reach E;

- it takes a diagonal edge $(i - 1, j) \to (i, j \pm 1)$ only if the horizontal edge $(i - 1, j) \to (i, j)$ is closed.

Lemma 4.1. *The path $\pi^\times$ is a geodesic between $\mathbf{0}$ and $(n, n\varepsilon\lambda)$ for the Cross Model. Moreover, $\pi^\times$ depends only on the trajectories of particles.*

Proof. The second assertion is clear by construction. Besides, the path always goes through vertices which are just above a particle and below an empty edge. Thus, when the first coordinate is zero, it is necessarily at the origin, since this is the only site on the first column which satisfies this property.

Writing
$$\pi^\times = (x_0 = 0, x_1, \dots, x_L = E),$$

we have to prove that for each i the length of the edge (x_{i-1}, x_i) is equal to $D^\times(x_i) - D^\times(x_{i-1})$.

- By construction, if we had to take at the first stage r vertical edges, these edges would lead to a site which is at distance $D^\times(n, n\varepsilon\lambda) - r$ from the origin.

- When the path takes an horizontal edge (x_{i-1}, x_i) (case C above), this edge is open and then $D^\times(x_i) = D^\times(x_{i-1}) + 1$.

- The case of a diagonal edge (x_{i-1}, x_i) remains (cases A,B above). We treat case A. Set $\ell = D^\times(x_i)$. Since there is a particle on the edge $(x_i, x_i - (0, 1))$, then $D^\times(x_i - (0, 1)) = \ell + 1$. Since σ_p has jumped then $D^\times(x_{i-1}) = \ell + 1 - 3$.

$\square$

4.2   How to bypass bad edges

The aim of this section is to construct from the path $\pi^\times$ obtained by Lemma 4.1 an open path of $\mathbb{Z}^2$ which is barely longer than $\pi^\times$.

Recall that $\pi^\times$ can take either horizontal, vertical or diagonal edges. Since we want to construct an open path on $\mathbb{Z}^2$, we need to replace its diagonal edges and its final closed vertical edges by detours of open edges.

We begin by doing a transformation which enables us to replace the diagonal edges of $\pi^\times$ without changing the length of the path. If $\pi^\times$ takes a diagonal edge $(i,j) \to (i+1, j \pm 1)$ then we replace this edge by the path $(i,j) \to (i, j \pm 1) \to (i+1, j \pm 1)$:
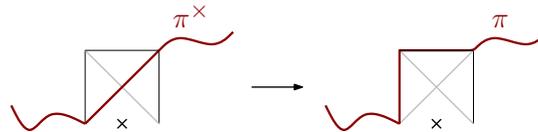
We denote the new path by π. We denote by $\mathcal{K}$ the set of edges which are either a vertical edge of $\pi^\times$ or an edge that appears in π but not in $\pi^\times$. Notice that $\pi^\times, \pi$ and $\mathcal{K}$ depends only on the TASEP. The new path is a path on $\mathbb{Z}^2$ and it just remains to bypass its closed edges. We call those closed edges the bad edges of π and denote the set of all bad edges by $\mathcal{B}$. By construction, $\mathcal{B}$ is a subset of $\mathcal{K}$. We shall also write $K = |\mathcal{K}|$ and $B = |\mathcal{B}|$.

Lemma 4.2. *For ε small enough, for all n large enough, we have $\mathbb{E}(K) \leq 2n\varepsilon$.*

Proof. The sum of the number of diagonal edges in $\pi^\times$ and of the number of final vertical edges, by definition of the Cross Model, is equal to $D^\times(n, n\varepsilon\lambda) - n$. Corollary 2.6 implies that $\mathbb{E}\left((D^\times(n, n\varepsilon\lambda) - n)/n\right)$ converges to a limit which is strictly less than ε for small enough ε. The lemma follows from the fact that K is at most $2(D^\times(n, n\varepsilon\lambda) - n)$. $\qquad\square$

Consider the dual graph $(\mathbb{Z}^2)^\star$ of $\mathbb{Z}^2$ and associate to each edge $e \in \mathbb{Z}^2$ the unique edge $e^\star$ of the dual which crosses e. We say that $e^\star$ is open (resp. closed) if e is open (resp. closed). For each (closed) bad edge e, consider the set $\mathcal{C}^\star(e)$ defined by

$$\mathcal{C}^\star(e) = \{\text{closed edges of } (\mathbb{Z}^2)^\star \text{ connected to } e^\star \text{ by a path of closed edges}\}.$$

Define its boundary

$$\Delta\mathcal{C}^\star(e) = \{\text{open edges of } (\mathbb{Z}^2)^\star \text{ which share at least one vertex with an edge of } \mathcal{C}^\star(e)\}.$$

What happens around a bad edge e:
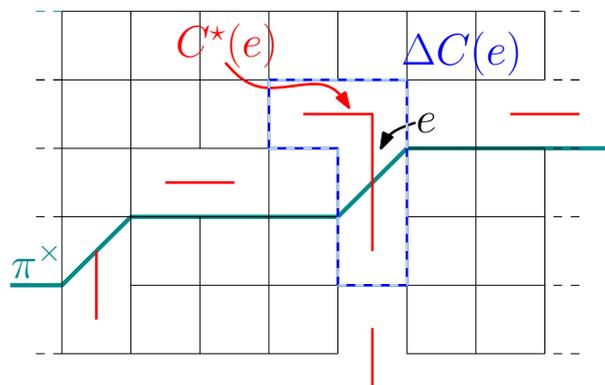
We denote by $\Delta\mathcal{C}(e)$ the set of (open) edges associated to $\Delta\mathcal{C}^{\star}(e)$ in the initial graph, and set

$$\Delta\mathcal{C}(\pi) = \bigcup_{e \in \mathcal{B}} \Delta\mathcal{C}(e).$$

The following topological proposition will be crucial to build from $\pi^{\times}$ a short open path in the percolation cluster.

Proposition 4.3. *On the event $\{\mathbf{0} \leftrightarrow (n, n\lambda\varepsilon) \leftrightarrow \infty\}$ there exists an open path in $\mathbb{Z}^2$ from $\mathbf{0}$ to $(n, n\lambda\varepsilon)$ which only uses edges of π or of $\Delta\mathcal{C}(\pi)$.*

Proof. The proof is purely topological and is postponed to Appendix. □

In view of this proposition, it remains to bound the cardinality of $\Delta\mathcal{C}(\pi)$ to get an upper bound of the length of the geodesic between $\mathbf{0}$ and $(n, n\lambda\varepsilon)$. Let us describe the probability distribution of the set of open edges conditional on the position of particles. An edge will be called *unbiased* if, conditional on the position of the particles which are inherited from the Cross Model, it is open with probability $1 - \varepsilon$ independently from all the states of the other edges. An edge which is not unbiased is called *biased*.

In the analysis of the state of an edge, four cases may arise.

Case 1. Vertical edges are all unbiased by definition of the Cross Model.

Case 2. The two configurations of particles leading to an unbiased horizontal edge are the following:
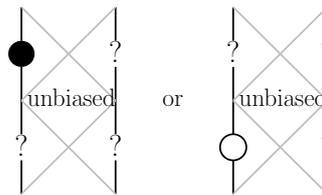
Indeed, the state of the horizontal edge has no influence on the motion of particles.

Case 3. The following configuration leads to a closed edge:
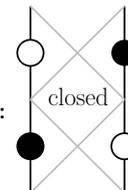
Case 4. The following configuration leads to an open edge:
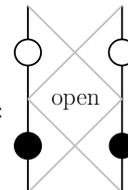
Denote by $\mathcal{T}$ the σ-field generated by the particles ($\mathcal{T}$ stands for TASEP). Thus, conditional on $\mathcal{T}$, some edges are open, some edges are closed and the other edges are unbiased. Now we use the previous facts to prove the following two lemmas:

Lemma 4.4. *The edges of $\mathcal{K}$ are unbiased.*

Proof. Vertical edges of $\mathcal{K}$ are unbiased. This is Case 1 above. Let us now consider an horizontal edge of $\mathcal{K}$ associated with a rising diagonal edge of $\pi^{\times}$. By construction of $\pi^{\times}$ this corresponds to the following situation:

This is the second situation in Case 2 in the discussion just above. The case of an horizontal edge of $\mathcal{K}$ associated with a downward diagonal of $\pi^\times$ corresponds to the first situation in Case 2:

$\square$

As a consequence of Lemma 4.4, conditional on $\mathcal{T}$, the number B of bad edges is a binomial random variable with parameters (K, ε). Using Lemma 4.2 we thus get that, for small enough ε and for large enough n, $\mathbb{E}(B) \leq \mathbb{E}(K)\varepsilon \leq 2n\varepsilon^2$. The third item of the following lemma will enable us to bound the size of the detour associated with each bad edge $e \in \mathcal{B}$. Items 1 and 2 are intermediate steps in the proof of Item 3.

We say that $e^\star$ in $(\mathbb{Z}^2)^\star$ is unbiased if its dual edge e is.

Lemma 4.5. 1. *If an edge $e^\star$ is biased, its six neighbouring edges are unbiased.*

2. *If an animal $\mathcal{A}$ of $(\mathbb{Z}^2)^\star$ (i.e. a connected subset of edges in $(\mathbb{Z}^2)^\star$) contains an edge of $\mathcal{K}$, then it contains at least $\max(1, |\mathcal{A}|/7)$ unbiased edges.*

3. *If e belongs to $\mathcal{K}$, then*

$$\mathbb{E}\left(|C^\star(e)|1_{\{e \in \mathcal{B}\}}\Big|\mathcal{T}\right) \leq C\varepsilon$$

for small enough ε where C is an absolute constant.

Proof. Item 1. Take a biased edge $e^\star$. It corresponds either to Case 3 above or to Case 4. As the proofs are identical, let us consider Case 3. It corresponds to the following situation, where we draw in red the dual edges of the six neighbouring edges of $e^\star$:

Four of them are vertical and thus unbiased. The horizontal edge above corresponds to the second situation of Case 2, whereas the horizontal edge below corresponds to the first situation of Case 2.

Item 2. Let $\mathcal{A}$ be an animal of $(\mathbb{Z}^2)^\star$ and assume that $\mathcal{A}$ contains an edge of $\mathcal{K}$. As all edges of $\mathcal{K}$ are unbiased by Lemma 4.4, the required lower bound is straightforward if the cardinality of $\mathcal{A}$ is 1. Let us assume that the cardinality of $\mathcal{A}$ is at least 2. By Item 1, we can then construct a mapping which associates to a biased edge of $\mathcal{A}$ one of its unbiased neighbours in $\mathcal{A}$. At most 6 biased edges are mapped to the same unbiased

edge (this is not optimal, one could replace 6 by 2). Thus, $\mathcal{A}$ contains at least $|\mathcal{A}|/7$ unbiased edges.

Item 3. Let us condition on $\mathcal{T}$ and let $e \in \mathcal{K}$.

$$\mathbb{E}\left(|C^{\star}(e)|1_{\{e\in\mathcal{B}\}}\Big|\mathcal{T}\right) = \sum_{k\geq 1}\mathbb{P}\left(|C^{\star}(e)|1_{\{e\in\mathcal{B}\}} \geq k\Big|\mathcal{T}\right).$$

On the latter event there is an animal of size k containing $e^{\star}$ and whose all edges are closed. There are less than 15^k animals of cardinality k containing the edge $e^{\star}$ (this can be proven for instance by adapting slightly the arguments of (4.24) p.81 in [6]). Using Item 2 we get:

$$\mathbb{E}\left(|C^{\star}(e)|1_{\{e\in\mathcal{B}\}}\Big|\mathcal{T}\right) \leq \sum_{k\geq 1} 15^k \varepsilon^{\max(1,k/7)} \leq C\varepsilon$$

for an absolute constant C and for small enough ε. $\qquad\square$

We are now in position to conclude the proof of Theorem 1.1 by giving an upper bound for $D(\mathbf{0},(n,n\lambda\varepsilon))$.

Proof of Theorem 1.1. Since an edge in $\mathcal{C}^{\star}(e)$ is connected to less than 6 open edges,

$$|\Delta\mathcal{C}^{\star}(e)| \leq 6|\mathcal{C}^{\star}(e)|.$$

Therefore,

$$|\Delta\mathcal{C}(\pi)| \leq \sum_{e\in\mathcal{B}}|\Delta\mathcal{C}(e)| = \sum_{e\in\mathcal{B}}|\Delta\mathcal{C}^{\star}(e)| \leq 6\sum_{e\in\mathcal{B}}|\mathcal{C}^{\star}(e)|.$$

Using Lemma 4.5 and Lemma 4.2 we thus get, for small enough ε and large enough n,

$$\mathbb{E}\left(|\Delta\mathcal{C}(\pi)|\right) \leq 6\mathbb{E}\left(\sum_{e\in\mathcal{K}}\mathbb{E}\left(|C^{\star}(e)|1_{\{e\in\mathcal{B}\}}\Big|\mathcal{T}\right)\right) \leq 6C\varepsilon\mathbb{E}(K) \leq 12Cn\varepsilon^2. \qquad (4.1)$$

Set $\mathcal{E}_n = \{\mathbf{0} \leftrightarrow (n,n\lambda\varepsilon) \leftrightarrow \infty\}$. We deduce from Proposition 4.3 that on the event $\mathcal{E}_n$,

$$D(\mathbf{0},(n,n\lambda\varepsilon)) \leq |\pi| + |\Delta\mathcal{C}(\pi)| = D^{\times}(n,n\lambda\varepsilon) + |\Delta\mathcal{C}(\pi)|.$$

This yields, for all $A > 0$,

$$\mathbb{P}(D(\mathbf{0},(n,n\lambda\varepsilon)) \geq nf(\lambda,\varepsilon) + An\varepsilon^2, \mathcal{E}_n)$$
$$\leq \mathbb{P}(D^{\times}(n,n\lambda\varepsilon) \geq nf(\lambda,\varepsilon) + An\varepsilon^2/2) + \mathbb{P}(|\Delta\mathcal{C}(\pi)| \geq An\varepsilon^2/2)$$
$$\leq \mathbb{P}(D^{\times}(n,n\lambda\varepsilon) \geq nf(\lambda,\varepsilon) + An\varepsilon^2/2) + \frac{\mathbb{E}(|\Delta\mathcal{C}(\pi)|)}{An\varepsilon^2/2}.$$

From Corollary 2.6 and (4.1) we get, for small enough ε,

$$\limsup_{n\to\infty}\mathbb{P}(D(\mathbf{0},(n,n\lambda\varepsilon)) \geq nf(\lambda,\varepsilon) + An\varepsilon^2, \mathcal{E}_n) \leq 24C/A.$$

Note that for ε small enough, $\mathbb{P}(\mathcal{E}_n) \geq \mathbb{P}(\mathbf{0} \leftrightarrow \infty)^2 \geq 1/2$. Since we know that $D(\mathbf{0},(n,n\lambda\varepsilon))/n$ converges almost surely to the constant $\mu_{1-\varepsilon}(\lambda)$, we get, for $A > 48C$,

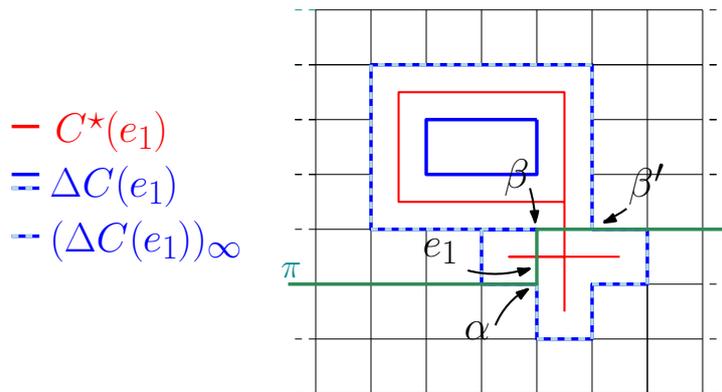$$\mu_{1-\varepsilon}(\lambda) \leq f(\lambda,\varepsilon) + A\varepsilon^2.$$

$\qquad\square$

Appendix: Proof of Proposition 4.3

Proof. Let e_1 be an edge in $\mathcal{B}$. We denote by $(\Delta\mathcal{C}(e_1))_\infty$ the set of edges e of $\Delta\mathcal{C}(e_1)$ for which there exists a path in $\mathbb{Z}^2$ (which may take open or closed edges) such that

- the edge e is the first edge of the path,
- the path does not cross $\mathcal{C}^\star(e_1)$,
- the path goes to infinity.

In other words, $(\Delta\mathcal{C}(e_1))_\infty$ is composed of the edges of $\Delta\mathcal{C}(e_1)$ that are not in the interior of $\mathcal{C}^\star(e_1)$.

A first observation is that the set $(\Delta\mathcal{C}(e_1))_\infty$ is connected. Take it for granted for a while and let us prove that there exists a path from $\mathbf{0}$ to $(n, n\lambda\varepsilon)$ which only uses edges of $\pi \setminus \{e_1\}$ or edges of $(\Delta\mathcal{C}(e_1))_\infty$. Let (α, β) be the first edge of π crossing $\mathcal{C}^\star(e_1)$. One can check that α is necessarily the extremity of an edge of $(\Delta\mathcal{C}(e_1))_\infty$. Following the path π backwards from $(n, n\lambda\varepsilon)$, we define similarly another site $\beta' \in (\Delta\mathcal{C}(e_1))_\infty$. The origin $\mathbf{0}$ is connected to α by π, α is connected to β' by $(\Delta\mathcal{C}(e_1))_\infty$, and β' is connected to $(n, n\lambda\varepsilon)$ by π. We proceed recursively for the next edges of $\mathcal{B}$.

It remains to prove that $(\Delta\mathcal{C}(e_1))_\infty$ is connected. Let $\Sigma(\mathcal{C}^\star(e_1)) \subset \mathbb{Z}^2$ be the circuit[2] surrounding $\mathcal{C}^\star(e_1)$ as constructed in Proposition 11.2 of [6] . By construction, $\Sigma(\mathcal{C}^\star(e_1)) \subset (\Delta\mathcal{C}(e_1))_\infty$. Assume that $(\Delta\mathcal{C}(e_1))_\infty$ is not connected and let H be a connected component which does not contain $\Sigma(\mathcal{C}^\star(e_1))$. The set H being a connected component of $(\Delta\mathcal{C}(e_1))_\infty$, an edge in ΔH can not be in $\Delta\mathcal{C}(e_1)$. This implies in particular that either

(i) $\Sigma(H) \subset \mathcal{C}^\star(e_1)$

(ii) $\Sigma(H)$ and $\mathcal{C}^\star(e_1)$ have no site in common.

The first possibility contradicts the connection of H with infinity, whereas the second one contradicts the connectivity of $\mathcal{C}^\star(e_1)$. □

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[2]$\Sigma(\mathcal{C}^\star(e_1))$ is the unique circuit containing $\mathcal{C}^\star(e_1)$ in its interior and with the property that every edge of $\Sigma(\mathcal{C}^\star(e_1))$ crosses an edge of $\Delta\mathcal{C}(e_1)$).

References

[1] A. Auffinger, M.Damron. Differentiability at the edge of the percolation cone and related results in first-passage percolation. *Probability Theory and Related Fields*, vol. 156 (2013), n.1-2 p.193-227 MR-3055257

[2] A-L. Basdevant, N. Enriquez et L. Gerin. Distances in the highly supercritical percolation cluster. *Annals of Probability*, vol.41 (2013) n.6 p.4342-4358. MR-3161477

[3] N.D. Blair-Stahn. First passage percolation and competition models (2010). arXiv:1005.0649.

[4] R. Durrett. Oriented percolation in two dimensions. *Annals of Probability*, vol.12 (1984), n.4 p.999-1040. MR-0757768

[5] O. Garet and R. Marchand. Asymptotic shape for the chemical distance and first-passage percolation on the infinite Bernoulli cluster. *ESAIM Probab. Stat.* vol.8 (2004), MR-2085613 p.169-199.

[6] G. Grimmett. *Percolation*. Springer-Verlag, Berlin, 2d edition (1999). MR-1707339

[7] J. M. Hammersley and D.J.A. Welsh. First passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. (1965) MR-0198576 *Proc. Internat. Res. Semin., Statist. Lab., Univ. California*, Berkeley, Calif. p.61-110.

[8] W. Jockusch, J.Propp, P.Shor. Random domino tilings and the arctic circle theorem (1995). arXiv:math/9801068.

[9] K. Johansson. Shape fluctuations and random matrices. *Communications in Mathematical Physics*, vol.209 (2000) n.2 p.437-476. MR-1737991

[10] T. Kriecherbauer and J.Krug. A pedestrian's view on interacting particle systems, KPZ universality and random matrices. *Journal of Physics A*, vol.43 (2010) n.40 403001. MR-2725545

[11] R. Marchand. Strict inequalities for the time constant in first passage percolation, *Annals of Applied Probability*, vol.12 (2002) n.3 p.1001-1038. MR-1925450

[12] T. Seppäläinen. Lecture Notes on the Corner Growth Model (2008). Available at `http://www.math.wisc.edu/~seppalai/`.