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# Reflected BSDEs with monotone generator*

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**Abstract**

We give necessary and sufficient condition for existence and uniqueness of $\mathbb{L}^p$-solutions of reflected BSDEs with continuous barrier, generator monotone with respect to $y$ and Lipschitz continuous with respect to $z$, and with data in $\mathbb{L}^p$, $p \geq 1$. We also prove that the solutions may be approximated by the penalization method.

**Keywords:** Reflected backward stochastic differential equation; monotone generator; $\mathbb{L}^p$-solutions.
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## 1 Introduction

Let $B$ be a standard $d$-dimensional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t\}$ denote the augmentation of the natural filtration generated by $B$. In the present paper we study the problem of existence, uniqueness and approximation of $\mathbb{L}^p$-solutions of reflected backward stochastic differential equations (RBSDEs for short) with monotone generator of the form

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T dK_s - \int_t^T Z_s\, dB_s, \quad t \in [0, T], \\ Y_t \geq L_t, \quad t \in [0, T], \\ K \text{ is continuous, increasing, } K_0 = 0,\ \int_0^T (Y_t - L_t)\, dK_t = 0. \end{cases} \tag{1.1}$$

Here $\xi$ is an $\mathcal{F}_T$-measurable random variable called the terminal condition, $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the generator (or coefficient) of the equation and an $\{\mathcal{F}_t\}$-adapted continuous proces $L = \{L_t, t \in [0, T]\}$ such that $L_T \leq \xi$ $P$-a.s. is called the obstacle (or barrier). A solution of (1.1) is a triple $(Y, Z, K)$ of $\{\mathcal{F}_t\}$-progressively measurable processes having some integrability properties depending on assumptions imposed on the data $\xi, f, L$ and satisfying (1.1) $P$-a.s.

Equations of the form (1.1) were introduced in El Karoui et al. [6]. At present it is widely recognized that they provide a useful and efficient tool for studying problems in different mathematical fields, such as mathematical finance, stochastic control and game theory, partial differential equations and others (see, e.g., [4, 6, 7, 8, 10]).

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In [6] existence and uniqueness of square-integrable solutions of (1.1) are proved under the assumption that $\xi$, $\int_0^T |f(t,0,0)|\,dt$ and $L^*_T = \sup_{t\le T}|L_t|$ are square-integrable, $f$ satisfies the linear growth condition and is Lipschitz continuous with respect to both variables $y$ and $z$. These assumptions are too strong for many interesting applications. Therefore many attempts have been made to prove existence and uniqueness of solutions of RBSDEs under less restrictive assumptions on the data. Roughly speaking one can distinguish here two types of results: for RBSDEs with less regular barriers (see, e.g., [15]) and for equations with continuous barriers whose generators or terminal conditions satisfy weaker assumptions than in [6]. We are interested in the second direction of investigation of (1.1).

In the paper we consider $\mathbb{L}^p$-integrable data with $p \ge 1$ and we assume that the generator is continuous and monotone in $y$ and Lipschitz continuous with respect to $z$. Assumptions of that type were considered in [1, 9, 12, 16] but it is worth mentioning that the case where the generator is monotone and at the same time the data are $\mathbb{L}^p$-integrable for some $p \in [1,2)$ was considered previously only in [1, 16] (to be exact, in [1] the author considers the case $p \in (1,2)$ but for generalized RBSDEs). Let us also mention that in the case $p = 2$ existence and uniqueness results are known for equations with generators satisfying even weaker regularity conditions. For instance, in [13] continuous generators satisfying the linear growth conditions are considered, in [17] it is assumed that the generator is left-Lipschitz continuous and possibly discontinuous in $y$, and in [11] equations with generators satisfying the superlinear growth condition with respect to $y$, the quadratic growth condition with respect to $z$ and with data ensuring boundedness of the first component $Y$ are considered. In all these papers except for [16] the authors consider the so-called general growth condition which says that

$$|f(t,y,0)| \le |f(t,0,0)| + \varphi(|y|), \quad t \in [0,T], y \in \mathbb{R}, \tag{1.2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function or continuous function which is bounded on bounded subsets of $\mathbb{R}$. In [16] weaker than (1.2) condition of the form

$$\forall_{r>0} \quad \sup_{|y|\le r} |f(\cdot,y,0) - f(\cdot,0,0)| \in \mathbb{L}^1(0,T). \tag{1.3}$$

is assumed. Condition (1.3) seems to be the best possible growth condition on $f$ with respect to $y$. It was used earlier in the paper [3] devoted to $\mathbb{L}^p$-solutions of usual (non-reflected) BSDEs with monotone generators. Similar condition is widely used in the theory of partial differential equations (see [2] and the references given there). Let us point out, however, that in contrast to the case of usual BSDEs with monotone generators, in general assumption (1.2) (or (1.3)) together with $\mathbb{L}^p$-integrability of the data (integrability of $\xi$, $L^*_T$, $\int_0^T |f(t,0,0)|\,dt$ in our case) do not guarantee existence of $\mathbb{L}^p$-integrable solutions of (1.1). For existence some additional assumptions relating the growth of $f$ with that of the barrier is required. In [1, 12] existence of solutions is proved under the assumption that $E|\varphi(\sup_{t\le T} e^{\mu t} L^+_t)|^2 < +\infty$, where $\varphi$ is the function of condition (1.2) and $\mu$ is the monotonicity coefficient of $f$. In [16] it is shown that it suffices to assume that

$$E(\int_0^T |f(t, \sup_{s\le t} L^+_t, 0)|\,dt)^p\,dt < +\infty. \tag{1.4}$$

Condition (1.4) is still not the best possible. In our main result of the paper we give a necessary and sufficient condition for existence and uniqueness of $\mathbb{L}^p$-integrable solution of RBSDE (1.1) under the assumptions that the data are $\mathbb{L}^p$-integrable, $f$ is monotone in $y$ and Lipschitz continuous in $z$ and (1.3) is satisfied. Moreover, our condition is not only weaker than (1.4) but at the same time much easier to check than

(1.4) in case of very important in applications Markov type RBSDEs with obstacles of the form $L = h(\cdot, X)$, where $h : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function and $X$ is a Hunt process associated with some Markov semigroup. In the case of Markov RBSDEs which appear for instance in applications to variational problems for PDEs (see, e.g., [6, 10]) our condition can be formulated in terms of $f, h$ only. We prove the main result for $p \geq 1$. Moreover, we show that for $p \geq 1$ a unique solution of RBSDE (1.1) can be approximated via penalization. The last result strengthens the corresponding result in [16] proved in case $p > 1$ for general generators and in case $p = 1$ for generators not depending on $z$.

In the last part of the paper we study (1.1) in the case where $\xi$, $L^{+,*}$, $\int_0^T |f(t,0,0)|\,dt$ are $\mathbb{L}^p$-integrable for some $p \geq 1$ but our weaker form of (1.4) is not satisfied. We have already mentioned, that then there are no $\mathbb{L}^p$-integrable solutions of (1.1). We show that still there exist solutions of (1.1) having weaker regularity properties.

The paper is organized as follows. Section 2 contains notation and main hypotheses used in the paper. In Section 3 we show basic a priori estimates for solutions of BSDEs. In Section 4 we prove comparison results as well as some useful results on càdlàg regularity of monotone limits of semimartingales and uniform estimates of monotone sequences. In Section 5 we prove our main existence and uniqueness result for $p > 1$, and in Section 6 for $p = 1$. Finally, in Section 7 we deal with nonintegrable solutions.

## 2 Notation and hypotheses

Let $B = \{B_t, t \geq 0\}$ be a standard $d$-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t, t \geq 0\}$ be the augmented filtration generated by $B$. In the whole paper all notions whose definitions are related to some filtration are understood with respect to the filtration $\{\mathcal{F}_t\}$.

Given a stochastic process $X$ on $[0,T]$ with values in $\mathbb{R}^n$ we set $X_t^* = \sup_{0\leq s\leq t} |X_s|$, $t \in [0,T]$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. By $\mathcal{S}$ we denote the set of all progressively measurable continuous processes. For $p > 0$ we denote by $\mathcal{S}^p$ the set of all processes $X \in \mathcal{S}$ such that

$$\|X\|_{\mathcal{S}^p} = (E \sup_{t\in[0,T]} |X_t|^p)^{1\wedge 1/p} < +\infty.$$

$M$ is the set of all progressively measurable processes $X$ such that

$$P(\int_0^T |X_t|^2\,dt < +\infty) = 1$$

and for $p > 0$, $M^p$ is the set of all processes $X \in M$ such that

$$(E(\int_0^T |X_t|^2\,dt)^{p/2})^{1\wedge 1/p} < +\infty.$$

For $p, q > 0$, $\mathbb{L}^{p,q}(\mathcal{F})$ (resp. $\mathbb{L}^p(\mathcal{F}_T)$) denotes the set of all progressively measurable processes ($\mathcal{F}_T$ measurable random variables) $X$ such that

$$(E(\int_0^T |X_t|^p\,dt)^{q/(1\wedge 1/p)})^{1\wedge 1/q} < +\infty \quad \Big(\text{resp. } (E|X|^p)^{1/p} < +\infty\Big).$$

For brevity we denote $\mathbb{L}^{p,p}(\mathcal{F})$ by $\mathbb{L}^p(\mathcal{F})$. By $\mathbb{L}^1(0,T)$ we denote the space of Lebesgue integrable real valued functions on $[0,T]$.

$\mathcal{M}_c$ is the set of all continuous martingales (resp. local martingales) and $\mathcal{M}_c^p$, $p \geq 1$, is the set of all martingales $M \in \mathcal{M}_c$ such that $E(\langle M\rangle_T)^{p/2} < +\infty$. $\mathcal{V}_c$ (resp. $\mathcal{V}_c^+$) is the set of all continuous progressively measurable processes of finite variation (resp. increasing processes) and $\mathcal{V}_c^p$ (resp. $\mathcal{V}_c^{+,p}$) is the set of all processes $V \in \mathcal{V}_c$ (resp. $V \in \mathcal{V}_c^+$) such that $E|V|_T^p < +\infty$. We put $\mathcal{H}_c^p = \mathcal{M}_c^p + \mathcal{V}_c^p$.

For a given measurable process $Y$ of class (D) we denote

$$\|Y\|_1 = \sup\{E|Y_\tau|, \tau \in \mathcal{T}\}.$$

In what follows $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function with respect to $Prog \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$, where $Prog$ denotes the $\sigma$-field of progressive subsets of $[0,T] \times \Omega$.

In the whole paper all equalities and inequalities between random elements are understood to hold $P$-a.s.

Let $p \geq 1$. In the paper we consider the following hypotheses.

(H1) $E|\xi|^p + E(\int_0^T |f(t,0,0)|\,dt)^p < +\infty$.

(H2) There exists $\lambda > 0$ such that $|f(t,y,z) - f(t,y,z')| \leq \lambda|z - z'|$ for every $t \in [0,T], y \in \mathbb{R}, z, z' \in \mathbb{R}^d$.

(H3) There exists $\mu \in \mathbb{R}$ such that $(f(t,y,z) - f(t,y',z))(y - y') \leq \mu(y - y')^2$ for every $t \in [0,T], y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d$.

(H4) For every $(t,z) \in [0,T] \times \mathbb{R}^d$ the mapping $\mathbb{R} \ni y \to f(t,y,z)$ is continuous.

(H5) For every $r > 0$ the mapping $[0,T] \ni t \to \sup_{|y| \leq r} |f(t,y,0) - f(t,0,0)|$ belongs to $\mathbb{L}^1(0,T)$.

(H6) $L$ is a continuous, progressively measurable process such that $L_T \leq \xi$.

(H7) There exists a semimartingale $X$ such that $X \in \mathcal{H}_c^p$ for some $p > 1$, $X_t \geq L_t$, $t \in [0,T]$ and $E(\int_0^T f^-(s,X_s,0)\,ds)^p < +\infty$.

(H7*) There exists a semimartingale $X$ of class (D) such that $X \in \mathcal{V}_c^1 + \mathcal{M}_c^q$ for every $q \in (0,1)$, $X_t \geq L_t$, $t \in [0,T]$ and $E\int_0^T f^-(s,X_s,0)\,ds < +\infty$.

(A) There exist $\mu \in \mathbb{R}$ and $\lambda \geq 0$ such that

$$\hat{y}f(t,y,z) \leq f_t + \mu|y| + \lambda|z|$$

for every $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, where $\hat{y} = \mathbf{1}_{\{y \neq 0\}} \frac{y}{|y|}$ and $\{f_t; t \in [0,T]\}$ is a nonnegative progressively measurable process.

(Z) There exist $\alpha \in (0,1)$, $\gamma \geq 0$ and a nonnegative process $g \in \mathbb{L}^1(\mathcal{F})$ such that

$$|f(t,y,z) - f(t,y,0)| \leq \gamma(g_t + |y| + |z|)^\alpha$$

for every $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$.

## 3 A priori estimates

In this section $K$ denotes an arbitrary but fixed process of the class $\mathcal{V}_c^+$ such that $K_0 = 0$.

The following version of Itô's formula will be frequently used in the paper.

**Proposition 3.1.** *Let $p \geq 1$ and let $X$ be a progressively measurable process of the form*

$$X_t = X_0 + \int_0^t dK_s + \int_0^t Z_s\,dB_s, \quad t \in [0,T],$$

*where $Z \in M$. Then there is $L \in \mathcal{V}_c^+$ such that*

$$|X_t|^p - |X_0|^p = p\int_0^t |X_s|^{p-1}\hat{X}_s\,dK_s + p\int_0^t |X_s|^{p-1}\hat{X}_s\,dB_s + c(p)\int_0^t \mathbf{1}_{\{X_s \neq 0\}}|X_s|^{p-2}|Z_s|^2\,ds + L_t\mathbf{1}_{\{p=1\}}$$

*with $c(p) = p(p-1)/2$.*

*Proof.* The proof is a matter of slight modification of the proof of [3, Lemma 2.2]. □

**Definition 3.2.** *We say that a pair* $(Y,Z)$ *of progressively measurable processes is a solution of* $\mathrm{BSDE}(\xi, f+dK)$ *iff* $Z\in M$*, the mapping* $[0,T]\ni t\mapsto f(t,Y_t,Z_t)$ *belongs to* $\mathbb{L}^1(0,T)$*,* $P$*-a.s. and*

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\,ds+\int_t^T dK_s-\int_t^T Z_s\,dB_s,\quad t\in[0,T]. \tag{3.1}$$

**Lemma 3.3.** *Let* $(Y,Z)$ *be a solution of* $\mathrm{BSDE}(\xi,f+dK)$*. Assume that* (H3) *is satisfied and there exists a progressively measurable process* $X$ *such that* $X_t\ge Y_t$*,* $t\in[0,T]$ *and the mappings* $[0,T]\ni t\mapsto X_t^+$*,* $[0,T]\ni t\mapsto f^-(t,X_t,0)$ *belong to* $\mathbb{L}^1(0,T)$*,* $P$*-a.s.*

(i) *If* (H2) *is satisfied then for every stopping time* $\tau\le T$ *and* $a\ge\mu$*,*

$$\int_0^\tau e^{at}dK_t\le|e^{a\tau}Y_\tau|+|Y_0|+\int_0^\tau e^{as}Z_s\,dB_s+\lambda\int_0^\tau e^{as}|Z_s|\,ds$$
$$+\int_0^\tau e^{as}f^-(s,X_s,0)\,ds+\int_0^\tau a^+e^{as}X_s^+\,ds.$$

(ii) *If* (Z) *is satisfied then for every stopping time* $\tau\le T$ *and* $a\ge\mu$*,*

$$\int_0^\tau e^{at}dK_t\le|e^{a\tau}Y_\tau|+|Y_0|+\int_0^\tau e^{as}Z_s\,dB_s+\gamma\int_0^\tau e^{as}(g_s+|Y_s|+|Z_s|)^\alpha\,ds$$
$$+\int_0^\tau e^{as}f^-(s,X_s,0)\,ds+\int_0^\tau a^+e^{as}X_s^+\,ds.$$

*Proof.* Assume that $\mu\le 0$. Then $f^-(s,Y_s,0)\le f^-(s,X_s,0)$, $s\in[0,T]$ and from (3.1) and (H2) it follows that

$$K_\tau\le -Y_\tau+Y_0+\int_0^\tau Z_s\,dB_s+\lambda\int_0^\tau|Z_s|\,ds-\int_0^\tau f(s,Y_s,0)\,ds,$$

which implies (i) with $a=0$. Now, let $a\ge\mu$ and let $\tilde Y_t=e^{at}Y_t$, $\tilde Z_t=e^{at}Z_t$ and $\tilde\xi=e^{aT}\xi$, $\tilde f(t,y,z)=e^{at}f(t,e^{-at}y,e^{-at}z)-ay$, $d\tilde K_t=e^{at}\,dK_t$. Then $\tilde f$ satisfies (H3) with $\mu=0$ and by Itô's formula,

$$\tilde Y_t=\tilde\xi+\int_t^T\tilde f(s,\tilde Y_s,\tilde Z_s)\,ds+\int_t^T d\tilde K_s-\int_t^T\tilde Z_s\,dB_s,\quad t\in[0,T],$$

from which in the same manner as before we obtain (i) for $a\ge\mu$.

To prove (ii) let us observe that from (3.1) and (Z) it follows immediately that

$$K_\tau\le -Y_\tau+Y_0+\int_0^\tau Z_s\,dB_s+\gamma\int_0^\tau(g_s+|Y_s|+|Z_s|)^\alpha\,ds-\int_0^\tau f(s,Y_s,0)\,ds.$$

Therefore repeating arguments from the proof of (i) we get (ii). □

**Lemma 3.4.** *Assume* (A) *and let* $(Y,Z)$ *be a solution of* $\mathrm{BSDE}(\xi,f+dK)$*. If* $Y\in\mathcal{S}^p$ *for some* $p>0$ *and*

$$E(\int_0^T X_s^+\,ds)^p+E(\int_0^T f^-(s,X_s,0)\,ds)^p+E(\int_0^T|f(s,0,0)|\,ds)^p<+\infty$$

*for some progressively measurable process* $X$ *such that* $X_t\ge Y_t$*,* $t\in[0,T]$*, then* $Z\in M^p$ *and there exists* $C$ *depending only on* $\lambda,p,T$ *such that for every* $a\ge\mu+\lambda^2$*,*

$$E\Big((\int_0^T e^{2as}|Z_s|^2\,ds)^{p/2}+(\int_0^T e^{as}\,dK_s)^p\Big)\le CE\Big(\sup_{t\le T}e^{apt}|Y_t|^p$$
$$+(\int_0^T e^{as}|f(s,0,0)|\,ds)^p+(\int_0^T e^{as}f^-(s,X_s,0)\,ds)^p+(\int_0^T a^+e^{as}X_s^+\,ds)^p\Big).$$

*Proof.* By standard arguments we may assume that $\mu+\lambda^2\le 0$ and take $a=0$. For each $k\in\mathbb{N}$ let us consider the stopping time

$$\tau_k=\inf\{t\in[0,T];\int_0^t|Z_s|^2\,ds\ge k\}\wedge T. \tag{3.2}$$

Then as in the proof of Eq. (5) in [3] we get

$$(\int_0^{\tau_k}|Z_s|^2\,ds)^{p/2}\le c_p\Big(|Y_T^*|^p+(\int_0^T f_s\,ds)^p+|\int_0^{\tau_k}Y_sZ_s\,dB_s|^{p/2}+(\int_0^{\tau_k}|Y_s|\,dK_s)^{p/2}\Big),$$

and hence, repeating arguments following Eq. (5) in [3] we show that

$$E(\int_0^{\tau_k}|Z_s|^2\,ds)^{p/2}\le c_pE\Big(|Y_T^*|^p+(\int_0^T f_s\,ds)^p+(\int_0^{\tau_k}|Y_s|\,dK_s)^{p/2}\Big). \tag{3.3}$$

By Lemma 3.3 and the Burkholder-Davis-Gundy inequality,

$$EK_{\tau_k}^p\le c'(p,\lambda,T)E\Big(|Y_T^*|^p+(\int_0^{\tau_k}|Z_s|^2\,ds)^{p/2}+(\int_0^T f^-(s,X_s,0)\,ds)^p\Big). \tag{3.4}$$

Moreover, applying Young's inequality we conclude from (3.3) that for every $\alpha>0$,

$$\begin{aligned}&E(\int_0^{\tau_k}|Z_s|^2\,ds)^{p/2}\\&\quad\le c''(p,\alpha)E\Big(|Y_T^*|^p+(\int_0^T f_s\,ds)^p+(\int_0^T f^-(s,X_s,0)\,ds)^p\Big)+\alpha EK_{\tau_k}^p.\end{aligned} \tag{3.5}$$

Taking $\alpha=(2c'(p,\lambda,T))^{-1}$ and combining (3.4) with (3.5) we obtain

$$E(\int_0^{\tau_k}|Z_s|^2\,ds)^{p/2}\le C(p,\lambda,T)E\Big(|Y_T^*|^p+(\int_0^T f_s\,ds)^p+(\int_0^T f^-(s,X_s,0)\,ds)^p\}\Big).$$

Applying Fatou's lemma we conclude from the above inequality and (3.4) that

$$E(\int_0^T|Z_s|^2\,ds)^{p/2}+EK_T^p\le CE\Big(|Y_T^*|^p+(\int_0^T f_s\,ds)^p+(\int_0^T f^-(s,X_s,0)\,ds)^p\Big),$$

which is the desired estimate. □

**Remark 3.5.** *Observe that if $f$ does not depend on $z$ then the constant $C$ of Lemma 3.4 depends only on $p$. This follows from the fact that in this case $c'$ in the key inequality (3.4) depends only on $p$.*

**Proposition 3.6.** *Assume that* (A) *is satisfied and*

$$E(\int_0^T f^-(s,X_s,0)\,ds)^p+E(\int_0^T|f(s,0,0)|\,ds)^p<+\infty$$

*for some $p>1$ and $X^+\in\mathcal{S}^p$ such that $X_t\ge Y_t$, $t\in[0,T]$. Then if $(Y,Z)$ is a solution of* $\mathrm{BSDE}(\xi,f+dK)$ *such that $Y\in\mathcal{S}^p$, then there exists $C$ depending only on $\lambda,p,T$ such that for every $a\ge\mu+\lambda^2/[1\wedge(p-1)]$ and every stopping time $\tau\le T$,*

$$\begin{aligned}&E\sup_{t\le\tau}e^{apt}|Y_t|^p+E(\int_0^\tau e^{2as}|Z_s|^2\,ds)^{p/2}+E(\int_0^\tau e^{as}\,dK_s)^p\\&\quad\le CE\Big(e^{ap\tau}|Y_\tau|^p+(\int_0^\tau e^{as}|f(s,0,0)|\,ds)^p+\sup_{t\le\tau}|e^{at}X_t^+|^p\\&\qquad+(\int_0^\tau e^{as}f^-(s,X_s,0)\,ds)^p+(\int_0^\tau a^+e^{as}X_s^+\,ds)^p\Big).\end{aligned}$$

*Assume additionally that $f$ does not depend on $z$. If $p=1$ and $X^+, Y$ are of class* (D) *then for every $a \ge \mu$,*

$$\|e^{a\cdot}Y\|_1 + E\int_0^T e^{as}\,dK_s \le E\Big(e^{aT}|\xi| + \int_0^T e^{as}|f(s,0)|\,ds + \int_0^T e^{as}f^-(s,X_s)\,ds + \int_0^T a^+e^{as}X_s^+\,ds\Big) + \|e^{a\cdot}X^+\|_1\,.$$

*Proof.* To shorten notation we prove the proposition in the case where $\tau = T$. The proof of the general case requires only minor technical changes. Moreover, by the change of variables used at the beginning of the proof of Lemma 3.3 we can reduce the proof to the case where $a=0$ and $\mu+\lambda^2/[1\wedge(p-1)]\le 0$. Therefore we will assume that $a,\mu,\lambda$ satisfy the last two conditions.

By Itô's formula (see Proposition 3.1),

$$|Y_t|^p + c(p)\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\,ds = |\xi|^p + p\int_t^T |Y_s|^{p-1}\hat{Y}_s f(s,Y_s,Z_s)\,ds + p\int_t^T |Y_s|^{p-1}\hat{Y}_s\,dK_s - p\int_t^T |Y_s|^{p-1}\hat{Y}_s Z_s\,dB_s, \quad t\in[0,T].$$

By the same method as in the proof of Eq. (6) in [3] we deduce from the above inequality that

$$|Y_t|^p + \frac{c(p)}{2}\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\,ds \le H - p\int_t^T |Y_s|^{p-1}\hat{Y}_s Z_s\,dB_s + p\int_t^T |Y_s|^{p-1}\hat{Y}_s\,dK_s, \quad t\in[0,T], \tag{3.6}$$

where $H = |\xi|^p + \int_0^T |Y_s|^{p-1}f_s\,ds$. Since the mapping $\mathbb{R}\ni y\mapsto |y|^{p-1}\hat{y}$ is increasing,

$$\int_t^T |Y_s|^{p-1}\hat{Y}_s\,dK_s \le \int_t^T |X_s^+|^{p-1}\hat{X}_s^+\,dK_s, \quad t\in[0,T].$$

From this and (3.6),

$$|Y_t|^p + \frac{c(p)}{2}\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\,ds \le H' - p\int_t^T |Y_s|^{p-1}\hat{Y}_s Z_s\,dB_s, \tag{3.7}$$

where $H' = |\xi|^p + p\int_0^T |Y_s|^{p-1}f_s\,ds + p\int_0^T |X_s^+|^{p-1}\,dK_s$. As in the proof of [3, Proposition 3.2] (see (7) and the second inequality following (8) in [3]), using the Burkholder-Davis-Gundy inequality we conclude from (3.7) that

$$E|Y_T^*|^p \le d_p EH'. \tag{3.8}$$

Applying Young's inequality we get

$$pd_pE\int_0^T |Y_s|^{p-1}f_t\,dt \le pd_pE(|Y_T^*|^{p-1}\int_0^T f_t\,dt) \le \frac{1}{4}E|Y_T^*|^p + d_p'E(\int_0^T f_t\,dt)^p \tag{3.9}$$

and

$$pd_pE\int_0^T |X_t^+|^{p-1}\,dK_t \le d'(p,\alpha)E|X_T^{+,*}|^p + \alpha EK_T^p\,. \tag{3.10}$$

By Lemma 3.3, there exists $d(p,\lambda,T)>0$ such that

$$EK_T^p\le d(p,\lambda,T)E\Big(|Y_T^*|^p+(\int_0^T|Z_s|^2\,ds)^{p/2}+(\int_0^T f^-(s,X_s,0)\,ds)^p\Big).$$

From this and Lemma 3.4 we see that there exists $c(p,\lambda,T)>0$ such that

$$EK_T^p\le c(p,\lambda,T)E\Big(|Y_T^*|^p+(\int_0^T f_s\,ds)^p+(\int_0^T f^-(s,X_s,0)\,ds)^p\Big). \tag{3.11}$$

Put $\alpha=(4c(p,\lambda,T))^{-1}$. Then from (3.8)–(3.11) it follows that there is $C(p,\lambda,T)$ such that

$$\begin{aligned}E|Y_T^*|^p\le C(p,\lambda,T)E\Big(&|\xi|^p+(\int_0^T|f(s,0,0)|\,ds)^p\\&+(\int_0^T f^-(s,X_s,0)\,ds)^p+\sup_{t\le T}|X_t^+|^p\Big).\end{aligned}$$

Hence, by (3.11) and Lemma 3.4,

$$\begin{aligned}E|Y_\tau^*|^p+E(\int_0^\tau|Z_s|^2\,ds)^{p/2}+EK_T^p\le CE\Big(&|Y_\tau|^p+(\int_0^\tau|f(s,0,0)|\,ds)^p+|X_\tau^{+,*}|^p\\&+(\int_0^\tau f^-(s,X_s,0)\,ds)^p\Big).\end{aligned}$$

From this the first assertion follows. Now suppose that $f$ does not depend on $z$. As in the first part of the proof we may assume that $\mu\le0$ and $a=0$. Applying Itô's formula (see Proposition 3.1) we conclude that for any stopping times $\sigma\le\tau\le T$,

$$|Y_\sigma|\le|Y_\tau|+\int_\sigma^\tau f(s,Y_s)\hat{Y}_s\,ds+\int_\sigma^\tau\hat{Y}_s\,dK_s-\int_\sigma^\tau Z_s\hat{Y}_s\,dB_s. \tag{3.12}$$

Let us define $\tau_k$ by (3.2). Then $\int_0^{\tau_k\wedge\cdot}Z_s\hat{Y}_s\,dB_s$ is a uniformly integrable martingale. Using this, the fact that $Y$ is of class (D) and monotonicity of $f$ with respect to $y$ we deduce from (3.12) that $|Y_\sigma|\le E(|\xi|+\int_0^T|f(s,0)|\,ds+K_T|\mathcal{F}_\sigma)$, hence that

$$\|Y\|_1\le E(|\xi|+\int_0^T|f(s,0)|\,ds+K_T). \tag{3.13}$$

On the other hand, $-f(t,Y_t)\le-f(t,X_t)$ for $t\in[0,T]$ since $Y_t\le X_t$, $t\in[0,T]$. Therefore

$$\begin{aligned}K_\tau&=Y_0-Y_\tau-\int_0^\tau f(s,Y_s)\,ds+\int_0^\tau Z_s\,dB_s\\&\le X_0-Y_\tau-\int_0^\tau f(s,X_s)\,ds+\int_0^\tau Z_s\,dB_s.\end{aligned}$$

Taking $\tau=\tau_k$ and using the fact that $Y$ is of class (D) we deduce from the above inequality that

$$EK_T\le EX_0^++E|\xi|+E\int_0^T f^-(s,X_s)\,ds.$$

Combining this with (3.13) we get the desired result. □

**Remark 3.7.** *If $f$ does not depend on $z$ then the constant $C$ of the first assertion of Proposition 3.6 depends only on $p$. To see this it suffices to observe that if $f$ does not depend on $z$ then the constant $c$ in the key inequality (3.11) depends only on $p$ (see Remark 3.5).*

## 4 Some useful tools

We begin with a useful comparison result for solutions of (3.1) with $K \equiv 0$.

**Proposition 4.1.** *Let* $(Y^1, Z^1), (Y^2, Z^2)$ *be solutions of* $\mathrm{BSDE}(\xi^1, f^1)$, $\mathrm{BSDE}(\xi^2, f^2)$, *respectively. Assume that* $(Y^1 - Y^2)^+ \in \mathcal{S}^q$ *for some* $q > 1$. *If* $\xi^1 \le \xi^2$ *and for a.e.* $t \in [0,T]$ *either*

$$\mathbf{1}_{\{Y_t^1 > Y_t^2\}}(f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^1, Z_t^1)) \le 0, \quad f^2 \text{ satisfies } (\mathrm{H2}), (\mathrm{H3}) \tag{4.1}$$

*or*

$$\mathbf{1}_{\{Y_t^1 > Y_t^2\}}(f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2)) \le 0, \quad f^1 \text{ satisfies } (\mathrm{H2}), (\mathrm{H3}) \tag{4.2}$$

*is satisfied then* $Y_t^1 \le Y_t^2$, $t \in [0,T]$.

*Proof.* We show the proposition in case (4.1) is satisfied. If (4.2) is satisfied, the proof is analogous. Without loss of generality we may assume that $\mu \le 0$. By the Itô-Tanaka formula, for every $p \in (1,q)$ and every stopping time $\tau \le T$,

$$\begin{aligned}
&|(Y^1_{t\wedge\tau} - Y^2_{t\wedge\tau})^+|^p + \frac{p(p-1)}{2}\int_{t\wedge\tau}^{\tau} \mathbf{1}_{\{Y_s^1 \neq Y_s^2\}} |(Y_s^1 - Y_s^2)^+|^{p-2} |Z_s^1 - Z_s^2|^2\, ds \\
&\quad = |(Y^1_\tau - Y^2_\tau)^+|^p + p\int_{t\wedge\tau}^{\tau} |(Y_s^1 - Y_s^2)^+|^{p-1}(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2))\, ds \\
&\quad\quad - p\int_{t\wedge\tau}^{\tau} |(Y_s^1 - Y_s^2)^+|^{p-1}(Z_s^1 - Z_s^2)\, dB_s.
\end{aligned} \tag{4.3}$$

By (4.1),

$$\begin{aligned}
&\mathbf{1}_{\{Y_t^1 > Y_t^2\}}(f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2)) \\
&\quad = \mathbf{1}_{\{Y_t^1 > Y_t^2\}}(f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^1, Z_t^1)) \\
&\quad\quad + \mathbf{1}_{\{Y_t^1 > Y_t^2\}}(f^2(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2)) \\
&\quad \le \mathbf{1}_{\{Y_t^1 > Y_t^2\}}(f^2(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^1)) \\
&\quad\quad + \mathbf{1}_{\{Y_t^1 > Y_t^2\}}(f^2(t, Y_t^2, Z_t^1) - f^2(t, Y_t^2, Z_t^2)) \\
&\quad \le \lambda \mathbf{1}_{\{Y_t^1 > Y_t^2\}} |Z_t^1 - Z_t^2|.
\end{aligned}$$

From this, (4.3) and Young's inequality,

$$\begin{aligned}
&|(Y^1_{t\wedge\tau} - Y^2_{t\wedge\tau})^+|^p + \frac{p(p-1)}{2}\int_{t\wedge\tau}^{\tau} \mathbf{1}_{\{Y_s^1 \neq Y_s^2\}} |(Y_s^1 - Y_s^2)^+|^{p-2} |Z_s^1 - Z_s^2|^2\, ds \\
&\quad \le |(Y^1_\tau - Y^2_\tau)^+|^p + p\lambda\int_{t\wedge\tau}^{\tau} |(Y_s^1 - Y_s^2)^+|^{p-1} |Z_s^1 - Z_s^2|\, ds \\
&\quad\quad - p\int_{t\wedge\tau}^{\tau} |(Y_s^1 - Y_s^2)^+|^{p-1}(Z_s^1 - Z_s^2)\, dB_s \\
&\quad \le |(Y^1_\tau - Y^2_\tau)^+|^p + \frac{p\lambda^2}{p-1}\int_{t\wedge\tau}^{\tau} |(Y_s^1 - Y_s^2)^+|^{p}\, ds \\
&\quad\quad + \frac{p(p-1)}{4}\int_{t\wedge\tau}^{\tau} \mathbf{1}_{\{Y_s^1 \neq Y_s^2\}} |(Y_s^1 - Y_s^2)^+|^{p-2} |Z_s^1 - Z_s^2|^2\, ds \\
&\quad\quad - p\int_{t\wedge\tau}^{\tau} |(Y_s^1 - Y_s^2)^+|^{p-1}(Z_s^1 - Z_s^2)\, dB_s.
\end{aligned}$$

Let $\tau_k = \inf\{t \in [0,T]; \int_0^t |(Y_s^1 - Y_s^2)^+|^{2(p-1)} |Z_s^1 - Z_s^2|^2\, ds \ge k\} \wedge T$. From the above estimate it follows that

$$E|(Y^1_{t\wedge\tau_k} - Y^2_{t\wedge\tau_k})^+|^p \le E|(Y^1_{\tau_k} - Y^2_{\tau_k})^+|^p + \frac{p\lambda^2}{p-1} E\int_{t\wedge\tau_k}^{\tau_k} |(Y_s^1 - Y_s^2)^+|^p\, ds.$$

Since$(Y^1-Y^2)^+\in\mathcal{S}^q$, letting $k\to+\infty$ and using the assumptions we get

$$E|(Y^1_t-Y^2_t)^+|^p\le\frac{p\lambda^2}{p-1}E\int_t^T|(Y^1_s-Y^2_s)^+|^p\,ds,\quad t\in[0,T].$$

By Gronwall's lemma, $E|(Y^1_t-Y^2_t)^+|^p=0$, $t\in[0,T]$, from which the desired result follows. □

**Lemma 4.2.** *Assume that* $\{(X^n,Y^n,K^n)\}$ *is a sequence of real valued càdlàg progressively measurable processes such that*

(a) $Y^n_t=-K^n_t+X^n_t$, $t\in[0,T]$, $K^n$*-increasing,* $K^n_0=0$,

(b) $Y^n_t\uparrow Y_t$, $t\in[0,T]$, $Y^1,Y$ *are of class* (D),

(c) *There exists a càdlàg process* $X$ *such that for some subsequence* $\{n'\}$, $X^{n'}_\tau\to X_\tau$ *weakly in* $\mathbb{L}^1(\mathcal{F}_T)$ *for every stopping time* $\tau\le T$.

*Then* $Y$ *is càdlàg and there exists a càdlàg increasing process* $K$ *such that* $K^{n'}_\tau\to K_\tau$ *weakly in* $\mathbb{L}^1(\mathcal{F}_T)$ *for every stopping time* $\tau\le T$ *and*

$$Y_t=-K_t+X_t,\quad t\in[0,T].$$

*Proof.* From (b) it follows that $Y^{n'}_\tau\to Y_\tau$ weakly in $\mathbb{L}^1(\mathcal{F}_T)$ for every stopping time $\tau\le T$. Set $K_t=X_t-Y_t$. By the above and (c), $K^{n'}_\tau\to K_\tau$ weakly in $\mathbb{L}^1(\Omega)$ for every stopping time $\tau\le T$. If $\sigma,\tau$ are stopping times such that $\sigma\le\tau\le T$ then $K_\sigma\le K_\tau$ since $K^n_\sigma\le K^n_\tau$, $n\in\mathbb{N}$. Therefore $K$ is increasing. The fact that $Y,K$ are càdlàg processes follows easily from [14, Lemma 2.2]. □

In what follows we say that a sequence $\{\tau_k\}$ of stopping times is stationary if

$$P(\liminf_{k\to+\infty}\{\tau_k=T\})=1.$$

**Lemma 4.3.** *Assume that* $\{Y^n\}$ *is a nondecreasing sequence of continuous processes such that* $\sup_{n\ge1}E|Y^{n,*}_T|^q<+\infty$ *for some* $q>0$. *Then there exists a stationary sequence* $\{\tau_k\}$ *of stopping times such that* $Y^{n,*}_{\tau_k}\le k\vee|Y^n_0|$, $P$*-a.s. for every* $k\in\mathbb{N}$.

*Proof.* Set $V^n_t=\sup_{0\le s\le t}(Y^n_s-Y^1_s)$. Then $V^n$ is nonnegative and $V^n\in\mathcal{V}^+_c$. Since $\{Y^n\}$ is nondecreasing, there exists an increasing process $V$ such that $V^n_t\uparrow V_t$, $t\in[0,T]$. By Fatou's lemma,

$$EV^q_T\le\liminf_{n\to+\infty}E|V^n_T|^q\le c(q)\sup_{n\ge1}E|Y^{n,*}_T|^q<\infty.$$

Now, set $V'_t=\inf_{t<t'\le T}V_{t'}$, $t\in[0,T]$ and then $\tau_k=\inf\{t\in[0,T];Y^{1,*}_t+V'_t>k\}\wedge T$. It is known that $V'$ is a progressively measurable càdlàg process. Since $V_T$ is integrable, the sequence $\{\tau_k\}$ is stationary. From the above it follows that if $\tau_k>0$ then

$$Y^{n,*}_{\tau_k}=Y^{n,*}_{\tau_k-}\le V'_{\tau_k-}+Y^{1,*}_{\tau_k-}\le k,\quad k\in\mathbb{N},$$

and the proof is complete. □

**Lemma 4.4.** *If* $\{Z^n\}$ *is a sequence of progressively measurable processes such that* $\sup_{n\ge1}E(\int_0^T|Z^n_t|^2\,dt)^{p/2}<\infty$ *for some* $p>1$, *then there exists* $Z\in M^p$ *and a subsequence* $\{n'\}$ *such that for every stopping time* $\tau\le T$, $\int_0^\tau Z^{n'}_t\,dB_t\to\int_0^\tau Z_t\,dB_t$ *weakly in* $\mathbb{L}^p(\mathcal{F}_T)$.

*Proof.* Since $\{Z^n\}$ is bounded in $\mathbb{L}^{2,p}(\mathcal{F})$ and the space $\mathbb{L}^{2,p}(\mathcal{F})$ is reflexive, there exists a subsequence (still denoted by $\{n\}$) and $Z \in \mathbb{L}^{2,p}(\mathcal{F})$ such that $Z^n \to Z$ weakly in $\mathbb{L}^{2,p}(\mathcal{F})$. It is known that if $\xi \in \mathbb{L}^{p'}(\mathcal{F}_T)$, where $p' = p/(p-1)$, then there exists $\eta \in \mathbb{L}^{2,p'}(\mathcal{F}) = (\mathbb{L}^{2,p}(\mathcal{F}))^*$ such that

$$\xi = E\xi + \int_0^T \eta_t \, dB_t. \tag{4.4}$$

Let $f \in (\mathbb{L}^p(\mathcal{F}_T))^*$. Then there exists $\xi \in \mathbb{L}^{p'}(\mathcal{F}_T)$ such that $f(\zeta) = E\zeta\xi$ for every $\zeta \in \mathbb{L}^p(\mathcal{F}_T)$. Let $\eta \in \mathbb{L}^{2,p'}(\mathcal{F})$ be such that (4.4) is satisfied. Without loss of generality we may assume that $E\xi = 0$. Then by Itô's isometry,

$$\begin{aligned} f(\int_0^T Z^n_t \, dB_t) &= E\xi \int_0^T Z^n_t \, dB_t = E \int_0^T \eta_t \, dB_t \int_0^T Z^n_t \, dB_t \\ &= E \int_0^T \eta_t Z^n_t \, dt \to E \int_0^T \eta_t Z_t = f(\int_0^T Z_t \, dB_t). \end{aligned}$$

Since the same reasoning applies to the sequence $\{\mathbf{1}_{\{\cdot \le \tau\}} Z^n\}$ in place of $\{Z^n\}$, the lemma follows. □

## 5 Existence and uniqueness results for $p > 1$

First we recall the definition of a solution $(Y, Z, K)$ of (1.1). Note that a priori we do not impose any integrability conditions on the processes $Y, Z, K$.

**Definition 5.1.** *We say that a triple* $(Y, Z, K)$ *of progressively measurable processes is a solution of* $\mathrm{RBSDE}(\xi, f, L)$ *iff*

(a) $K$ *is an increasing continuous process,* $K_0 = 0$*,*
(b) $Z \in M$ *and the mapping* $[0, T] \ni t \mapsto f(t, Y_t, Z_t)$ *belongs to* $\mathbb{L}^1(0, T)$*,* $P$*-a.s.,*
(c) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dK_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T]$,
(d) $Y_t \ge L_t$, $t \in [0, T]$, $\int_0^T (Y_t - L_t) \, dK_t = 0$.

**Proposition 5.2.** *Let* $(Y^1, Z^1, K^1), (Y^2, Z^2, K^2)$ *be solutions of* $\mathrm{RBSDE}(\xi^1, f^1, L^1)$ *and* $\mathrm{RBSDE}(\xi^2, f^2, L^2)$*, respectively. Assume that* $(Y^1 - Y^2)^+ \in \mathcal{S}^q$ *for some* $q > 1$*. If* $\xi^1 \le \xi^2$*,* $L^1_t \le L^2_t$*,* $t \in [0, T]$*, and either* (4.1) *or* (4.2) *is satisfied then* $Y^1_t \le Y^2_t$*,* $t \in [0, T]$*.*

*Proof.* Assume that (4.1) is satisfied. Let $q > 1$ be such that $(Y^1 - Y^2)^+ \in \mathcal{S}^q$. Without loss of generality we may assume that $\mu \le 0$. By the Itô-Tanaka formula, for $p \in (1, q)$ and every stopping time $\tau \le T$,

$$\begin{aligned} &|(Y^1_{t\wedge\tau} - Y^2_{t\wedge\tau})^+|^p + \frac{p(p-1)}{2} \int_{t\wedge\tau}^\tau \mathbf{1}_{\{Y^1 \ne Y^2_s\}} |(Y^1_s - Y^2_s)^+|^{p-2} |Z^1_s - Z^2_s|^2 \, ds \\ &= |(Y^1_\tau - Y^2_\tau)^+|^p + p \int_{t\wedge\tau}^\tau |(Y^1_s - Y^2_s)^+|^{p-1} (f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)) \, ds \\ &\quad + p \int_{t\wedge\tau}^\tau |(Y^1_s - Y^2_s)^+|^{p-1} \, (dK^1_s - dK^2_s) \\ &\quad - p \int_{t\wedge\tau}^\tau |(Y^1_s - Y^2_s)^+|^{p-1} (Z^1_s - Z^2_s) \, dB_s. \end{aligned} \tag{5.1}$$

By monotonicity of the function $x \mapsto \hat{x}|x|^{p-1}$, condition (d) of the definition of a solution of reflected BSDE and the fact that $L^1_t \le L^2_t$ for $t \in [0, T]$,

$$\begin{aligned} \int_{t\wedge\tau}^\tau |(Y^1_s - Y^2_s)^+|^{p-1} \, (dK^1_s - dK^2_s) &\le \int_{t\wedge\tau}^\tau |(Y^1_s - Y^2_s)^+|^{p-1} \, dK^1_s \\ &\le \int_{t\wedge\tau}^\tau |(Y^1_s - L^1_s)^+|^{p-1} \, dK^1_s = 0. \end{aligned}$$

Combining this with (5.1) we get estimate (4.3) in Proposition 4.1. Therefore repeating arguments following (4.3) in the proof of that proposition we obtain the desired result. The proof in case (4.2) is satisfied is analogous and therefore left to the reader. □

**Proposition 5.3.** *If* $f$ *satisfies* (H2)*,* (H3) *then there exists at most one solution* $(Y,Z,K)$ *of* $\mathrm{RBSDE}(\xi,f,L)$ *such that* $Y\in\mathcal{S}^p$ *for some* $p>1$*.*

*Proof.* Follows immediately from Proposition 5.2 and uniqueness of the Doob-Meyer decomposition of semimartingales. □

**Theorem 5.4.** *Let* $p>1$*.*

(i) *Assume* (H1)–(H6)*. Then there exists a solution* $(Y,Z,K)$ *of* $\mathrm{RBSDE}(\xi,f,L)$ *such that* $(Y,Z,K)\in\mathcal{S}^p\otimes M^p\otimes\mathcal{V}_c^{+,p}$ *iff* (H7) *is satisfied.*

(ii) *Assume* (H1)–(H7)*. For* $n\in\mathbb{N}$ *let* $(Y^n,Z^n)$ *be a solution of the BSDE*

$$Y_t^n=\xi+\int_t^T f(s,Y_s^n,Z_s^n)\,ds+\int_t^T dK_s^n-\int_t^T Z_s^n\,dB_s,\ t\in[0,T] \tag{5.2}$$

*with*

$$K_t^n=\int_0^t n(Y_s^n-L_s)^-\,ds \tag{5.3}$$

*such that* $(Y^n,Z^n)\in\mathcal{S}^p\otimes M^p$*. Then*

$$E\sup_{t\le T}|Y_t^n-Y_t|^p+E\sup_{t\le T}|K_t^n-K_t|^p+E(\int_0^T|Z_t^n-Z_t|^2dt)^{p/2}\to 0 \tag{5.4}$$

*as* $n\to+\infty$*.*

*Proof.* Without loss of generality we may assume that $\mu\le 0$. Assume that there is a solution $(Y,Z,K)\in\mathcal{S}^p\otimes M^p\otimes\mathcal{V}_c^{+,p}$ of $\mathrm{RBSDE}(\xi,f,L)$. Then by [3, Remark 4.3],

$$E(\int_0^T|f(s,Y_s,Z_s)|\,ds)^p\le cE\Big(|\xi|^p+(\int_0^T f_s\,ds)^p+K_T^p\Big), \tag{5.5}$$

which in view of (H2) and the fact that $Y_t\ge L_t$, $t\in[0,T]$ shows (H7). Conversely, let us assume that (H1)–(H7) are satisfied. Let $(Y^n,Z^n)$ be a solution of (5.2) such that $(Y^n,Z^n)\in\mathcal{S}^p\otimes M^p$. We will show that there exists a process $\overline{X}\in\mathcal{H}_c^p$ such that $\overline{X}_t\ge Y_t^n$, $t\in[0,T]$ for every $n\in\mathbb{N}$. Since $X\in\mathcal{H}_c^p$, there exist $M\in\mathcal{M}_c^p$ and $V\in\mathcal{V}_c^p$ such that $X=V+M$. By the representation property of Brownian filtration, there exists $Z'\in M^p$ such that

$$X_t=X_T-\int_t^T dV_s-\int_t^T Z'_s\,dB_s,\quad t\in[0,T].$$

The above identity can be rewritten in the form

$$\begin{aligned}X_t=X_T&+\int_t^T f(s,X_s,Z'_s)\,ds-\int_t^T(f^+(s,X_s,Z'_s)\,ds+dV_s^+)\\&+\int_t^T(f^-(s,X_s,Z'_s)\,ds+dV_s^-)-\int_t^T Z'_s\,dB_s,\quad t\in[0,T].\end{aligned}$$

By [3, Theorem 4.2], there exists a unique solution $(\overline{X},\overline{Z})\in\mathcal{S}^p\otimes M^p$ of the BSDE

$$\overline{X}_t=\xi\vee X_T+\int_t^T f(s,\overline{X}_s,\overline{Z}_s)\,ds+\int_t^T(f^-(s,X_s,Z'_s)\,ds+dV_s^-)-\int_t^T\overline{Z}_s\,dB_s.$$

By Proposition 4.1, $\overline{X}_t \ge X_t \ge L_t$, $t\in[0,T]$. Hence

$$\overline{X}_t=\xi\vee X_T+\int_t^T f(s,\overline{X}_s,\overline{Z}_s)\,ds+\int_t^T n(\overline{X}_s-L_s)^-\,ds \\ +\int_t^T (f^-(s,X_s,Z'_s)\,ds+dV^-_s)-\int_t^T \overline{Z}_s\,dB_s,\quad t\in[0,T],$$

so using once again Proposition 4.1 we see that $\overline{X}_t\ge Y^n_t$, $t\in[0,T]$. By [3, Remark 4.3], $E(\int_0^T|f(s,\overline{X}_s,0)|\,ds)^p<\infty$. Hence, by Lemma 3.3 and Proposition 3.6,

$$E|Y^{n,*}_T|^p+E(\int_0^T|Z^n_s|^2\,ds)^{p/2}+E|K^n_T|^p \\ \le C(p,\lambda,T)E\bigg(|\xi|^2+(\int_0^T f_s\,ds)^p+(\int_0^T|f(s,\overline{X}_s,0)|\,ds)^p\bigg). \tag{5.6}$$

From this and [3, Remark 4.3],

$$E(\int_0^T|f(s,Y^n_s,Z^n_s)|\,ds)^p\le C^{'}(p,\lambda,T). \tag{5.7}$$

By Proposition 4.1 there exists a progressively measurable process $Y$ such that $Y^n_t\uparrow Y_t$, $t\in[0,T]$. Using the monotone convergence of $Y^n$, (H3)–(H5), (5.6), (5.7) and the Lebesgue dominated convergence theorem we conclude that

$$E(\int_0^T|f(s,Y^n_s,0)-f(s,Y_s,0)|\,ds)^p\to 0 \tag{5.8}$$

Moreover, by (H2) and (5.6),

$$\sup_{n\ge1}E\int_0^T|f(s,Y^n_s,Z^n_s)-f(s,Y^n_s,0)|^p\,ds<\infty.$$

It follows in particular that there exists a process $\eta\in\mathbb{L}^p(\mathcal{F})$ such that

$$\int_0^\tau(f(s,Y^n_s,Z^n_s)-f(s,Y^n_s,0))\,ds\to\int_0^\tau\eta_s\,ds$$

weakly in $\mathbb{L}^1(\mathcal{F}_T)$ for every stopping time $\tau\le T$. Consequently, by Lemmas 4.2 and 4.4, $Y$ is a càdlàg process and there exist $Z\in M^p$ and a càdlàg increasing process $K$ such that $K_0=0$ and

$$Y_t=\xi+\int_t^T f(s,Y_s,0)\,ds+\int_t^T\eta_s\,ds+\int_t^T dK_s-\int_t^T Z_s\,dB_s,\quad t\in[0,T]. \tag{5.9}$$

From (5.2), (5.6), (5.7) and the pointwise convergence of the sequence $\{Y^n\}$ one can deduce that $E\int_0^T(Y_s-L_s)^-\,ds=0$, which when combined with (H6) and the fact that $Y$ is càdlàg implies that $Y_t\ge L_t$, $t\in[0,T]$. From this, the monotone character of the convergence of the sequence $\{Y^n\}$ and Dini's theorem we conclude that

$$E|(Y^n-L)^{-,*}_T|^p\to 0. \tag{5.10}$$

By Proposition 3.1, for $n,m\in\mathbb{N}$ we have

$$
\begin{aligned}
&|Y^n_t-Y^m_t|^p+c(p)\int_t^T|Y^n_s-Y^m_s|^{p-2}\mathbf{1}_{\{Y^n_s-Y^m_s\neq 0\}}|Z^n_s-Z^m_s|^2\,ds\\
&\quad=p\int_t^T|Y^n_s-Y^m_s|^{p-1}\widehat{Y^n_s-Y^m_s}(f(s,Y^n_s,Z^n_s)-f(s,Y^m_s,Z^m_s))\,ds\\
&\qquad+p\int_t^T|Y^n_s-Y^m_s|^{p-1}\widehat{Y^n_s-Y^m_s}(dK^n_s-dK^m_s)\\
&\qquad-p\int_t^T|Y^n_s-Y^m_s|^{p-1}\widehat{Y^n_s-Y^m_s}(Z^n_s-Z^m_s)\,dB_s,\quad t\in[0,T].
\end{aligned}
\tag{5.11}
$$

By monotonicity of the function $\mathbb{R}\ni x\mapsto|x|^{p-1}\hat{x}$,

$$
\int_t^T|Y^n_s-Y^m_s|^{p-1}\widehat{Y^n_s-Y^m_s}\,dK^n_s\le\int_t^T|(Y^m_s-L_s)^-|^{p-1}(\widehat{Y^m_s-L_s})^-\,dK^n_s \tag{5.12}
$$

and

$$
-\int_t^T|Y^n_s-Y^m_s|^{p-1}\widehat{Y^n_s-Y^m_s}\,dK^m_s\le\int_t^T|(Y^n_s-L_s)^-|^{p-1}(\widehat{Y^n_s-L_s})^-\,dK^m_s. \tag{5.13}
$$

By (H2), (H3), (5.11)–(5.13) and Hölder's inequality,

$$
\begin{aligned}
&E|Y^n_t-Y^m_t|^p+c(p)E\int_t^T|Y^n_s-Y^m_s|^{p-2}\mathbf{1}_{\{Y^n_s-Y^m_s\neq 0\}}|Z^n_s-Z^m_s|^2\,ds\\
&\quad\le p\lambda E\int_0^T|Y^n_s-Y^m_s|^{p-1}|Z^n_s-Z^m_s|\,ds+(E|(Y^n-L)^{-,*}_T|^p)^{(p-1)/p}(E|K^m_T|^p)^{1/p}\\
&\qquad+(E|(Y^m-L)^{-,*}_T|^p)^{(p-1)/p}(E|K^n_T|^p)^{1/p}.
\end{aligned}
\tag{5.14}
$$

Since

$$
\begin{aligned}
p\lambda|Y^n_s-Y^m_s|^{p-1}|Z^n_s-Z^m_s|&\le\frac{p\lambda^2}{1\wedge(p-1)}|Y^n_s-Y^m_s|^p\\
&\quad+\frac{c(p)}{2}\mathbf{1}_{\{Y^n_s-Y^m_s\neq 0\}}|Y^n_s-Y^m_s|^{p-2}|Z^n_s-Z^m_s|^2,
\end{aligned}
$$

from (5.14) we get

$$
\begin{aligned}
&E|Y^n_t-Y^m_t|^p+\frac{c(p)}{2}E\int_t^T|Y^n_s-Y^m_s|^{p-2}\mathbf{1}_{\{Y^n_s-Y^m_s\neq 0\}}|Z^n_s-Z^m_s|^2\,ds\\
&\quad\le c(p,\lambda)E\int_0^T|Y^n_s-Y^m_s|^p\,ds+(E|(Y^n-L)^{-,*}_T|^p)^{(p-1)/p}(E|K^m_T|^p)^{1/p}\\
&\qquad+(E|(Y^m-L)^{-,*}_T|^p)^{(p-1)/p}(E|K^n_T|^p)^{1/p}\equiv I_{n,m}.
\end{aligned}
$$

From the above, (5.6), (5.10) and the monotone convergence of $\{Y^n\}$ we get

$$
\lim_{n,m\to+\infty}I^{n,m}=0 \tag{5.15}
$$

which implies that

$$
\lim_{n,m\to+\infty}E\int_0^T|Y^n_s-Y^m_s|^{p-2}\mathbf{1}_{\{Y^n_s\neq Y^m_s\}}|Z^n_s-Z^m_s|^2\,ds=0. \tag{5.16}
$$

From (5.11) one can also conclude that

$$
\begin{aligned}
&E\sup_{0\le t\le T}|Y^n_t-Y^m_t|^p\\
&\qquad\le c'(p,\lambda)\Big(I^{n,m}+E\sup_{0\le t\le T}|\int_t^T|Y^n_s-Y^m_s|^{p-1}\widehat{Y^n_s-Y^m_s}(Z^n_s-Z^m_s)\,dB_s|\Big).
\end{aligned}
$$

Using the Burkholder-Davis-Gundy inequality and then Young's inequality we deduce from the above that

$$
E\sup_{0\le t\le T}|Y^n_t-Y^m_t|^p\le c''(p,\lambda)\Big(I^{n,m}+E\int_0^T\mathbf{1}_{\{Y^n_s\neq Y^m_s\}}|Y^n_s-Y^m_s|^{p-2}|Z^n_s-Z^m_s|^2\,ds\Big).
$$

Hence, by (5.15) and (5.16),

$$
\lim_{n,m\to+\infty}E\sup_{0\le t\le T}|Y^n_t-Y^m_t|^p=0, \tag{5.17}
$$

which implies that $Y\in\mathcal{S}^p$. Our next goal is to show that

$$
\lim_{n,m\to+\infty}E(\int_0^T|Z^n_t-Z^m_t|^2\,dt)^{p/2}=0. \tag{5.18}
$$

By Itô's formula applied to $|Y^n-Y^m|^2$, (H2) and (H3),

$$
\begin{aligned}
\int_0^T|Z^n_t-Z^m_t|^2\,dt&\le 2\lambda\int_0^T|Y^n_t-Y^m_t||Z^n_t-Z^m_t|\,dt+2\int_0^T|Y^n_t-Y^m_t|\,dK^n_t\\
&\quad+2\int_0^T|Y^n_t-Y^m_t|\,dK^m_t+\sup_{0\le t\le T}|\int_t^T(Z^n_s-Z^m_s)(Y^n_s-Y^m_s)\,dB_s|.
\end{aligned}
$$

Hence, by the Burkholder-Davis-Gundy inequality and Young's inequality,

$$
\begin{aligned}
E(\int_0^T|Z^n_t-Z^m_t|^2\,dt)^{p/2}&\le C(p,\lambda)\Big(E|(Y^n-Y^m)^*_T|^p\\
&\quad+(E|(Y^n-Y^m)^*_T|^p)^{1/2}(E|K^n_T|^p)^{1/2}+(E|(Y^n-Y^m)^*_T|^p)^{1/2}(E|K^m_T|^p)^{1/2}\Big).
\end{aligned}
$$

From the above inequality, (5.6) and (5.17) we get (5.18). From (5.18) and (5.9) it follows immediately that

$$
Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\,ds+\int_t^T dK_s-\int_t^T Z_s\,dB_s,\quad t\in[0,T],
$$

which implies that $K$ is continuous. In fact, by (5.6), $K\in\mathcal{V}^{+,p}_c$. Moreover, from (5.2), (5.7), (5.8) (5.17), (5.18) and (H2) we deduce that

$$
\lim_{n,m\to+\infty}E\sup_{0\le t\le T}|K^n_t-K^m_t|^p=0. \tag{5.19}
$$

Since $\int_0^T(Y^n_t-L_t)\,dK^n_t\le 0$, it follows from (5.17), (5.19) that $\int_0^T(Y_t-L_t)\,dK_t\le 0$, which when combined with the fact that $Y_t\ge L_t$, $t\in[0,T]$ shows that

$$
\int_0^T(Y_t-L_t)\,dK_t=0.
$$

Thus the triple $(Y,Z,K)$ is a solution of RBSDE$(\xi,f,L)$, which completes the proof of (i). Assertion (ii) follows from (5.17)–(5.19). □

**Remark 5.5.** *Let $p>1$ and let assumptions (H1)–(H3) hold. If $(Y,Z,K)$ is a solution of RBSDE$(\xi,f,L)$ such that $(Y,Z)\in\mathcal{S}^p\otimes M^p$ then from [3, Remark 4.3] it follows immediately that*

$$E(\int_0^T|f(s,Y_s,Z_s)|\,ds)^p<+\infty \text{ iff } EK_T^p<+\infty.$$

*Moreover, if there exists $X\in\mathcal{H}_c^p$ such that $E(\int_0^T f^-(s,X_s,0)\,ds)^p<+\infty$ then*

$$E(\int_0^T \mathbf{1}_{\{Y_s\le X_s\}}\,dK_s)^p<+\infty. \tag{5.20}$$

*Indeed, since $X\in\mathcal{H}_c^p$, there exist $M\in\mathcal{M}_c^p$ and $V\in\mathcal{V}_c^p$ such that $X_t=X_0+M_t+V_t$, $t\in[0,T]$. Let $L^0(Y-X)$ denote the local time of $Y-X$ at 0. By (H2), (H3) and the Itô-Tanaka formula applied to $(Y-X)^-$,*

$$\begin{aligned}\int_0^T \mathbf{1}_{\{Y_s\le X_s\}}\,dK_s&=(Y_T-X_T)^--(Y_0-X_0)^--\int_0^T\mathbf{1}_{\{Y_s\le X_s\}}f(s,Y_s,Z_s)\,ds\\&\quad-\int_0^T\mathbf{1}_{\{Y_s\le X_s\}}dV_s-\frac12\int_0^T dL_s^0(Y-X)-\int_0^T\mathbf{1}_{\{Y_s\le X_s\}}Z_s\,dB_s\\&\quad+\int_0^T\mathbf{1}_{\{Y_s\le X_s\}}\,dM_s\\&\le 2Y_T^*+2X_T^*-\int_0^T\mathbf{1}_{\{Y_s\le X_s\}}f(s,X_s,0)\,ds+\lambda\int_0^T|Z_s|\,ds\\&\quad+\int_0^T d|V|_s-\int_0^T\mathbf{1}_{\{Y_s\le X_s\}}Z_s\,dB_s+\int_0^T\mathbf{1}_{\{Y_s\le X_s\}}\,dM_s,\end{aligned}$$

*from which one can easily get (5.20).*

We close this section with an example which shows that assumption (1.4) is not necessary for existence of $p$-integrable solutions of reflected BSDEs.

**Example 5.6.** *Let $V_t=\exp(|B_t|^4)$, $t\in[0,T]$. Observe that*

$$P(\int_0^T V_t\,dt<+\infty)=1,\quad E\int_a^T V_t\,dt=+\infty,\quad a\in(0,T).$$

*Now, set $\xi\equiv0$, $f(t,y)=-(y-(T-t))^+V_t$, $L_t=T-t$, $t\in[0,T]$. Then $\xi,f,L$ satisfy (H1)–(H7) with $p=2$. On the other hand,*

$$E\int_0^T f^-(t,L_t^*)\,dt=E\int_0^T f^-(t,T)\,dt=E\int_0^T tV_t\,dt\ge aE\int_a^T V_t\,dt=+\infty.$$

## 6 Existence and uniqueness results for $p=1$

We first prove uniqueness.

**Proposition 6.1.** *If $f$ satisfies* (H2), (H3) *and* (Z) *then there exists at most one solution $(Y,Z,K)$ of* RBSDE$(\xi,f,L)$ *such that $Y$ is of class* (D) *and $Z\in\bigcup_{\beta>\alpha}M^\beta$.*

*Proof.* Without loss of generality we may assume that $\mu\le0$. Let triples $(Y^1,Z^1,K^1)$, $(Y^2,Z^2,K^2)$ be two solutions to RBSDE$(\xi,f,L)$. By Proposition 5.2 it suffices to prove that $|Y^1-Y^2|\in\mathcal{S}^p$ for some $p>1$. Write $Y=Y^1-Y^2$, $Z=Z^1-Z^2$, $K=K^1-K^2$

and $\tau_k = \inf\{t \in [0,T]; \int_0^t (|Z_s^1|^2 + |Z_s^2|^2)\,ds > k\} \wedge T$. Then by the Itô formula (see [3, Corollary 2.3]),

$$|Y_{t\wedge\tau_k}| \le |Y_{\tau_k}| + \int_{t\wedge\tau_k}^{\tau_k} \hat{Y}_s(f(s,Y_s^1,Z_s^1) - f(s,Y_s^2,Z_s^2))\,ds \\ + \int_{t\wedge\tau_k}^{\tau_k} \hat{Y}_s\,dK_s - \int_{t\wedge\tau_k}^{\tau_k} \hat{Y}_s Z_s\,dB_s, \quad t\in[0,T].$$

By the minimality property (d) of the reaction measures $K^1, K^2$ in the definition of a solution of RBSDE$(\xi,f,L)$, $\int_0^T \hat{Y}_s\,dK_s \le 0$. Hence

$$|Y_{t\wedge\tau_k}| \le |Y_{\tau_k}| + \int_{t\wedge\tau_k}^{\tau_k} \hat{Y}_s(f(s,Y_s^1,Z_s^1) - f(s,Y_s^2,Z_s^2))\,ds - \int_{t\wedge\tau_k}^{\tau_k} \hat{Y}_s Z_s\,dB_s \\ \le |Y_{\tau_k}| + \int_0^T |f(s,Y_s^1,Z_s^1) - f(s,Y_s^1,Z_s^2)|\,ds - \int_{t\wedge\tau_k}^{\tau_k} \hat{Y}_s Z_s\,dB_s$$

for $t\in[0,T]$, the last inequality being a consequence of (H3). Consequently,

$$|Y_{t\wedge\tau_k}| \le E^{\mathcal{F}_t}(|Y_{\tau_k}| + \int_0^T |f(s,Y_s^1,Z_s^1) - f(s,Y_s^1,Z_s^2)|\,ds), \quad t\in[0,T].$$

Since $Y$ is of class (D), letting $k\to+\infty$ we conclude from the above that

$$|Y_t| \le E^{\mathcal{F}_t}(\int_0^T |f(s,Y_s^1,Z_s^1) - f(s,Y_s^1,Z_s^2)|\,ds), \quad t\in[0,T].$$

By (Z),

$$|Y_t| \le 2\gamma E^{\mathcal{F}_t}(\int_0^T (g_s + |Y^1|_s + |Z^1|_s + |Z_s^2|)^\alpha\,ds).$$

From this it follows that $|Y| \in \mathcal{S}^p$ for some $p>1$, which proves the proposition. □

**Remark 6.2.** *A brief inspection of the proof of Proposition 6.1 reveals that if $f$ does not depend on $z$ and satisfies (H2) then there exits at most one solution $(Y,Z,K)$ of RBSDE$(\xi,f,L)$ such that $Y$ is of class (D).*

**Remark 6.3.** *If (H1), (H3), (Z) are satisfied and $(Y,Z)$ is a unique solution of BSDE$(\xi,f)$ such that $Y$ is of class (D) and $Z\in\mathbb{L}^\alpha(\mathcal{F})$ then*

$$E\int_0^T |f(s,Y_s,Z_s)|\,ds < +\infty.$$

*Indeed, by Proposition 3.1, for every stopping time $\tau\le T$,*

$$|Y_{t\wedge\tau}| \le |Y_\tau| + \int_{t\wedge\tau}^{\tau} \hat{Y}_s f(s,Y_s,Z_s)\,ds - \int_{t\wedge\tau}^{\tau} \hat{Y}_s Z_s\,dB_s, \quad t\in[0,T].$$

*Hence*

$$-\int_{t\wedge\tau}^{\tau} \hat{Y}_s(f(s,Y_s,Z_s) - f(s,0,Z_s))\,ds \le |Y_\tau| - |Y_{t\wedge\tau}| + \int_{t\wedge\tau}^{\tau} |f(s,0,Z_s)|\,ds \\ - \int_{t\wedge\tau}^{\tau} \hat{Y}_s Z_s\,dB_s.$$

*By the above inequality, (H3) (without loss of generality we may assume that $\mu \le 0$) and (Z), for $t \in [0,T]$ we have*

$$E\int_{t\wedge\tau_k}^{\tau} |f(s,Y_s,Z_s)-f(s,0,Z_s)|\,ds \le E|Y_{\tau_k}| + E\int_{t\wedge\tau_k}^{\tau}(g_s+|Z_s|+|Y_s|)^\alpha\,ds + \int_{t\wedge\tau_k}^{\tau} f_s\,ds,$$

*where $\tau_k$ is defined by (3.2). Since $Y$ is of class (D), letting $k\to+\infty$ we obtain*

$$E\int_0^T |f(s,Y_s,Z_s)-f(s,0,Z_s)|\,ds \le E|\xi| + \gamma E\int_0^T (g_s+|Z_s|+|Y_s|)^\alpha\,ds + \int_0^T f_s\,ds.$$

*Using once again (Z) we conclude from the above that*

$$E\int_0^T |f(s,Y_s,Z_s)|\,ds \le E|\xi| + 2\gamma E\int_0^T (g_s+|Z_s|+|Y_s|)^\alpha\,ds + 2\int_0^T f_s\,ds < +\infty.$$

**Theorem 6.4.** *Let $p=1$.*

(i) *Assume* (H1)–(H6), (Z). *Then there exists a solution $(Y,Z,K)$ of* $\mathrm{RBSDE}(\xi,f,L)$ *such that $Y$ is of class* (D), *$K\in\mathcal{V}_c^{+,1}$ and $Z\in\bigcap_{q<1}M^q$ iff* (H7*) *is satisfied.*

(ii) *Assume* (H1)–(H6), (H7*) *and for $n\in\mathbb{N}$ let $(Y^n,Z^n)$ be a solution of* (5.2) *such that $(Y^n,Z^n)\in\mathcal{S}^q\otimes M^q$, $q\in(0,1)$, and $Y^n$ is of class* (D). *Let $K^n$ be defined by* (5.3). *Then for every $q\in(0,1)$,*

$$E\sup_{t\le T}|Y^n_t-Y_t|^q + E\sup_{t\le T}|K^n_t-K_t|^q + E(\int_0^T |Z^n_t-Z_t|^2\,dt)^{q/2}\to 0$$

*as $n\to+\infty$.*

*Proof.* (i) Necessity. By Remark 6.3, if there is a solution $(Y,Z,K)$ of $\mathrm{BSDE}(\xi,f,L)$ such that $(Y,Z)\in\mathcal{S}^q\otimes M^q$, $q\in(0,1)$, $K\in\mathcal{V}_c^{+,1}$ and $Y$ is of class (D) then (H7*) is satisfied with $X=Y$.

Sufficiency. We first show that the sequence $\{Y^n\}$ is nondecreasing. To this end, let us put $f_n(t,y,z)=f(t,y,z)+n(y-L_t)^-$. Since the exponential change of variable described at the beginning of the proof of Lemma 3.3 does not change the monotonicity of the sequence $\{Y^n\}$, we may and will assume that the mapping $\mathbb{R}\ni y\mapsto f_n(t,y,0)$ is nonincreasing. By the Itô-Tanaka formula, for every stopping time $\tau\le T$,

$$\begin{aligned}&(Y^n_{t\wedge\tau}-Y^{n+1}_{t\wedge\tau})^+ + \frac12\int_{\tau\wedge t}^{\tau} dL^0_s(Y^n-Y^{n+1})\\ &\quad=(Y^n_\tau-Y^{n+1}_\tau)^+ + \int_{t\wedge\tau}^{\tau}\mathbf{1}_{\{Y^n_s>Y^{n+1}_s\}}(f_n(s,Y^n_s,Z^n_s)-f_{n+1}(s,Y^{n+1}_s,Z^{n+1}_s))\,ds\\ &\qquad-\int_{t\wedge\tau}^{\tau}\mathbf{1}_{\{Y^n_s>Y^{n+1}_s\}}(Z^n_s-Z^{n+1}_s)\,dB_s.\end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ on both sides of the above equality with $\tau$ replaced by $\tau_k=\inf\{t\in[0,T];\int_0^t|Z^n_s-Z^{n+1}_s|^2\,ds\ge k\}\wedge T$, letting $k\to+\infty$ and using the fact that $Y$ is of class (D) we obtain

$$(Y^n_t-Y^{n+1}_t)^+ \le E^{\mathcal{F}_t}\int_t^T \mathbf{1}_{\{Y^n_s>Y^{n+1}_s\}}(f_n(s,Y^n_s,Z^n_s)-f_{n+1}(s,Y^{n+1}_s,Z^{n+1}_s))\,ds. \tag{6.1}$$

From the above inequality and the fact that $f_n \le f_{n+1}$ we get

$$\begin{aligned}
&\int_t^T \mathbf{1}_{\{Y_s^n > Y_s^{n+1}\}} (f_n(s, Y_s^n, Z_s^n) - f_{n+1}(s, Y_s^n, Z_s^n))\, ds \\
&\quad + \int_t^T \mathbf{1}_{\{Y_s^n > Y_s^{n+1}\}} (f_{n+1}(s, Y_s^n, Z_s^n) - f_{n+1}(s, Y_s^{n+1}, Z_s^{n+1}))\, ds \\
&\le \int_t^T \mathbf{1}_{\{Y_s^n > Y_s^{n+1}\}} (f_{n+1}(s, Y_s^n, Z_s^n) - f_{n+1}(s, Y_s^{n+1}, Z_s^{n+1}))\, ds \\
&= \int_t^T \mathbf{1}_{\{Y_s^n > Y_s^{n+1}\}} (f_{n+1}(s, Y_s^n, Z_s^n) - f_{n+1}(s, Y_s^n, 0))\, ds \\
&\quad + \int_t^T \mathbf{1}_{\{Y_s^n > Y_s^{n+1}\}} (f_{n+1}(s, Y_s^n, 0) - f_{n+1}(s, Y_s^{n+1}, 0))\, ds \\
&\quad + \int_t^T \mathbf{1}_{\{Y_s^n > Y_s^{n+1}\}} (f_{n+1}(s, Y_s^{n+1}, 0) - f_{n+1}(s, Y_s^{n+1}, Z_s^{n+1}))\, ds.
\end{aligned}$$

Since $f_n(t,y,z) - f_n(t,y,z') = f(t,y,z) - f(t,y,z')$ for every $t \in [0,T]$, $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, using the monotonicity of $f_{n+1}$ and assumption (Z) we conclude from the above and (6.1) that for $t \in [0,T]$,

$$(Y_t^n - Y_t^{n+1})^+ \le 2\gamma E^{\mathcal{F}_t} \int_0^T (g_s + |Y_s^n| + |Z_s^n| + |Y_s^{n+1}| + |Z_s^{n+1}|)^\alpha\, ds.$$

Since $(Y^n, Z^n) \in \mathcal{S}^q \otimes M^q$ for every $q \in (0,1)$, $n \in \mathbb{N}$, it follows from the above estimate that $(Y^n - Y^{n+1})^+ \in \mathcal{S}^p$ for some $p > 1$. Hence, by Proposition 4.1, $Y_t^n \le Y_t^{n+1}$, $t \in [0,T]$. Write

$$Y_t = \lim_{n \to +\infty} Y_t^n, \quad t \in [0,T].$$

We are going to show that there is a process $\overline{X}$ of class (D) such that $\overline{X} \in \mathcal{V}_c^1 + \mathcal{M}_c^q$ for $q \in (0,1)$ and $\overline{X}_t \ge Y_t$, $t \in [0,T]$. Indeed, since $X$ from assumption (H7*) belongs to $\mathcal{V}_c^1 + \mathcal{M}_c^q$ for $q \in (0,1)$, there exist $M \in \mathcal{M}_c^q$ and $V \in \mathcal{V}_c^1$ such that $X = V + M$. By the representation property of the Brownian filtration there exists $Z' \in M^q$ such that

$$X_t = X_T - \int_t^T dV_s - \int_t^T Z'_s\, dB_s, \quad t \in [0,T],$$

which we can write in the form

$$\begin{aligned}
X_t = X_T &+ \int_t^T f(s, X_s, Z'_s)\, ds - \int_t^T (f^+(s, X_s, Z'_s)\, ds + dV_s^+) \\
&+ \int_t^T (f^-(s, X_s, Z'_s)\, ds + dV_s^-) - \int_t^T Z'_s\, dB_s, \quad t \in [0,T].
\end{aligned}$$

By [3, Theorem 6.3] and Remark 6.3 there exists a unique solution $(\overline{X}, \overline{Z})$ of the BSDE

$$\overline{X}_t = \xi \vee X_T + \int_t^T f(s, \overline{X}_s, \overline{Z}_s)\, ds + \int_t^T (f^-(s, X_s, Z'_s)\, ds + dV_s^-) - \int_t^T \overline{Z}_s\, dB_s$$

such that $(\overline{X}, \overline{Z}) \in \bigcap_{q<1} \mathcal{S}^q \otimes M^q$, $\overline{X}$ is of class (D) and

$$E \int_0^T |f(t, \bar{X}_t, \bar{Z}_t)|\, dt < +\infty. \tag{6.2}$$

As in the proof of the fact that $(Y^n-Y^{n+1})^+\in\mathcal{S}^p$ one can show that for every stopping time $\tau\le T$,

$$\begin{aligned}(X_{t\wedge\tau}-\overline{X}_{t\wedge\tau})^+&\le(X_\tau-\overline{X}_\tau)^++\int_{t\wedge\tau}^{\tau}\mathbf{1}_{\{X_s>\overline{X}_s\}}(f(s,X_s,Z'_s)-f(s,\overline{X}_s,\overline{Z}_s))\,ds\\&\quad-2\int_{t\wedge\tau}^{\tau}\mathbf{1}_{\{X_s>\overline{X}_s\}}(Z'_s-\overline{Z}_s)\,dB_s\\&\le(X_\tau-\overline{X}_\tau)^++2\gamma\int_{t\wedge\tau}^{\tau}(g_s+|X_s|+|\bar{X}_s|+|Z'_s|+|Z_s|)^\alpha\,ds\\&\quad-2\int_{t\wedge\tau}^{\tau}\mathbf{1}_{\{X_s>\overline{X}_s\}}(Z'_s-\overline{Z}_s)\,dB_s.\end{aligned}$$

Let $\tau_k=\inf\{t\in[0,T];\int_0^t(|Z'_s|^2+|\overline{Z}_s|^2)\,ds\ge k\}\wedge T$. Then

$$(X_{t\wedge\tau_k}-\overline{X}_{t\wedge\tau_k})^+\le E^{\mathcal{F}_t}(X_{\tau_k}-\overline{X}_{\tau_k})^++2\gamma E^{\mathcal{F}_t}\int_0^T(g_s+|X_s|+|\bar{X}_s|+|Z'_s|+|Z_s|)^\alpha\,ds.$$

Since $X,\overline{X}$ are of class (D), letting $k\to+\infty$ we get

$$(X_t-\overline{X}_t)^+\le2\gamma E^{\mathcal{F}_t}\int_0^T(g_s+|X_s|+|\bar{X}_s|+|Z'_s|+|Z_s|)^\alpha\,ds.$$

Therefore $(X-\overline{X})^+\in\mathcal{S}^p$ for some $p>1$ since $Z',\overline{Z}\in M^q$, $X,\bar{X}\in\mathcal{S}^q$, $q\in(0,1)$. Consequently, by Proposition 4.1, $X_t\le\overline{X}_t$, $t\in[0,T]$. Thus

$$\begin{aligned}\overline{X}_t&=\xi\vee X_T+\int_t^T f(s,\overline{X}_s,\overline{Z}_s)\,ds+\int_t^T n(\overline{X}_s-L_s)^-\,ds\\&\quad+\int_t^T(f^-(s,X_s,Z'_s)\,ds+dV_s^-)-\int_t^T\overline{Z}_s\,dB_s,\quad t\in[0,T].\end{aligned}$$

As in the case of the process $(X-\overline{X})^+$ one can show that $(Y^n-\overline{X})^+\in\mathcal{S}^p$ for some $p>1$. Hence, by Proposition 4.1, $Y_t^n\le\overline{X}_t$, $t\in[0,T]$ for every $n\in\mathbb{N}$. Furthermore, since $Y^1,\overline{X}\in\mathcal{S}^q$, $q\in(0,1)$, we have

$$\sup_{n\ge1}E|Y_T^{n,*}|^q<+\infty. \tag{6.3}$$

It follows in particular that $\sup_{n\ge1}|Y_0^n|<\infty$ since $Y_0^n$ are deterministic. Moreover, by Lemma 4.3, there exists a stationary sequence $\{\sigma_k^1\}$ of stopping times such that for every $k\in\mathbb{N}$,

$$\sup_{n\ge1}|Y_{\sigma_k^1}^{n,*}|\le k\vee(\sup_{n\ge1}|Y_0^n|)<+\infty. \tag{6.4}$$

Set

$$\sigma_k^2=\inf\{t\in[0,T],\min\{Y_t^{1,*},\overline{X}_t^{+,*},\int_0^t f^-(s,\overline{X}_s,0)\,ds,\int_0^t|f(s,0,0)|\,ds\}>k\}\wedge T$$

and $\tau_k=\sigma_k^1\wedge\sigma_k^2$. It is easy to see that the sequence $\{\tau_k\}$ is stationary. Using this and the fact that $Y_{\tau_k}^n$, $f$, $L$ satisfy the assumptions of Theorem 5.4 on the interval $[0,\tau_k]$ one can show that there exist $Y,K\in\mathcal{S},Z\in M$ such that $K$ is increasing, $K_0=0$ and

$$\sup_{0\le t\le T}|Y_t^n-Y_t|+\sup_{0\le t\le T}|K_t^n-K_t|+\int_0^T|Z_s^n-Z_s|^2\,ds\to0\text{ in probability }P \tag{6.5}$$

as $n\to+\infty$. Moreover, one can show that $Y_t\ge L_t$, $t\in[0,T]$,

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\,ds+\int_t^T dK_s-\int_t^T Z_s\,dB_s,\quad t\in[0,T] \tag{6.6}$$

and

$$\int_0^T (Y_s-L_s)\,dK_s=0. \tag{6.7}$$

Accordingly, the triple $(Y,Z,K)$ is a solution of RBSDE$(\xi,f,L)$. The proof of (6.5)–(6.7) runs as the proof of Theorem 5.4 (see the reasoning following (5.6) with $p=2$), the only difference being in the fact that now we consider equations on $[0,\tau_k]$ with terminal values depending on $n$. However, using (6.4) and the pointwise convergence of $\{Y^n\}$ allows overcome this difficulty. Since $Y^1_t\le Y_t\le\overline{X}_t$, $t\in[0,T]$, and $Y^1,X^+$ are of class (D), it follows that $Y$ is of class (D). By Lemma 3.4 for every $q\in(0,1)$,

$$\sup_{n\ge1}E\big((\int_0^T|Z^n_t|^2\,dt)^{q/2}+|K^n_T|^q\big)<+\infty. \tag{6.8}$$

From this and (6.5) we conclude that $Z\in\bigcap_{q<1}M^q$ and $E|K_T|^q<\infty$ for $q\in(0,1)$. To see that $EK_T<\infty$ let us define $\tau_k$ by (3.2). Then by (6.6),

$$K_{\tau_k}=Y_0-Y_{\tau_k}-\int_0^{\tau_k}f(s,Y_s,Z_s)\,ds+\int_0^{\tau_k}Z_s\,dB_s. \tag{6.9}$$

Since $Y$ is of class (D), using Fatou's lemma, (H2), (Z) and the fact that $Y_t\le\overline{X}_t$, $t\in[0,T]$ we conclude from (6.9) that

$$EK_T\le EY_0^++E\xi^-+E\int_0^T f^-(s,\overline{X}_s,0)\,ds+\gamma E\int_0^T(g_s+|Y_s|+|Z_s|)^\alpha\,ds.$$

Hence $EK_T<\infty$, because by (6.2) and (H2), $E\int_0^T|f(s,\overline{X}_s,0)|\,ds<+\infty$.

(ii) Convergence of $\{Y^n\}$ in $\mathcal{S}^q$ for $q\in(0,1)$ follows from (6.3) and (6.5). The desired convergence of $\{Z^n\}$ and $\{K^n\}$ follows from (6.5) and (6.8). □

**Remark 6.5.** *An important class of generators satisfying (H1)–(H5) together with (Z) are generators satisfying (H1)–(H5) which are bounded or not depending on $z$. Another class which share these properties are generators of the form*

$$f(t,y,z)=g(t,y)+c(1+|z|)^q,$$

*where $q\in[0,\alpha]$ and $g$ is a progressively measurable function satisfying (H1)–(H5).*

**Remark 6.6.** *Let assumptions (H1)–(H3), (Z) hold and let $(Y,Z,K)$ be a solution of RBSDE$(\xi,f,L)$ such that $Y$ is of class (D) and $Z\in\bigcup_{\beta>\alpha}M^\beta$. Then from Remark 3.5 it follows immediately that*

$$E(\int_0^T|f(s,Y_s,Z_s)|\,ds)<+\infty\text{ iff }EK_T<+\infty.$$

*If, in addition, there exists a continuous semimartingale $X$ such that (H7*) is satisfied then*

$$E\int_0^T\mathbf{1}_{\{Y_s\le X_s\}}dK_s<+\infty.$$

*To prove the last estimate let us put* $\tau_k = \inf\{t \in [0,T]; \langle M\rangle_t + \int_0^t |Z_s|^2\,ds > k\} \wedge T$*. By the Itô-Tanaka formula and (H2), (H3),*

$$\int_0^{\tau_k} \mathbf{1}_{\{Y_s \le X_s\}}\,dK_s = (Y_{\tau_k} - X_{\tau_k})^- - (Y_0 - X_0)^- - \int_0^{\tau_k} \mathbf{1}_{\{Y_s\le X_s\}} f(s,Y_s,Z_s)\,ds$$
$$- \int_0^{\tau_k} \mathbf{1}_{\{Y_s\le X_s\}} dV_s - \frac{1}{2}\int_0^{\tau_k} dL^0_s(Y-X)$$
$$- \int_0^{\tau_k} \mathbf{1}_{\{Y_s\le X_s\}} Z_s\,dB_s + \int_0^{\tau_k} \mathbf{1}_{\{Y_s\le X_s\}}\,dM_s.$$

*Hence*

$$E\int_0^{\tau_k} \mathbf{1}_{\{Y_s\le X_s\}}\,dK_s \le E|Y_{\tau_k}| + EX^+_{\tau_k} + E\int_0^T \mathbf{1}_{\{Y_s\le X_s\}} f^-(s,X_s,0)\,ds$$
$$+ \gamma E\int_0^T (g_s + |Z_s| + |Y_s|)^\alpha\,ds + E\int_0^T d|V|_s.$$

*Since* $(Y-X)^-$ *is of class (D), letting* $k\to+\infty$ *in the above inequality we get the desired result.*

## 7 Nonintegrable solutions of reflected BSDEs

In this section we examine existence and uniqueness of solutions of reflected BSDEs in the case where the data satisfy (H1)–(H6) (resp. (H1)–(H6), (Z) for $p=1$) but (H7) (resp. (H7*) in case $p=1$) is not satisfied. In view of Theorems 5.4 and 6.4 in that case there is neither a solution $(Y,Z,K)$ in the space $\mathcal{S}^p\otimes M^p\otimes\mathcal{V}^{p,+}_c$ if $p>1$ nor a solution in the space $\mathcal{S}^q\otimes M^q\otimes\mathcal{V}^{1,+}_c$, $q\in(0,1)$ with $Y$ of class (D) if $p=1$. We will show that nevertheless there exists a solution with weaker integrability properties. Before proving our main result let us note that in [6, 9, 13] reflected BSDEs with generator $f$ such that $|f(t,y,z)|\le M(|f(t,0,0)|+|y|+|z|)$ for some $M\ge 0$ are considered. In case $p=2$ it is proved there that if we assume that $\xi,\int_0^T|f(s,0,0)|\,ds\in\mathbb{L}^2(\mathcal{F}_T)$, $L$ is continuous and $L^+\in\mathcal{S}^2$ then there exists a solution $(Y,Z,K)\in\mathcal{S}^2\otimes M^2\otimes\mathcal{V}^{+,2}_c$ of (1.1) (see [6] for the case of Lipschitz continuous generator and [9, 13] for continuous generator). We would like to stress that although in [6, 9, 13] condition (H7) is not explicitly stated, it is satisfied, because if $f$ satisfies the linear growth condition and $L^+\in\mathcal{S}^2$ then

$$E(\int_0^T f^-(t,L^{+,*}_t,0)\,dt)^2 \le 2M^2T^2 + 2T^2E|L^{+,*}_T|^2 < +\infty$$

and $L_t\le L^{+,*}_t$, $t\in[0,T]$, $L^{+,*}\in\mathcal{V}^{+,2}_c$.

**Theorem 7.1.** *Let* (H1)–(H6) *(resp.* (H1)–(H6), (Z)*) be satisfied and* $L^+\in\mathcal{S}^p$ *for some* $p>1$ *(resp.* $L^+$ *is of class* (D)*). Then there exists a solution* $(Y,Z,K)\in\mathcal{S}^p\otimes M\otimes\mathcal{V}^+_c$ *(resp.* $(Y,Z,K)\in\mathcal{S}^q\otimes M\otimes\mathcal{V}^+_c$*,* $q\in(0,1)$ *such that* $Y$ *is of class* (D)*) of the* RBSDE$(\xi,f,L)$*.*

*Proof.* We first assume that $p=1$. By [3, Theorem 6.3] there exists a unique solution $(Y^n,Z^n)\in\bigcap_{q<1}\mathcal{S}^q\otimes M^q$ of (5.2) such that $Y^n$ is of class (D). By Proposition 6.4 (see also the reasoning used at the beginning of the proof of Theorem 6.4), for every $n\in\mathbb{N}$, $Y^n_t\le Y^{n+1}_t$ and $Y^n_t\le\bar Y^n_t$, $t\in[0,T]$, where $(\bar Y^n,\bar Z^n)\in\bigcap_{q<1}\mathcal{S}^q\otimes M^q$ is a solution of the BSDE

$$\bar Y^n_t = \xi + \int_t^T f^+(s,\bar Y^n_s,\bar Z^n_s)\,ds + \int_t^T n(\bar Y^n_s - L_s)^-\,ds - \int_t^T \bar Z^n_s\,dB_s,\quad t\in[0,T]$$

such that $\bar{Y}^n$ is of class (D). Hence

$$|Y^n_t| \le |Y^1_t| + |\bar{Y}^n_t|, \quad t \in [0,T]. \tag{7.1}$$

Put

$$R_t(L) = \operatorname*{ess\,sup}_{t\le\tau\le T} E(L_\tau|\mathcal{F}_t).$$

It is known (see [4, 5]) that $R(L)$ has a continuous version (still denoted by $R(L)$) such that $R(L)$ is a supermartingale of class (D) majorizing the process $L$. Moreover, by the Doob-Meyer decomposition theorem there exist a uniformly integrable continuous martingale $M$ and a process $V \in \mathcal{V}^{+,1}_c$ such that $R(L) = M + V$. In particular, by [3, Lemma 6.1], $R(L) \in \mathcal{M}^q_c + \mathcal{V}^{+,1}_c$ for every $q \in (0,1)$. Therefore the data $\xi, f^+, L$ satisfy assumptions (H1)–(H6), (Z) and (H7*) with $X = R(L)$. Hence, by Theorem 6.4, there exists a unique solution $(\bar{Y}, \bar{Z}, \bar{K}) \in \mathcal{S}^q \otimes M^q \otimes \mathcal{V}^{+,1}_c$, $q \in (0,1)$, of the RBSDE$(\xi, f^+, L)$ such that $\bar{Y}$ is of class (D) and

$$\bar{Y}^n_t \nearrow \bar{Y}_t, \quad t \in [0,T].$$

By the above and (7.1),

$$|Y^n_t| \le |Y^1_t| + |\bar{Y}_t|, \quad t \in [0,T]. \tag{7.2}$$

Put $Y_t = \sup_{n\ge1} Y^n_t$, $t \in [0,T]$ and

$$\tau_k = \inf\{t \in [0,T]; \int_0^t f^-(s, R_s(L), 0)\,ds > k\} \wedge T.$$

Then $f, L$ satisfy assumptions (H1)–(H6), (Z) and (H7*) with $X = R(L)$ on each interval $[0,\tau_k]$. Therefore analysis similar to that in the proof of (5.4), but applied to the equation

$$Y^n_{t\wedge\tau_k} = Y^n_{\tau_k} + \int_{t\wedge\tau_k}^{\tau_k} f(s, Y^n_s, Z^n_s)\,ds + \int_{t\wedge\tau_k}^{\tau_k} n(Y^n_s - L_s)^-\,ds - \int_{t\wedge\tau_k}^{\tau_k} Z^n_s\,dB_s \tag{7.3}$$

instead of (5.2), shows that for every $k \in \mathbb{N}$,

$$E\sup_{0\le t\le\tau_k} |Y^n_t - Y^m_t|^q + E(\int_0^{\tau_k} |Z^n_s - Z^m_s|^2\,ds)^{q/2} + E\sup_{0\le t\le\tau_k} |K^n_t - K^m_t|^q \to 0 \tag{7.4}$$

as $n,m \to +\infty$, where $K^n_t = \int_0^t n(Y^n_s - L_s)^-\,ds$. (The only difference between the proof of (7.4) and (5.4) is caused by the fact that in (7.3) the terminal condition $Y^n_{\tau_k}$ depends on $n$. But in view of (7.2), monotonicity of the sequence $\{Y^n\}$ and integrability of $Y^1, \bar{Y}$ the dependence of $Y^n_{\tau_k}$ on $n$ presents no difficulty). Since the sequence $\{\tau_k\}$ is stationary, from (7.3), (7.4) we conclude that there exist $K \in \mathcal{V}^+_c$ and $Z \in M$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds + \int_t^T dK_s - \int_t^T Z_s\,dB_s, \quad t \in [0,T]$$

and (7.4) holds with $(Y, Z, K)$ in place of $(Y^n, Z^n, K^n)$. From the properties of the sequence $\{(Y^n, Z^n, K^n)\}$ on $[0,\tau_k]$ proved in Theorem 6.4 it follows that

$$Y_t \ge L_t, \quad t \in [0,\tau_k], \quad \int_0^{\tau_k} (Y_s - L_s)\,ds = 0$$

for $k \in \mathbb{N}$. By stationarity of the sequence $\{\tau_k\}$ this implies that

$$Y_t \ge L_t, \quad t \in [0,T], \quad \int_0^T (Y_s - L_s)\,ds = 0.$$

Accordingly, the triple $(Y,Z,K)$ is a solution of RBSDE$(\xi,f,L)$.

In case $p>1$ the proof is similar. As a matter of fact it is simpler because instead of considering the Snell envelope $R(L)$ of the process $L$ it suffices to consider the process $L^{+,*}$. □

**Remark 7.2.** *From Proposition 6.1 it follows that the solution obtained in Theorem 7.1 is unique in its class for $p>1$. In case $p=1$ it is unique in its class if $f$ does not depend on $z$ (see Remark 6.2).*

The next example shows that in general the process $K$ of Theorem 7.1 may be non-integrable for any $q>0$.

**Example 7.3.** *Let $f(t,y)=-y^+\exp(|B_t|^4)$, $L_t\equiv 1$, $\xi\equiv 1$. Then $\xi,f,L$ satisfy (H1)–(H6) and $L\in\mathcal{S}^p$ for every $p\geq 1$. So by Theorem 7.1 and Proposition 5.2 there exists a unique solution $(Y,Z,K)\in\mathcal{S}^2\otimes M\otimes\mathcal{V}_c^+$ of the RBSDE$(\xi,f,L)$. Observe that $EK_T^q=+\infty$ for any $q>0$. Of course, to check this it suffices to consider the case $q\in(0,1]$. Aiming for a contradiction, suppose that $q\in(0,1]$ and $EK_T^q<+\infty$. Then by [3, Lemma 3.1], $Z\in M^q$, which implies that $E(\int_0^T f^-(t,Y_t)\,dt)^q<+\infty$. On the other hand, since $Y_t\geq 1$ for $t\in[0,T]$, it follows that*

$$E(\int_0^T f^-(t,Y_t)\,dt)^q\geq E\int_0^T (f^-(t,1))^q\,dt=E\int_0^T\exp(q|B_t|^4)\,dt=+\infty.$$

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