Mixing and hitting times for finite Markov chains

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Abstract

Let $0 < \alpha < 1/2$. We show that that the mixing time of a continuous-time Markov chain on a finite state space is about as large as the largest expected hitting time of a subset of the state space with stationary measure $\geq \alpha$. Suitably modified results hold in discrete time and/or without the reversibility assumption. The key technical tool in the proof is the construction of random set A such that the hitting time of A is a light-tailed stationary time for the chain. We note that essentially the same results were obtained independently by Peres and Sousi.

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1   Introduction

The present paper is a contribution to the general quantitative theory of finite-state Markov chains that was started in [2] and further developed in [4]. The gist of those papers is that the so-called mixing time of a Markov chain is fundamentally related, in a precise quantitative sense, to hitting times and other quantities of interest. Our main achievement is to add a new equivalent quantity to this list by showing that mixing times nearly coincide with maximum hitting times of large sets in the state space.

Remark 1.1 (Important remark). *The results in this paper were proven (but not made public) around May 2010. In July 2011 we learned that extremely similar results for discrete-time chains have been proven independently by Peres and Sousi [8]. We then decided to submit our results, in the hope that our ideas might also be found useful and interesting. We will discuss their results at several points in our paper. Here we just mention that the main difference between the papers is the construction of the stopping time in Lemma 2.1 (see Section 1.1).*

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Mixing and hitting for Markov chains

We need to introduce some notions before we clarify what we mean; [3] and [5] are our main references for the involved concepts. In this paper E will always denote the finite state space of a continuous-time Markov chain with generator Q, with transition rates $q(x, y)$ ($x, y \in E$, $x \neq y$). Most of the time Q and E will be implicit in our notation. The trajectories of the chain are denoted by $\{X_t\}_{t \geq 0}$, and the law of $\{X_t\}_{t \geq 0}$ started from $x \in E$ or from a probability distribution μ over E are denoted by $\mathbb{P}_x$ or $\mathbb{P}_\mu$ (respectively) . For $t \geq 0$, we write:

$$p_t(x, y) \equiv \mathbb{P}_x\left(X_t = y\right) \quad (x, y \in E)$$

for the transition probability from x to y at time t. In what follows we will always assume that Q is irreducible, which implies that it has a unique stationary distribution π and:

$$\forall (x, y) \in E^2 \; : \; \lim_{t \to +\infty} p_t(x, y) = \pi(y).$$

We can measure the *rate* of this convergence after we introduce a metric over probability distributions. We choose the total variation metric:

$$d_{\mathrm{TV}}(\mu, \nu) = \max_{A \subset E} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{a \in E} |\mu(a) - \nu(a)| \quad (\mu, \nu \text{ prob. measures over } E)$$

and define the mixing time of Q as:

$$\mathrm{T}_{\mathrm{mix}}^Q(\delta) = \inf\{t \geq 0 \; : \; \forall x \in E, \, d_{\mathrm{TV}}(p_t(x, \cdot), \pi(\cdot)) \leq \delta\}.$$

Finally, given $\emptyset \neq A \subset E$, we may define the hitting time of A as:

$$H_A \equiv \inf\{t \geq 0 \; : \; X_t \in A\}.$$

Results for reversible chains. Recall that Q is *reversible* if $\pi(x)q(x, y) = \pi(y)q(y, x)$ for all distinct $x, y \in E$. In this setting, Aldous proved:

Theorem 1.2 (Aldous, [2]). *There exist universal (ie. chain independent) constants $c_-, c_+ > 0$ such that for any irreducible, reversible, finite-state-space Markov chain in continuous time with generator Q:*

$$c_- \, \mathrm{T}_{\mathrm{hit}}^Q \leq \mathrm{T}_{\mathrm{mix}}^Q(1/4) \leq c_+ \, \mathrm{T}_{\mathrm{hit}}^Q$$

where $\mathrm{T}_{\mathrm{hit}}^Q \equiv \sup\{\pi(A) \, \mathbb{E}_x\left[H_A\right] \; : \; x \in E, \emptyset \neq A \subset E\}$.

Notice that $\mathrm{T}_{\mathrm{hit}}^Q = 1$ if Q consists of iid jumps at rate 1 between states in E, so $\mathrm{T}_{\mathrm{hit}}^Q$ can be viewed as a measure of how "non-iid" the chain is. Informally, the mixing time is another measure of "non-iid-ness", and the Theorem shows that these two measures are quantitatively related in a very strong sense. We emphasize that Theorem 1.2 is part of a much larger family of universal inequalities for reversible Markov chains; see [2] for details.

In this paper we prove a stronger form of Theorem 1.2. Given $\alpha > 0$, let:

$$\mathrm{T}_{\mathrm{hit}}^Q(\alpha) \equiv \sup\{\mathbb{E}_x\left[H_A\right] \; : \; x \in E, \emptyset \neq A \subset E, \pi(A) \geq \alpha\}.$$

Unlike $\mathrm{T}_{\mathrm{hit}}^Q$, only "large enough" sets are considered in this definition. We prove in Section 4 that:

Theorem 1.3. *For any $0 < \alpha < 1/2$ there exist constants $C_+(\alpha), C_-(\alpha) > 0$ depending only on α such that, for any irreducible continuous-time Markov chain as above:*

$$C_-(\alpha) \, \mathrm{T}_{\mathrm{hit}}^Q(\alpha) \leq \mathrm{T}_{\mathrm{mix}}^Q(1/4) \leq C_+(\alpha) \, \mathrm{T}_{\mathrm{hit}}^Q(\alpha).$$

Although similar to Theorem 1.2, the intuitive content of Theorem 1.3 seems different: instead of measures of non-iid-ness, we have a statement that says that mixing times are about as large as the expected time necessary to hit any large set, which is quite reasonable. Theorem 1.3 should also be easier to use in applications. The condition $\alpha < 1/2$ is discussed in Section 1.1.

Remark 1.4. *Theorem 1.3 also holds in discrete time if $p_1(x, x) \geq 1/2$ for all $x \in E$ (use [5, Theorem 20.3]). Peres and Sousi [8] have shown that $p_1(x, x) \geq \beta > 0$ for any fixed $\beta > 0$ also suffices. Some lower bound on $p_1(x, x)$ is necessary; otherwise there are counterexamples such as large complete bipartite graphs with an edge added to one of the parts.*

Results for non-reversible chains. Theorem 1.3 and the main results of [2] only apply to reversible chains; counterexamples can be found in that paper. Aldous, Lóvasz and Winkler [4] developed a quantitative theory in the general case using a different notion of mixing time. Let $M_1([0,t])$ be the set of all probability measures over $[0,t]$ and define:

$$\mathrm{T}_{\mathrm{rmix}}^Q(\delta) \equiv \inf \left\{ t \geq 0 \; : \; \begin{array}{l} \exists \mu \in M_1([0,t]), \, \forall x \in E, \\ d_{\mathrm{TV}}\left(\int_{[0,t]} p_s(x,\cdot)\, \mu(ds), \pi \right) \leq \delta \end{array} \right\}.$$

In discrete time, one replaces $M_1([0,t])$ with the set $M_1(\{0,\ldots,t\})$ of all probability measures over $\{0,\ldots,t\}$. Aldous, Lóvasz and Winkler [4] proved an analogue to Theorem 1.2 for arbitrary Markov chains in discrete time, where $\mathrm{T}_{\mathrm{rmix}}$ replaces $\mathrm{T}_{\mathrm{mix}}$ (their method can also be applied in continuous time). We prove an analogue of Theorem 1.3 in this setting:

Theorem 1.5. *For any $\alpha \in (0,1/2)$ there exist $C_-'(\alpha) > 0, C_+'(\alpha)$ such that for any irreducible finite-state Markov chain Q in continuous time:*

$$C_-'(\alpha)\,\mathrm{T}_{\mathrm{hit}}^Q(\alpha) \leq \mathrm{T}_{\mathrm{rmix}}^Q(1/4) \leq C_+'(\alpha)\,\mathrm{T}_{\mathrm{hit}}^Q(\alpha).$$

Remark 1.6. *Our proof can be easily adapted to discrete time. Peres and Sousi [8] have proved a variant of Theorem 1.5 where $\mathrm{T}_{\mathrm{rmix}}^Q(1/4)$ is replaced by another notion of time-averaged mixing, with μ a geometric distribution with success probability $1/t$.*

1.1 Discussion of the results

Outside of potential applications to bounding mixing, Theorems 1.3 and 1.5 seem conceptually interesting. They show that mixing times are *natural* in that they are strongly related to hitting times, a quantity of intrinsic interest. For instance, we have the following immediate corollary of Theorem 1.5.

Corollary 1.7. *There exists some universal $C > 0$ such that for any irreducible Markov chain in discrete or continuous time,*

$$\forall x \in E, \, \forall \emptyset \neq A \subset V \; : \; \mathbb{E}_x\left[H_A\right] \leq \frac{C \sup_{B \subset V,\, \pi(B) \geq 1/3} \sup_{y \in E} \mathbb{E}_y\left[H_B\right]}{\pi(A)}.$$

We omit the proof, which follows from $\mathrm{T}_{\mathrm{hit}}^Q \leq c\,\mathrm{T}_{\mathrm{rmix}}(1/4) \leq c'\,\mathrm{T}_{\mathrm{hit}}^Q(1/3)$ (with $c, c' > 0$ universal). This result says that one may control the hitting times of small sets via those of large sets.sOther applications of (slight variants of) our theorems are considered in [8].

The limitation $\alpha < 1/2$ is not clearly necessary for the Theorems to hold. However, Peres [7] noted that one cannot allow $\alpha > 1/2$. In that case one may contradict the two theorems by connecting two complete graphs K_n by a single edge. In this case $\mathrm{T}_{\mathrm{hit}}(\alpha) = O(n)$ whenever $\alpha > 1/2$, since any set A with $\pi(A) \geq \alpha$ occupies a constant proportion of the mass of each clique. However, mixing requires crossing the connecting edge, so $\mathrm{T}_{\mathrm{mix}}(1/4) = \Omega\left(\mathrm{T}_{\mathrm{rmix}}(1/4)\right) = \Omega\left(n^2\right)$. The intersting question is then:

Question 1. *What happens when $\alpha = 1/2$?*

In 2009 Peres conjectured that $\mathrm{T}_{\mathrm{hit}}^Q(1/2)$ is also "equivalent up to universal constant factors" to $\mathrm{T}_{\mathrm{mix}}^Q(1/4)$ (for lazy and reversible Q) [1]. We prove this result in an upcoming paper with Griffiths, Kang and Patel.

1.2 Steps of the proof

The main step in the proof is Lemma 2.1, proven in Section 2. We construct there a randomized stopping time T, which depends on the initial distribution, such that X_T has the stationary distribution. This stopping rule is the hitting time of a randomly chosen subset $A \subset E$, where the possible values of A form a chain $A_1 \supset A_2 \supset \cdots \supset A_n$. We will see that this property property implies that we can control the tail of H_A via $\mathrm{T}_{\mathrm{hit}}^Q(\alpha)$. We note that this stopping time was outlined in eg. [6, Theorem 4.9], but it is not explicit anywhere. Moreover, T is minimal in some sense (cf. Remark 2.3). Peres and Sousi [8] prove similar results via another minimal stopping rule, the so-called *filling rule* that was also employed in [2, 4]). We believe that our construction provides an interesting alternative point of view.

Ater the construction of T, our paper continues with the proof of Theorem 1.5, proven in Section 3. The elegant argument we use employs Lemma 2.1 together with a simple coupling devised in the survey [6]. The proof of Theorem 1.3 in Section 4 follows a convoluted computation in [2], which we reproduce in order to get the sharp form we need. An Appendix presents a simple lower bound of $\mathrm{T}_{\mathrm{rmix}}^Q(\alpha/2)$ in terms of $\mathrm{T}_{\mathrm{hit}}^Q(\alpha)$.

1.3 Acknowledgements

We thank Yuval Peres for the counterexample in Section 1.1 [7] and both him and Perla Sousi for presenting [8] to us.

2 A special stationary stopping time

We use the notation in Section 1. Recall that a *randomized stopping time* for this chain is a $[0, +\infty)$-valued random variable T such that for all $t \geq 0$ the event $\{T \leq t\}$ is measurable relative to the σ-field generated by $\{X_s\}_{s \leq t}$ and an independent random variable U.

Lemma 2.1. *Suppose μ_0 is a probability measure over E. Then there exists a randomized stopping time T with*

$$\mathbb{P}_{\mu_0}(X_T = \cdot) = \pi(\cdot) \text{ and } \mathbb{P}_{\mu_0}(T > t) \leq \epsilon + \frac{T_{\text{hit}}^Q(\epsilon)}{t} \text{ for all } \epsilon \in (0, 1). \tag{2.1}$$

Remark 2.2. *The same result works (with a slightly different proof) if π is replaced by another target distribution μ_1 over E and π substitutes μ_1 in the definition of $\mathrm{T}_{\text{hit}}^Q(\epsilon)$.*

Remark 2.3. *Although we do not use this, one can show that $\mathbb{E}_{\mu_0}[T]$ is minimal among all randomized stopping times with $\mathbb{P}_\mu(X_T = \cdot) = \pi(\cdot)$. This is because our T has a halting state [6, Theorem 4.5].*

Remark 2.4. *We note from the definitions that $\mathrm{T}_{\text{hit}}^Q(\epsilon) \leq \mathrm{T}_{\text{hit}}^Q/\epsilon$. We may plug this into Lemma 2.1 and optimize over ϵ to deduce:*

$$\mathbb{P}_{\mu_0}(T > t) \leq \sqrt{\frac{\mathrm{T}_{\text{hit}}^Q}{t}}.$$

Aldous [2] proves a similar bound for a different stopping time, which he uses to prove Theorem 1.2. The same proof would go through with our own T. Another proof of Theorem 1.2 is presented in [8]

Proof. Let $n \equiv |E|$ denote the cardinality of E. The idea in the proof is to find a chain of subsets $E = A_1 \supset A_2 \supset \cdots \supset A_n \neq \emptyset$ and numbers $p_1, \ldots, p_n \geq 0$ with $\sum_i p_i = 1$. We then define a random A that equals A_i with probability p_i and define $T = H_A$. We will then show that if $\{X_t\}_{t \geq 0}$ is a realization $\mathbb{P}_{\mu_0}$ that is independent from A, then $\text{Law}(X_T) = \pi$. The tail behavior of $T = H_A$ will follow automatically from the construction.

Notation. For any set $\emptyset \neq S \subset E$, let $\rho_S(\cdot) = \mathbb{P}_{\mu_0}(X_{H_S} = \cdot)$ denote the harmonic measure on S for the chain started from μ_0. The irreducibility of the chain implies that $H_S < +\infty$ $\mathbb{P}_{\mu_0}$-a.s. and therefore ρ_S is a probability measure over E with support in S.

Inductive construction of (A_i, p_i): Set $A_1 = E$ and choose $a_1 \in A_1$ so that $\rho_{A_1}(a_1)/\pi(a_1)$ is the maximum of $\rho_{A_1}(a)/\pi(a)$ over all $a \in A_1$. Since the π-weighted average of such ratios satisfies:

$$\sum_{a \in A_1} \pi(a) \left(\frac{\rho_{A_1}(a)}{\pi(a)} \right) = \sum_{a \in A_1} \rho_{A_1}(a) = 1,$$

the maximal value must satisfy $\rho_{A_1}(a_1)/\pi(a_1) \geq 1$. We then choose $p_1 = \pi(a_1)/\rho_{A_1}(a_1)$ and note that $p_1 \in [0, 1]$, $p_1 \rho_{A_1}(a_1) = \pi(a_1)$ and $p_1 \rho_{A_1}(a)/\pi(a) \leq 1$ for all other $a \in E \backslash \{a_1\}$.

Assume inductively that we have chosen distinct elements $a_1, \ldots, a_k \in E$ and numbers $0 \leq p_1, \ldots, p_k \leq 1$ such that if $A_i = E \backslash \{a_j : 1 \leq j < i\}$ $(1 \leq i \leq k)$, we have the following properties:

1. for all $1 \leq j \leq k$, $\sum_{i=1}^k p_i \rho_{A_i}(a_j) = \pi(a_j)$;
2. moreover, for $a \in E \backslash \{a_1, \ldots, a_k\}$, $\sum_{i=1}^k p_i \rho_{A_i}(a) \leq \pi(a)$.

Assume also that $k < n$, so that $A_{k+1} = E \backslash \{a_1, \ldots, a_k\}$ is non-empty. We will prove that one may choose (p_{k+1}, a_{k+1}) so as to preserve these properties for one further step. The following claim is the key:

Claim 2.5. *The set $\mathcal{P}_{k+1} \subset [0, 1] \times A_{k+1}$ of all (p, a) with $\sum_{i=1}^k p_i \rho_{A_i}(a) + p\,\rho_{A_{k+1}}(a) = \pi(a)$ is non-empty.*

Given the claim, we choose a pair $(p_{k+1}, a_{k+1}) \in \mathcal{P}_{k+1}$ with *minimum* value of the first coordinate. Let us show that condition 2. above remains valid for $a \in E \backslash \{a_1, \ldots, a_{k+1}\}$. Any a violating 2 would have to satisfy:

$$\sum_{i=1}^{k} p_i \rho_{A_i}(a) \leq \pi(a) < p_{k+1} \rho_{A_{k+1}}(a) + \sum_{i=1}^{k} p_i \rho_{A_i}(a),$$

and this would imply that there is some $0 \leq p < p_{k+1}$ with:

$$p \, \rho_{A_{k+1}}(a) + \sum_{i=1}^{k} p_i \rho_{A_i}(a) = \pi(a) \text{ (ie. } (p, a) \in \mathcal{P}_{k+1}),$$

which would contradict the minimality of p_{k+1}.

To prove that condition 1. also remains valid, we simply observe that it certainly holds for a_{k+1} and that it also holds for a_i, $i < k+1$, because $a_i \notin A_{k+1}$ and therefore $\rho_{A_{k+1}}(a_i) = 0$. Hence such a choice of p_{k+1}, a_{k+1} preserves the induction hypothesis for one more step.

We now prove the Claim. Notice that:

$$\frac{\sum_{a \in A_{k+1}} \pi(a) \left(\frac{\rho_{A_{k+1}}(a)}{\pi(a)} \right)}{\sum_{a \in A_{k+1}} \pi(a)} \geq \frac{\sum_{a \in A_{k+1}} \pi(a) \left(\frac{\rho_{A_{k+1}}(a)}{\pi(a)} \right)}{\sum_{a \in E} \pi(a)} = \sum_{a \in A_{k+1}} \rho_{A_{k+1}}(a) = 1.$$

Since the first term in the LHS is an average, there must exist some $a \in A_{k+1}$ with $\rho_{A_{k+1}}(a) \geq \pi(a)$, whence:

$$\sum_{i=1}^{k} p_i \rho_{A_i}(a) + \rho_{A_{k+1}}(a) \geq \pi(a).$$

Moreover, the inductive assumption 2. implies that $\sum_{i=1}^{k} p_i \rho_{A_i}(a) \leq \pi(a)$, so there exists some $p \in [0, 1]$ with

$$\sum_{i=1}^{k} p_i \rho_{A_i}(a) + p \rho_{A_{k+1}}(a) = \pi(a),$$

which proves the claim.

Analysis of the construction. Carrying the induction to its end at $k = n$ implies that there exist $p_1, \ldots, p_n \in [0, 1]$ and an ordering $a_1, \ldots, a_n$ of the elements of E such that, if $A_i \equiv E \backslash \{a_j : 1 \leq j < i\}$, then:

$$\forall 1 \leq i \leq n, \pi(a_i) = \sum_{j=1}^{n} p_j \rho_{A_j}(a_i) = \sum_{j=1}^{i} p_j \rho_{A_j}(a_i)$$

(the last identity in the RHS follows from $a_i \notin A_j$ for $j > i$).

These are the only facts about the construction we will use in the remainder of the analysis. We now prove some consequences of these facts. First notice that:

$$\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} p_j \sum_{i=1}^{n} \rho_{A_j}(a_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_j \rho_{A_j}(a_i) = \sum_{i=1}^{n} \pi(a_i) = 1,$$

which implies that the p_i form a probability distribution over $\{1, \ldots, n\}$. Moreover, the same line of reasoning implies that for all $k \in \{1, \ldots, n\}$:

$$\sum_{j=1}^{k} p_j \geq \sum_{j=1}^{k} p_j \left(\sum_{i=1}^{k} \rho_{A_j}(a_i) \right) = \sum_{i=1}^{k} \sum_{j=1}^{k} p_j \rho_{A_j}(a_i) = \sum_{i=1}^{k} \pi(a_i) = 1 - \pi(A_{k+1}), \qquad (2.2)$$

where $A_{n+1} = \emptyset$ by definition.

We now define our randomized stopping time as $T = H_A$, where the choice of A is independent of the realization of the chain and $\mathbb{P}(A = A_i) = p_i$, $1 \leq i \leq n$. Notice that $A \neq \emptyset$, hence $T < +\infty$ almost surely. Moreover, it is easy to check that $\mathbb{P}_{\mu_0}(X_T = \cdot) = \pi(\cdot)$, as desired.

To finish, we bound the upper tail of T. Given $\epsilon \in (0,1)$, let $j(\epsilon)$ be the largest $j \in [n+1]$ with $\pi(A_j) \geq \epsilon$ (recall our convention $A_{n+1} = \emptyset$). Since the A_i's form a decreasing chain, (2.2) implies:

$$\mathbb{P}_{\mu_0}\left(\pi(A) \geq \epsilon\right) = \sum_{i=1}^{j(\epsilon)} \mathbb{P}\left(A = A_i\right) = \sum_{i=1}^{j(\epsilon)} p_i \geq 1 - \pi(A_{j(\epsilon)+1}) \geq 1 - \epsilon.$$

Moreover, $j \leq j(\epsilon)$ imples $A_j \supset A_{j(\epsilon)}$. We deduce:

$$\begin{aligned}
\mathbb{P}_{\mu_0}\left(T > t\right) &\leq& \mathbb{P}_{\mu_0}\left(\pi(A) < \epsilon\right) + \mathbb{P}_{\mu_0}\left(H_A > t \mid \pi(A) \geq \epsilon\right) \\
&\leq& \epsilon + \mathbb{P}_{\mu_0}\left(H_{A_{j(\epsilon)}} > t\right) \\
&\leq& \epsilon + \frac{\mathbb{E}_{\mu_0}\left[H_{A_{j(\epsilon)}}\right]}{t} \leq \epsilon + \frac{\mathrm{T}_{\mathrm{hit}}^Q(\epsilon)}{t}.
\end{aligned}$$

$\square$

3 Mixing of non-reversible chains

In this section we prove Theorem 1.5.

Proof of Theorem 1.5. The lower bound on $\mathrm{T}_{\mathrm{rmix}}^Q(\alpha)$ follows easily from the ideas in [4]. We give a proof in the Appendix for completeness. For the upper bound, we proceed as follows. Define:

$$\overline{d}_r(t) = \inf_{\mu \in M_1([0,t])} \sup_{x,z \in E} d_{\mathrm{TV}}\left(\int_0^t p_s(x,\cdot)\,\mu(ds), \int_0^t p_s(z,\cdot)\,\mu(ds)\right).$$

Claim 3.1. *For all $t \geq 0$,*
$$\overline{d}_r(kt) \leq \overline{d}_r(t)^k.$$

Proof of the Claim. A standard compactness argument shows that there exists a measure μ which achieves the infimum in the definition of $\overline{d}_r(t)$. Let M be the discrete time Markov chain whose transition probabilities are given by:

$$m(x,y) \equiv \int_0^t p_s(x,y)\,\mu(ds), \ (x,y) \in E^2. \tag{3.1}$$

Define:

$$\overline{d}_M(k) \equiv \sup_{(x,y) \in E^2} d_{\mathrm{TV}}(m_k(x,\cdot), m_k(y,\cdot))$$

where m_t is the transition probability for t steps of m. Notice that $\overline{d}_M(1) = \overline{d}_r(t)$ by the choice of μ. Moreover, $\overline{d}_r(kt) \leq \overline{d}_M(k)$ because k steps of M correspond to replacing t and μ and t in (3.1) with kt and μ^{*k} (respectively; μ^{*k} is the k-fold convolution of μ with itself). Lemma 4.12 in [5] implies that

$$\overline{d}_r(kt) \leq \overline{d}_M(k) \leq \overline{d}_M(1)^k = \overline{d}_r(t)^k.$$

$\square$

Notice that $\overline{d}_r(t) \leq 1/4$ implies $\mathrm{T}_{\mathrm{rmix}}^Q(1/4) \leq t$. We will spend most of the rest of the proof proving that for all irreducible Markov chains Q,

$$\textbf{Goal:} \ \overline{d}_r\left(c(\alpha)\,\mathrm{T}_{\mathrm{hit}}^Q(\alpha)\right) \leq 1 - \delta(\alpha), \tag{3.2}$$

where $c(\alpha), \delta(\alpha) > 0$ depend only on $\alpha \in (0, 1/2)$. Applying the Claim with $t = c(\alpha)\,\mathrm{T}_{\mathrm{hit}}^Q(\alpha)$ and $k = k(\alpha)$ such that $(1 - \delta(\alpha))^k \leq 1/4$ we may then deduce that

$$\mathrm{T}_{\mathrm{rmix}}^Q(\alpha) \leq C_+(\alpha)\,\mathrm{T}_{\mathrm{hit}}^Q(\alpha) \text{ where } C_+(\alpha) = k(\alpha)\,c(\alpha) \text{ depends only on } \alpha,$$

which is the desired result.

Given $x, z \in E$, we let $\{X_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ denote trajectories of Q started from x and z (respectively). Let T_x, T_z be obtained from Lemma 2.1 for $\mu_0 = \delta_x$ and δ_z (resp.). Clearly,

$$\mathrm{Law}(X_{T_x}) = \mathrm{Law}(Z_{T_z}) = \pi.$$

Sample $\mathcal{U}$ uniformly from $[0, t]$ and independently from the two chains. The Markov property and the stationarity of π imply:

$$\text{Law}(X_{T_x + \mathcal{U}}) = \text{Law}(Z_{T_z + \mathcal{U}}) = \pi.$$

Now fix some $t \geq 0$ and define

$$\mathcal{U}_x \equiv (T_x + \mathcal{U}) \mod t \text{ and } \mathcal{U}_z = (T_z + \mathcal{U}) \mod t.$$

Notice that $\mathcal{U}_x$ is uniform over $[0, t]$, independently from $\{X_t\}_{t \geq 0}$, and similarly for $\mathcal{U}_z$. Hence:

$$\text{Law}(X_{\mathcal{U}_x}) = \int_0^t p_s(x, \cdot) \, \mu(ds) \text{ and } \text{Law}(Z_{\mathcal{U}_z}) = \int_0^t p_s(z, \cdot) \, \mu(ds),$$

where μ is uniform over $[0, t]$. Therefore,

$$
\begin{aligned}
d_{\text{TV}} \left(\int_0^t p_s(x, \cdot) \, \mu(ds), \int_0^t p_s(z, \cdot) \, \mu(ds) \right) &= d_{\text{TV}}(\text{Law}(X_{\mathcal{U}_x}), \text{Law}(Z_{\mathcal{U}_z})) \\
&\leq d_{\text{TV}}(\text{Law}(X_{\mathcal{U}_x}), \text{Law}(X_{T_x + \mathcal{U}})) \\
&\quad + d_{\text{TV}}(\text{Law}(Z_{\mathcal{U}_z}), \text{Law}(Z_{T_z + \mathcal{U}}))
\end{aligned}
\tag{3.3}
$$

by the triangle inequality and the previous remarks. We now show that:

$$d_{\text{TV}}(\text{Law}(X_{\mathcal{U}_x}), \text{Law}(X_{T_x + \mathcal{U}})) \leq \alpha + 2 \sqrt{\frac{\text{T}_{\text{hit}}^Q(\alpha)}{t}}. \tag{3.4}$$

This is of course trivial if $t < \text{T}_{\text{hit}}^Q(\alpha)$, so we assume the opposite is true. The coupling characterization of total variation distance implies that for any $\lambda \in (0, 1)$:

$$
\begin{aligned}
d_{\text{TV}}(\text{Law}(X_{\mathcal{U}_x}), \text{Law}(X_{T_x + \mathcal{U}})) &\leq \mathbb{P}_x (X_{\mathcal{U}_x} \neq X_{T_x + \mathcal{U}}) \\
&\leq \mathbb{P}_x (\mathcal{U} > t - T_x) \\
&\leq \mathbb{P}_x (T_x > \lambda t) + \mathbb{P}((1 - \lambda)t \leq \mathcal{U} \leq t) \\
(\text{use Lemma 2.1}) \quad &= \alpha + \frac{\text{T}_{\text{hit}}^Q(\alpha)}{\lambda t} + \lambda
\end{aligned}
$$

Choosing $\lambda = \sqrt{\text{T}_{\text{hit}}^Q(\alpha)/t}$ gives (3.4). We plug this and the corresponding statement for $Z_{T_x + \mathcal{U}}$ into (3.3) to deduce:

$$d_{\text{TV}} \left(\int_0^t p_s(x, \cdot) \, \mu(ds), \int_0^t p_s(z, \cdot) \, \mu(ds) \right) \leq 2\alpha + 4 \sqrt{\frac{\text{T}_{\text{hit}}^Q(\alpha)}{t}}.$$

Now recall that $\alpha < 1/2$ and take

$$t = t(\alpha) \equiv \frac{64 \, \text{T}_{\text{hit}}^Q(\alpha)}{(1 - 2\alpha)^2}.$$

For this value of t, we have:

$$d_{\text{TV}} \left(\int_0^t p_s(x, \cdot) \, \mu(ds), \int_0^t p_s(z, \cdot) \, \mu(ds) \right) \leq \frac{1 + 2\alpha}{2}.$$

Since x, z are arbitrary, we deduce (3.2) with $c(\alpha) = 64/(1 - 2\alpha)^2$ and $\delta(\alpha) = (1 - 2\alpha)/2$. □

4 Mixing of reversible chains

We now prove Theorem 1.3.

Proof of Theorem 1.3. Notice that $\text{T}_{\text{mix}}^Q(\alpha) \geq \text{T}_{\text{rmix}}^Q(\alpha)$, so the lower bound in the Appendix also applies here. For the upper bound, we first define:

$$\overline{d}(t) \equiv \sup_{x, z \in E} d_{\text{TV}}(p_t(x, \cdot), p_t(z, \cdot)).$$

It is well-known that $\overline{d}$ is submultiplicative [3, Chapter 2] and that $\overline{d}(t) \leq 1/4$ implies $\mathrm{T}^Q_{\mathrm{mix}}(1/4) \leq t$. In light of this, we need to show that:

$$\textbf{Goal: } \overline{d}\left(c(\alpha)\,\mathrm{T}^Q_{\mathrm{hit}}(\alpha)\right) \leq 1 - \delta(\alpha), \tag{4.1}$$

where $c(\alpha), \delta(\alpha) > 0$ depend only on $\alpha \in (0, 1/2)$.

Basic definitions for the proof. Let $U > L > 0$ (we will choose their values later). Fix a pair $x, z \in E$ and let $\{X_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ denote trajectories of Q started from x and z (respectively). Also let T_x, T_z be the randomized stopping times given by Lemma 2.1 for the X and Z processes, and define η_x, η_z to be the probability distributions of (X_{T_x}, T_x) and (Z_{T_z}, T_z) over $E \times [0, +\infty)$. Finally, we let $f_x(a) \equiv \mathbb{P}_x\left(X_{T_x} = a, T_x \leq L\right)$ and $f_z(a) = \mathbb{P}_z\left(Z_{T_z} = a, T_z \leq L\right)$ $(a \in E)$.

Estimating total variation distance. Recall:

$$d_{\mathrm{TV}}(p_t(x, \cdot), p_t(z, \cdot)) = \frac{1}{2} \sum_{a \in E} |p_t(x, a) - p_t(z, a)|$$

Notice that:

$$p_t(x, a) = \mathbb{P}_x\left(X_t = a, T_x \leq L\right) + \mathbb{P}_x\left(X_t = a, T_x > L\right),$$

and similarly for $p_t(z, a)$. Therefore,

$$
\begin{aligned}
d_{\mathrm{TV}}(p_t(x, \cdot), p_t(z, \cdot)) &\leq \frac{1}{2} \sum_{a \in E} \left|\mathbb{P}_x\left(X_t = a, T_x \leq L\right) - \mathbb{P}_z\left(Z_t = a, T_z \leq L\right)\right| \\
&\quad + \frac{1}{2} \sum_{a \in E} \left|\mathbb{P}_x\left(X_t = a, T_x > L\right) - \mathbb{P}_z\left(Z_t = a, T_z > L\right)\right| \\
&\leq \frac{1}{2} \sqrt{\sum_{a \in E} \frac{\left(\mathbb{P}_x\left(X_t = a, T_x \leq L\right) - \mathbb{P}_z\left(Z_t = a, T_z \leq L\right)\right)^2}{\pi(a)}} \\
&\quad + \frac{1}{2} \sum_{a \in E} \left|\mathbb{P}_x\left(X_t = a, T_x > L\right) - \mathbb{P}_z\left(Z_t = a, T_z > L\right)\right|.
\end{aligned}
\tag{4.2}
$$

where the last line uses the Cauchy Schwartz inequality. We may further bound:

$$
\begin{aligned}
\sum_{a \in E} \left|\mathbb{P}_x\left(X_t = a, T_x > L\right) - \mathbb{P}_z\left(Z_t = a, T_z > L\right)\right| &\leq \sum_{a \in E} \mathbb{P}_x\left(X_t = a, T_x > L\right) \\
&\quad + \sum_{a \in E} \mathbb{P}_z\left(Z_t = a, T_z > L\right) \\
&\leq \mathbb{P}_x\left(T_x > L\right) + \mathbb{P}_z\left(T_z > L\right),
\end{aligned}
$$

and plugging this into (4.2) gives the inequality:

$$
\begin{aligned}
d_{\mathrm{TV}}(p_t(x, \cdot), p_t(z, \cdot)) &\leq \frac{1}{2} \sqrt{\sum_{a \in E} \frac{\left(\mathbb{P}_x\left(X_t = a, T_x \leq L\right) - \mathbb{P}_z\left(Z_t = a, T_z \leq L\right)\right)^2}{\pi(a)}} \\
&\quad + \frac{\mathbb{P}_x\left(T_x > L\right) + \mathbb{P}_z\left(T_z > L\right)}{2}.
\end{aligned}
\tag{4.3}
$$

Averaging. Our next step is to average the LHS and RHS of (4.3) over $t \in [L, U]$. Since $d_{\mathrm{TV}}(p_t(x, \cdot), p_t(z, \cdot))$ is decreasing in t [5], the distance at time $t = U$ is at most this average. We use concavity to move the averaging inside the square root and deduce:

$$
\begin{aligned}
d_{\mathrm{TV}}(p_U(x, \cdot), p_U(z, \cdot)) &\leq \frac{1}{U - L} \int_L^U d_{\mathrm{TV}}(p_t(x, \cdot), p_t(z, \cdot))\, dt \\
&\leq \frac{1}{2} \sqrt{\frac{1}{U - L} \int_L^U \sum_{a \in E} \frac{\left(\mathbb{P}_x\left(X_t = a, T_x \leq L\right) - \mathbb{P}_z\left(Z_t = a, T_z \leq L\right)\right)^2}{\pi(a)}} \\
&\qquad\qquad\qquad\qquad\qquad + \frac{\mathbb{P}_x\left(T_x > L\right) + \mathbb{P}_z\left(T_z > L\right)}{2}.
\end{aligned}
\tag{4.4}
$$

The term inside the square root. Define $E_L \equiv E \times [0, L]$. By the strong Markov property:

$$
\mathbb{P}_x \left(X_t = a, T_x \le L \right)^2 = \left(\int_{E_L} p_{t-s}(u, a) \, d\eta_x(u, s) \right)^2
$$
$$
= \int_{E_L} \int_{E_L} p_{t-s}(u, a) p_{t-s'}(u', a) \, d\eta_x(u, s) d\eta_x(u', s').
$$

By reversibility, we may rewrite the integrand in the RHS as

$$
p_{t-s}(u, a) \pi(a) p_{t-s'}(a, u') / \pi(u'),
$$

which implies that:

$$
\sum_{a \in E} \frac{\mathbb{P}_x \left(X_t = a, T_x \le L \right)^2}{\pi(a)}
$$
$$
= \int_{E_L} \int_{E_L} \left(\sum_{a \in E} \frac{p_{t-s}(u, a) p_{t-s'}(a, u')}{\pi(u')} \right) d\eta_x(u, s) d\eta_x(u', s')
$$
$$
= \int_{E_L} \int_{E_L} \frac{p_{2t-s'-s}(u, u')}{\pi(u')} \, d\eta_x(u, s) d\eta_x(u', s').
$$

Integrating over t (with the change of variables $t' = 2t - s - s'$), we find that:

$$
\frac{1}{U - L} \int_L^U \sum_{a \in E} \frac{\mathbb{P}_x \left(X_t = a, T_x \le L \right)^2}{\pi(a)} \, dt
$$
$$
= \int_{E_L} \int_{E_L} \left(\frac{1}{2U - 2L} \int_{2L-s-s'}^{2U-s-s'} \frac{p_{t'}(u, u')}{\pi(u')} \, dt' \right) d\eta_x(u, s) d\eta_x(u', s')
$$
$$
\le \int_{E_L} \int_{E_L} \left(\frac{1}{2U - 2L} \int_0^{2U} \frac{p_{t'}(u, u')}{\pi(u')} \, dt' \right) d\eta_x(u, s) d\eta_x(u', s') \quad (4.5)
$$

where the last inequality follows from the fact that $[2L - s - s', 2U - s - s'] \subset [0, 2U]$, which holds for all s, s' in the range considered. With this the bracketed term becomes independent of s, which may be integrated out. Since:

$$
\int_{\{u\} \times [0, L]} d\eta_x(u, s) = f_x(u) \le \pi(u),
$$

we obtain:

$$
\frac{1}{U - L} \int_L^U \sum_{a \in E} \frac{\mathbb{P}_x \left(X_t = a, T_x \le L \right)^2}{\pi(a)} \, dt
$$
$$
\le \sum_{u, u' \in E} \frac{f_x(u) f_x(u')}{\pi(u')} \left(\frac{1}{2U - 2L} \int_0^{2U} p_{t'}(u, u') \, dt' \right)
$$
$$
\le \sum_{u, u' \in E} \frac{f_x(u) f_x(u')}{\pi(u')} \left(\frac{1}{2U - 2L} \int_{2L}^{2U} p_w(u, u') \, dw \right)
$$
$$
+ \sum_{u, u' \in E} \frac{\pi(u)}{2U - 2L} \int_0^{2L} p_w(u, u') \, dw
$$
$$
\le \sum_{u, u' \in E} \frac{f_x(u) f_x(u')}{\pi(u')} \left(\frac{1}{2U - 2L} \int_{2L}^{2U} p_w(u, u') \, dw \right) + \frac{L}{U - L}, \quad (4.6)
$$

as well as a similar bound for z. On the other hand, starting from the formula:

$$
\mathbb{P}_x \left(X_t = a, T_x \le L \right) \mathbb{P}_z \left(Z_t = a, T_z \le L \right)
$$
$$
= \int_{E_L} \int_{E_L} p_{t-s}(u, z) p_{t-s'}(u', z) \, d\eta_x(u, s) d\eta_z(u', s')
$$

averaging over $t \in [L, U]$ and using $[2L - s - s', 2L + 2U - s - s'] \supset [2L, 2U]$, we may obtain:

$$\frac{1}{U - L} \int_L^U \sum_{a \in E} \frac{\mathbb{P}_x \left(X_t = a, T_x \leq L \right) \mathbb{P}_z \left(Z_t = a, T_z \leq L \right)}{\pi(z)} \, dt$$
$$\geq \sum_{u, u' \in E} \frac{f_x(u) f_z(u')}{\pi(u')} \left(\frac{1}{2U - 2L} \int_{2L}^{2U} p_w(u, u') \, dw \right).$$

Combining these bounds we obtain

$$\sum_{z \in E} \frac{1}{U - L} \int_L^U \frac{\left(\mathbb{P}_x \left(X_t = z, T_x \leq L \right) - \mathbb{P}_x \left(Z_t = z, T_z \leq L \right) \right)^2}{\pi(z)}$$
$$\leq \sum_{u, u' \in E} (f_x(u) - f_z(u)) \left(\frac{f_x(u') - f_z(u')}{\pi(u')} \right) \left(\frac{1}{2U} \int_{2L}^{2U} p_w(u, u') \, dw \right) + \frac{2L}{U - L}.$$

To bound the sum in the RHS, we notice again that $f_x(\cdot), f_z(\cdot) \leq \pi(\cdot)$, and also that for all $u \in E$, $\sum_{u'} p_w(u, u') = 1$. Hence

$$\sum_{u, u' \in E} (f_x(u) - f_z(u)) \left(\frac{f_x(u') - f_z(u')}{\pi(u')} \right) \left(\frac{1}{2U - 2L} \int_{2L}^{2U} p_w(u, u') \, dw \right)$$
$$\leq \sum_{u \in E} |f_x(u) - f_z(u)|.$$

Now recall that

$$f_x(u) = \mathbb{P}_x \left(X_{T_x} = u, T_x \leq L \right) = \pi(u) - \mathbb{P}_x \left(X_{T_x} = u, T_x > L \right)$$

and similarly for z, so that

$$\sum_{u \in E} |f_x(u) - f_z(u)| = \sum_{a \in E} |\mathbb{P}_x \left(X_{T_x} = a, T_x > L \right) - \mathbb{P}_z \left(Z_{T_z} = a, T_z > L \right)|.$$

We deduce that the term inside the square root in (4.4) is bounded by:

$$\frac{1}{U - L} \int_L^U \sum_{a \in E} \frac{\left(\mathbb{P}_x \left(X_t = a, T_x \leq L \right) - \mathbb{P}_z \left(Z_t = a, T_z \leq L \right) \right)^2}{\pi(z)} \, dt$$
$$\leq \sum_{a \in E} |\mathbb{P}_x \left(X_{T_x} = a, T_x > L \right) - \mathbb{P}_z \left(Z_{T_z} = a, T_z > L \right)| + \frac{2L}{U - L}$$
$$\leq \sum_{a \in E} \mathbb{P}_x \left(X_{T_x} = a, T_x > L \right) + \sum_{a \in E} \mathbb{P}_z \left(Z_{T_z} = a, T_z > L \right) + \frac{2L}{U - L}$$
$$\leq \mathbb{P}_x \left(T_x > L \right) + \mathbb{P}_z \left(T_z > L \right) + \frac{2L}{U - L}.$$

Wrapping up. We now plug this previous inequality into (4.4) to deduce:

$$d_{\mathrm{TV}}(p_U(x, \cdot), p_U(z, \cdot))$$
$$\leq \frac{1}{2} \sqrt{\mathbb{P}_x \left(T_x > L \right) + \mathbb{P}_z \left(T_z > L \right) + \frac{2L}{U - L}}$$
$$+ \frac{\mathbb{P}_x \left(T_x > L \right) + \mathbb{P}_z \left(T_z > L \right)}{2}.$$

If the quantity inside the square root is < 1, we get another upper bound:

$$d_{\mathrm{TV}}(p_U(x, \cdot), p_U(z, \cdot)) \leq \sqrt{\mathbb{P}_x \left(T_x > L \right) + \mathbb{P}_z \left(T_z > L \right) + \frac{2L}{U - L}} \tag{4.7}$$

Now by Lemma 2.1

$$\mathbb{P}_x \left(T_x > L \right) + \mathbb{P}_z \left(T_z > L \right) \leq 2\alpha + 2 \frac{\mathrm{T}_{\mathrm{hit}}^Q(\alpha)}{L}$$

so choosing

$$L = \frac{8 T_{\text{hit}}^Q(\alpha)}{1 - 2\alpha} \text{ and } U = \left[\frac{8}{1 - 2\alpha} + \left(\frac{8}{1 - 2\alpha}\right)^2\right] T_{\text{hit}}^Q(\alpha)$$

we obtain:

$$\mathbb{P}_x\left(T_x > L\right) + \mathbb{P}_z\left(T_z > L\right) + \frac{2L}{U - L} \leq \frac{1 + 2\alpha}{2} < 1.$$

Thus the condition for (4.7) is satisfied, and we have the bound:

$$d_{\text{TV}}(p_U(x, \cdot), p_U(z, \cdot)) \leq \sqrt{\frac{1 + 2\alpha}{2}}.$$

Since $x, z \in E$ are arbitrary, we deduce:

$$\overline{d}\left(\left[\frac{8}{1 - 2\alpha} + \left(\frac{8}{1 - 2\alpha}\right)^2\right] T_{\text{hit}}^Q(\alpha)\right) \leq 1 - \left(1 - \sqrt{\frac{1 + 2\alpha}{2}}\right),$$

which has the form requested in (4.1). $\qquad\qquad\square$

A Appendix: the lower bound

In this section we prove the lower bound part of the main theorems. As above, Q is a irreducible continuous-time Markov chain with state space E and stationary distribution π. The trajectories of the chain are denoted by $\{X_t\}_{t \geq 0}$

Proposition A.1. *For any* $\alpha \in (0, 1)$, $T_{\text{hit}}^Q(\alpha) \leq c(\alpha) T_{\text{rmix}}^Q(1/4)$ *where* $c(\alpha) > 0$ *depends only on* α.

Proof. It follows from Claim 3.1 that:

$$T_{\text{rmix}}^Q(1/2^k) \leq k\, T_{\text{rmix}}^Q(1/4).$$

In particular,

$$T_{\text{rmix}}^Q(\alpha) \leq (\log_2(1/\alpha) + 1)\, T_{\text{rmix}}^Q(1/4).$$

Thus it suffices to show that $T_{\text{hit}}^Q(\alpha) \leq (2/\alpha)\, T_{\text{rmix}}^Q(\alpha/2)$.

Fix $A \subset V$ with measure $\pi(A) \geq \alpha$ and $x \in V$. By the definition of $T_{\text{rmix}}^Q(\alpha/2)$ and a simple compactness argument, there exists a distribution supported on $[0, T_{\text{rmix}}^Q(\alpha/2)]$ such that if $\mathcal{U}$ has this distribution and is independent from $\{X_t\}_t$,

$$d_{\text{TV}}(\text{Law}(X_{\mathcal{U}}), \pi) \leq \alpha/2.$$

As a result,

$$\mathbb{P}_x\left(X_{\mathcal{U}} \notin A\right) \leq 1 - \pi(A) + d_{\text{TV}}(\text{Law}(X_{\mathcal{U}}), \pi) \leq 1 - \frac{\alpha}{2}.$$

Since $\mathcal{U}$ is supported in $[0, T_{\text{rmix}}^Q(\alpha/2)]$,

$$\{H_A \geq T_{\text{rmix}}^Q(\alpha/2)\} \subset \{X_{\mathcal{U}} \notin A\},$$

and we deduce:

$$\forall x \in V, \forall A \subset V \text{ with } \pi(A) \geq \alpha \,:\, \mathbb{P}_x\left(H_A \geq T_{\text{rmix}}^Q(\alpha/2)\right) \leq 1 - \frac{\alpha}{2}. \qquad (A.1)$$

Let us use this to show that $\mathbb{E}_x[H_A] \leq (2/\alpha)\, T_{\text{rmix}}^Q(\alpha/2)$ for all x and A as above. Let $k \in \mathbb{N}\setminus\{0\}$ and denote by Λ_k the law of $X_{(k-1)T_{\text{rmix}}^Q(\alpha/2)}$ conditioned on $\{H_A \geq (k-1)T_{\text{rmix}}^Q(\alpha/2)\}$. By (A.1),

$$\mathbb{P}_{\Lambda_k}\left(H_A \geq T_{\text{rmix}}^Q(\alpha/2)\right) \leq 1 - \frac{\alpha}{2},$$

whereas by the Markov property,

$$
\begin{aligned}
\mathbb{P}_x\left(H_A \geq k T_{\text{rmix}}^Q(\alpha/2)\right) &\leq \mathbb{P}_x\left(H_A \geq (k-1)T_{\text{rmix}}^Q(\alpha/2)\right) \mathbb{P}_{\Lambda_k}\left(H_A \geq T_{\text{rmix}}^Q(\alpha/2)\right) \\
&\leq \left(1 - \frac{\alpha}{2}\right) \mathbb{P}_x\left(H_A \geq (k-1)T_{\text{rmix}}^Q(\alpha/2)\right) \\
(\text{...induction...}) &\leq \left(1 - \frac{\alpha}{2}\right)^k
\end{aligned}
$$

We deduce:

$$\frac{\mathbb{E}_x\left[H_A\right]}{\mathrm{T}_{\mathrm{rmix}}^{Q}(\alpha/2)} \le \sum_{k\ge 0}\left(1-\frac{\alpha}{2}\right)^k = \frac{2}{\alpha}$$

Since $x \in V$ and $A \subset V$ with $\pi(A) \ge \alpha$ were arbitrary, this finishes the proof. $\qquad\square$

References

[1] List of open problems from AIM Workshop on Algorithmic Convex Geometry. `http://www.aimath.org/WWN/convexgeometry/convexgeometry.pdf`. Compiled by Navin Goyal (2009).

[2] Aldous, D.: Some Inequalities for Reversible Markov Chains. *J. London Math. Soc. (2)* **25**, (1982), 564-576. MR-0657512

[3] Aldous, D. and Fill, J.A.: Reversible Markov Chains and Random Walks on Graphs. Book draft available from `http://www.stat.berkeley.edu/~aldous/`.

[4] Aldous, D., Lovász, L. and Winkler, P.: Mixing times for uniformly ergodic Markov chains. *Stochastic Process. Appl* **71** (2), (1997), 165-185. MR-1484158

[5] Levin, D., Peres, Y., and Wilmer, E.: Markov Chains and Mixing Times. With a chapter by James G. Propp and David B. Wilson. *American Mathematical Society*, Providence, RI, 2009. xviii+371 pp. MR-2466937

[6] Lovász, L. and Winkler, P.: Mixing of random walks and other diffusions on a graph. *Surveys in combinatorics*, 1995 (Stirling), 119–154, London Math. Soc. Lecture Note Ser., 218, Cambridge Univ. Press, Cambridge, 1995. MR-1358634

[7] Peres,Y.: Personal communication (2011).

[8] Peres, Y. and Sousi, P.: Mixing times are hitting times of large sets, arXiv:1108.0133.