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Optimal regularity for semilinear stochastic partial differential equations with multiplicative noise*

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Abstract

This paper deals with the spatial and temporal regularity of the unique Hilbert space valued mild solution to a semilinear stochastic parabolic partial differential equation with nonlinear terms that satisfy global Lipschitz conditions and certain linear growth bounds. It is shown that the mild solution has the same optimal regularity properties as the stochastic convolution. The proof is elementary and makes use of existing results on the regularity of the solution, in particular, the Hölder continuity with a non-optimal exponent.

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1 Introduction

Consider the following semilinear stochastic partial differential equation (SPDE)

$$\begin{aligned} \mathrm{d}X(t) + [AX(t) + F(X(t))]\,\mathrm{d}t &= G(X(t))\,\mathrm{d}W(t), \quad \text{for } 0 \le t \le T, \\ X(0) &= X_0, \end{aligned} \tag{1.1}$$

where the mild solution X takes values in a Hilbert space H. The linear operator $A\colon D(A) \subset H \to H$ is self-adjoint, positive definite with compact inverse and $-A$ is the generator of an analytic semigroup $E(t) = \mathrm{e}^{-tA}$ on H. For example, let $-A$ be the Laplacian with homogeneous Dirichlet boundary conditions and $H = L^2(\mathcal{D})$ for some bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with smooth boundary $\partial\mathcal{D}$ or a convex domain with polygonal boundary. The nonlinear operators F and G are assumed to be globally Lipschitz continuous in the appropriate sense and $W\colon [0,T] \times \Omega \to U$ denotes a standard Q-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in some Hilbert space U.

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Our aim is to study the spatial and temporal regularity properties of the unique mild solution $X\colon [0,T]\times\Omega\to H$. The spatial regularity is measured in terms of the domains $\dot{H}^r := D(A^{\frac{r}{2}})$, $r\geq 0$, of fractional powers of the operator A. If $-A$ is the Laplacian, these domains coincide with standard Sobolev spaces, for example, $\dot{H}^1 = H^1_0(\mathcal{D})$ or $\dot{H}^2 = H^1_0(\mathcal{D})\cap H^2(\mathcal{D})$ (c.f. [10, Th. 6.4] or [16, Ch. 3]). The regularity in time is expressed by the Hölder exponent.

Assuming only that the semigroup $E(t)$ is analytic and that F and G satisfy appropriate global Lipschitz conditions on H one may show that (1.1) has a unique mild solution X and that for every $\gamma\in[0,1)$ the solution X maps into $\dot{H}^\gamma\subset H$ and is $\frac{\gamma}{2}$-Hölder continuous with respect to the norm $\big(\mathbf{E}\big[\|\cdot\|_H^p\big]\big)^{\frac{1}{p}}$, $p\in[2,\infty)$, see, for example, [6] and [7].

The border case $\gamma=1$ is of special interest in numerical analysis. For example, if one is analyzing an approximation scheme based on a finite element method, the spatial regularity determines the order of convergence. Hence, a suboptimal regularity result leads to a suboptimal estimate of the order of convergence (c.f. [16]).

The border case can be handled by making the additional assumption on the semigroup $E(t)=\mathrm{e}^{-tA}$ that the generator is self-adjoint with compact inverse. Under this assumption the optimal regularity of stochastic convolutions of the form

$$W_A^\Phi(t)=\int_0^t E(t-\sigma)\Phi(\sigma)\,\mathrm{d}W(\sigma),$$

is studied in [6, Prop. 6.18] and [3]. Here Φ is a stochastically square integrable ($p=2$) process with values in the set of Hilbert-Schmidt operators. If, for $r\geq 0$, the process Φ is regular enough so that the process $t\mapsto A^{\frac{r}{2}}\Phi(t)$ is still stochastically square integrable, then the convolution is a stochastic process, which is square integrable with values in $\dot{H}^{r+1}$. There exist some generalizations of this result, for instance, to Banach space valued integrands [5], to the case $p>2$ [18], and to Lévy noise [2].

The recent paper [7] extends this type of higher regularity result to the nonlinear problem (1.1) by introducing an appropriate linear growth assumption for G on the space $\dot{H}^r$ for some $r\in[0,1)$ (see (2.3) below). It is shown that X maps into $\dot{H}^{r+\gamma}$ for $\gamma\in[0,1)$. The border case $\gamma=1$ is not included because no additional assumption is made on the analytic semigroup.

The purpose of the present paper is to fill this gap. We therefore assume that the semigroup is generated by a self-adjoint operator with compact inverse and we complement the global Lipschitz assumptions for F,G on H by a linear growth bound for G on $\dot{H}^r$. Our main results are presented in Theorems 3.1, 4.1, and 4.2. The proofs are based on a very careful use of the smoothing property of the semigroup $E(t)=\mathrm{e}^{-tA}$ (see Lemma 3.2), and on the Hölder continuity of X with a suboptimal exponent (see Lemmas 3.4 and 3.5).

Our regularity result for the mild solution of (1.1) coincides with the optimal regularity property of the stochastic convolution but with the restriction $r<1$. In this sense we understand our result to be optimal.

Evolution equations of the form (1.1) are also studied by other authors. We refer to [6, 9, 14, 19] and the references therein. Further related results are [20], where conditions for spatial C^∞-regularity are given, and [17], which provides conditions for the existence of strong solutions to (1.1).

This paper consists of four additional sections. In the next section we give a more precise formulation of our assumptions. In Section 3 we are concerned with the spatial regularity of the mild solution. The proof is divided into several lemmas, which contain the key ideas of proof. The lemmas are also useful in the proof of the temporal Hölder continuity in Section 4. The proof of continuity in the border case requires an additional argument in form of Lebesgue's dominated convergence theorem. This technique is also developed in Section 4. The last section briefly reviews our results in the special case of additive noise and gives an example in which the spatial regularity results are indeed optimal.

2 Preliminaries

In this section we present the general form of the SPDE we are interested in. After introducing some notation we state our assumptions and cite the result on existence, uniqueness and regularity of a mild solution from [7].

By H we denote a separable Hilbert space $(H,(\cdot,\cdot),\|\cdot\|)$. Further, let $A\colon D(A)\subset H\to H$ be a densely defined, linear, self-adjoint, positive definite operator, which is not necessarily bounded but with compact inverse. Hence, there exists an increasing sequence of real numbers $(\lambda_n)_{n\geq 1}$ and an orthonormal basis $(e_n)_{n\geq 1}$ in H such that $Ae_n=\lambda_n e_n$ and

$$0<\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_n(\to\infty).$$

The domain of A is characterized by

$$D(A)=\Big\{x\in H\,:\,\sum_{n=1}^{\infty}\lambda_n^2(x,e_n)^2<\infty\Big\}.$$

Thus, $-A$ is the generator of an analytic semigroup of contractions, which is denoted by $E(t)=\mathrm{e}^{-At}$.

By $W\colon[0,T]\times\Omega\to U$ we denote a Q-Wiener process with values in a separable Hilbert space $(U,(\cdot,\cdot)_U,\|\cdot\|_U)$. While our underlying probability space is $(\Omega,\mathcal{F},P)$, we assume that the Wiener process is adapted to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_t\subset\mathcal{F}$ for all $t\in[0,T]$. The covariance operator $Q\colon U\to U$ is linear, bounded, self-adjoint, positive semidefinite and trace-class, that is

$$\mathrm{Tr}(Q)=\sum_{m=1}^{\infty}(e_m,Qe_m)_U<\infty$$

for an arbitrary orthonormal basis $(e_m)_{m\in\mathbb{N}}$ of U.

We study the regularity properties of a stochastic process $X\colon[0,T]\times\Omega\to H$, $T>0$, which is the mild solution to the stochastic partial differential equation (1.1). Thus, X satisfies the equation

$$X(t)=E(t)X_0-\int_0^t E(t-\sigma)F(X(\sigma))\,\mathrm{d}\sigma+\int_0^t E(t-\sigma)G(X(\sigma))\,\mathrm{d}W(\sigma)\qquad(2.1)$$

for all $0\leq t\leq T$.

In order to formulate our assumptions and main result we introduce the notion of fractional powers of the linear operator A. For any $r\in\mathbb{R}$ the operator $A^{\frac{r}{2}}$ is given by

$$A^{\frac{r}{2}}x=\sum_{n=1}^{\infty}\lambda_n^{\frac{r}{2}}x_ne_n$$

for all

$$x\in D(A^{\frac{r}{2}})=\Big\{x=\sum_{n=1}^{\infty}x_ne_n\,:\,(x_n)_{n\geq1}\subset\mathbb{R}\text{ with }\|x\|_r^2:=\|A^{\frac{r}{2}}x\|^2=\sum_{n=1}^{\infty}\lambda_n^rx_n^2<\infty\Big\}.$$

By defining $\dot{H}^r := D(A^{\frac{r}{2}})$ together with the norm $\|x\|_r$ for $r \in \mathbb{R}$, $\dot{H}^r$ becomes a Hilbert space.

Instead of defining $\dot{H}^{-r}$ for $r > 0$ as above one can also work with the dual space $(\dot{H}^r)'$. But, as it is shown, for example in [8, Th. B.8], our spaces $\dot{H}^{-r}$ are isometrically isomorphic to $(\dot{H}^r)'$ for $r > 0$. Therefore, the results in our paper are independent of the way $\dot{H}^{-r}$ is defined. We prefer to work with the spaces $\dot{H}^r$ for $r \in \mathbb{R}$, since the identical spectral structure for all $r \in \mathbb{R}$ allows for a simple extension of the operator $A^{-\frac{r}{2}}$ seen as a mapping from H to $\dot{H}^r$ to a mapping from $\dot{H}^{-r}$ to H.

As usual [6, 12] we introduce the separable Hilbert space $U_0 := Q^{\frac{1}{2}}(U)$ with the inner product $(u_0, v_0)_{U_0} := (Q^{-\frac{1}{2}} u_0, Q^{-\frac{1}{2}} v_0)_U$ with $Q^{-\frac{1}{2}}$ denoting the pseudoinverse. The diffusion operator G maps H into L_2^0, where L_2^0 denotes the space of all Hilbert-Schmidt operators $\Phi\colon U_0 \to H$ with norm

$$\|\Phi\|_{L_2^0}^2 := \sum_{m=1}^{\infty} \|\Phi \psi_m\|^2.$$

Here $(\psi_m)_{m\geq 1}$ is an arbitrary orthonormal basis of U_0 (for details see, for example, Proposition 2.3.4 in [12]). Further, $L_{2,r}^0$ denotes the set of all Hilbert-Schmidt operators $\Phi\colon U_0 \to \dot{H}^r$ together with the norm $\|\Phi\|_{L_{2,r}^0} := \|A^{\frac{r}{2}} \Phi\|_{L_2^0}$.

Let $r \in [0,1)$, $p \in [2,\infty)$ be given. As in [7, 13] we make the following additional assumptions.

Assumption 2.1. *There exists a constant C such that*

$$\|G(x) - G(y)\|_{L_2^0} \leq C\|x - y\| \quad \forall x, y \in H \tag{2.2}$$

and we have that $G(\dot{H}^r) \subset L_{2,r}^0$ and

$$\|G(x)\|_{L_{2,r}^0} \leq C\,(1 + \|x\|_r) \quad \forall x \in \dot{H}^r. \tag{2.3}$$

Assumption 2.2. *The nonlinearity F maps H into $\dot{H}^{-1+r}$. Furthermore, there exists a constant C such that*

$$\|F(x) - F(y)\|_{-1+r} \leq C\|x - y\| \quad \forall x, y \in H. \tag{2.4}$$

Assumption 2.3. *The initial value $X_0\colon \Omega \to \dot{H}^{r+1}$ is an $\mathcal{F}_0$-measurable random variable with $\mathbf{E}\left[\|X_0\|_{r+1}^p\right] < \infty$.*

Under the above conditions Theorem 1 in [7] states that for every $\gamma \in [r, r+1)$ and $T > 0$ there exists an up to modification unique mild solution $X\colon [0,T]\times\Omega \to \dot{H}^\gamma$ to (1.1) of the form (2.1), which satisfies

$$\sup_{t\in[0,T]} \mathbf{E}\left[\|X(t)\|_\gamma^p\right] < \infty.$$

Moreover, the solution process is continuous with respect to $\left(\mathbf{E}\left[\|\cdot\|_\gamma^p\right]\right)^{\frac{1}{p}}$ and fulfills

$$\sup_{t_1,t_2\in[0,T], t_1\neq t_2} \frac{\left(\mathbf{E}\left[\|X(t_1) - X(t_2)\|_s^p\right]\right)^{\frac{1}{p}}}{|t_1 - t_2|^{\min(\frac{1}{2},\frac{\gamma-s}{2})}} < \infty$$

for every $s \in [0,\gamma]$.

The aim of this paper is to show that these results on the spatial regularity and the temporal Hölder continuity also hold with $\gamma = r+1$. For $r = 0$ we also prove that the solution process remains continuous with respect to the norm $\left(\mathbf{E}\left[\|\cdot\|_1^p\right]\right)^{\frac{1}{p}}$.

Remarks 2.4. *1. Actually, Theorem* 1 *in [7] assumes that* $F\colon H \to H$ *is globally Lipschitz, which is slightly stronger than Assumption 2.2. That Assumption 2.2 is sufficient can be proved by just following the given proof line by line and making the appropriate changes where ever* F *comes into play.*

2. The linear growth bound (2.3) *follows from* (2.2) *when* $r = 0$.

3. Assumption 2.3 can be relaxed to $X_0\colon \Omega \to H$ *being an* $\mathcal{F}_0$*-measurable random variable with* $\mathbf{E}\left[\|X_0\|^p\right] < \infty$. *But, as it is known from deterministic PDE theory, this will lead to a singularity at* $t = 0$.

4. The framework is quite general. More explicit examples and a detailed discussion of Assumption 2.1 can be found in [7]. We also refer to the discussion in [13] for further examples, references and a related result for temporal regularity.

3 Spatial regularity

In this section we deal with the spatial regularity of the mild solution. Our result is given by the following theorem. For a more convenient notation we set $\|\cdot\|_{L^p(\Omega;\mathcal{H})} := \left(\mathbf{E}\left[\|\cdot\|_{\mathcal{H}}^p\right]\right)^{\frac{1}{p}}$ for any Hilbert space $\mathcal{H}$. Also, if applied to an operator, the norm $\|\cdot\|$ is understood as the operator norm for bounded, linear operators from H to H.

Theorem 3.1 (Spatial regularity)**.** *Let* $r \in [0, 1)$, $p \in [2, \infty)$. *Given the assumptions of Section 2 the unique mild solution* X *in* (2.1) *satisfies*

$$\mathbf{P}\big[X(t) \notin \dot{H}^{r+1}\big] = 0$$

for all $t \in [0, T]$. *In particular, there exists a constant* $C > 0$ *depending on* p, r, A, F, G, T *and the Hölder continuity constant of* X *with respect to the norm* $\|\cdot\|_{L^p(\Omega;H)}$ *such that*

$$\sup_{t\in[0,T]} \big(\mathbf{E}\big[\|X(t)\|_{r+1}^p\big]\big)^{\frac{1}{p}} \quad \leq \quad \big(\mathbf{E}\big[\|X_0\|_{r+1}^p\big]\big)^{\frac{1}{p}} \;+\; C\Big(1 \;+\; \sup_{t\in[0,T]} \big(\mathbf{E}\big[\|X(t)\|_{r}^p\big]\big)^{\frac{1}{p}}\Big).$$

Before we prove the theorem we introduce several useful lemmas. The first states some well known facts on the semigroups $(E(t))_{t\in[0,T]}$. The parts *(i)*, *(ii)* and *(iv)* hold true for analytic semigroups in general, while we use an orthonormal eigenbasis of the generator for the proof of *(iii)*. For a proof of *(i)* and *(ii)* we refer to [11, Ch. 2.6, Th. 6.13]. Since parts *(iii)*, *(iv)* are not readily found in the literature, we provide proofs here.

Lemma 3.2. *Let the infinitesimal generator* $-A$ *of the semigroup* $(E(t))_{[0,\infty)}$ *be self-adjoint with compact inverse. Then the following properties hold true:*

(i) For any $\mu \geq 0$ *there exists a constant* $C = C(\mu)$ *such that*

$$\|A^{\mu} E(t)\| \leq C t^{-\mu} \quad \text{for } t > 0.$$

(ii) For any $0 \leq \nu \leq 1$ *there exists a constant* $C = C(\nu)$ *such that*

$$\|A^{-\nu}(E(t) - I)\| \leq C t^{\nu} \text{ for } t \geq 0.$$

(iii) For any $0 \leq \rho \leq 1$ *there exists a constant* $C = C(\rho)$ *such that*

$$\int_{\tau_1}^{\tau_2} \|A^{\frac{\rho}{2}} E(\tau_2 - \sigma)x\|^2 \,\mathrm{d}\sigma \leq C(\tau_2 - \tau_1)^{1-\rho} \|x\|^2 \text{ for all } x \in H,\ 0 \leq \tau_1 < \tau_2.$$

(iv) For any $0 \leq \rho \leq 1$ *there exists a constant* $C = C(\rho)$ *such that*

$$\Big\|A^{\rho} \int_{\tau_1}^{\tau_2} E(\tau_2 - \sigma)x \,\mathrm{d}\sigma\Big\| \leq C(\tau_2 - \tau_1)^{1-\rho} \|x\| \text{ for all } x \in H,\ 0 \leq \tau_1 < \tau_2.$$

Proof. For the proof of *(iii)* we use the expansion of $x \in H$ in terms of the eigenbasis $(e_n)_{n\geq 1}$ of the operator A. By Parseval's identity we get

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \big\|A^{\frac{\rho}{2}}E(\tau_2-\sigma)x\big\|^2 \,\mathrm{d}\sigma &= \int_{\tau_1}^{\tau_2} \Big\| \sum_{n=1}^{\infty} A^{\frac{\rho}{2}}E(\tau_2-\sigma)(x,e_n)e_n\Big\|^2 \,\mathrm{d}\sigma \\ &= \sum_{n=1}^{\infty}\int_{\tau_1}^{\tau_2} (x,e_n)^2\lambda_n^{\rho}e^{-2\lambda_n(\tau_2-\sigma)}\,\mathrm{d}\sigma \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(x,e_n)^2\lambda_n^{\rho-1}\big(1-e^{-2\lambda_n(\tau_2-\tau_1)}\big).\end{aligned}$$

By the boundedness of the function $x \mapsto x^{\rho-1}(1-\mathrm{e}^{-x})$ for $x \in [0,\infty)$ and $\rho \in [0,1]$ there exists a constant $C = C(\rho) > 0$ such that

$$\int_{\tau_1}^{\tau_2} \big\|A^{\frac{\rho}{2}}E(\tau_2-\sigma)x\big\|^2 \,\mathrm{d}\sigma \leq C(\rho)(\tau_2-\tau_1)^{1-\rho}\sum_{n=1}^{\infty}(x,e_n)^2,$$

which completes the proof of *(iii)*.

The following proof of *(iv)* also works for analytic semigroups in general. By [11, Ch. 1.2, Th. 2.4 (ii)] we first note that

$$\begin{aligned}\Big\|A^{\rho}\int_{\tau_1}^{\tau_2} E(\tau_2-\sigma)x \,\mathrm{d}\sigma\Big\| &= \Big\|A^{\rho-1}A\int_{0}^{\tau_2-\tau_1} E(\sigma)x \,\mathrm{d}\sigma\Big\| \\ &= \big\|A^{\rho-1}\big(E(\tau_2-\tau_1)-I\big)x\big\|.\end{aligned}$$

Then, *(iv)* follows from *(ii)*. □

The next lemma is a special case of [6, Lem. 7.2] and will be needed to estimate stochastic integrals.

Lemma 3.3. *For any $p \geq 2$, $0 \leq \tau_1 < \tau_2 \leq T$, and for any L_2^0-valued predictable process $\Phi(t)$, $t \in [\tau_1,\tau_2]$, which satisfies*

$$\mathbf{E}\Big[\Big(\int_{\tau_1}^{\tau_2} \big\|\Phi(\sigma)\big\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big)^{\frac{p}{2}}\Big] < \infty,$$

we have

$$\mathbf{E}\Big[\Big\|\int_{\tau_1}^{\tau_2} \Phi(\sigma)\,\mathrm{d}W(\sigma)\Big\|^p\Big] \leq C(p)\mathbf{E}\Big[\Big(\int_{\tau_1}^{\tau_2} \big\|\Phi(\sigma)\big\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big)^{\frac{p}{2}}\Big].$$

Here the constant can be chosen to be

$$C(p) = \Big(\frac{p}{2}(p-1)\Big)^{\frac{p}{2}}\left(\frac{p}{p-1}\right)^{p(\frac{p}{2}-1)}.$$

The following two lemmas contain our main idea of proof and yield the key estimates. The applied technique is already known in the literature for Bochner integrals consisting of a convolution with an analytic semigroup, for example in [4, Prop. 3] and [15, p. 157].

Lemma 3.4. *Let $s \in [0, r+1]$, $p \geq 2$, and Y be a predictable stochastic process on $[0,T]$ which maps into $\dot{H}^r$ with $\sup_{\sigma\in[0,T]} \|A^{\frac{r}{2}}Y(\sigma)\|_{L^p(\Omega;H)} < \infty$. Then there exists a constant $C = C(p,r,s,A,G)$ such that, for all $\tau_1,\tau_2 \in [0,T]$ with $\tau_1 < \tau_2$,*

$$\begin{aligned}&\Big(\mathbf{E}\Big[\Big(\int_{\tau_1}^{\tau_2} \big\|A^{\frac{s}{2}}E(\tau_2-\sigma)G(Y(\tau_2))\big\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{p}} \\ &\qquad\leq C\Big(1+\sup_{\sigma\in[0,T]}\big\|A^{\frac{r}{2}}Y(\sigma)\big\|_{L^p(\Omega;H)}\Big)(\tau_2-\tau_1)^{\min(\frac{1}{2},\frac{1+r-s}{2})}. \qquad (3.1)\end{aligned}$$

If, in addition, for some $\delta > \frac{r}{2}$ there exists C_δ such that

$$\|Y(t_1) - Y(t_2)\|_{L^p(\Omega;H)} \le C_\delta |t_2 - t_1|^\delta \text{ for all } t_1, t_2 \in [0,T],$$

then we also have, with $C = C(p,s,G,C_\delta)$, that

$$\Big(\mathbf{E}\Big[\Big(\int_{\tau_1}^{\tau_2} \big\|A^{\frac{s}{2}} E(\tau_2-\sigma)\big(G(Y(\sigma)) - G(Y(\tau_2))\big)\big\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{p}} \le \frac{C}{\sqrt{1+2\delta-s}} (\tau_2-\tau_1)^{\frac{1+2\delta-s}{2}}. \quad (3.2)$$

In particular, with $C = C(T,\delta,p,r,s,A,G,C_\delta)$ it holds that

$$\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma) G(Y(\sigma)) \,\mathrm{d}W(\sigma) \Big\|_{L^p(\Omega;H)} \le C\Big(1 + \sup_{\sigma\in[0,T]} \big\|A^{\frac{r}{2}} Y(\sigma)\big\|_{L^p(\Omega;H)}\Big) (\tau_2-\tau_1)^{\min(\frac{1}{2},\frac{1+r-s}{2})}. \quad (3.3)$$

Proof. First note that, for $0 \le \tau_1 < \tau_2 \le T$ fixed, the mapping $[\tau_1,\tau_2] \ni \sigma \mapsto A^{\frac{s}{2}} E(\tau_2 - \sigma) G(Y(\sigma))$ is a predictable L_2^0-valued process. Hence, Lemma 3.3 is applicable and gives

$$\begin{aligned}
&\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma) G(Y(\sigma)) \,\mathrm{d}W(\sigma) \Big\|_{L^p(\Omega;H)} \\
&\le C(p) \Big\| \Big(\int_{\tau_1}^{\tau_2} \big\|A^{\frac{s}{2}} E(\tau_2-\sigma) G(Y(\sigma))\big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&\le C(p) \Big\| \Big(\int_{\tau_1}^{\tau_2} \big\|A^{\frac{s}{2}} E(\tau_2-\sigma) G(Y(\tau_2))\big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + C(p) \Big\| \Big(\int_{\tau_1}^{\tau_2} \big\|A^{\frac{s}{2}} E(\tau_2-\sigma)\big(G(Y(\sigma)) - G(Y(\tau_2))\big)\big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&=: S_1 + S_2.
\end{aligned}$$

In the second step we just used the triangle inequality. Now we deal with both summands separately. In the first term S_1 the time in $G(Y(\tau_2))$ is fixed. We also notice that $\eta := s - r - \max(0, s-r) \le 0$ and, hence, $A^{\frac{\eta}{2}}$ is a bounded linear operator on H. Furthermore, since $s \in [0, r+1]$ we have $\rho := \max(0, s-r) \in [0,1]$ and Lemma 3.2 *(iii)* is applicable. By writing $s = \eta + \rho + r$, we get

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \big\|A^{\frac{s}{2}} E(\tau_2-\sigma) G(Y(\tau_2))\big\|_{L_2^0}^2 \,\mathrm{d}\sigma \\
&= \int_{\tau_1}^{\tau_2} \sum_{m=1}^{\infty} \big\|A^{\frac{s}{2}} E(\tau_2-\sigma) G(Y(\tau_2))\varphi_m\big\|^2 \,\mathrm{d}\sigma \\
&\le \sum_{m=1}^{\infty} \int_{\tau_1}^{\tau_2} \big\|A^{\frac{\eta}{2}}\big\|^2 \big\|A^{\frac{\rho}{2}} E(\tau_2-\sigma) A^{\frac{r}{2}} G(Y(\tau_2))\varphi_m\big\|^2 \,\mathrm{d}\sigma \\
&\le C(s,r) \big\|A^{\frac{\eta}{2}}\big\|^2 \big\|A^{\frac{r}{2}} G(Y(\tau_2))\big\|_{L_2^0}^2 (\tau_2-\tau_1)^{\min(1,1+r-s)},
\end{aligned}$$

where $(\varphi_m)_{m\ge1}$ denotes an orthonormal basis of U_0. We also used that $1 - \rho = 1 - \max(0, s-r) = \min(1, 1+r-s)$. Finally, by Assumption 2.1 we conclude

$$S_1 \le C(p,r,s,A,G)\Big(1 + \sup_{\sigma\in[0,T]} \big\|A^{\frac{r}{2}} Y(\sigma)\big\|_{L^p(\Omega;H)}\Big) (\tau_2-\tau_1)^{\min(\frac{1}{2},\frac{1+r-s}{2})}.$$

This proves (3.1). For S_2 we first make use of the fact that $\|B\Phi\|_{L_2^0} \le \|B\|\|\Phi\|_{L_2^0}$ and then apply Lemma 3.2 *(i)* followed by (2.2) to get

$$\begin{aligned} S_2 &\le C(p,s,G)\Big\| \Big(\int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{-s} \|Y(\sigma)-Y(\tau_2)\|^2 \,\mathrm{d}\sigma \Big)^{\frac12} \Big\|_{L^p(\Omega;\mathbb{R})} \\ &= C(p,s,G)\Big(\Big\| \int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{-s} \|Y(\sigma)-Y(\tau_2)\|^2 \,\mathrm{d}\sigma \Big\|_{L^{p/2}(\Omega;\mathbb{R})} \Big)^{\frac12} \\ &\le C(p,s,G)\Big(\int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{-s} \|Y(\sigma)-Y(\tau_2)\|^2_{L^p(\Omega;H)} \,\mathrm{d}\sigma \Big)^{\frac12}. \end{aligned}$$

By the Hölder continuity of Y we arrive at

$$\begin{aligned} S_2 &\le C(p,s,G,C_\delta)\Big(\int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{-s+2\delta} \,\mathrm{d}\sigma \Big)^{\frac12} \\ &\le \frac{C(p,s,G,C_\delta)}{\sqrt{1+2\delta-s}} (\tau_2-\tau_1)^{\frac{1+2\delta-s}{2}}. \end{aligned}$$

This shows (3.2). Combination of the estimates for S_1 and S_2 yields (3.3) by using $(\tau_2-\tau_1)^{2\delta-r} \le T^{2\delta-r}$. □

Lemma 3.5. *Let $s \in [0, r+1]$, $p \ge 2$, and Y be a stochastic process on $[0,T]$ which maps into H with $\sup_{\sigma\in[0,T]} \|Y(\sigma)\|_{L^p(\Omega;H)} < \infty$. Then there exists a constant $C = C(r,s,F)$ such that, for all $\tau_1,\tau_2 \in [0,T]$ with $\tau_1 < \tau_2$,*

$$\begin{aligned} &\Big\| A^{\frac{s}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma) F(Y(\tau_2)) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\qquad \le C\Big(1 + \sup_{\sigma\in[0,T]} \|Y(\sigma)\|_{L^p(\Omega;H)}\Big) (\tau_2-\tau_1)^{\frac{1+r-s}{2}}. \end{aligned} \tag{3.4}$$

If, in addition, for some $\delta > 0$ there exists C_δ such that

$$\|Y(t_1)-Y(t_2)\|_{L^p(\Omega;H)} \le C_\delta |t_2-t_1|^\delta \text{ for all } t_1,t_2 \in [0,T],$$

then we also have, with $C = C(r,s,F,C_\delta)$, that

$$\begin{aligned} &\Big\| A^{\frac{s}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma) \big(F(Y(\tau_2)) - F(Y(\sigma))\big) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\qquad \le \frac{C}{1+r-s+2\delta} (\tau_2-\tau_1)^{\frac{1+r-s+2\delta}{2}}. \end{aligned} \tag{3.5}$$

In particular, with $C = C(T,\delta,r,s,F,C_\delta)$ it holds that

$$\begin{aligned} &\Big\| A^{\frac{s}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma) F(Y(\sigma)) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\qquad \le C\Big(1 + \sup_{\sigma\in[0,T]} \|Y(\sigma)\|_{L^p(\Omega;H)}\Big) (\tau_2-\tau_1)^{\frac{1+r-s}{2}}. \end{aligned} \tag{3.6}$$

Proof. As in the previous lemma the main idea is to use the Hölder continuity of Y to estimate the left-hand side in (3.6). We have

$$\begin{aligned} &\Big\| A^{\frac{s}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma) F(Y(\sigma)) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\quad \le \Big\| A^{\frac{s}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma) F(Y(\tau_2)) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\qquad + \Big\| A^{\frac{s}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma) \big(F(Y(\tau_2)) - F(Y(\sigma))\big) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)}. \end{aligned}$$

Therefore, if we show (3.4) and (3.5) then (3.6) follows immediately by using $(\tau_2-\tau_1)^\delta \leq T^\delta$.

For (3.4) first note that the random variable $A^{\frac{-1+r}{2}}F(X(\tau_2))$ takes values in H almost surely. Hence, we can apply Lemma 3.2 *(iv)*. Together with Assumption 2.2 this yields

$$\begin{aligned}
&\Big\| A^{\frac{s}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma)F(Y(\tau_2))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\quad \leq \Big\| A^{\frac{s+1-r}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma)A^{\frac{-1+r}{2}}F(Y(\tau_2))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\quad \leq C(r,s)(\tau_2-\tau_1)^{\frac{1+r-s}{2}} \big\| A^{\frac{-1+r}{2}}F(Y(\tau_2)) \big\|_{L^p(\Omega;H)} \\
&\quad \leq C(r,s,F)\Big(1+\sup_{\sigma\in[0,T]}\|Y(\sigma)\|_{L^p(\Omega;H)}\Big)(\tau_2-\tau_1)^{\frac{1+r-s}{2}}.
\end{aligned}$$

Finally, again by Lemma 3.2 and Assumption 2.2, we show (3.5):

$$\begin{aligned}
&\Big\| A^{\frac{s}{2}} \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma)\big(F(Y(\tau_2))-F(Y(\sigma))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\quad \leq \int_{\tau_1}^{\tau_2} \big\| A^{\frac{s+1-r}{2}}E(\tau_2-\sigma)A^{\frac{-1+r}{2}}\left(F(Y(\tau_2))-F(Y(\sigma))\right) \big\|_{L^p(\Omega;H)}\,\mathrm{d}\sigma \\
&\quad \leq C(r,s,F)\int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{\frac{r-s-1}{2}} \, \|Y(\tau_2)-Y(\sigma)\|_{L^p(\Omega;H)}\,\mathrm{d}\sigma \\
&\quad \leq C(r,s,F,C_\delta)\int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{\frac{r-s-1+2\delta}{2}}\,\mathrm{d}\sigma = \frac{2C(r,s,F,C_\delta)}{1+r-s+2\delta}(\tau_2-\tau_1)^{\frac{1+r-s+2\delta}{2}}.
\end{aligned}$$

This completes the proof. □

Now we are well prepared for the proof of Theorem 3.1.

Proof of Theorem 3.1. By taking norms in (2.1) we get, for $t\in[0,T]$,

$$\begin{aligned}
\left(\mathbf{E}\left[\|X(t)\|_{r+1}^p\right]\right)^{\frac{1}{p}} &= \|A^{\frac{r+1}{2}}X(t)\|_{L^p(\Omega;H)} \\
&\leq \|A^{\frac{r+1}{2}}E(t)X_0\|_{L^p(\Omega;H)} \\
&\quad + \Big\| A^{\frac{r+1}{2}} \int_0^t E(t-\sigma)F(X(\sigma))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\quad + \Big\| A^{\frac{r+1}{2}} \int_0^t E(t-\sigma)G(X(\sigma))\,\mathrm{d}W(\sigma) \Big\|_{L^p(\Omega;H)} \\
&=: I + II + III.
\end{aligned}$$

The first term is well-known from deterministic theory and can be estimated by

$$\|A^{\frac{r+1}{2}}E(t)X_0\|_{L^p(\Omega;H)} \leq \|A^{\frac{r+1}{2}}X_0\|_{L^p(\Omega;H)} < \infty,$$

since $X_0\colon \Omega \to \dot{H}^{r+1}$ by Assumption 2.3.

We recall that, by Theorem 1 in [7], the mild solution X is an $\dot{H}^r$-valued predictable stochastic process which is δ-Hölder continuous for any $0<\delta<\frac{1}{2}$ with respect to the norm $\|\cdot\|_{L^p(\Omega;H)}$. We choose $\delta := \frac{r+1}{4}$ so that $0 \leq \frac{r}{2} < \delta < \frac{1}{2}$. Hence, we can apply Lemmas 3.4 and 3.5 with $Y=X$.

For the second term we apply (3.6) with $\tau_1=0$, $\tau_2=t$, $s=r+1$ and $Y=X$. This yields

$$II \leq C\Big(1+\sup_{\sigma\in[0,T]}\|X(\sigma)\|_{L^p(\Omega;H)}\Big) < \infty.$$

For the last term we apply (3.3) with the same parameters as above:

$$III \leq C\Big(1 + \sup_{\sigma\in[0,T]} \big\|A^{\frac{r}{2}} X(\sigma)\big\|_{L^p(\Omega;H)}\Big) < \infty.$$

Note that $\sup_{\sigma\in[0,T]} \|X(\sigma)\|_{L^p(\Omega;H)} \leq \|A^{-\frac{r}{2}}\| \sup_{\sigma\in[0,T]} \|A^{\frac{r}{2}} X(\sigma)\|_{L^p(\Omega;H)}$ is finite because of Theorem 1 in [7]. □

4 Regularity in time

This section is devoted to the temporal regularity of the mild solution. Our main result is summarized in the following theorem. For the border case $s = r+1$ we refer to Theorem 4.2 below.

Theorem 4.1 (Temporal regularity). *Let $r \in [0,1)$, $p \in [2,\infty)$. Under the assumptions of Section 2 the unique mild solution X to* (1.1) *is Hölder continuous with respect to $\big(\mathbf{E}\,[\|\cdot\|_s^p]\big)^{\frac{1}{p}}$ and satisfies*

$$\sup_{t_1,t_2\in[0,T], t_1\neq t_2} \frac{\big(\mathbf{E}\,[\|X(t_1)-X(t_2)\|_s^p]\big)^{\frac{1}{p}}}{|t_1-t_2|^{\min(\frac{1}{2},\frac{1+r-s}{2})}} < \infty \tag{4.1}$$

for every $s \in [0, r+1)$.

Proof. Let $0 \leq t_1 < t_2 \leq T$ be arbitrary. By using the mild formulation (2.1) we get

$$\begin{aligned}
\big(\mathbf{E}\,[\|X(t_1)-X(t_2)\|_s^p]\big)^{\frac{1}{p}} &= \big\|A^{\frac{s}{2}}(X(t_1)-X(t_2))\big\|_{L^p(\Omega;H)} \\
&\leq \big\|A^{\frac{s}{2}}(E(t_1)-E(t_2))X_0\big\|_{L^p(\Omega;H)} \\
&\quad + \Big\|A^{\frac{s}{2}}\int_{t_1}^{t_2} E(t_2-\sigma)F(X(\sigma))\,\mathrm{d}\sigma\Big\|_{L^p(\Omega;H)} \\
&\quad + \Big\|A^{\frac{s}{2}}\int_0^{t_1} \big(E(t_2-\sigma)-E(t_1-\sigma)\big)F(X(\sigma))\,\mathrm{d}\sigma\Big\|_{L^p(\Omega;H)} \\
&\quad + \Big\|A^{\frac{s}{2}}\int_{t_1}^{t_2} E(t_2-\sigma)G(X(\sigma))\,\mathrm{d}W(\sigma)\Big\|_{L^p(\Omega;H)} \\
&\quad + \Big\|A^{\frac{s}{2}}\int_0^{t_1} \big(E(t_2-\sigma)-E(t_1-\sigma)\big)G(X(\sigma))\,\mathrm{d}W(\sigma)\Big\|_{L^p(\Omega;H)} \\
&=: T_1+T_2+T_3+T_4+T_5.
\end{aligned} \tag{4.2}$$

We estimate the five terms separately. The term T_1 is estimated by

$$\begin{aligned}
T_1 &= \Big\|A^{\frac{s-r-1}{2}}(I-E(t_2-t_1))A^{\frac{r+1}{2}}E(t_1)X_0\Big\|_{L^p(\Omega;H)} \\
&\leq C\big\|A^{\frac{r+1}{2}}X_0\big\|_{L^p(\Omega;H)}(t_2-t_1)^{\frac{1+r-s}{2}},
\end{aligned}$$

where we used Lemma 3.2 *(ii)* and Assumption 2.3.

As in the proof of Theorem 3.1 we choose the Hölder exponent $\delta := \frac{r+1}{4}$ so that $\frac{r}{2} < \delta < \frac{1}{2}$ and we can apply Lemmas 3.4 and 3.5 with $Y = X$.

The term T_2 coincides with (3.6) and we have

$$T_2 \leq C\Big(1+\sup_{\sigma\in[0,T]}\|X(\sigma)\|_{L^p(\Omega;H)}\Big)(t_2-t_1)^{\frac{1+r-s}{2}}.$$

For the third term we also apply Lemma 3.2 *(ii)* before we use (3.6):

$$\begin{aligned}
T_3 &= \left\| A^{\frac{s-r-1}{2}} \left(E(t_2-t_1) - I\right) A^{\frac{r+1}{2}} \int_0^{t_1} E(t_1-\sigma) F(X(\sigma))\,\mathrm{d}\sigma \right\|_{L^p(\Omega;H)} \\
&\le C(t_2-t_1)^{\frac{1+r-s}{2}} \left\| A^{\frac{r+1}{2}} \int_0^{t_1} E(t_1-\sigma) F(X(\sigma))\,\mathrm{d}\sigma \right\|_{L^p(\Omega;H)} \\
&\le C\Big(1 + \sup_{\sigma\in[0,T]} \|X(\sigma)\|_{L^p(\Omega;H)}\Big)(t_2-t_1)^{\frac{1+r-s}{2}}.
\end{aligned}$$

The fourth term is estimated analogously by using (3.3) instead of (3.6). We get

$$T_4 \le C\Big(1 + \sup_{\sigma\in[0,T]} \big\|A^{\frac{r}{2}} X(\sigma)\big\|_{L^p(\Omega;H)}\Big)(t_2-t_1)^{\min(\frac12, \frac{1+r-s}{2})}.$$

Finally, for the last term we use Lemma 3.3 first. Since, for $0 \le t_1 < t_2 \le T$ fixed, the function $[0,t_1] \ni \sigma \mapsto A^{\frac{s}{2}} \left(E(t_2-\sigma) - E(t_1-\sigma)\right) G(X(\sigma))$ is a predictable stochastic process Lemma 3.3 can be applied. Then, by using Lemma 3.2 *(ii)* with $\nu = \frac{1+r-s}{2}$ and Lemma 3.4 with $s = r+1$ we get

$$\begin{aligned}
T_5 &\le C \Big\| \Big(\int_0^{t_1} \big\| A^{\frac{s-r-1}{2}} (E(t_2-t_1) - I) A^{\frac{r+1}{2}} E(t_1-\sigma) G(X(\sigma)) \big\|^2_{L_2^0} \,\mathrm{d}\sigma \Big)^{\frac12} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&\le C(t_2-t_1)^{\frac{1+r-s}{2}} \Big(\Big\| \Big(\int_0^{t_1} \big\| A^{\frac{r+1}{2}} E(t_1-\sigma) G(X(t_1)) \big\|^2_{L_2^0} \,\mathrm{d}\sigma \Big)^{\frac12} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + \Big\| \Big(\int_0^{t_1} \big\| A^{\frac{r+1}{2}} E(t_1-\sigma) \big(G(X(\sigma)) - G(X(t_1)) \big) \big\|^2_{L_2^0} \,\mathrm{d}\sigma \Big)^{\frac12} \Big\|_{L^p(\Omega;\mathbb{R})} \Big) \\
&\le C(t_2-t_1)^{\frac{1+r-s}{2}} \Big(1 + \sup_{\sigma\in[0,T]} \big\| A^{\frac{r}{2}} X(\sigma) \big\|_{L^p(\Omega;H)} \Big).
\end{aligned}$$

Altogether, this proves (4.1) and the Hölder continuity of X with respect to the norm $\|A^{\frac{s}{2}} \cdot \|_{L^p(\Omega;H)}$ for all $s \in [0, r+1)$. ☐

The temporal regularity of X with respect to the norm $\big(\mathbf{E}[\|\cdot\|^p_{r+1}\big)^{\frac1p}$ is more involved. For the case $r = 0$ we can prove the following result.

Theorem 4.2. *Let $r = 0$ and $p \in [2,\infty)$. Under the assumptions of Section 2 the unique mild solution X to* (1.1) *is continuous with respect to $\big(\mathbf{E}\,[\|\cdot\|_1^p]\big)^{\frac1p}$.*

Before we begin the proof we analyze the continuity properties of the semigroup in the deterministic context.

Lemma 4.3. *Let $0 \le \tau_1 < \tau_2 \le T$. Then we have*

(i)

$$\lim_{\tau_2-\tau_1\to 0} \int_{\tau_1}^{\tau_2} \big\| A^{\frac12} E(\tau_2-\sigma) x \big\|^2 \,\mathrm{d}\sigma = 0 \quad \textit{for all } x \in H,$$

(ii)

$$\lim_{\tau_2-\tau_1\to 0} \Big\| A \int_{\tau_1}^{\tau_2} E(\tau_2-\sigma) x \,\mathrm{d}\sigma \Big\| = 0 \quad \textit{for all } x \in H.$$

Proof. As in the proof of Lemma 3.2 we use the orthogonal expansion of $x \in H$ with respect to the eigenbasis $(e_n)_{n\ge1}$ of the operator A. Thus, for *(i)* we get, as in the proof of Lemma 3.2 *(iii)*,

$$\int_{\tau_1}^{\tau_2} \Big\| A^{\frac12} E(\tau_2-\sigma) x \Big\|^2 \,\mathrm{d}\sigma = \frac12 \sum_{n=1}^{\infty} (x,e_n)^2 \left(1 - \mathrm{e}^{2\lambda_n(\tau_2-\tau_1)}\right).$$

We apply Lebesgue's dominated convergence theorem. Note that the sum is dominated by $\frac{1}{2}\|x\|^2$ for all $\tau_2 - \tau_1 \geq 0$. Moreover, for every $n \geq 1$ we have

$$\lim_{\tau_2-\tau_1\to 0} \big(1 - \mathrm{e}^{2\lambda_n(\tau_2-\tau_1)}\big)(x, e_n)^2 = 0.$$

Hence, Lebesgue's theorem gives us *(i)*. The same argument also yields the second case, since

$$\Big\| A \int_{\tau_1}^{\tau_2} E(\tau_2 - \sigma) x \,\mathrm{d}\sigma \Big\|^2 = \sum_{n=1}^{\infty} (x, e_n)^2 \big(1 - \mathrm{e}^{\lambda_n(\tau_2-\tau_1)}\big)^2.$$

The proof is complete. □

Proof of Theorem 4.2. We must show that $\lim_{t_2-t_1\to 0^+} \|X(t_2) - X(t_1)\|_{L^p(\Omega;\dot{H}^1)} = 0$ with either t_1 or t_2 fixed. As already demonstrated in the proof of Lemma 4.3 we use Lebesgue's dominated convergence theorem. Let $0 \leq t_1 < t_2 \leq T$. We consider again the terms T_i, $i = 1, \dots, 5$, in (4.2) but now with $s = 1$.

For T_1 continuity follows immediately: For almost every $\omega \in \Omega$ we get that $X_0(\omega) \in \dot{H}^1$. Thus, for every fixed $\omega \in \Omega$ with this property we have

$$\lim_{t_2-t_1\to 0} \|(E(t_2) - E(t_1)) A^{\frac{1}{2}} X_0(\omega)\| = 0$$

by the strong continuity of the semigroup. We also have that

$$\|(E(t_2) - E(t_1)) A^{\frac{1}{2}} X_0(\omega)\| \leq \|A^{\frac{1}{2}} X_0(\omega)\|,$$

where the latter is an element of $L^p(\Omega;\mathbb{R})$ as a function of $\omega \in \Omega$ by Assumption 2.3. Hence, Lebesgue's theorem is applicable and yields $\lim_{t_2-t_1\to 0} T_1 = 0$.

In order to treat the right and left limits simultaneously in the remaining terms, we compute the limits as $t_1 \to t_3$ and $t_2 \to t_3$ for fixed but arbitrary $t_3 \in [t_1, t_2]$.

In the case of T_2 we get

$$\begin{aligned} T_2 \leq{} & \Big\| A \int_{t_1}^{t_2} E(t_2 - \sigma) A^{-\frac{1}{2}} \big(F(X(\sigma)) - F(X(t_2))\big) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ & + \Big\| A \int_{t_1}^{t_2} E(t_2 - \sigma) A^{-\frac{1}{2}} \big(F(X(t_2)) - F(X(t_3))\big) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ & + \Big\| A \int_{t_1}^{t_2} E(t_2 - \sigma) A^{-\frac{1}{2}} F(X(t_3)) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)}. \end{aligned} \tag{4.3}$$

Because of (3.5), where we can choose $s = r + 1 = 1$ and $\delta = \frac{1}{4} > 0$, the limit $t_2 - t_1 \to 0$ of the first summand is 0. For the second summand in (4.3) we apply Lemma 3.2 *(iv)* with $\rho = 1$, and Assumption 2.2 with $r = 0$. Then we derive

$$\begin{aligned} & \Big\| A \int_{t_1}^{t_2} E(t_2 - \sigma) A^{-\frac{1}{2}} \big(F(X(t_2)) - F(X(t_3))\big) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ & \quad \leq C \big\| A^{-\frac{1}{2}} (F(X(t_2)) - F(X(t_3))) \big\|_{L^p(\Omega;H)} \\ & \quad \leq C \|X(t_2) - X(t_3)\|_{L^p(\Omega;H)} \end{aligned}$$

and the limit $t_2 \to t_3$ of this term vanishes by (4.1) with $s = 0$.

For the last summand in (4.3) we again apply Lemma 3.2 *(iv)* with $\rho = 1$ and obtain, for almost every $\omega \in \Omega$,

$$\begin{aligned} \Big\| A \int_{t_1}^{t_2} E(t_2 - \sigma) A^{-\frac{1}{2}} F(X(t_3, \omega)) \,\mathrm{d}\sigma \Big\| & \leq C \|A^{-\frac{1}{2}} F(X(t_3, \omega))\| \\ & \leq C \big(1 + \|X(t_3, \omega)\|\big), \end{aligned}$$

which belongs to $L^p(\Omega;\mathbb{R})$ for all $t_3 \in [0,T]$. By Lemma 4.3 *(ii)* it also holds that

$$\lim_{\substack{t_1\to t_3\\ t_2\to t_3}} \Big\| A \int_{t_1}^{t_2} E(t_2-\sigma)A^{-\frac{1}{2}} F(X(t_3,\omega))\,\mathrm{d}\sigma \Big\| = 0$$

for almost all $\omega \in \Omega$. Then Lebesgue's dominated convergence theorem yields that this term vanishes, which completes the proof for T_2.

Next, we take care of T_3, which is estimated by

$$\begin{aligned} T_3 \le{}& \Big\| A^{\frac{1}{2}} \int_0^{t_1} \big(E(t_2-\sigma)-E(t_1-\sigma)\big)\big(F(X(\sigma))-F(X(t_1))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &+ \Big\| A^{\frac{1}{2}} \int_0^{t_1} \big(E(t_2-\sigma)-E(t_1-\sigma)\big)\big(F(X(t_1))-F(X(t_3))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &+ \Big\| A^{\frac{1}{2}} \int_0^{t_1} \big(E(t_2-\sigma)-E(t_1-\sigma)\big)F(X(t_3))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)}. \end{aligned} \tag{4.4}$$

For the first summand in (4.4) we get by Lemma 3.2 *(ii)*

$$\begin{aligned} &\Big\| A^{\frac{1}{2}} \int_0^{t_1} \big(E(t_2-\sigma)-E(t_1-\sigma)\big)\big(F(X(\sigma))-F(X(t_1))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\le \int_0^{t_1} \big\| A^{-\frac{\eta}{2}}\big(E(t_2-t_1)-I\big)A^{\frac{1+\eta}{2}}E(t_1-\sigma)\big(F(X(\sigma))-F(X(t_1))\big)\big\|_{L^p(\Omega;H)}\,\mathrm{d}\sigma \\ &\le C(t_2-t_1)^{\frac{\eta}{2}} \int_0^{t_1} (t_1-\sigma)^{-\frac{2+\eta}{2}} \big\| A^{-\frac{1}{2}}\big(F(X(\sigma))-F(X(t_1))\big)\big\|_{L^p(\Omega;H)}\,\mathrm{d}\sigma, \end{aligned} \tag{4.5}$$

where $\eta \in (0,2]$. We continue the estimate by applying Assumption 2.2 and the Hölder continuity of X with exponent $\frac{1}{2}$ with respect to the norm $\|\cdot\|_{L^p(\Omega;H)}$ as it was shown in (4.1) with $s=0$. This gives

$$\begin{aligned} &\Big\| A \int_0^{t_1} \big(E(t_2-\sigma)-E(t_1-\sigma)\big)A^{-\frac{1}{2}}\big(F(X(\sigma))-F(X(t_1))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\quad\le C(t_2-t_1)^{\frac{\eta}{2}} \int_0^{t_1} (t_1-\sigma)^{-\frac{2+\eta-1}{2}}\,\mathrm{d}\sigma = C\frac{2}{1-\eta} t_1^{\frac{1-\eta}{2}} (t_2-t_1)^{\frac{\eta}{2}}. \end{aligned}$$

Therefore, in the limit $t_2 - t_1 \to 0$ this term is zero as long as $\eta \in (0,1)$.

For the second summand in (4.4) we apply Lemma 3.2 *(iv)* with $\rho = 1$ and get

$$\begin{aligned} &\Big\| A^{\frac{1}{2}} \int_0^{t_1} \big(E(t_2-\sigma)-E(t_1-\sigma)\big)F(X(t_1))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &= \Big\| A \int_0^{t_1} E(t_1-\sigma)\big(E(t_2-t_1)-I\big)A^{-\frac{1}{2}}F(X(t_1))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\le C \Big\| \big(E(t_2-t_1)-I\big)A^{-\frac{1}{2}}F(X(t_1)) \Big\|_{L^p(\Omega;H)} \\ &\le C \Big\| \big(E(t_2-t_1)-I\big)A^{-\frac{1}{2}}\big(F(X(t_1))-F(X(t_3))\big) \Big\|_{L^p(\Omega;H)} \\ &\quad + C \Big\| \big(E(t_2-t_1)-I\big)A^{-\frac{1}{2}}F(X(t_3)) \Big\|_{L^p(\Omega;H)}. \end{aligned}$$

By Assumption 2.2 and (4.1) it holds true that

$$\begin{aligned} &\Big\| \big(E(t_2-t_1)-I\big)A^{-\frac{1}{2}}\big(F(X(t_1))-F(X(t_3))\big) \Big\|_{L^p(\Omega;H)} \\ &\quad\le C\big\|X(t_1)-X(t_3)\big\|_{L^p(\Omega;H)} \le C|t_1-t_3|^{\frac{1}{2}}. \end{aligned}$$

Hence, this term vanishes in the limit $t_1 \to t_3$. Therefore, the proof for T_3 is complete, if we can show that

$$\begin{aligned}
&\lim_{\substack{t_1\to t_3\\ t_2\to t_3}} \big\| \big(E(t_2-t_1)-I\big)A^{-\frac{1}{2}}F(X(t_3))\big\|_{L^p(\Omega;H)} \\
&\quad = \lim_{\substack{t_1\to t_3\\ t_2\to t_3}} \Big\| A\int_{t_1}^{t_2} E(t_2-\sigma)A^{-\frac{1}{2}}F(X(t_3))\,\mathrm{d}\sigma\Big\|_{L^p(\Omega;H)} = 0.
\end{aligned}$$

This is true by an application of Lebesgue's dominated convergence theorem. In order to apply this theorem, we obtain a dominating function for almost every $\omega\in\Omega$ by

$$\big\| \big(E(t_2-t_1)-I\big)A^{-\frac{1}{2}}F(X(t_3,\omega))\big\| \le C\big(1+\|X(t_3,\omega)\|\big).$$

Further, Lemma 4.3 *(ii)* yields

$$\lim_{\substack{t_1\to t_3\\ t_2\to t_3}} \Big\| A\int_{t_1}^{t_2} E(t_2-\sigma)A^{-\frac{1}{2}}F(X(t_3,\omega))\,\mathrm{d}\sigma\Big\| = 0$$

for almost every $\omega\in\Omega$. Altogether, this shows $\lim_{\substack{t_1\to t_3\\ t_2\to t_3}} T_3 = 0$.

For T_4, one has to use Lemma 3.3, which yields

$$\begin{aligned}
T_4 &\le C\Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}}E(t_2-\sigma)G(X(\sigma))\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\Big\|_{L^p(\Omega;\mathbb{R})} \\
&\le C\Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}}E(t_2-\sigma)\big(G(X(\sigma))-G(X(t_2))\big)\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\Big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + C\Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}}E(t_2-\sigma)\big(G(X(t_2))-G(X(t_3))\big)\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\Big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + C\Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}}E(t_2-\sigma)G(X(t_3))\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\Big\|_{L^p(\Omega;\mathbb{R})}.
\end{aligned} \tag{4.6}$$

The limit $t_2-t_1\to 0$ of the first summand is 0 because of (3.2), where we again choose $s=1$ and $\delta=\frac{1}{4}>\frac{r}{2}=0$. As before, we discuss the simultaneous limits $t_1\to t_3$ and $t_2\to t_3$ for the remaining summands in (4.6).

By Lemma 3.2 *(iii)* it holds for the second summand in (4.6) that

$$\begin{aligned}
&\Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}}E(t_2-\sigma)\big(G(X(t_2))-G(X(t_3))\big)\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\Big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad = \Big\| \Big(\sum_{m=1}^{\infty}\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}}E(t_2-\sigma)\big(G(X(t_2))-G(X(t_3))\big)\varphi_m\big\|^2\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\Big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad \le C\big\| G(X(t_2))-G(X(t_3))\big\|_{L^p(\Omega;L_2^0)} \\
&\quad \le C\big\| X(t_2)-X(t_3)\big\|_{L^p(\Omega;H)}.
\end{aligned} \tag{4.7}$$

Consequently, this term also vanishes as $t_2\to t_3$ by (4.1).

Next we come to the third summand in (4.6). By Lemma 3.2 *(iii)* with $\rho=1$ and Assumption 2.1 we obtain for almost every $\omega\in\Omega$

$$\begin{aligned}
&\Big(\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}}E(t_2-\sigma)G(X(t_3,\omega))\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}} \\
&\quad = \Big(\sum_{m=1}^{\infty}\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}}E(t_2-\sigma)G(X(t_3,\omega))\varphi_m\big\|^2\,\mathrm{d}\sigma\Big)^{\frac{1}{2}} \\
&\quad \le C\Big(\sum_{m=1}^{\infty} \big\| G(X(t_3,\omega))\varphi_m\big\|^2\Big)^{\frac{1}{2}} \le C\big(1+\big\|X(t_3,\omega)\big\|\big),
\end{aligned}$$

where $(\varphi_m)_{m\geq 1}$ is an arbitrary orthonormal basis of U_0 and the last term belongs to $L^p(\Omega;\mathbb{R})$. In order to apply Lebesgue's Theorem it remains to discuss the pointwise limit. For this Lemma 4.3 *(i)* yields

$$\begin{aligned}&\lim_{\substack{t_1\to t_3\\ t_2\to t_3}}\int_{t_1}^{t_2}\big\|A^{\frac12}E(t_2-\sigma)G(X(t_3,\omega))\big\|^2_{L_2^0}\,\mathrm{d}\sigma\\ &\quad=\sum_{m=1}^{\infty}\lim_{\substack{t_1\to t_3\\ t_2\to t_3}}\int_{t_1}^{t_2}\big\|A^{\frac12}E(t_2-\sigma)G(X(t_3,\omega))\varphi_m\big\|^2\,\mathrm{d}\sigma=0.\end{aligned}$$

In fact, the interchanging of summation and taking the limit is justified by a further application of Lebesgue's Theorem. Altogether, this proves the desired result for T_4.

The estimate of T_5 works similarly as for T_3. We apply Lemma 3.3 and get

$$\begin{aligned}T_5\leq C\Big\|\Big(\int_0^{t_1}\big\|A^{\frac12}\big(E(t_2-\sigma)-E(t_1-\sigma)\big)\big(G(X(\sigma))-G(X(t_1))\big)\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac12}\Big\|_{L^p(\Omega;\mathbb{R})}\\ +C\Big\|\Big(\int_0^{t_1}\big\|A^{\frac12}\big(E(t_2-\sigma)-E(t_1-\sigma)\big)G(X(t_1))\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac12}\Big\|_{L^p(\Omega;\mathbb{R})}.\end{aligned}\tag{4.8}$$

By using a similar technique as for (4.5) the first summand in (4.8) is estimated by

$$\begin{aligned}&\Big\|\Big(\int_0^{t_1}\big\|A^{\frac12}\big(E(t_2-\sigma)-E(t_1-\sigma)\big)\big(G(X(\sigma))-G(X(t_1))\big)\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac12}\Big\|^2_{L^p(\Omega;\mathbb{R})}\\ &=\Big\|\int_0^{t_1}\big\|A^{\frac12}\big(E(t_2-\sigma)-E(t_1-\sigma)\big)\big(G(X(\sigma))-G(X(t_1))\big)\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big\|_{L^{p/2}(\Omega;\mathbb{R})}\\ &\leq C(t_2-t_1)^{\eta}\Big\|\int_0^{t_1}(t_1-\sigma)^{-1-\eta}\,\|G(X(\sigma))-G(X(t_1))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big\|_{L^{p/2}(\Omega;\mathbb{R})}\\ &\leq C(t_2-t_1)^{\eta}\int_0^{t_1}(t_1-\sigma)^{-1-\eta}\,\|X(\sigma)-X(t_1)\|^2_{L^p(\Omega;H)}\,\mathrm{d}\sigma\\ &\leq C(t_2-t_1)^{\eta}\frac{1}{1-\eta}t_1^{1-\eta}.\end{aligned}$$

For the first inequality we applied Lemma 3.2 *(i)* and *(ii)* with an arbitrary parameter $\eta\in(0,1)$. Then we used (2.2) and the $\frac12$-Hölder continuity of X. It follows that the summand vanishes in the limit $t_2-t_1\to 0$.

For the second summand in (4.8) it holds that

$$\begin{aligned}&\Big\|\Big(\int_0^{t_1}\big\|A^{\frac12}\big(E(t_2-\sigma)-E(t_1-\sigma)\big)G(X(t_1))\big\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac12}\Big\|_{L^p(\Omega;\mathbb{R})}\\ &=\Big\|\Big(\sum_{m=1}^{\infty}\int_0^{t_1}\big\|A^{\frac12}E(t_1-\sigma)\big(E(t_2-t_1)-I\big)G(X(t_1))\varphi_m\big\|^2\,\mathrm{d}\sigma\Big)^{\frac12}\Big\|_{L^p(\Omega;\mathbb{R})}\\ &\leq C\big\|\big(E(t_2-t_1)-I\big)G(X(t_1))\big\|_{L^p(\Omega;L_2^0)}\\ &\leq C\big\|G(X(t_1))-G(X(t_3))\big\|_{L^p(\Omega;L_2^0)}+\big\|\big(E(t_2-t_1)-I\big)G(X(t_3))\big\|_{L^p(\Omega;L_2^0)},\end{aligned}$$

where we used Lemma 3.2 *(iii)*. By Assumption 2.1 and (4.1) it holds that

$$\lim_{t_1\to t_3}\big\|G(X(t_1))-G(X(t_3))\big\|_{L^p(\Omega;L_2^0)}=0.$$

Since, as above, Lebesgue's dominated convergence theorem yields that

$$\lim_{\substack{t_1\to t_3\\ t_2\to t_3}}\big\|\big(E(t_2-t_1)-I\big)G(X(t_3))\big\|_{L^p(\Omega;L_2^0)}=0$$

the proof for T_5 is complete. This completes the proof of the theorem. □

Remark 4.4. *If one wants to extend the result of Theorem 4.2 to general $r \in [0,1)$ it is not hard to adapt the given arguments for all terms T_i, $i \in \{1,2,3,5\}$.*

For T_4, however, the situation is more delicate. This becomes apparent in the discussion of (4.7)*, which for $r \in (0,1)$ is equal to*

$$\begin{aligned}&\Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{1+r}{2}} E(t_2-\sigma)\big(G(X(t_2)) - G(X(t_3))\big)\big\|_{L_2^0} \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|^2_{L^p(\Omega;\mathbb{R})} \\ &\quad \le C \big\| A^{\frac{r}{2}}\big(G(X(t_2)) - G(X(t_3))\big)\big\|_{L^p(\Omega;L_2^0)}.\end{aligned}$$

Unlike the case $r = 0$ we do not want to assume that $x \mapsto A^{\frac{r}{2}}G(x)$ is globally Lipschitz continuous. Therefore, we cannot directly conclude that this term vanishes in the limit $t_2 \to t_3$.

However, for $p \in (2,\infty]$, it is enough to assume that the mapping $x \mapsto A^{\frac{r}{2}}G(x)$ is continuous. In order to show this we use a generalized version of Lebesgue's dominated convergence theorem (see [1, 1.23]), which allows a t_2-dependent family of dominating functions.

In fact, by the linear growth condition (2.3) *we obtain, for almost all $\omega \in \Omega$,*

$$\big\| A^{\frac{r}{2}}\big(G(X(t_2,\omega)) - G(X(t_3,\omega))\big)\big\|_{L_2^0} \le C\big(1 + \|A^{\frac{r}{2}}X(t_2,\omega)\| + \|A^{\frac{r}{2}}X(t_3,\omega)\|\big),$$

where, by (4.1)*, the family of dominating functions converges:*

$$\big(1 + \|A^{\frac{r}{2}}X(t_2)\| + \|A^{\frac{r}{2}}X(t_3)\|\big) \to \big(1 + 2\|A^{\frac{r}{2}}X(t_3)\|\big) \quad \text{in } L^p(\Omega,\mathbb{R}) \text{ as } t_2 \to t_3.$$

Further, by (4.1) *with $p \in (2,\infty)$ Kolmogorov's continuity theorem [6, Th. 3.3] yields that there exists a continuous version of the process $t \mapsto A^{\frac{r}{2}}X(t)$. Hence, under the additional assumption that $x \mapsto A^{\frac{r}{2}}G(x)$ is continuous we obtain, for almost all $\omega \in \Omega$,*

$$\lim_{t_2 \to t_3} \big\| A^{\frac{r}{2}}\big(G(X(t_2,\omega)) - G(X(t_3,\omega))\big)\big\|_{L_2^0} = 0.$$

By the generalized version of the dominated convergence theorem (see [1, 1.23]) we conclude that the unique mild solution X to (1.1) *is continuous with respect to $\big(\mathbf{E}[\|\cdot\|^p_{r+1}]\big)^{\frac{1}{p}}$.*

The case $p = 2$ remains as an open problem.

5 Additive noise and optimal regularity

In this section we briefly review the assumptions and our results in the case of additive noise, that is, we consider the case where $G \in L_2^0$ is independent of x. Then the SPDE (1.1) has the form

$$\begin{aligned}\mathrm{d}X(t) + [AX(t) + F(X(t))]\,\mathrm{d}t &= G\,\mathrm{d}W(t), \quad \text{for } 0 \le t \le T,\\ X(0) &= X_0.\end{aligned} \tag{5.1}$$

For related regularity results in this special case we refer to [6, Ch. 5].

Since now G is a fixed bounded linear operator Assumption 2.1 is simplified to

Assumption 5.1 (Additive noise)**.** *The Hilbert-Schmidt operator G satisfies*

$$\|G\|_{L^0_{2,r}} = \|A^{\frac{r}{2}}G\|_{L_2^0} < \infty. \tag{5.2}$$

Recall that the covariance operator Q of the Wiener process W is incorporated into the norm $\|\cdot\|_{L_2^0}$. If, for example, $H = U$ and G is the identity $I\colon H \to H$, then (5.2) reads as follows

$$\|I\|_{L^0_{2,r}} = \sum_{m=1}^{\infty} \big\| A^{\frac{r}{2}} Q^{\frac{1}{2}} \varphi_m \big\|^2 < \infty,$$

where $(\varphi_m)_{m\geq 1}$ denotes an arbitrary orthonormal basis of the Hilbert space H. This is a common assumption on the covariance operator Q (see [6]). In particular, for $r = 0$ this condition becomes $\|I\|_{L_2^0} = \mathrm{Tr}(Q) < \infty$.

Our result for additive noise is summarized by the following corollary.

Corollary 5.2 (Additive noise). *If the Assumptions 2.2, 2.3 and 5.1 hold for some $r \in [0,1]$, $p \in [2,\infty)$, then the unique mild solution $X\colon [0,T]\times\Omega \to H$ to (5.1) takes values in $\dot{H}^{r+1}$. Moreover, for every $s \in [0, r+1]$, the solution process is continuous with respect to $\big(\mathbf{E}\left[\|\cdot\|_s^p\right]\big)^{\frac{1}{p}}$ and fulfills*

$$\sup_{t_1,t_2\in[0,T],t_1\neq t_2} \frac{\big(\mathbf{E}\left[\|X(t_1)-X(t_2)\|_s^p\right]\big)^{\frac{1}{p}}}{|t_1-t_2|^{\min(\frac{1}{2},\frac{r+1-s}{2})}} < \infty.$$

We stress that the case $r = 1$ is now included. In fact, the only place, where $r < 1$ is required, is the estimate (3.2) and its consequences. But in the case of additive noise the left-hand side of this estimate is equal to zero and we avoid this problem. The same is true for the proof of continuity, where the critical terms vanish analogously (c.f. the proof of Theorems 4.1 and 4.2).

In order to motivate why we speak of optimal spatial regularity we conclude this section with the following example, where our results turn out to be sharp. Without loss of generality we restrict our discussion to the case $p = 2$. For $p > 2$ one may use the results on the optimal regularity of the stochastic convolution from [18] or [2].

Example 5.3. *Let $H = L^2(0,1)$ be the space of all square integrable real-valued functions which are defined on the unit interval $(0,1)$. Further, assume that $-A$ is the Laplacian with Dirichlet boundary conditions. In this situation the orthonormal eigenbasis $(e_k)_{k\geq 1}$ of $-A$ is explicitly known to be*

$$\lambda_k = k^2\pi^2 \quad \textit{and} \quad e_k(y) = \sqrt{2}\sin(k\pi y) \quad \textit{for all } k \geq 1,\ y \in (0,1).$$

Consider the SPDE

$$\begin{aligned} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t &= G\,\mathrm{d}W(t), \quad \textit{for } 0 \leq t \leq T, \\ X(0) &= 0. \end{aligned} \tag{5.3}$$

We choose the operator G to be the identity on H, and W to be a Q-Wiener process on H, where the covariance operator $Q\colon H \to H$ is given by

$$Qe_1 = 0, \quad Qe_k = \frac{1}{k\log(k)^2}e_k \quad \textit{for all } k \geq 2.$$

Then we have

$$\|G\|_{L^0_{2,r}} = \sum_{k=2}^{\infty}\left\|A^{\frac{r}{2}}Q^{\frac{1}{2}}e_k\right\|^2 = \sum_{k=2}^{\infty}\lambda_k^r\frac{1}{k\log(k)^2} = \pi^{2r}\sum_{k=2}^{\infty}\frac{k^{2r}}{k\log(k)^2}.$$

Since this series converges only with $r = 0$, Assumption 5.1 is satisfied only for $r = 0$. Corollary 5.2 yields that the mild solution X to (5.3) takes values in $\dot{H}^1$. In the following we show that this result cannot be improved.

In our example the mild formulation (2.1) reads

$$X(t) = \int_0^t E(t-\sigma)G\,\mathrm{d}W(\sigma).$$

Hence, by the Itô-isometry for the stochastic integral we have

$$\begin{aligned}\mathbf{E}\big[\big\|A^{\frac{1+r}{2}}X(t)\big\|^2\big] &= \int_0^t \big\|A^{\frac{1+r}{2}}E(t-\sigma)G\big\|_{L_2^0}^2 \,\mathrm{d}\sigma \\ &= \int_0^t \sum_{k=2}^{\infty} \lambda_k^{1+r}\,\mathrm{e}^{-2\lambda_k(t-\sigma)}\frac{1}{k\log(k)^2}\,\mathrm{d}\sigma \\ &= \frac{1}{2}\sum_{k=2}^{\infty}\lambda_k^r\big(1-\,\mathrm{e}^{-2\lambda_k t}\big)\frac{1}{k\log(k)^2} \\ &\geq \frac{1}{2}\pi^{2r}\big(1-\,\mathrm{e}^{-2\lambda_1 t}\big)\sum_{k=2}^{\infty}\frac{k^{2r}}{k\log(k)^2} = \infty \quad \textit{for all } t>0,\ r>0.\end{aligned}$$

Thus, $X(t) \notin L^2(\Omega;\dot{H}^{1+r})$ *for* $r > 0$*.*

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