DOI: 10.1214/EJP.v14-637

Electronic Journal of Probability

Vol. 14 (2009), Paper no. 31, pages 865–894.

Journal URL
http://www.math.washington.edu/~ejpecp/

On the asymptotic behaviour of increasing self-similar Markov processes*

Maria Emilia Caballero
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, CU
04510 México, D.F. México
E-mail: marie@matem.unam.mx

Víctor Rivero
Centro de Investigación en Matemáticas A.C.
Calle Jalisco s/n
36240 Guanajuato, Gto. México.
E-mail: rivero@cimat.mx

Abstract

It has been proved by Bertoin and Caballero [8] that a $1/\alpha$-increasing self-similar Markov process X is such that $t^{-1/\alpha}X(t)$ converges weakly, as $t \to \infty$, to a degenerate random variable whenever the subordinator associated to it via Lamperti's transformation has infinite mean. Here we prove that $\log(X(t)/t^{1/\alpha})/\log(t)$ converges in law to a non-degenerate random variable if and only if the associated subordinator has Laplace exponent that varies regularly at 0. Moreover, we show that $\liminf_{t\to\infty} \log(X(t))/\log(t) = 1/\alpha$, a.s. and provide an integral test for the upper functions of $\{\log(X(t)), t \geq 0\}$. Furthermore, results concerning the rate of growth of the random clock appearing in Lamperti's transformation are obtained. In particular, these allow us to establish estimates for the left tail of some exponential functionals of subordinators. Finally, some of the implications of these results in the theory of self-similar fragmentations are discussed.

Key words: Dynkin-Lamperti Theorem, Lamperti's transformation, law of iterated logarithm, subordinators, weak limit theorem.

AMS 2000 Subject Classification: Primary 60G18.

Submitted to EJP on November 15, 2007, final version accepted February 10, 2009.

*This research was funded by CONCyTEG (Council of Science and Technology of the state of Guanajuato, México) and partially by the project PAPIITT-IN120605, UNAM

1 Introduction

Let $X = \{X(t), t \geq 0\}$ be an increasing positive self-similar Markov process with càdlàg paths, (pssMp) viz. X is a $]0,\infty[$ valued strong Markov process that fulfills the scaling property: there exists an $\alpha > 0$ such that for every $c > 0$

$$\left(\{cX(tc^{-\alpha}), t \geq 0\}, \mathbb{P}_x\right) \stackrel{\text{Law}}{=} \left(\{X(t), t \geq 0\}, \mathbb{P}_{cx}\right), \qquad x \in]0,\infty[,$$

where $\mathbb{P}_y$ denotes the law of the process X with starting point $y > 0$. We will say that X is an increasing $1/\alpha$-pssMp.

A stable subordinator of parameter $\beta \in]0,1[$ is a classical example of increasing pssMp and its index of self-similarity is $1/\beta$. Another example of this class of processes appears in the theory of extremes. More precisely, let $Y_\beta = \{Y_\beta(t), t \geq 0\}$ be a stable Lévy process of parameter $\beta \in]0,2[$, with non-negative jumps and so its Lévy measure has the form $a\beta x^{-1-\beta}$, $x > 0$, for some $a > 0$. The increasing process X_β defined as

$$X_\beta(t) := \text{the largest jump in } [0,t] \text{ of the process } Y_\beta, \quad t \geq 0,$$

has the strong Markov property because the jumps of Y_β form a Poisson point process with intensity measure $a\beta x^{-1-\beta}$, $x > 0$, and inherits the scaling property from Y_β, with a self-similarity index $1/\beta$. In fact, the processes X_β belongs to the class of extremal process whose Q-function has the form $Q(x) = cx^{-b}$, for $x > 0$ and $Q(x) = \infty$ otherwise, for some $c, b > 0$, see e.g. [21] for further results concerning this and other related processes. In our specific example $c = a$ and $0 < b = \beta < 2$. Furthermore, according to [21] Proposition 3 an extremal process with Q function as above with $b \geq 2$, which is an increasing pssMp, can be constructed by taking the largest jump in $[0,t]$ of the process $(Y_{1/b})^{1/2b}$ for $t \geq 0$. Some asymptotic results for these processes were obtained in [22] Section 5.

Another example of an increasing pssMp is that of the reciprocal of the process of a tagged fragment which appeared recently in the theory of self-similar fragmentations, see [7] Section 3.3 or Section 7 below where some of our main results are applied to this class of processes.

It is well known that by means of a transformation due to Lamperti [20] any increasing positive self-similar Markov processes can be transformed into a subordinator and vice-versa. By a subordinator we mean a càdlàg real valued process with independent and stationary increments, that is, a Lévy process with increasing paths. To be more precise about Lamperti's transformation, given an increasing $1/\alpha$-pssMp X we define a new process ξ by

$$\xi_t = \log\left(\frac{X(\gamma_t)}{X(0)}\right), \qquad t \geq 0,$$

where $\{\gamma_t, t \geq 0\}$ denotes the inverse of the additive functional

$$\int_0^t (X(s))^{-\alpha} \mathrm{d}s, \quad t \geq 0.$$

The process $\xi = \{\xi_t, t \geq 0\}$ defined this way is a subordinator started from 0, and we denote by $\mathbf{P}$ its law. Reciprocally, given a subordinator ξ and $\alpha > 0$, the process constructed in the following way is an increasing $1/\alpha$-pssMp. For $x > 0$, we denote by $\mathbb{P}_x$ the law of the process

$$x \exp\{\xi_{\tau(t/x^\alpha)}\}, \qquad t \geq 0,$$

where $\{\tau(t), t \geq 0\}$ is the inverse of the additive functional

$$C_t := \int_0^t \exp\{\alpha \xi_s\} \mathrm{d}s, \qquad t \geq 0. \tag{1}$$

So for any $x > 0$, $\mathbb{P}_x$, is the law of an $1/\alpha$-pssMp started from $x > 0$. We will refer to any of these transformations as Lamperti's transformation.

In a recent paper Bertoin and Caballero [8] studied the problem of existence of entrance laws at 0+ for an increasing pssMp. They established that if the subordinator $(\xi, \mathbf{P})$ (which is assumed to be non arithmetic) associated to $(X, \mathbb{P})$ via Lamperti's transformation has finite mean $m := \mathbf{E}(\xi_1) < \infty$, then there exists a non-degenerate probability measure $\mathbb{P}_{0+}$ on the space of paths that are right continuous and left limited which is the limit in the sense of finite dimensional laws of $\mathbb{P}_x$ as $x \to 0+$. Using the scaling and Markov properties it is easy to see that the latter result is equivalent to the weak convergence of random variables

$$t^{-1/\alpha} X(t) \xrightarrow[t \to \infty]{\text{Law}} Z, \tag{2}$$

where X is started at 1 and Z is a non-degenerate random variable. The law of Z will be denoted by μ, and it is the probability measure defined by

$$\mu(f) := \mathbb{E}_{0+}\left(f(X(1))\right) = \frac{1}{\alpha m} \mathbf{E}\left(f\left(\left(\frac{1}{I}\right)^{1/\alpha}\right) \frac{1}{I}\right), \tag{3}$$

for any measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$; where I is the exponential functional

$$I := \int_0^\infty \exp\{-\alpha \xi_s\} \mathrm{d}s,$$

associated to the subordinator ξ; see the Remark on page 202 in [8], and [10] where the analogous result for more general self-similar Markov processes is obtained. The fact that I is finite a.s. is a consequence of the fact that ξ_t tends to infinity as $t \to \infty$ at least with a linear rate owing to the law of large numbers for subordinators, see e.g. [11] Theorem 1. Besides, it is important to mention that in [8] the case of an arithmetic subordinator was not studied for sake of brevity. However, the analogous result can be obtained with the same techniques but using instead the arithmetic renewal theorem and tacking limits over well chosen sequences.

The following result complements the latter.

Proposition 1. *Let $\{X(t), t \geq 0\}$ be an increasing $1/\alpha$-pssMp. Assume that the subordinator ξ, associated to X via Lamperti's transformation is non arithmetic and has finite mean, $m = \mathbf{E}(\xi_1) < \infty$. Then*

$$\frac{1}{\log(t)} \int_0^t f(s^{-1/\alpha} X(s)) \frac{\mathrm{d}s}{s} \xrightarrow[t \to \infty]{} \mu(f), \quad \mathbb{P}_{0+}\text{-a.s.}$$

for every function $f \in L^1(\mu)$. Furthermore,

$$\frac{\log(X(t))}{\log(t)} \xrightarrow[t \to \infty]{} 1/\alpha, \qquad \mathbb{P}_1\text{-a.s.}$$

In fact, the results of the previous proposition are not new.

The first assertion can be obtained as a consequence of an ergodic theorem for self-similar processes due to Csáki and Földes [17], and the second assertion has been obtained in [6]. However, we provide a proof of these results for ease of reference.

A study of the short and large time behaviour of X under $\mathbb{P}_{0+}$ has been done in [22] and [16].

In [8] the authors also proved that if the subordinator $(\xi, \mathbf{P})$ has infinite mean then the convergence in law in (2) still holds but Z is a degenerate random variable equal to ∞ a.s. The main purpose of this paper is to study in this setting the rate at which $t^{-1/\alpha}X(t)$ tends to infinity as the time grows.

Observe that the asymptotic behaviour of $(X, \mathbb{P})$ at large times is closely related to the large jumps of it, because it is so for the subordinator $(\xi, \mathbf{P})$. So, for our purposes it will be important to have some information about the large jumps of $(\xi, \mathbf{P})$ or equivalently about those of $(X, \mathbb{P})$. Such information will be provided by the following assumption. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the Laplace exponent of $(\xi, \mathbf{P})$, viz.

$$\phi(\lambda) := -\log\left(\mathbf{E}(e^{-\lambda\xi_1})\right) = d\lambda + \int_{]0,\infty[} (1 - e^{-\lambda x})\Pi(\mathrm{d}x), \qquad \lambda \geq 0,$$

where $d \geq 0$ and Π is a measure over $]0, \infty[$ such that $\int (1 \wedge x)\Pi(\mathrm{d}x) < \infty$, which are called the drift term and Lévy measure of ξ, respectively. We will assume that ϕ is regularly varying at 0, i.e.

$$\lim_{\lambda \to 0} \frac{\phi(c\lambda)}{\phi(\lambda)} = c^{\beta}, \qquad c > 0,$$

for some $\beta \in [0, 1]$, which will be called the index of regular variation of ϕ. In the case where $\beta = 0$, it is said that the function ϕ is slowly varying. It is known that ϕ is regularly varying at 0 with an index $\beta \in]0, 1[$ if and only if the right tail of the Lévy measure Π is regularly varying with index $-\beta$, viz.

$$\lim_{x \to \infty} \frac{\Pi]cx, \infty[}{\Pi]x, \infty[} = c^{-\beta}, \qquad c > 0. \tag{4}$$

Well known examples of subordinators whose Laplace exponent is regularly varying are the stable subordinators and the gamma subordinator. A quite rich but less known class of subordinators whose Laplace exponent is regularly varying at 0 is that of tempered stable subordinators, see [24] for background on tempered stable laws. In this case, the drift term is equal to 0, and the Lévy measure Π_δ has the form $\Pi_\delta(\mathrm{d}x) = x^{-\delta-1}q(x)\mathrm{d}x$, $x > 0$, where $\delta \in]0, 1[$ and $q : \mathbb{R}^+ \to \mathbb{R}^+$ is a completely monotone function such that $\int_0^1 x^{-\delta}q(x)\mathrm{d}x < \infty$. By l'Hôpital's rule, for Π_δ to be such that the condition (4) is satisfied it is necessary and sufficient that q be regularly varying at infinity with index $-\lambda$ and such that $0 < \lambda + \delta < 1$.

We have all the elements to state our first main result.

Theorem 1. *Let $\{X(t), t \geq 0\}$ be a positive $1/\alpha$-self-similar Markov process with increasing paths. The following assertions are equivalent:*

(i) The subordinator ξ, associated to X via Lamperti's transformation, has Laplace exponent $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, which is regularly varying at 0 with an index $\beta \in [0, 1]$.

(ii) Under $\mathbb{P}_1$ the random variables $\left\{\log(X(t)/t^{1/\alpha})/\log(t), t > 1\right\}$ converge weakly as $t \to \infty$ towards a random variable V.

(iii) *For any $x > 0$, under $\mathbb{P}_x$ the random variables $\left\{\log(X(t)/t^{1/\alpha})/\log(t), t > 1\right\}$ converge weakly as $t \to \infty$ towards a random variable V.*

In this case, the law of V is determined in terms of the value of β as follows: $V = 0$ a.s. if $\beta = 1$; $V = \infty$, a.s. if $\beta = 0$, and if $\beta \in]0, 1[$, its law has a density given by

$$\frac{\alpha^{1-\beta} 2^{\beta} \sin(\beta\pi)}{\pi} v^{-\beta}(2 + \alpha v)^{-1} \mathrm{d}v, \qquad v > 0.$$

We will see in the proof of Theorem 1 that under the assumption of regular variation of ϕ at 0, the asymptotic behaviour of $X(t)$ is quite irregular. Namely, it is not of order t^a for any $a > 0$, see Remark 5. This justifies our choice of smoothing the paths of X by means of the logarithm.

Observe that the case where the underlying subordinator is arithmetic is not excluded in Theorem 1. This is possible as the proof of this Theorem uses among other tools the Dynkin-Lamperti Theorem for subordinators which in turn does not exclude the case of arithmetic subordinators, see e.g. [4] Section 3.1.2, and Corollary 1 in [23]. Moreover, we can find some similarities between the Dynkin-Lamperti Theorem and our Theorem 1. For example, the conclusions of the former hold if and only if one of the conditions of the latter hold; both theorems describe the asymptotic behaviour of ξ at a sequence of stopping times, those appearing in the former are the first passage times above a barrier, while in the latter they are given by $\tau(\cdot)$. It shall be justified in Section 8 that in fact both families of stopping times bear similar asymptotic behaviour.

The equivalence between (ii) and (iii) in Theorem 1 is a simple consequence of the scaling property. Another simple consequence of the scaling property is that: *if there exists a normalizing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $x > 0$, under $\mathbb{P}_x$, the random variables $\left\{\log(X(t)/t^{1/\alpha})/h(t), t > 0\right\}$ converge weakly as $t \to \infty$ towards a non-degenerate random variable V whose law does not depend on x, then the function h is slowly varying at infinity.* Hence, in the case where the Laplace exponent is not regularly varying at 0 it is natural to ask if there exists a function h that grows faster or slower than $\log(t)$ and such that $\log(X(t)/t^{1/\alpha})/h(t)$ converges in law to a non-degenerate random variable. The following result answers this question negatively.

Theorem 2. *Assume that the Laplace exponent of ξ is not regularly varying at 0 with a strictly positive index and let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function that varies slowly at ∞. If $h(t)/\log(t)$ tends to 0 or ∞, as $t \to \infty$, and the law of $\log(X(t)/t^{1/\alpha})/h(t)$, under $\mathbb{P}_1$, converges weakly to a real valued random variable, as $t \to \infty$, then the limiting random variable is degenerate.*

Now, observe that in the case where the underlying subordinator has finite mean, Proposition 1 provides some information about the rate of growth of the random clock $(\tau(t), t \geq 0)$ because it is equal to the additive functional $\int_0^t (X(s))^{-\alpha} \mathrm{d}s$, $t \geq 0$ under $\mathbb{P}_1$. In the case where ϕ is regularly varying at 0 with an index in $[0, 1[$ it can be verified that

$$\frac{1}{\log(t)} \int_0^t (X(s))^{-\alpha} \mathrm{d}s \xrightarrow[t\to\infty]{} 0, \qquad \mathbb{P}_1\text{-a.s.}$$

see Remark 4 below. Nevertheless, in the latter case we can establish an estimate of the Darling-Kac type for the functional $\int_0^t (X(s))^{-\alpha} \mathrm{d}s$, $t \geq 0$, which provides some insight about the rate of growth of the random clock. This is the content of the following result.

Proposition 2. *The following conditions are equivalent:*

(i) ϕ *is regularly varying at* 0 *with an index* $\beta \in [0,1]$.

(ii) The law of $\phi\left(\frac{1}{\log(t)}\right)\int_0^t (X(s))^{-\alpha}\mathrm{d}s$, *under* $\mathbb{P}_1$, *converges in distribution, as* $t \to \infty$, *to a random variable* $\alpha^{-\beta}W$, *where* W *is a random variable that follows a Mittag-Leffler law of parameter* $\beta \in [0,1]$.

(iii) For some $\beta \in [0,1]$, $\mathbb{E}_1\left(\left(\phi\left(\frac{1}{\log(t)}\right)\int_0^t (X(s))^{-\alpha}\mathrm{d}s\right)^n\right)$ *converges towards* $\alpha^{-\beta n} n!/\Gamma(1+n\beta)$, *for* $n = 0, 1, \dots$, *as* $t \to \infty$.

Before continuing with our exposition about the asymptotic results for $\log(X)$ let us make a digression to remark that this result has an interesting consequence for a class of random variables introduced by Bertoin and Yor[9] that we explain next. Recently, they proved that there exists a $\mathbb{R}^+$ valued random variable R_ϕ associated to $I_\phi := \int_0^\infty \exp\{-\alpha\xi_s\}\mathrm{d}s$, such that

$$R_\phi I_\phi \stackrel{\text{Law}}{=} \mathbf{e}(1), \quad \text{where } \mathbf{e}(1) \text{ follows an exponential law of parameter } 1.$$

The law of R_ϕ is completely determined by its entire moments, which in turn are given by

$$\mathbf{E}(R_\phi^n) = \prod_{k=1}^n \phi(\alpha k), \qquad \text{for } n = 1, 2, \dots$$

Corollary 1. *Assume that* ϕ *is regularly varying at* 0 *with index* $\beta \in [0,1]$. *The following estimates*

$$\mathbf{E}\left(1_{\{R_\phi > s\}}\frac{1}{R_\phi}\right) \sim \frac{1}{\alpha^\beta \Gamma(1+\beta)\phi(1/\log(1/s))}, \quad \mathbf{P}(R_\phi < s) = o\left(\frac{s}{\alpha^\beta \Gamma(1+\beta)\phi(1/\log(1/s))}\right),$$

as $s \to 0$, *hold. If furthermore, the function* $\lambda/\phi(\lambda)$, $\lambda > 0$, *is the Laplace exponent of a subordinator then*

$$\mathbf{E}\left(1_{\{I_\phi > s\}}\frac{1}{I_\phi}\right) \sim \frac{\alpha^\beta \log(1/s)\phi(1/\log(1/s))}{\Gamma(2-\beta)}, \quad \mathbf{P}(I_\phi < s) = o\left(\frac{\alpha^\beta s \log(1/s)\phi\left(1/\log(1/s)\right)}{\Gamma(2-\beta)}\right),$$

as $s \to 0$.

It is known, [25] Theorem 2.1, that a Laplace exponent ϕ is such that the function $\lambda/\phi(\lambda)$ is the Laplace exponent of a subordinator if and only if the renewal measure of ξ has a decreasing density; see also [19] Theorem 2.1 for a sufficient condition on the Lévy measure for this to hold. The relevance of the latter estimates relies on the fact that in the literature about the subject there are only a few number of subordinators for which estimates for the left tail of I_ϕ are known.

In the following theorem, under the assumption that (i) in Theorem 1 holds, we obtain a law of iterated logarithm for $\{\log(X(t)), t \geq 0\}$ and provide an integral test to determine the upper functions for it.

Theorem 3. *Assume that the condition (i) in Theorem 1 above holds with* $\beta \in]0,1[$. *We have the following estimates of* $\log(X(t))$.

(a) $\liminf_{t\to\infty} \frac{\log(X(t))}{\log(t)} = 1/\alpha, \qquad \mathbb{P}_1\text{-a.s.}$

(b) Let $g :]e,\infty[\to \mathbb{R}^+$ *be the function defined by*

$$g(t) = \frac{\log\left(\log(t)\right)}{\varphi\left(t^{-1}\log\left(\log(t)\right)\right)}, \qquad t > e,$$

with φ *the inverse of* ϕ. *For* $f : \mathbb{R}^+ \to (0,\infty)$ *increasing function with positive increase, i.e.* $0 < \liminf_{t\to\infty} \frac{f(t)}{f(2t)}$, *we have that*

$$\limsup_{t\to\infty} \frac{\log(X(t))}{f\left(\log(t)\right)} = 0, \qquad \mathbb{P}_1\text{-a.s.} \tag{5}$$

whenever

$$\int^{\infty} \phi\left(1/f(g(t))\right) \mathrm{d}t < \infty, \tag{6}$$

and

$$\limsup_{t\to\infty} \frac{\log(X(t))}{f\left(\log(t)\right)} = \infty, \qquad \mathbb{P}_1\text{-a.s.} \tag{7}$$

whenever, for some $\varepsilon > 0$

$$\int^{\infty} \phi\left(1/f((g(t))^{1+\varepsilon})\right) \mathrm{d}t = \infty. \tag{8}$$

Remark 1. Observe that in the case where the Laplace exponent varies regularly at 0 with index 1, then Theorem 1 implies that

$$\frac{\log(X(t))}{\log(t)} \xrightarrow[t\to\infty]{\text{Probability}} 1/\alpha.$$

Proposition 1 says that the finiteness of the mean of the underlying subordinator is a sufficient condition for this to hold. A question that remains open is to show whether this condition is also necessary.

Remark 2. In the case where ϕ is slowly varying at 0, Theorem 1 implies that

$$\frac{\log(X(t))}{\log(t)} \xrightarrow[t\to\infty]{\text{Probability}} \infty.$$

In the proof of Theorem 2 it will be seen that if $h : \mathbb{R}^+ \to]0,\infty[$ is a function such that $\log(t)/h(t) \to 0$ as $t \to \infty$, then

$$\frac{\log(X(t))}{h(t)} \xrightarrow[t\to\infty]{\text{Probability}} 0,$$

which is a weak analogue of Theorem 3.

Remark 3. Observe that the local behaviour of X, when started at a strictly positive point, is quite similar to that of the underlying subordinator. This is due to the elementary fact

$$\frac{\tau(t)}{t} \xrightarrow[t\to 0+]{} 1, \qquad \mathbb{P}_1\text{-a.s.}$$

So, for short times the behaviour of ξ is not affected by the time change, which is of course not the case for large times. Using this fact and known results for subordinators, precisely Theorem 3 in [3] Section III.3, it is straightforward to prove the following Proposition which is the short time analogue of our Theorem 1. We omit the details of the proof.

Proposition 3. *Let $\{X(t), t \geq 0\}$ be a positive $1/\alpha$-self-similar Markov process with increasing paths. The following conditions are equivalent:*

(i) The subordinator ξ, associated to X via Lamperti's transformation, has Laplace exponent $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, which is regularly varying at ∞ with an index $\beta \in]0,1[$.

(ii) There exists an increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that under $\mathbb{P}_1$ the random variables $\{h(t)\log(X(t)), t > 0\}$ converge weakly as $t \to 0$ towards a non-degenerate random variable

(iii) There exists an increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that under $\mathbb{P}_1$ the random variables $h(t)(X(t)-1), t > 0$ converge weakly as $t \to 0$ towards a non-degenerate random variable

In this case, the limit law is a stable law with parameter β, and $h(t) \sim \varphi(1/t)$, as $t \to 0$, with φ the inverse of ϕ.

It is also possible to obtain a short time analogue of Theorem 3, which is a simple translation for pssMp of results such as those appearing in [3] Section III.4.

The rest of this paper is mainly devoted to prove the results stated before. The paper is organized so that each subsequent Section contains a proof: in Section 2 we prove Proposition 1, in Section 3 the first Theorem, in Section 4 the proof of Theorem 2 is given, Section 5 is devoted to Proposition 2 and Section 6 to Theorem 3. Furthermore, in Section 7 we establish some interesting consequences of our main results to self-similar fragmentation theory. Finally, Section 8 is constituted of a comparison of the results obtained here with the known results describing the behaviour of the underlying subordinator.

2 Proof of Proposition 1

Assume that the mean of ξ is finite, $m := \mathbf{E}(\xi_1) < \infty$. According to the Theorem 1 in [8] there exists a measure $\mathbb{P}_{0+}$ on the space of càdlàg paths defined over $]0,\infty[$ that takes only positive values, under which the canonical process is a strong Markov process with the same semigroup as X.

Its entrance law can be described in terms of the exponential functional $I = \int_0^\infty \exp\{-\alpha\xi_s\}\mathrm{d}s$, by the formula

$$\mathbb{E}_{0+}\left(f(X(t))\right) = \frac{1}{\alpha m}\mathbf{E}\left(f\left((t/I)^{1/\alpha}\right)\frac{1}{I}\right), \qquad t > 0,$$

for any measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$. This formula is a consequence of (3) and the scaling property. A straightforward consequence of the scaling property is that the process of the Ornstein-Uhlenbeck type U defined by

$$U_t = e^{-t/\alpha}X(e^t), \qquad t \in \mathbb{R},$$

under $\mathbb{E}_{0+}$ is a strictly stationary process. This process has been studied by Carmona, Petit and Yor [15] and by Rivero in [22]. Therein it is proved that U is a positive recurrent and strong Markov process. Observe that the law of U_0 under $\mathbb{E}_{0+}$ is given by the probability measure μ defined in (3). By the ergodic theorem we have that

$$\frac{1}{t}\int_0^t f(U_s)\mathrm{d}s \xrightarrow[t\to\infty]{} \mathbb{E}_{0+}\left(f(U_0)\right) = \mu(f), \quad \mathbb{P}_{0+}\text{-a.s.}$$

for every function $f \in L^1(\mu)$. Observe that a change of variables $u = e^s$ allows us to deduce that

$$\frac{1}{\log(t)}\int_1^t f(u^{-1/\alpha}X(u))\frac{\mathrm{d}u}{u} = \frac{1}{\log(t)}\int_0^{\log(t)} f(U_s)\mathrm{d}s \xrightarrow[t\to\infty]{} \mathbb{E}_{0+}\left(f(U_0)\right), \quad \mathbb{P}_{0+}\text{-a.s.}$$

Now to prove the second assertion of Proposition 1 we use the well known fact that

$$\lim_{t\to\infty}\frac{\xi_t}{t} = m, \qquad \mathbf{P}\text{-a.s.}$$

So, to prove the result it will be sufficient to establish that

$$\tau(t)/\log(t) \xrightarrow[t\to\infty]{} 1/m\alpha, \qquad \mathbf{P}\text{-a.s.} \tag{9}$$

Indeed, if this is the case, then

$$\frac{\log(X(t))}{\log(t)} = \frac{\xi_{\tau(t)}}{\tau(t)}\frac{\tau(t)}{\log(t)} \xrightarrow[t\to\infty]{} m/\alpha m, \qquad \mathbf{P}\text{-a.s.}$$

Now, a simple consequence of Lamperti's transformation is that under $\mathbb{P}_1$

$$\tau(t) = \int_0^t (X(s))^{-\alpha}\mathrm{d}s = \int_0^t \left(s^{-1/\alpha}X(s)\right)^{-\alpha}\frac{\mathrm{d}s}{s}, \qquad t \geq 0.$$

So, the result just proved applied to the function $f(x) = x^{-\alpha}$, $x > 0$, leads to

$$\frac{1}{\log(1+t)}\int_1^{1+t}\left(u^{-1/\alpha}X(u)\right)^{-\alpha}\frac{\mathrm{d}u}{u} \xrightarrow[t\to\infty]{} 1/\alpha m, \quad \mathbb{P}_{0+}\text{-a.s.}$$

Denote by $\mathscr{H}$ the set were the latter convergence holds. By the Markov property it is clear that

$$\mathbb{P}_{0+}\left(\mathbb{P}_{X(1)}\left(\frac{1}{\log(1+t)}\int_0^t\left(u^{-1/\alpha}X(u)\right)^{-\alpha}\frac{\mathrm{d}u}{u} \nrightarrow 1/\alpha m\right)\right) = \mathbb{P}_{0+}(\mathscr{H}^c) = 0.$$

So for $\mathbb{P}_{0+}$–almost every $x > 0$,

$$\mathbb{P}_x\left(\frac{1}{\log(1+t)}\int_0^t\left(u^{-1/\alpha}X(u)\right)^{-\alpha}\frac{\mathrm{d}u}{u} \xrightarrow[t\to\infty]{} 1/\alpha m\right) = 1.$$

For such an x, it is a consequence of the scaling property that

$$\frac{1}{\log(1+t)}\int_0^t\left(u^{-1/\alpha}xX(ux^{-\alpha})\right)^{-\alpha}\frac{\mathrm{d}u}{u} \xrightarrow[t\to\infty]{} 1/\alpha m, \qquad \mathbb{P}_1\text{-a.s.}$$

Therefore, by making a change of variables $s = ux^{-\alpha}$ and using the fact that $\frac{\log(1+tx^{-\alpha})}{\log(t)} \to 1$, as $t \to \infty$, we prove that (9) holds. In view of the previous comments this concludes the proof of the second assertion in Proposition 1.

Remark 4. In the case where the mean is infinite, $\mathbf{E}(\xi_1) = \infty$, we can still construct a measure N with all but one of the properties of $\mathbb{P}_{0+}$; the missing property is that N is not a probability measure, it is in fact a σ-finite, infinite measure. The measure N is constructed following the methods used by Fitzsimmons [18].

The details of this construction are beyond the scope of this note so we omit them. Thus, using results from the infinite ergodic theory (see e.g. [1] Section 2.2) it can be verified that

$$\frac{1}{\log(t)} \int_0^t f(s^{-1/\alpha} X(s)) \frac{\mathrm{d}s}{s} \xrightarrow[t\to\infty]{} 0, \quad N\text{-a.s.}$$

for every function f such that $N(|f(X(1))|) = \mathbf{E}(|f(I^{-1/\alpha})|I^{-1}) < \infty$; in particular for $f(x) = x^{-\alpha}$, $x > 0$. The latter holds also under $\mathbb{P}_1$ because of the Markov and self-similarity properties.

3 Proof of Theorem 1

The proof of Theorem 1 follows the method of proof in [8]. So, here we will first explain how the auxiliary Lemmas and Corollaries in [8] can be extended in our setting and then we will apply those facts to prove the claimed results.

We start by introducing some notation. We define the processes of the age and rest of life associated to the subordinator ξ,

$$(A_t, R_t) = (t - \xi_{L(t)-}, \xi_{L(t)} - t), \qquad t \geq 0,$$

where $L(t) = \inf\{s > 0 : \xi_s > t\}$. The methods used by Bertoin and Caballero are based on the fact that if the mean $\mathbf{E}(\xi_1) < \infty$ then the random variables (A_t, R_t) converge weakly to a non-degenerate random variable (A, R) as the time tends to infinity. In our setting, $\mathbf{E}(\xi_1) = \infty$, the random variables (A_t, R_t) converge weakly towards (∞, ∞). Nevertheless, if the Laplace exponent ϕ is regularly varying at 0 then $(A_t/t, R_t/t)$ converge weakly towards a non-degenerate random variable (U, O) (see e.g. Theorem 3.2 in [4] where the result is established for A_t/t and the result for $(A_t/t, R_t/t)$ can be deduced therefrom by elementary arguments as in Corollary 1 in [23]; for sake of reference the limit law of the latter is described in Lemma 2 below). This fact, known as the Dynkin-Lamperti Theorem, is the clue to solve our problem.

The following results can be proved with little effort following [8]. For $b > 0$, let T_b be the first entry time into $]b, \infty[$ for X, viz. $T_b = \inf\{s > 0 : X(s) > b\}$.

Lemma 1. *Fix $0 < x < b$. The distribution of the pair $(T_b, X(T_b))$ under $\mathbb{P}_x$ is the same as that of*

$$\left(b^\alpha \exp\{-\alpha A_{\log(b/x)}\} \int_0^{L(\log(b/x))} \exp\{-\alpha \xi_s\} \mathrm{d}s, b \exp\{R_{\log(b/x)}\} \right).$$

This result was obtained in [8] as Corollary 5 and is still true under our assumptions because the proof holds without any hypothesis on the mean of the underlying subordinator. Now, using the latter result, the arguments in the proof of Lemma 6 in [8], the Dynkin-Lamperti Theorem for subordinators and arguments similar to those provided in the proof of Corollary 7 in [8] we deduce the following result.

Lemma 2. *Assume that the Laplace exponent ϕ of the subordinator ξ is regularly varying at 0 with index $\beta \in [0,1]$.*

i) Let $F : \mathbf{D}_{[0,s]} \to \mathbb{R}$ and $G : \mathbb{R}_+^2 \to \mathbb{R}$ be measurable and bounded functions. Then

$$\lim_{t\to\infty} \mathbf{E}\left(F\left(\xi_r, r \le s\right) G\left(\frac{A_t}{t}, \frac{R_t}{t}\right)\right) = \mathbf{E}\left(F(\xi_r, r \le s)\right) \mathbf{E}(G(U,O)),$$

where (U,O) is a $[0,1]\times[0,\infty]$ valued random variable whose law is determined as follows: if $\beta = 0$ (resp. $\beta = 1$), it is the Dirac mass at $(1,\infty)$ (resp. at $(0,0)$). For $\beta \in]0,1[$, it is the distribution with density

$$p_\beta(u,w) = \frac{\beta \sin \beta\pi}{\pi}(1-u)^{\beta-1}(u+w)^{-1-\beta}, \qquad 0<u<1, w>0.$$

ii) As t tends to infinity the triplet

$$\left(\int_0^{L(t)} \exp\{-\alpha\xi_s\}\mathrm{d}s, \frac{A_t}{t}, \frac{R_t}{t}\right)$$

converges in distribution towards

$$\left(\int_0^{\infty} \exp\{-\alpha\xi_s\}\mathrm{d}s, U, O\right),$$

where ξ is independent of the pair (U,O) which has the law specified in (i).

We have the necessary tools to prove Theorem 1.

Proof of Theorem 1. Let $c > -1$, and $b(x) = e^{c\log(1/x)}$, for $0 < x < 1$. In the case where $\beta = 1$ we will furthermore assume that $c \neq 0$ owing that in this setting 0 is a point of discontinuity for the distribution of U. The elementary relations

$$\log(b(x)/x) = (c+1)\log(1/x), \qquad \log\left(b(x)/x^2\right) = (c+2)\log(1/x), \qquad 0<x<1,$$

will be useful. The following equality in law follows from Lemma 1

$$\left(\frac{\log\left(T_{b(x)/x}\right)}{\log(1/x)}, \frac{\log\left(X\left(T_{b(x)/x}\right)\right)}{\log(1/x)}\right) \stackrel{\text{Law}}{=}$$
$$\left(\frac{\alpha\log(b(x)/x) - \alpha A_{\log\left(b(x)/x^2\right)} + \log\left(\int_0^{L\left(\log\left(b(x)/x^2\right)\right)} \exp\{-\alpha\xi_s\}\mathrm{d}s\right)}{\log(1/x)}, \frac{\log(b(x)/x) + R_{\log\left(b(x)/x^2\right)}}{\log(1/x)}\right), \tag{10}$$

for all $0 < x < 1$. Moreover, observe that the random variable $\int_0^{L(r)} \exp\{-\alpha\xi_s\}\mathrm{d}s$ converges almost surely to $\int_0^\infty \exp\{-\alpha\xi_s\}\mathrm{d}s$, as $r \to \infty$; and that for any $t > 0$ fixed,

$$\mathbb{P}_1\left(\frac{\log\left(xX(tx^{-\alpha})\right)}{\log(1/x)} > c\right) = \mathbb{P}_1\left(xX(tx^{-\alpha}) > b(x)\right), \qquad 0<x<1,$$

$$\begin{aligned}\mathbb{P}_1(T_{b(x)/x} < tx^{-\alpha}) &\leq \mathbb{P}_1(xX(tx^{-\alpha}) > b(x)) \\ &\leq \mathbb{P}_1(T_{b(x)/x} \leq tx^{-\alpha}) \leq \mathbb{P}_1(xX(tx^{-\alpha}) \geq b(x)), \quad 0 < x < 1.\end{aligned} \tag{11}$$

Thus, under the assumption of regular variation at 0 of ϕ, the equality in law in (10) combined with the result in Lemma 2-(ii) leads to the weak convergence

$$\left(\frac{\log(T_{b(x)/x})}{\log(1/x)}, \frac{\log\left(X\left(T_{b(x)/x}\right)\right)}{\log(1/x)}\right) \xrightarrow[x\to 0+]{\mathrm{D}} (\alpha\left[c+1-(c+2)U\right], c+1+(c+2)O).$$

As a consequence we get

$$\mathbb{P}_1(T_{b(x)/x} < tx^{-\alpha}) = \mathbb{P}_1\left(\frac{\log\left(T_{b(x)/x}\right)}{\log(1/x)} < \frac{\log(t)}{\log(1/x)} + \alpha\right) \xrightarrow[x\to 0+]{} \mathbf{P}\left(\frac{c}{c+2} < U\right),$$

for $c > -1$. In view of the first two inequalities in (11) this shows that for any $t > 0$ fixed

$$\mathbb{P}_1\left(\frac{\log\left(xX(tx^{-\alpha})\right)}{\log(1/x)} > c\right) \xrightarrow[x\to\ 0+]{} \mathbf{P}\left(\frac{c}{c+2} < U\right), \tag{12}$$

for $c > -1$, and we have so proved that (i) implies (ii).

Next, we prove that (ii) implies (i). If (ii) holds then

$$\mathbb{P}_1\left(\frac{\log\left(xX(tx^{-\alpha})\right)}{\log(1/x)} > c\right) \xrightarrow[x\to\ 0+]{} \mathbf{P}(V > c),$$

for every $c > -1$ point of continuity of the distribution of V. Using this and the second and third inequalities in (11) we obtain that

$$\mathbb{P}_1\left(\frac{\log\left(T_{b(x)/x}\right)}{\log(1/x)} < \frac{\log(t)}{\log(1/x)} + \alpha\right) \xrightarrow[x\to 0+]{} \mathbf{P}(c < V).$$

Owing to the equality in law (10) we have that

$$\begin{aligned}&\mathbf{P}(c < V) \\ &= \lim_{x\to 0+} \mathbf{P}\left(\frac{\alpha\log(b(x)/x) - \alpha A_{\log(b(x)/x^2)} + \log\left(\int_0^{L(\log(b(x)/x^2))} \exp\{-\alpha\xi_s\}\mathrm{d}s\right)}{\log(1/x)} < \frac{\log(t)}{\log(1/x)} + \alpha\right) \\ &= \lim_{x\to 0+} \mathbf{P}\left(\alpha(c+1) - \frac{\alpha(c+2)A_{\log(b(x)/x^2)}}{\log\left(b(x)/x^2\right)} < \alpha\right) \\ &= \lim_{z\to\infty} \mathbf{P}\left(\frac{A_z}{z} > \frac{c}{c+2}\right)\end{aligned} \tag{13}$$

So we can ensure that if (ii) holds then A_z/z converges weakly, as $z \to \infty$, which is well known to be equivalent to the regular variation at 0 of the Laplace exponent ϕ, see e.g. [4] Theorem 3.2 or [3] Theorem III.6. Thus we have proved that (ii) implies (i).

To finish, observe that if (i) holds with $\beta = 0$, it is clear that $V = \infty$ a.s. given that in this case $U = 1$ a.s. In the case where (i) holds with $\beta \in]0, 1]$ it is verified using (12) and elementary calculations that V has the law described in Theorem 1. □

Remark 5. Observe that if in the previous proof we replace the function b by $b'(x,a)=ae^{c\log(1/x)}$, for $a>0$, $c>-1$ and $0<x<1$, then

$$\mathbb{P}_1(x^{1+c}X(x^{-\alpha})>a)=\mathbb{P}_x(X(1)>b'(x,a))=\mathbb{P}_1\left(\frac{\log\left(xX(x^{-\alpha})\right)}{\log(1/x)}>c+\frac{\log a}{\log(1/x)}\right),$$

and therefore its limit does not depend on a, as x goes to $0+$. That is for each $c>-1$ we have the weak convergence under $\mathbb{P}_1$ of the random variables

$$x^{1+c}X(x^{-\alpha})\xrightarrow[x\to 0]{\mathrm{D}}Y(c),$$

and $Y(c)$ is an $\{0,\infty\}$-valued random variable whose law is given by

$$\mathbb{P}(Y(c)=\infty)=\mathbb{P}\left(\frac{c}{c+2}<U\right),\qquad \mathbb{P}(Y(c)=0)=\mathbb{P}\left(\frac{c}{c+2}\geq U\right).$$

Therefore, we can ensure that the asymptotic behaviour of $X(t)$ is not of the order t^a for any $a>0$, as $t\to\infty$.

4 Proof of Theorem 2

Assume that the Laplace exponent of ξ is not regularly varying at 0 with a strictly positive index. Let $h:\mathbb{R}^+\to]0,\infty[$ be an increasing function such that $h(t)\to\infty$ as $t\to\infty$ and varies slowly at infinity; and define $f(x)=h(x^{-\alpha})$, $0<x<1$. Assume that h, and so f, are such that

$$\frac{\log(xX(x^{-\alpha}))}{f(x)}\xrightarrow[x\to 0+]{\text{Law}}V,$$

where V is an a.s. non-degenerate, finite and positive valued random variable. For c a continuity point of V let $b_c(x)=\exp\{cf(x)\}$, $0<x<1$. We have that

$$\mathbb{P}_1\left(\frac{\log(xX(x^{-\alpha}))}{f(x)}>c\right)\xrightarrow[x\to 0+]{}\mathbf{P}(V>c).$$

Arguing as in the proof of Theorem 1 it is proved that the latter convergence implies that

$$\mathbb{P}_1\left(\frac{\log\left(T_{b_c(x)/x}\right)}{\log(1/x)}\leq\alpha\right)\xrightarrow[x\to 0+]{}\mathbf{P}(V>c).$$

Using the identity in law (10) and arguing as in equation (13) it follows that the latter convergence implies that

$$\begin{aligned}\mathbf{P}(V>c)&=\lim_{x\to 0+}\mathbf{P}\left(\frac{A_{\log(b_c(x)/x^2)}}{f(x)}\geq c\right)\\&=\lim_{x\to 0+}\mathbf{P}\left(\frac{A_{\log(b_c(x)/x^2)}}{\log(b_c(x)/x^2)}\left(c+\frac{2\log(1/x)}{f(x)}\right)\geq c\right),\end{aligned}\tag{14}$$

where the last equality follows from the definition of b_c.

Now, assume that $\frac{\log(t)}{h(t)} \to 0$, as $t \to \infty$, or equivalently that $\frac{\log(1/x)}{f(x)} \to 0$, as $x \to 0+$. It follows that

$$\begin{aligned}\mathbf{P}(V > c) &= \lim_{x\to 0+} \mathbf{P}\left(\frac{A_{\log(b_c(x)/x^2)}}{\log(b_c(x)/x^2)} \geq 1\right) \\ &= \lim_{z\to\infty} \mathbf{P}\left(\frac{A_z}{z} \geq 1\right),\end{aligned}$$

owing that by hypothesis $\log(b_c(x)/x^2)$ is a strictly decreasing function. Observe that this equality holds for any $c > 0$ point of continuity of V. Making c first tend to infinity and then to 0+, respectively, and using that V is a real valued random variable it follows that

$$\mathbf{P}(V = \infty) = 0 = \lim_{z\to\infty} \mathbf{P}\left(\frac{A_z}{z} \geq 1\right) = \mathbf{P}(V > 0).$$

Which implies that $V = 0$ a.s. this in turn is a contradiction to the fact that V is a non-degenerate random variable.

In the case where $\frac{\log(t)}{h(t)} \to \infty$, as $t \to \infty$, or equivalently $\frac{\log(1/x)}{f(x)} \to \infty$, as $x \to 0+$, we will obtain a similar contradiction. Indeed, let $l_c : \mathbb{R}^+ \to \mathbb{R}^+$ be the function $l_c(x) = \log(b_c(x)/x^2)$, for $x > 0$. This function is strictly decreasing and so its inverse l_c^{-1} exists. Observe that by hypothesis $\log(b_c(x)/x^2)/f(x) = c + \frac{2\log(1/x)}{f(x)} \to \infty$ as $x \to 0$, thus $z/f\left(l_c^{-1}(z)\right) \to \infty$ as $z \to \infty$. So, for any $\epsilon > 0$, it holds that $f\left(l_c^{-1}(z)\right)/z < \epsilon$, for every z large enough. It follows from the first equality in equation (14) that

$$\begin{aligned}\mathbf{P}(V \geq c) &= \lim_{z\to\infty} \mathbf{P}\left(\frac{A_z}{z}\frac{z}{f(l_c^{-1}(z))} \geq c\right) \\ &\geq \lim_{z\to\infty} \mathbf{P}\left(\frac{A_z}{z} \geq c\epsilon\right),\end{aligned}$$

for any c point of continuity of the distribution of V. So, by replacing c by c/ϵ, making ϵ tend to 0+, and using that V is finite a.s. it follows that

$$\frac{A_z}{z} \xrightarrow[z\to\infty]{\text{Law}} 0.$$

By the Dynkin Lamperti Theorem it follows that the Laplace exponent ϕ of the underlying subordinator ξ, is regularly varying at 0 with index 1. This is a contradiction to our assumption that the Laplace exponent of ξ is not regularly varying at 0 with a strictly positive index.

5 Proof of Proposition 2

We will start by proving that (i) is equivalent to

(i') For any $r > 0$, $\dfrac{\log\left(\int_0^{r/\phi(1/t)} \exp\{\alpha\xi_s\}\mathrm{d}s\right)}{\alpha t} \xrightarrow[t\to\infty]{\text{Law}} \widetilde{\xi}_r$, with $\widetilde{\xi}$ a stable subordinator of parameter β, whenever $\beta \in]0,1[$, and in the case where $\beta = 0$, respectively $\beta = 1$, we have that

$\widetilde{\xi}_r = \infty 1_{\{\mathbf{e}(1)<r\}}$, respectively $\widetilde{\xi}_r = r$ a.s. where $\mathbf{e}(1)$ denotes an exponential random variable with parameter 1.

Indeed, using the time reversal property for Lévy processes we obtain the equality in law

$$\int_0^{r/\phi(1/t)} \exp\{\alpha\xi_s\}\mathrm{d}s = \exp\{\alpha\xi_{r/\phi(1/t)}\} \int_0^{r/\phi(1/t)} \exp\{-\alpha(\xi_{r/\phi(1/t)} - \xi_s)\}\mathrm{d}s$$
$$\stackrel{\text{Law}}{=} \exp\{\alpha\xi_{r/\phi(1/t)}\} \int_0^{r/\phi(1/t)} \exp\{-\alpha\xi_s\}\mathrm{d}s.$$

Given that the random variable $\int_0^\infty \exp\{-\alpha\xi_s\}\mathrm{d}s$ is finite $\mathbf{P}$-a.s. we deduce that

$$\int_0^{r/\phi(1/t)} \exp\{-\alpha\xi_s\}\mathrm{d}s \xrightarrow[t\to\infty]{} \int_0^\infty \exp\{-\alpha\xi_s\}\mathrm{d}s < \infty \qquad \mathbf{P}\text{-a.s.}$$

These two facts allow us to conclude that as $t \to \infty$, the random variable

$$\log\left(\int_0^{r/\phi(1/t)} \exp\{\alpha\xi_s\}\mathrm{d}s\right)/\alpha t$$

converges in law if and only if $\xi_{r/\phi(1/t)}/t$ does. The latter convergence holds if and only if ϕ is regularly varying at 0 with an index $\beta \in [0,1]$. In this case both sequences of random variables converge weakly towards $\widetilde{\xi}_r$. To see this it suffices to observe that the weak convergence of the infinitely divisible random variable $\xi_{r/\phi(1/t)}/t$ holds if and only if its Laplace exponent converges pointwise towards the Laplace exponent of $\widetilde{\xi}_r$ as t tends to infinity. The former Laplace exponent is given by

$$-\log\left(\mathbf{E}\left(\exp\{-\lambda\xi_{r/\phi(1/t)}/t\}\right)\right) = -r\phi(\lambda/t)/\phi(1/t).$$

The rightmost term in this expression converges pointwise as $t \to \infty$ if and only if ϕ is regularly varying at 0 and in this case

$$\lim_{t\to\infty} r\phi(\lambda/t)/\phi(1/t) = r\lambda^\beta, \qquad \lambda \geq 0,$$

for some $\beta \in [0,1]$, see e.g. Theorem 1.4.1 and Section 8.3 in [12]. This proves the claimed fact as the Laplace exponent of $\widetilde{\xi}_r$ is given by $r\lambda^\beta$, $\lambda \geq 0$.

Let φ be the inverse of ϕ. Assume that (i), and so (i'), hold. To prove that (ii) holds we will use the following equalities valid for $\beta \in]0,1]$, for any $x>0$

$$\begin{aligned}
\mathbf{P}\left(\left(\alpha\widetilde{\xi}_1\right)^{-\beta} < x\right) &= \mathbf{P}\left(\alpha\widetilde{\xi}_1 > x^{-1/\beta}\right) \\
&= \mathbf{P}\left(\alpha\widetilde{\xi}_x > 1\right) \\
&= \lim_{t\to\infty} \mathbf{P}\left(\log\left(\int_0^{x/\phi(1/t)} \exp\{\alpha\xi_s\}\mathrm{d}s\right) > t\right) \\
&= \lim_{l\to\infty} \mathbf{P}\left(\int_0^l \exp\{\alpha\xi_s\}\,\mathrm{d}s > \exp\{1/\varphi(x/l)\}\right) \qquad (15)\\
&= \lim_{u\to\infty} \mathbf{P}\left(x\left(\phi\left(\frac{1}{\log(u)}\right)\right)^{-1} > \tau(u)\right) \\
&= \lim_{u\to\infty} \mathbb{P}_1\left(x > \phi\left(\frac{1}{\log(u)}\right)\int_0^u (X(s))^{-\alpha}\mathrm{d}s\right),
\end{aligned}$$

where the second equality is a consequence of the fact that $\widetilde{\xi}$ is self-similar with index $1/\beta$ and hence $x^{1/\beta}\widetilde{\xi}_1$ has the same law as $\widetilde{\xi}_x$. So, using the well known fact that $(\widetilde{\xi}_1)^{-\beta}$ follows a Mittag-Leffler law of parameter β, it follows therefrom that (i') implies (ii). Now, to prove that if (ii) holds then (i') does, simply use the previous equalities read from right to left. So, it remains to prove the equivalence between (i) and (ii) in the case $\beta=0$. In this case we replace the first two equalities in equation (15) by

$$\mathbf{P}(\mathbf{e}(1) < x) = \mathbf{P}(\alpha\widetilde{\xi}_x > 1),$$

and simply repeat the arguments above.

Given that the Mittag-Leffler distribution is completely determined by its entire moments the fact that (iii) implies (ii) is a simple consequence of the method of moments. Now we will prove that (i) implies (iii). Let $n\in\mathbb{N}$. To prove the convergence of the n-th moment of $\phi\left(\frac{1}{\log(t)}\right)\int_0^t (X(s))^{-\alpha}\mathrm{d}s$ to that of a multiple of a Mittag-Leffler random variable we will use the following identity, for $x,c>0$,

$$\begin{aligned}
\mathbb{E}_x\left(\left(c\int_0^t (X(s))^{-\alpha}\mathrm{d}s\right)^n\right) &= \mathbf{E}\left(\left(c\tau(tx^{-\alpha})\right)^n\right) \\
= \quad & c^n\int_0^\infty ny^{n-1}\,\mathbf{P}(\tau(tx^{-\alpha}) > y)\mathrm{d}y \\
= \quad & \int_0^\infty ny^{n-1}\,\mathbf{P}(\tau(tx^{-\alpha}) > y/c)\mathrm{d}y \\
= \quad & \int_0^\infty ny^{n-1}\,\mathbf{P}\left(\log(tx^{-\alpha}) > \alpha\xi_{y/c} + \log\int_0^{y/c}\exp\{-\alpha\xi_s\}\mathrm{d}s\right)\mathrm{d}y, \qquad (16)
\end{aligned}$$

where in the last equality we have used the time reversal property for Lévy processes. We use the notation $f_t(y) = \mathbf{P}\left(\log(tx^{-\alpha}) > \alpha\xi_{y/c} + \log\int_0^{y/c}\exp\{-\alpha\xi_s\}\mathrm{d}s\right)$ and we will prove that

$$\sup_{t>0}\left(\int_0^\infty ny^{n-1}f_t(y)dy\right) < \infty, \qquad \sup_{t>0}\left(\int_0^\infty (ny^{n-1}f_t(y))^2 dy\right) < \infty.$$

This will show that the family $\{ny^{n-1}f_t(y)\}_{t\geq 0}$ is uniformly integrable. To prove the first assertion observe that for any $t, y > 0$ such that $y > \phi(1/\log(t))$ we have

$$\log \int_0^{\frac{y}{\phi(1/\log(t))}} e^{-\alpha\xi_s} \mathrm{d}s \geq \log \int_0^1 e^{-\alpha\xi_s} \mathrm{d}s \geq -\alpha\xi_1,$$

and as a consequence

$$\left\{ \log(tx^{-\alpha}) \geq \alpha\xi_{y/\phi(1/\log(t))} + \log \int_0^{\frac{y}{\phi(1/\log(t))}} e^{-\alpha\xi_s} \mathrm{d}s \right\} \subseteq \left\{ \log(tx^{-\alpha}) \geq \alpha \left(\xi_{y/\phi(1/\log(t))} - \xi_1 \right) \right\}.$$

Using this, the fact that $\xi_{y/\phi(1/\log(t))} - \xi_1$ has the same law as $\xi_{\frac{y}{\phi(1/\log(t))}-1}$ and Markov's inequality it follows that the rightmost term in equation (16) is bounded from above by

$$\begin{aligned}
&(\phi(1/\log(t)))^n + \int_{\phi(1/\log(t))}^{\infty} ny^{n-1}\, \mathbf{P}\left(\log(tx^{-\alpha}) \geq \alpha\xi_{\frac{y}{\phi(1/\log(t))}-1} \right) \mathrm{d}y \\
&\leq (\phi(1/\log(t)))^n + \int_{\phi(1/\log(t))}^{\infty} ny^{n-1} \exp\left\{ -\frac{(y - \phi(1/\log(t)))\, \phi\left(\alpha/\log\left(tx^{-\alpha}\right)\right)}{\phi(1/\log(t))} \right\} \mathrm{d}y \\
&\leq (\phi(1/\log(t)))^n + n2^{n-1} \frac{(\phi(1/\log(t)))^n}{\phi\left(\alpha/\log\left(tx^{-\alpha}\right)\right)} + 2^{n-1}\Gamma(n+1) \left(\frac{\phi(1/\log(t))}{\phi(\alpha/\log(tx^{-\alpha}))} \right)^n.
\end{aligned}$$

The regular variation of ϕ implies that the rightmost term in this equation is uniformly bounded for large t.

Since

$$\int_0^\infty (ny^{n-1}f_t(y))^2 dy \leq \int_0^\infty (n^2y^{2n-2}f_t(y))dy$$

a similar bound can be obtained (for a different value of n) and this yields $\sup_t(\int_0^\infty (ny^{n-1}f_t(y))^2 dy) < \infty$

By hypothesis, we know that for $y > 0$, $(\log(t))^{-1}\xi_{y/\phi\left(\frac{1}{\log(t)}\right)} \xrightarrow[t\to\infty]{\text{Law}} \widetilde{\xi}_y$, and therefore

$$\begin{aligned}
&\mathbf{P}\left(\log(tx^{-\alpha}) > \alpha\xi_{y/\phi\left(\frac{1}{\log(t)}\right)} + \log \int_0^{y/\phi\left(\frac{1}{\log(t)}\right)} \exp\{-\alpha\xi_s\} \mathrm{d}s \right) \\
&\sim \mathbf{P}(1 > \alpha\widetilde{\xi}_y) \quad \text{as } t \to \infty.
\end{aligned} \tag{17}$$

Therefore, we conclude from (16), (17) and the uniform integrability that

$$\begin{aligned}
\mathbb{E}_x\left(\left(\phi\left(\frac{1}{\log(t)}\right) \int_0^t (X(s)^{-\alpha}\mathrm{d}s \right)^n \right) &\xrightarrow[t\to\infty]{} \int_0^\infty ny^{n-1}\, \mathbf{P}\left(1 > \alpha\widetilde{\xi}_y\right) \mathrm{d}y \\
&= \begin{cases} \int_0^\infty ny^{n-1}\, \mathbf{P}\left(\mathbf{e}(1) > y\right) \mathrm{d}y, & \text{if } \beta = 0, \\ \int_0^\infty ny^{n-1}\, \mathbf{P}\left(1 > \alpha y^{1/\beta}\widetilde{\xi}_1\right) \mathrm{d}y, & \text{if } \beta \in]0,1], \end{cases} \\
&= \begin{cases} n!, & \text{if } \beta = 0, \\ \mathbf{E}\left(\left(\alpha^{-\beta}\widetilde{\xi}_1^{-\beta} \right)^n \right), & \text{if } \beta \in]0,1], \end{cases}
\end{aligned}$$

for any $x > 0$. We have proved that (i) implies (iii) and thus finished the proof of Proposition 2.

Proof of Corollary 1. It has been proved in [9] that the law of R_ϕ is related to X by the following formula

$$\mathbb{E}_1\left((X(s))^{-\alpha}\right) = \mathbf{E}(e^{-sR_\phi}), \qquad s \geq 0.$$

It follows therefrom that

$$\mathbb{E}_1\left(\int_0^t (X(s))^{-\alpha} \mathrm{d}s\right) = \int_{[0,\infty[} \frac{1-e^{-tx}}{x} \mathbf{P}(R_\phi \in \mathrm{d}x), \qquad t \geq 0.$$

Moreover, the function $t \mapsto \mathbb{E}_1((X(t))^{-\alpha})$ is non-increasing. So, by (iii) in Proposition 2 it follows that

$$\int_0^t \mathbb{E}_1\left((X(s))^{-\alpha}\right) \mathrm{d}s \sim \frac{1}{\alpha^\beta \Gamma(1+\beta)\phi\left(\frac{1}{\log(t)}\right)}, \qquad t \to \infty.$$

Then, the monotone density theorem for regularly varying functions (Theorem 1.7.2 in [12]) implies that

$$\mathbb{E}_1\left((X(t))^{-\alpha}\right) = o\left(\frac{1}{\alpha^\beta \Gamma(1+\beta) t \phi\left(\frac{1}{\log(t)}\right)}\right), \qquad t \to \infty.$$

Given that $\mathbb{E}_1\left((X(t))^{-\alpha}\right) = \mathbf{E}(e^{-tR_\phi})$, for every $t \geq 0$, we can apply Karamata's Tauberian Theorem (Theorem 1.7.1' in [12]) to obtain the estimate

$$\mathbf{P}(R_\phi < s) = o\left(\frac{s}{\alpha^\beta \Gamma(1+\beta)\phi\left(\frac{1}{\log(1/s)}\right)}\right), \qquad s \to 0+.$$

Also applying Fubini's theorem and making a change of variables of the form $u = sR_\phi/t$ we obtain the identity

$$\begin{aligned}
\int_0^t \mathbb{E}_1((X(s))^{-\alpha})\mathrm{d}s &= \int_0^t \mathbf{E}(e^{-sR_\phi})\mathrm{d}s \\
&= \mathbf{E}\left(\frac{t}{R_\phi}\int_0^{R_\phi} e^{-tu}\mathrm{d}u\right) \\
&= t\int_0^\infty \mathrm{d}u e^{-tu}\,\mathbf{E}\left(1_{\{R_\phi > u\}}\frac{1}{R_\phi}\right), \qquad t > 0.
\end{aligned}$$

So using Proposition 2 and Karamata's Tauberian Theorem we deduce that

$$\mathbf{E}\left(1_{\{R_\phi > s\}}\frac{1}{R_\phi}\right) \sim \frac{1}{\alpha^\beta \Gamma(1+\beta)\phi(1/\log(1/s))}, \qquad s \to 0+.$$

The proof of the second assertion follows from the fact that I_ϕ has the same law as $\alpha^{-1}R_\theta$ where $\theta(\lambda) = \lambda/\phi(\lambda)$, $\lambda > 0$, for a proof of this fact see the final Remark in [9]. □

6 Proof of Theorem 3

The proof of the first assertion in Theorem 3 uses a well known law of iterated logarithm for subordinators, see e.g. Chapter III in [3]. The second assertion in Theorem 3 is reminiscent of, and its proof is based on, a result for subordinators that appears in [2]. But to use those results we need three auxiliary Lemmas. The first of them is rather elementary.

Recall the definition of the additive functional $\{C_t, t \geq 0\}$ in (1).

Lemma 3. *For every $c > 0$, and for every $f : \mathbb{R}^+ \to \mathbb{R}^+$, we have that*

$$\liminf_{s\to\infty} \frac{\xi_{\tau(s)}}{\log(s)} \leq c \iff \liminf_{s\to\infty} \frac{\xi_s}{\log(C_s)} \leq c,$$

and

$$\limsup_{s\to\infty} \frac{\xi_{\tau(s)}}{f(\log(s))} \geq c \iff \limsup_{s\to\infty} \frac{\xi_s}{f(\log(C_s))} \geq c$$

Proof. The proof of these assertions follows from the fact that the mapping $t \mapsto C_t$, $t \geq 0$ is continuous, strictly increasing and so bijective. □

Lemma 4. *Under the assumptions of Theorem 3 we have the following estimates of the functional $\log\left(C_t\right)$ as $t \to \infty$,*

$$\liminf_{t\to\infty} \frac{\log\left(C_t\right)}{g(t)} = \alpha\beta(1-\beta)^{(1-\beta)/\beta} =: \alpha c_\beta, \qquad \mathbf{P}\text{-}a.s., \tag{18}$$

$$\limsup_{t\to\infty} \frac{\log\left(C_t\right)}{\xi_t} = \alpha, \qquad \mathbf{P}\text{-}a.s. \tag{19}$$

and

$$\lim_{t\to\infty} \frac{\log\log(C_t)}{\log(g(t))} = 1, \qquad \mathbf{P}\text{-}a.s. \tag{20}$$

Proof. We will use the fact that if ϕ is regularly varying with an index $\beta \in]0,1[$, then

$$\liminf_{t\to\infty} \frac{\xi_t}{g(t)} = \beta(1-\beta)^{(1-\beta)/\beta} = c_\beta, \qquad \mathbf{P}\text{-a.s.} \tag{21}$$

A proof for this law of iterated logarithm for subordinators may be found in Theorem III.14 in [3]. Observe that

$$\log\left(C_t\right) \leq \log(t) + \alpha\xi_t, \qquad \forall t \geq 0,$$

so

$$\liminf_{t\to\infty} \frac{\log\left(C_t\right)}{g(t)} \leq \liminf_{t\to\infty} \left(\frac{\log(t)}{g(t)} + \frac{\alpha\xi_t}{g(t)}\right) = \alpha c_\beta, \qquad \mathbf{P}\text{-a.s.}$$

because g is a function that is regularly varying at infinity with an index $0 < 1/\beta$ and (21). For every $\omega \in \mathscr{B} := \{\liminf_{t\to\infty} \frac{\xi_t}{g(t)} = c_\beta\}$ and every $\epsilon > 0$ there exists a $t(\epsilon, \omega)$ such that

$$\xi_s(\omega) \geq (1-\epsilon)c_\beta g(s), \qquad s \geq t(\epsilon, \omega).$$

Therefore,

$$\int_0^t \exp\{\alpha\xi_s\}\mathrm{d}s \geq \int_{t(\epsilon,\omega)}^t \exp\{(1-\epsilon)\alpha c_\beta g(s)\}\mathrm{d}s, \qquad \forall t \geq t(\epsilon,\omega),$$

and by Theorem 4.12.10 in [12] we can ensure that

$$\lim_{t\to\infty} \frac{\log\left(\int_{t(\epsilon,\omega)}^t \exp\{(1-\epsilon)\alpha c_\beta g(s)\}\mathrm{d}s\right)}{(1-\epsilon)\alpha c_\beta g(t)} = 1.$$

This implies that for every $\omega \in \mathscr{B}$ and $\epsilon > 0$

$$\liminf_{t\to\infty} \frac{\log\left(C_t(\omega)\right)}{g(t)} \geq (1-\epsilon)\alpha c_\beta.$$

Thus, by making $\epsilon \to 0+$ we obtain that for every $\omega \in \mathscr{B}$

$$\liminf_{t\to\infty} \frac{\log\left(C_t(\omega)\right)}{g(t)} = \alpha c_\beta,$$

which finishes the proof of the first claim because $\mathbf{P}(\mathscr{B}) = 1$.

We will now prove the second claim. Indeed, as before we have that

$$\limsup_{t\to\infty} \frac{\log\left(C_t\right)}{\xi_t} \leq \limsup_{t\to\infty} \frac{\log(t) + \alpha\xi_t}{\xi_t} = \alpha, \qquad \mathbf{P}\text{-a.s.}$$

on account of the fact

$$\lim_{t\to\infty} \frac{\xi_t}{t} = \mathbf{E}(\xi_1) = \infty, \qquad \mathbf{P}\text{-a.s.}$$

Furthermore, it is easy to verify that for every $\omega \in \mathscr{B}$

$$\alpha c_\beta = \liminf_{t\to\infty} \frac{\log(C_t)(\omega)}{g(t)} \leq \left[\liminf_{t\to\infty} \frac{\xi_t(\omega)}{g(t)}\right] \left[\limsup_{t\to\infty} \frac{\log\left(C_t(\omega)\right)}{\xi_t(\omega)}\right],$$

and therefore that

$$\alpha \leq \limsup_{t\to\infty} \frac{\log\left(C_t\right)}{\xi_t}, \qquad \mathbf{P}\text{-a.s.}$$

This finishes the proof of the a.s. estimate in equation (19). Now to prove the estimate in (20) we observe that by (18) it follows that

$$\liminf_{t\to\infty} \frac{\log(\log(C_t))}{\log(g(t))} = 1, \qquad \mathbf{P}\text{-a.s.}$$

Now, Theorem III.13 in [3] and the regular variation of ϕ, imply that for $\epsilon > 0$,

$$\limsup_{t\to\infty} \frac{\xi_t}{t^{(1-\epsilon)/\beta}} = \infty, \qquad \lim_{t\to\infty} \frac{\xi_t}{t^{(1+\epsilon)/\beta}} = 0, \qquad \mathbf{P}\text{-a.s.}$$

Using the strong law of large numbers for subordinators we deduce that the former limsup is in fact a limit. The latter and former facts in turn imply that

$$\mathbf{P}\left(\log(\xi_t) \geq \frac{(1+\epsilon)}{\beta}\log(t),\ \text{i.o. } t \to \infty\right) = 0 = \mathbf{P}\left(\log(\xi_t) \leq \frac{(1-\epsilon)}{\beta}\log(t),\ \text{i.o. } t \to \infty\right).$$

Therefore, we obtain that

$$\frac{\log(\xi_t)}{\log(t)} = 1/\beta, \qquad \mathbf{P}\text{-a.s.}$$

Moreover, a consequence of the fact that g is a $1/\beta$-regularly varying function and Proposition 1.3.6 in [12] is the estimate

$$\lim_{t\to\infty} \frac{\log(g(t))}{\log(t)} = 1/\beta.$$

Using these facts together with (19) we infer that

$$\limsup_{t\to\infty} \log(\log(C_t))/\log(g(t)) = 1, \qquad \mathbf{P}\text{-a.s.}$$

□

Using Lemma 3 and the estimate (19) the first assertion in Theorem 3 is straightforward. To prove the second assertion in Theorem 3 we will furthermore need the following technical result.

Lemma 5. *Under the assumptions of (b) in Theorem 3 for any increasing function f with positive increase we have that*

$$\int^{\infty} \phi\left(1/f(g(t))\right) \mathrm{d}t < \infty \Longleftrightarrow \int^{\infty} \phi\left(1/f(cg(t))\right) \mathrm{d}t < \infty \quad \textit{for all} \quad c > 0$$

$$\Longleftrightarrow \int^{\infty} \phi\left(1/f(cg(t))\right) \mathrm{d}t < \infty, \quad \textit{for some} \quad c > 0 \tag{22}$$

Proof. Our argument is based on the fact that ϕ and g are functions of regular variation at 0, and ∞, respectively, with index β and $1/\beta$, respectively, and on the fact that f has positive increase. Let $c > 0$. We can assume that there is a constant constant $M > 0$ such that $M < \liminf_{s\to\infty} \frac{f(s)}{f(2s)}$. Thus for all t, s large enough we have the following estimates for g and ϕ

$$\frac{1}{2} \leq \frac{g(tc^{\beta})}{cg(t)} \leq 2, \qquad \frac{1}{2} \leq \frac{\phi(M/s)}{M^{\beta}\phi(1/s)} \leq 2.$$

Assume that the integral in the left side of the equation (22) is finite. It implies that the integral

$\int^{\infty} \phi\left(1/f(g(c^{\beta}t))\right) \mathrm{d}t$ is finite and so that

$$\begin{aligned}
\infty &> \int^{\infty} \phi\left(1/f(g(c^{\beta}t))\right) \mathrm{d}t \\
&\geq \int^{\infty} \phi\left(1/f(2cg(t))\right) \mathrm{d}t \\
&= \int^{\infty} \phi\left(\frac{f(cg(t))}{f(2cg(t))}\frac{1}{f(cg(t))}\right) \mathrm{d}t \\
&\geq \int^{\infty} \phi\left(M\frac{1}{f(cg(t))}\right) \mathrm{d}t \\
&= M^{\beta}\int^{\infty} \frac{\phi\left(M\frac{1}{f(cg(t))}\right)}{M^{\beta}\phi\left(\frac{1}{f(cg(t))}\right)}\phi\left(\frac{1}{f(cg(t))}\right) \mathrm{d}t \\
&\geq \frac{M^{\beta}}{2}\int^{\infty} \phi\left(\frac{1}{f(cg(t))}\right) \mathrm{d}t,
\end{aligned}$$

where to get the second inequality we used that f and ϕ are increasing and the estimate of g, in the fourth we used the fact that f has positive increase and in the sixth inequality we used the estimate of ϕ.

To prove that if the integral on the left side of equation (22) is not finite then that the one in the right is not finite either, we use that $\limsup_{s\to\infty}\frac{f(s)}{f(s/2)} < M^{-1}$, and the estimates provided above for g and ϕ, respectively. We omit the details. □

Now we have all the elements to prove the second claim of Theorem 3.

Proof of Theorem 3 (b). The proof of this result is based on Lemma 4 in [2] concerning the rate of growth of subordinators when the Laplace exponent is regularly varying at 0. Let f be a function such that the hypothesis in (b) in Theorem 3 is satisfied and the condition in (6) is satisfied. A consequence of Lemma 5 is that

$$\int^{\infty} \phi\left(1/f(\alpha c_{\beta} g(t))\right) \mathrm{d}t < \infty.$$

According to the Lemma 4 in [2] we have that

$$\limsup_{t\to\infty} \frac{\xi_t}{f(\alpha c_{\beta} g(t))} = 0, \qquad \mathbf{P}\text{-a.s.}$$

Let Ω_1 be the set of paths for which the latter estimate and the one in (18) hold. It is clear that $\mathbf{P}\left(\Omega_1\right) = 1$. On the other hand, for every $\omega \in \Omega_1$ there exists a $t_0(\omega, 1/2)$ such that

$$\alpha c_{\beta} g(s)/2 \leq \log\left(C_s(\omega)\right), \qquad \forall s \geq t_0(\omega, 1/2),$$

with c_{β} as in the proof of Lemma 4. Together with the fact $\limsup_{t\to\infty}\frac{f(t)}{f(t/2)} < \infty$, this implies that for $\omega \in \Omega_1$,

$$\limsup_{s\to\infty} \frac{\xi_s(\omega)}{f\left(\log\left(C_s(\omega)\right)\right)} \leq \limsup_{s\to\infty} \frac{\xi_s(\omega)}{f(\alpha c_{\beta} g(s))} \frac{f(\alpha c_{\beta} g(s))}{f\left(\alpha c_{\beta} g(s)/2\right)} = 0.$$

In this way we have proved that

$$\limsup_{s\to\infty} \frac{\xi_s}{f\left(\log\left(C_s\right)\right)} = 0, \qquad \mathbf{P}\text{-a.s.}$$

Using Lemma 3 we infer that

$$\limsup_{s\to\infty} \frac{\log(X(s))}{f\left(\log(s)\right)} = 0, \qquad \mathbb{P}_1\text{-a.s.}$$

Now, let f be an increasing function with positive increase such that (8) holds for some $\varepsilon > 0$. It is seen using the regular variation of g and elementary manipulations that the function $t \mapsto f(g(t)^{1+\varepsilon})$, is also an increasing function with positive increase. The integral test in Lemma 4 in [2] implies that

$$\limsup_{t\to\infty} \frac{\xi_t}{f((g(t))^{1+\varepsilon})} = \infty, \qquad \mathbf{P}\text{-a.s.}$$

On account of (20) we can ensure that for all t large enough

$$g(t)^{1-\varepsilon} \leq \log(C_t) \leq g(t)^{1+\varepsilon}.$$

These facts together imply that

$$\limsup_{t\to\infty} \frac{\xi_t}{f(\log(C_t))} = \infty, \qquad \mathbf{P}\text{-a.s.}$$

The proof of the claim in (7) follows from Lemma 3. □

7 An application to self-similar fragmentations

The main purpose of this section is to provide an application of our results into the theory of self-similar fragmentation processes, which are random models for the evolution of an object that splits as time goes on. Informally, a self-similar fragmentation is a process that enjoys both a fragmentation property and a scaling property. By fragmentation property, we mean that the fragments present at a time t will evolve independently with break-up rates depending on their masses. The scaling property specifies these mass-dependent rates. We will next make this definition precise and provide some background on fragmentation theory. We refer the interested reader to the recent book [7] for further details.

First, we introduce the set of non-negative sequence whose total sum is finite

$$\mathscr{S}^{\downarrow} = \left\{ \mathbf{s} = (s_i)_{i\in\mathbb{N}} : s_1 \geq s_2 \geq \cdots \geq 0,\ \sum_{i=1}^{\infty} s_i < \infty \right\}.$$

Let $Y = (Y(t), t \geq 0)$ be a $\mathscr{S}^{\downarrow}$-valued Markov process and for $r \geq 0$, denote by $\mathbb{Q}_r$ the law of Y started from the configuration $(r, 0, \ldots)$. It is said that Y is a self-similar fragmentation process if:

- for every $s, t \geq 0$ conditionally on $Y(t) = (x_1, x_2, \ldots)$, $Y(t+s)$, for $s \geq 0$, has the same law as the variable obtained by ranking in decreasing order the terms of the random sequences $Y^1(s), Y^2(s), \ldots$ where the random variables $Y^i(s)$ are independent with values in $\mathscr{S}^{\downarrow}$ and $Y^i(s)$ has the same law as $Y(s)$ under $\mathbb{Q}_{x_i}$, for each $i = 1, 2, \ldots$

- there exists some $\alpha \in \mathbb{R}$, called index of self-similarity, such that for every $r \geq 0$ the distribution under $\mathbb{Q}_1$ of the rescaled process $(rY(r^\alpha t), t \geq 0)$ is $\mathbb{Q}_r$.

Associated to Y there exists a characteristic triple (α, c, ν), where α is the index of self similarity, $c \geq 0$ is known as the erosion coefficient and ν is the so called dislocation measure, which is a measure over $\mathscr{S}^{\downarrow,*} := \left\{\mathbf{s} = (s_i)_{i\in\mathbb{N}} : s_1 \geq s_2 \geq \cdots \geq 0,\ \sum_{i=1}^{\infty} s_i \leq 1\right\}$ that does not charge $(1, 0, \ldots)$ such that

$$\int_{\mathscr{S}^{\downarrow,*}} \nu(\mathrm{d}s)(1 - s_1) < \infty.$$

In the sequel we will implicitly exclude the case when $\nu \equiv 0$. Here we will only consider self-similar fragmentations with self-similarity index $\alpha > 0$, no erosion rate $c = 0$, and such that

$$\nu\left(s \in \mathscr{S}^{\downarrow,*} : \sum_{i=1}^{\infty} s_i < 1\right) = 0,$$

which means that no mass can be lost when a sudden dislocation occurs.

In [6] Bertoin studied under some assumptions the long time behaviour of the process Y under $\mathbb{Q}_1$ via an empirical probability measure carried, at each t, by the components of $Y(t)$

$$\widetilde{\rho}_t(\mathrm{d}y) = \sum_{i\in\mathbb{N}} Y_i(t)\delta_{t^{1/\alpha}Y_i(t)}(\mathrm{d}y), \qquad t \geq 0. \tag{23}$$

To be more precise, he proved that if the function

$$\Phi(q) := \int_{\mathscr{S}^{\downarrow,*}} \left(1 - \sum_{i=1}^{\infty} s_i^{q+1}\right) \nu(\mathrm{d}s), \qquad q \geq 0,$$

is such that $m := \Phi'(0+) < \infty$, then the measure defined in (23) converges in probability to a deterministic measure, say $\widetilde{\rho}_\infty$, which is completely determined by the moments

$$\int_0^\infty x^{\alpha k} \widetilde{\rho}_\infty(\mathrm{d}x) = \frac{(k-1)!}{\alpha m \Phi(\alpha) \cdots \Phi(\alpha(k-1))}, \qquad k = 1, 2, \ldots$$

with the assumption that the quantity in the right-hand side equals $(\alpha m)^{-1}$, when $k = 1$. Bertoin proved this result by cleverly applying the results in [8] and the fact that there exists an increasing $1/\alpha$-pssMp, say $\widetilde{Z} = \left(\widetilde{Z}_t, t \geq 0\right)$ such that $\mathbb{Q}_r(\widetilde{Z}_0 = r) = 1$, and for any bounded and measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$

$$\mathbb{Q}_1\left(\widetilde{\rho}_t f\right) = \mathbb{Q}_1\left(\sum_{i=1}^{\infty} Y_i(t) f(t^{1/\alpha} Y_i(t))\right) = \mathbb{Q}_1\left(f\left(t^{1/\alpha}/\widetilde{Z}_t\right)\right), \qquad t \geq 0;$$

and that the process $\widetilde{Z}$ is an increasing $1/\alpha$–pssMp whose underlying subordinator has Laplace exponent Φ. In fragmentation theory the process $(1/\widetilde{Z}_t, t \geq 0)$ is called the *process of the tagged fragment*.

Besides, it can be viewed using the method of proof of Bertoin that if $\Phi'(0+) = \infty$, then the measure $\widetilde{\rho}_t$ converges in probability to the law of a random variable degenerate at 0. This suggests that in the latter case, to obtain further information about the repartition of the components of $Y(t)$ it would be convenient to study a different form of the empirical measure of Y. A suitable form of the empirical measure is given by the random probability measure

$$\rho_t(\mathrm{d}y) = \sum_{i=1}^{\infty} Y_i(t)\delta_{\{\log(Y_i(t))/\log(t)\}}(\mathrm{d}y), \qquad t \geq 0.$$

The arguments provided by Bertoin are quite general and can be easily modified to prove the following consequence of Theorem 1, we omit the details of the proof.

Corollary 2. *Let Y be a self-similar fragmentation with self-similarity index $\alpha > 0$, $c = 0$ and dislocation measure ν. Assume that $\nu\left(s \in \mathcal{S}^{\downarrow,*} : \sum_{i=1}^{\infty} s_i < 1\right) = 0$, and that the function Φ is regularly varying at 0 with an index $\beta \in [0,1]$. Then, as $t \to \infty$, the random probability measure $\rho_t(\mathrm{d}y)$ converges in probability towards the law of $-\alpha^{-1} - V$, where V is as in Theorem 1.*

To the best of our knowledge in the literature about self-similar fragmentation theory there is no example of self-similar fragmentation process whose dislocation measure is such that the hypotheses about the function Φ in Corollary 2 is satisfied. So, we will next extend a model studied by Brennan and Durrett [13; 14] to provide an example of such a fragmentation process. We will finish this section by providing a necessary condition for a dislocation measure to be such that the hypothesis of Corollary 2 is satisfied.

Example 1. In [13; 14] Brennan and Durrett studied a model that represents the evolution of a particle system in which a particle of size x waits an exponential time of parameter x^{α}, for some $\alpha > 0$, and then undergoes a binary dislocation into a left particle of size Ux and a right particle of size $(1-U)x$. It is assumed that U is a random variable that takes values in $[0,1]$ with a fixed distribution and whose law is independent of the past of the system. Assume that the particle system starts with a sole particle of size 1 and that we observe the size of the left-most particle and write l_t for its length at time $t \geq 0$. It is known that the process $X := \{X(t) = 1/l_t, t \geq 0\}$ is an increasing self-similar Markov process with self-similarity index $1/\alpha$, starting at 1, see e.g. [13; 14] or [8]. It follows from the construction that the subordinator ξ associated to X via Lamperti's transformation is a compound Poisson process with Lévy measure the distribution of $-\log(U)$. That is, the Laplace exponent of ξ has the form

$$\phi(\lambda) = \mathbb{E}\left(1 - U^{\lambda}\right), \qquad \lambda \geq 0.$$

In the case where $\mathbb{E}\left(-\log(U)\right) < \infty$, it has been proved in [13; 14] and [8] that l_t decreases as a power function of order $-1/\alpha$, and the weak limit of $t^{1/\alpha}l_t$ as $t \to \infty$ is $1/Z$, where Z is the random variable whose law is described in (2) and (3); so the limit law depends on the whole trajectory of the underlying subordinator. Whilst if the Laplace exponent ϕ is regularly varying at zero with an index $\beta \in]0,1[$, which holds if and only if $x \mapsto \mathbb{P}(-\log(U) > x)$ is regularly varying at infinity with index $-\beta$, and in particular the mean of $-\log(U)$ is not finite, we can use our results to deduce the asymptotic behaviour of X. Indeed, in this framework we have that

$$\frac{-\log(l_t)}{\log(t)} \xrightarrow[t\to\infty]{Law} V + \frac{1}{\alpha},$$

where V is a random variable whose law is described in Theorem 1. Besides, the first part of Theorem 3 implies that

$$\limsup_{t\to\infty} \frac{\log(l_t)}{\log(t)} = -1/\alpha, \qquad \text{a.s.}$$

The $\liminf$ can be studied using the second part of Theorem 3. Observe that the limit law of $-\log l_t/\log(t)$ depends only on the index of self-similarity and that one of regular variation of the right tail of $-\log(U)$.

Another interesting increasing pssMp arising in this model is that of the tagged fragment. It will be described below after we discuss a few generalities for this class of processes.

It is known, see [5] equation (8), that in general the dislocation measure, say ν, of a self-similar fragmentation process is related to the Lévy measure, say Π, of the subordinator associated via Lamperti's transformation to the process of the tagged fragment, through the formula

$$\Pi]x,\infty[=\int_{\mathscr{S}^{\downarrow,*}}\left(\sum_{i=1}^{\infty}s_i 1_{\{s_i<\exp(-x)\}}\right)\nu(\mathrm{d}s),\qquad x>0.$$

So the hypothesis of Corollary 2 is satisfied with an index $\beta\in]0,1[$ whenever ν is such that

- the function $x\mapsto\int_{\mathscr{S}^{\downarrow,*}}\left(\sum_i s_i 1_{\{s_i<\exp(-x)\}}\right)\nu(\mathrm{d}s)$, $x>0$, is regularly varying at infinity with an index $-\beta$.

In the particular case where ν is binary, that is when $\nu\{s\in\mathscr{S}^{\downarrow,*}:s_3>0\}=0$, the latter condition is equivalent to the condition

- the function $x\mapsto\int_0^{\exp(-x)}y\nu(s_2\in\mathrm{d}y)=\int_{1-\exp(-x)}^1(1-z)\nu(s_1\in\mathrm{d}z)$, $x>0$, is regularly varying at infinity with an index $-\beta$,

given that in this case s_1 is always $\geq 1/2$, and $\nu\{s_1+s_2\neq 1\}=0$, by hypothesis.

Example 2 (Continuation of Example 1)**.** In this model the fragmentation process is binary, the self-similarity index is α, the erosion rate $c=0$, and the associated dislocation measure is such that for any measurable and positive function $f:\mathbb{R}^{+,2}\to\mathbb{R}^+$

$$\int\nu(s_1\in\mathrm{d}y_1,s_2\in\mathrm{d}y_2)f(y_1,y_2)=\int_{[0,1]}\mathbb{P}(U\in\mathrm{d}y)\left(f(y,1-y)1_{\{y\geq 1/2\}}+f(1-y,y)1_{\{y<1/2\}}\right).$$

Therefore the Laplace exponent of the subordinator associated via Lamperti's transformation to the process of the tagged fragment is given by

$$\begin{aligned}\Phi(q)&=\int_{[0,1]}\mathbb{P}(U\in\mathrm{d}y)\left(1-(1-y)^{q+1}-y^{q+1}\right)\\&=\int_{]0,\infty[}\left(\mathbb{P}(-\log U\in\mathrm{d}z)+\mathbb{P}(-\log(1-U)\in\mathrm{d}z)\right)e^{-z}(1-e^{-qz}),\qquad q\geq 0.\end{aligned}$$

It follows that Φ is regularly varying at 0 with an index $\beta\in]0,1[$ if and only if

$$H(x):=\int_{]0,\infty[}\left(\mathbb{P}(-\log U\in\mathrm{d}z)+\mathbb{P}(-\log(1-U)\in\mathrm{d}z)\right)e^{-z}1_{\{z>x\}},\qquad x>0,$$

is regularly varying at infinity with index $-\beta$. Elementary calculations show that

$$H(x) = e^{-x} \int_0^1 dt\, \mathbb{P}\left(te^{-x} < U \leq e^{-x}\right) + e^{-x} \int_0^1 dt\, \mathbb{P}\left(te^{-x} < 1 - U \leq e^{-x}\right)$$

for $x > 0$. Hence, the function H is a regularly varying function at infinity if for instance

$$\lim_{\lambda \to 0} \frac{\mathbb{P}(t\lambda < U \leq \lambda)}{\lambda(\log(1/\lambda))^{\beta_1} L_1(\log(1/\lambda))} = 1 - t = \lim_{\lambda \to 0} \frac{\mathbb{P}(t\lambda < 1 - U \leq \lambda)}{\lambda(\log(1/\lambda))^{\beta_2} L_2(\log(1/\lambda))},$$

uniformly in $t \in]0, 1[$; where $0 < \beta_1, \beta_2 < 1$, and L_1, L_2 are slowly varying functions. In this case, H is regularly varying at infinity with an index $\beta_1 \wedge \beta_2$.

Alternatively, it may be seen using a dominated convergence argument that a sufficient condition for H to be regularly varying at infinity is that

$$\lim_{\lambda \to 0} \frac{\mathbb{P}(U \leq \lambda)}{\lambda(\log(1/\lambda))^{\beta_1} L_1(\log(1/\lambda))} = 1 = \lim_{\lambda \to 0} \frac{\mathbb{P}(1 - U \leq \lambda)}{\lambda(\log(1/\lambda))^{\beta_2} L_2(\log(1/\lambda))}, \tag{24}$$

with β_i, L_i, $i = 1, 2$, as above. It is worth mentioning that if this condition is satisfied the mean of $-\log(U)$ and $-\log(1 - U)$ is finite, respectively. However

$$\Phi'(0+) = \mathbb{E}(U \log(1/U)) + \mathbb{E}((1 - U)\log(1/(1 - U))) = \infty.$$

Hence the process of the leftmost particle and that of the tagged fragment bear different asymptotic behaviour. Indeed, if the condition (24) is satisfied then the process of the left-most particle $(l_t, t \geq 0)$ is such that $t^{1/\alpha} l_t$ converges in law as $t \to \infty$ to a non-degenerate random variable and

$$-\frac{\log(l_t)}{\log(t)} \xrightarrow[t \to \infty]{} \frac{1}{\alpha}, \qquad \text{a.s.}$$

Besides, in this case the process of the tagged fragment $F_t = 1/\widetilde{Z}_t, t \geq 0$, is not of order t^{-a} for any $a > 0$, in the sense described in Remark 5, and

$$-\frac{\log(F_t)}{\log(t)} \xrightarrow[t \to \infty]{\text{Law}} \frac{1}{\alpha} + V, \quad \text{and} \quad \liminf_{t \to \infty} \frac{-\log(F_t)}{\log(t)} = \frac{1}{\alpha}, \qquad \limsup_{t \to \infty} \frac{-\log(F_t)}{\log(t)} = \infty, \qquad \text{a.s.}$$

where V is a non-degenerate random variable whose law is described in Theorem 1.

Furthermore, the main result in [14] can be used because under assumption (24) the mean of $-\log(U)$ is finite. It establishes the almost sure convergence of the empirical measure

$$\frac{1}{N(t)} \sum_{i=1}^{N(t)} \delta_{t^{1/\alpha} Y_i(t)}(\mathrm{d}y),$$

as $t \to \infty$, where $N(t)$ denotes the number of fragments with positive size, and it is finite almost surely. The limit of the latter empirical measure is a deterministic measure characterized in terms of α and the law of U. Besides, as $\Phi'(0+) = \infty$ it follows from our discussion and Corollary 2 that

$$\widetilde{\rho}_t(\mathrm{d}y) = \sum_{i=1}^{\infty} Y_i(t) \delta_{t^{1/\alpha} Y_i(t)}(\mathrm{d}y) \xrightarrow[t \to \infty]{\text{Probability}} \delta_0(\mathrm{d}y)$$

$$\rho_t(\mathrm{d}y) = \sum_{i=1}^{\infty} Y_i(t) \delta_{\log(Y_i(t))/\log(t)}(\mathrm{d}y) \xrightarrow[t \to \infty]{\text{Probability}} \mathbb{P}(-\alpha^{-1} - V \in \mathrm{d}y),$$

where V is a non-degenerate random variable that follows the law described in Theorem 1.

8 Final comments

Lamperti's transformation tells us that under $\mathbb{P}_1$ the process $(\int_0^t (X(s))^{-\alpha} \mathrm{d}s, \log(X(t)), t \geq 0)$ has the same law as $(\tau(t), \xi_{\tau(t)}), t \geq 0)$ under $\mathbf{P}$. So, our results can be viewed as a study of how the time change τ modifies the asymptotic behaviour of the subordinator ξ. Thus, it may be interesting to compare our results with those known for subordinators in the case where the associated Laplace exponent is regularly varying at 0.

On the one hand, we used before that the regular variation of the Laplace exponent ϕ at 0 with an index $\beta \in]0,1]$ is equivalent to the convergence in distribution of $\varphi(1/t)\xi_t$ as $t \to \infty$ to a real valued random variable, with φ the right-continuous inverse of ϕ. On the other hand, Theorem 1 tells us that the former is equivalent to the convergence in distribution of $\xi_{\tau(t)}/\log(t)$, as $t \to \infty$, to a real valued random variable. Moreover, under the assumption of regular variation of ϕ with an index $\beta \in]0,1]$, we have that $\lim_{t\to\infty} \varphi(1/t)\log(t) = 0$. Thus we can conclude that the effect of $\tau(t)$ on ξ is to slow down its rate of growth, which is rather normal given that $\tau(t) \leq t$, for all $t \geq 0$, $\mathbf{P}$-a.s. Theorem 1 tells us the exact rate of growth of ξ_τ, in the sense of weak convergence. Furthermore, these facts suggest that $\varphi(1/\tau(t))$ and $\log(t)$ should have the same order, which is confirmed by Proposition 2. Indeed, using the regular variation of φ and the estimate in (ii) in Proposition 2 we deduce the following estimates in distribution

$$\varphi(1/\tau(t))\log(t) \sim \varphi(1/\tau(t))/\varphi(\phi(1/\log(t))) \sim (\alpha^{-\beta} W)^{-1/\beta}, \quad \text{as } t \to \infty,$$

where W follows a Mittag-Leffler law of parameter β. Observe also that if $\beta \in]0,1[$, $\tau(t)$ bears the same asymptotic behaviour as the first passage time for $e^{\alpha\xi}$ above t, $L_{\log(t)/\alpha} = \inf\{s \geq 0, e^{\alpha\xi_s} > t\}$. Indeed, it is known that under the present assumptions the process $\{t\xi_{u/\phi(1/t)}, u \geq 0\}$ converges in Skorohod's topology, as $t \to \infty$, towards a stable subordinator of parameter β, say $\{\widetilde{\xi}_t, t \geq 0\}$. This implies that $\phi(1/s)L_s$ converges weakly to the first passage time above the level 1 for $\widetilde{\xi}$, and the latter follows a Mittag-Leffler law of parameter $\beta \in]0,1[$. This plainly justifies our assertion owing to Proposition 2 and the fact that $\phi(1/\log(t))L_{\log(t)/\alpha}$ converges weakly towards a random variable $\alpha^{-\beta}\widetilde{W}$, where $\widetilde{W}$ follows a Mittag-Leffler law of parameter β.

Besides, we can obtain further information about the rate of growth of ξ when evaluated at stopping times of the form τ. Recall that if ϕ is regularly varying with an index $\beta \in]0,1[$ then

$$\liminf_{t\to\infty} \frac{\xi_t}{g(t)} = \beta(1-\beta)^{(1-\beta)/\beta}, \qquad \mathbf{P}\text{-a.s.},$$

where the function g is defined in Theorem 3, see e.g. Section III.4 in [3]. This Theorem also states that

$$\liminf_{t\to\infty} \frac{\xi_{\tau(t)}}{\log(t)} = \frac{1}{\alpha}, \qquad \mathbf{P}\text{-a.s.}$$

These, together with the fact that $\lim_{t\to\infty} \frac{\log(t)}{g(t)} = 0$, confirm that the rate of growth of $\xi_{\tau(\cdot)}$ is slower than that of ξ, but this time using a.s. convergence. The long time behaviour of $\log(t)/g(\tau(t))$ is studied in the proof of Theorem 3. The results on the upper envelop of ξ and that of ξ_τ can be discussed in a similar way. We omit the details.

Acknowledgement. We would like to thank B. Haas for suggesting us the application of our results to self-similar fragmentations and insightful discussions about the topic. We also appreciate the comments of the anonymous referees, thanks to them this paper has been improved in many aspects.

References

[1] J. Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. MR1450400

[2] J. Bertoin. Sample path behaviour in connection with generalized arcsine laws. *Probab. Theory Related Fields*, 103(3):317–327, 1995. MR1358080

[3] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. MR1406564

[4] J. Bertoin. Subordinators: examples and applications. In *Lectures on probability theory and statistics (Saint-Flour, 1997)*, volume 1717 of *Lecture Notes in Math.*, pages 1–91. Springer, Berlin, 1999. MR1746300

[5] J. Bertoin. Self-similar fragmentations. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(3):319–340, 2002. MR1899456

[6] J. Bertoin. The asymptotic behavior of fragmentation processes. *J. Eur. Math. Soc. (JEMS)*, 5(4):395–416, 2003. MR2017852

[7] J. Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006. MR2253162

[8] J. Bertoin and M.-E. Caballero. Entrance from 0+ for increasing semi-stable Markov processes. *Bernoulli*, 8(2):195–205, 2002. MR1895890

[9] J. Bertoin and M. Yor. On subordinators, self-similar Markov processes and some factorizations of the exponential variable. *Electron. Comm. Probab.*, 6:95–106 (electronic), 2001. MR1871698

[10] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, 17(4):389–400, 2002. MR1918243

[11] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probability Surveys*, 2:191–212, 2005. MR2178044

[12] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989. MR1015093

[13] M. D. Brennan and R. Durrett. Splitting intervals. *Ann. Probab.*, 14(3):1024–1036, 1986. MR0841602

[14] M. D. Brennan and R. Durrett. Splitting intervals. II. Limit laws for lengths. *Probab. Theory Related Fields*, 75(1):109–127, 1987. MR0879556

[15] P. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Bibl. Rev. Mat. Iberoamericana, pages 73–130. Rev. Mat. Iberoamericana, Madrid, 1997. MR1648657

[16] L. Chaumont and J. C. Pardo. The lower envelope of positive self-similar Markov processes. *Electron. J. Probab.*, 11:no. 49, 1321–1341 (electronic), 2006. MR2268546

[17] E. Csáki and A. Földes. On two ergodic properties of self-similar processes. In *Asymptotic methods in probability and statistics (Ottawa, ON, 1997)*, pages 97–111. North-Holland, Amsterdam, 1998. MR1660754

[18] P. Fitzsimmons. On the existence of recurrent extensions of self-similar markov processes. *Electronic Communications in Probability*, 11:230–241, 2006. MR2266714

[19] J. Hawkes. On the potential theory of subordinators. *Z. Warscheinlichkeitstheorie und Verw. Gebiete*, 33(2):113–132, 1975/76. MR0388553

[20] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972. MR0307358

[21] S. I. Resnick and M. Rubinovitch. The structure of extremal processes. *Advances in Appl. Probability*, 5:287–307, 1973. MR0350867

[22] V. Rivero. A law of iterated logarithm for increasing self–similar Markov processes. *Stochastics and Stochastics Reports*, 75(6):443–462, 2003. MR2029617

[23] V. Rivero. Sinaĭ's condition for real valued Lévy processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 43(3):299–319, 2007. MR2319699

[24] J. Rosiński. Tempering stable processes. *Stochastic Process. Appl.*, 117(6):677–707, 2007. MR2327834

[25] R. Song and Z. Vondraček. Potential theory of special subordinators and subordinate killed stable processes. *J. Theoret. Probab.*, 19(4):817–847, 2006. MR2279605