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Edgeworth expansions for a sample sum from a finite set of independent random variables*

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Abstract

Let $\{X_1, \cdots, X_N\}$ be a set of N independent random variables, and let S_n be a sum of n random variables chosen without replacement from the set $\{X_1, \cdots, X_N\}$ with equal probabilities. In this paper we give a one-term Edgeworth expansion of the remainder term for the normal approximation of S_n under mild conditions.

Key words: Edgeworth expansion, finite population, sampling without replacement.

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1 Introduction and main results

Let $\{X_1, \cdots, X_N\}$ be a set of independent random variables, $\mu_k = EX_k,\ 1 \le k \le N$. Let $R = (R_1, \cdots, R_N)$ be a random vector independent of $X_1, \cdots, X_N$, such that $P(R = r) = 1/N!$ for any permutation $r = (r_1, \cdots, r_N)$ of the numbers $1, \cdots, N$, and put $S_n = \sum_{j=1}^n X_{R_j}, 1 \le n \le N$, that is for a sum of n random variables chosen without replacement from the set $\{X_1, \cdots, X_N\}$ with equal probabilities.

In the situation that $X_k = \mu_k, 1 \le k \le N$, are (nonrandom) real numbers, the sample sum S_n has been studied by a number of authors. The asymptotic normality was established by Erdös and Rényi (1959) under quite general conditions. The rate in the Erdös and Rényi central limit theorem was studied by Bikelis (1969) and later Höglund (1978). An Edgeworth expansion was obtained by Robinson (1978), Bickel and van Zwet (1978), Schneller (1989), Babu and Bai (1996) and later Bloznelis (2000a, b). Extensions to U-statistics and, more generally, symmetric statistics can be found in Nandi and Sen (1963), Zhao and Chen (1987, 1990), Kokic and Weber (1990), Bloznelis and Götze (2000, 2001) and Bloznelis (2003).

In contrast to rich investigations for the case $X_k = \mu_k, 1 \le k \le N$, are (nonrandom) real numbers, there are only a few results concerned with the asymptotics of general S_n discussed in this paper. von Bahr (1972) showed that the distribution of $S_n/\sqrt{VarS_n}$ may be approximated by a normal distribution under certain mild conditions. The rate of the normal approximation has currently been established by Zhao, Wu and Wang (2004), in which the paper improved essentially earlier work by von Bahr (1972). Along the lines of Zhao, Wu and Wang (2004), this paper discusses Edgeworth expansions for the distribution of $S_n/\sqrt{VarS_n}$. Throughout the paper, let

$$\gamma_{12} = \frac{1}{N}\sum_{k=1}^N EX_k\, EX_k^2, \qquad \alpha_j = \frac{1}{N}\sum_{k=1}^N (EX_k)^j, \qquad \beta_j = \frac{1}{N}\sum_{k=1}^N E(X_k^j),$$

for $j = 1, 2, 3, 4$, and

$$p = n/N, \qquad q = 1 - p, \qquad b = 1 - p\alpha_2.$$

Theorem 1. *Suppose that $\alpha_1 = 0$ and $\beta_2 = 1$. Then, for all $1 \le n < N$,*

$$\begin{aligned} \sup_x &\left| P(S_n/\sqrt{nb} \le x) - G_n(x) \right| \\ &\le\ C\left(\Delta_{1n} + (nq)^{-1}\right) + 3\sqrt{nq}\,\log(nq)\,\exp\left\{-nq\,\delta_N\right\}, \end{aligned} \tag{1}$$

where C is an absolute constant,

$$G_n(x) = \Phi(x) - \frac{\beta_3 - 3p\gamma_{12} + 2p^2\alpha_3}{6\sqrt{n}b^{3/2}}\Phi'''(x),$$

with $\Phi(x)$ being a standard normal distribution,

$$\Delta_{1n} = (nb)^{-1}\alpha_4 + \frac{(nb^2)^{-1}}{N}\sum_{k=1}^N E(X_k - pEX_k)^4,$$

and

$$\delta_N = 1 - \sup_{\delta_0 b/(9\mathcal{L}_0)\le |t|\le 16\sqrt{nb}} \Big|\frac{1}{N}\sum_{k=1}^{N} E e^{itX_k}\Big|,$$

where $\mathcal{L}_0 = \frac{1}{N}\sum_k E|X_k|^3$ *and* δ_0 *is so small that* $192\delta_0^2 + 24\delta_0 \le 1-\cos(1/16)$.

Property (1) improves essentially a result of Mirakhmedov (1983). The related result in Theorem 1 of Mirakhmedov (1983) depends on $max_{1\le k\le N} EX_k^4$. Note that it is frequently the case that $N^{-1}\sum_{k=1}^N EX_k^4$ is bounded, but $\max_{1\le k\le N} EX_k^4$ tends to ∞. Also note that Corollary 1 of Mirakhmedov (1983) requires $\overline{\lim}_{t\to\infty}|Ee^{itX_k}| \le \epsilon < 1$. This condition is quite restrictive since it takes away the most interesting case that the X_k are all degenerate.

When $X_k = \mu_k, 1\le k\le N$, are (nonrandom) real numbers, it is readily seen that $\alpha_2 = 1$, $b = q$, $\alpha_3 = \beta_3 = \gamma_{12} = \frac{1}{N}\sum_{k=1}^N \mu_k^3$,

$$\Delta_{1N} + (nq)^{-1} \le 3\,(nq)^{-1}\,\frac{1}{N}\sum_{k=1}^{N}\mu_k^4.$$

In this case, the property (1) reduces to

$$\sup_x |P(S_n/\sqrt{nq} \le x) - G_{1n}(x)| \\ \le C\,(nq)^{-1}\,\frac{1}{N}\sum_{k=1}^{N}\mu_k^4 + 3\sqrt{nq}\,\log(nq)\,\exp\big\{-nq\,\delta_N\big\},$$

where $G_{1n} = \Phi(x) + \frac{p-q}{6\sqrt{nq}}\,\frac{1}{N}\sum_{k=1}^N \mu_k^3\,\Phi'''(x)$, which gives one of main results in Bloznelis (2000a, b).

We next give a result complementary to Theorem 1. The result is better than Theorem 1 under certain conditions such as some of the X_k's are non-degenerate random variables and q is close to 0.

Theorem 2. *Suppose that* $\alpha_1 = 0$ *and* $\beta_2 = 1$. *Then, for all* $1\le n\le N$,

$$\sup_x \Big|P(S_n/\sqrt{nb} \le x) - G_n(x)\Big| \le C\,\Delta_{2n} + 3\sqrt{n}\,\log n\,\exp\big\{-n\,\delta_{1N}\big\}, \tag{2}$$

where C *is an absolute constant,* $G_n(x)$, $\mathcal{L}_0$ *and* δ_0 *are defined as in Theorem 1,* $\Delta_{2n} = (nb^2)^{-1}\beta_4$ *and*

$$\delta_{1N} = 1 - \sup_{\delta_0 b/(9\mathcal{L}_0)\le |t|\le 16\sqrt{nb}} \frac{1}{N}\sum_{k=1}^{N}\big|Ee^{itX_k}\big|.$$

In the next section, we prove the main results. Throughout the paper we shall use $C, C_1, C_2, ...$ to denote absolute constants whose value may differ at each occurrence. Also, $I(A)$ denotes the indicator function of a set A, $\sharp(A)$ denotes the number of elements in the set A, $\sum_k$ denotes $\sum_{k=1}^N$, and $\prod_k$ denotes $\prod_{k=1}^N$. The symbol i will be used exclusively for $\sqrt{-1}$.

2 Proofs of Theorems

Let $\mu_k = EX_k$ and $\Psi(t) = E\exp\{itS_n/\sqrt{nb}\}$. Recall $\alpha_1 = 0$ and $\beta_2 = 1$. As in (4) of Zhao, Wu and Wang (2004),

$$\Psi(t) = [B_n(p)]^{-1} \int_{|\psi|\le \pi\sqrt{nq}} \prod_k E\rho_k(\psi, t)d\psi, \tag{3}$$

where $B_n(p) = \sqrt{2\pi nq}G_n(p),\ G_n(p) = \sqrt{2\pi}C_N^n p^n q^{N-n},\ X_k^* = X_k - p\mu_k$ and

$$\rho_k(\psi, t) = q\exp\left\{-\frac{ip\psi}{\sqrt{nq}} - \frac{ip\mu_k t}{\sqrt{nb}}\right\} + p\exp\left\{\frac{iq\psi}{\sqrt{nq}} + \frac{itX_k^*}{\sqrt{nb}}\right\}.$$

The main idea of the proofs is outlined as follows. We first provide the expansions and the basic properties for $\prod_k E\rho_k(\psi, t)$ in Lemmas 1–4. In Lemma 5, the idea in von Bahr (1972) is extended to give an expansion of $\Psi(t)$ for the case $n/N \ge 1/2$. The proofs of Theorems 1 and 2 are finally completed by virtue of the classical Esseen's smoothing lemma.

In the proofs of Lemmas 1–4, we assume that $\Delta_{1n} < 1/16$ and $nq > 256$, where Δ_{1n} is defined as in Theorem 1. Throughout this section, we also define,

$$h(\psi, t) \;=\; \prod_k E\rho_k(\psi, t) \quad \text{and} \quad g(\psi, t) = \Big(1 + \frac{i^3 f(\psi, t)}{6\sqrt{N}}\Big)e^{-(\psi^2+t^2)/2},$$

where $f(\psi, t) = A_0\psi^3 + 3A_1\psi t^2 + A_2 t^3$, with

$$A_0 = \frac{q-p}{\sqrt{pq}}, \qquad A_1 = \frac{(1-2p\alpha_2)\sqrt{pq}}{pb}, \qquad A_2 = \frac{\beta_3 - 3p\gamma_{12} + 2p^2\alpha_3}{p^{1/2}b^{3/2}}. \tag{4}$$

Lemma 1. *For $|\psi| \le (nq)^{1/4}/4$ and $|t| \le \Delta_{1n}^{-1/4}/4$, we have*

$$|h(\psi, t) - g(\psi, t)| \;\le\; C[\Delta_{1n} + (nq)^{-1}](s^4 + s^8)\exp\{-s^2/3\}, \tag{5}$$

where $s^2 = \psi^2 + t^2$.

Proof. Define a sequence of independent random vectors (U_k, V_k), $1 \le k \le N$, by the conditional distribution given X_k^* as follows:

$$P(U_k = -p/\sqrt{pq},\ V_k = -p\mu_k/\sqrt{pb}\,|\,X_k^*) = q,$$
$$P(U_k = q/\sqrt{pq},\ V_k = X_k^*/\sqrt{pb}\,|\,X_k^*) = p.$$

Let $W_k = \psi U_k + tV_k$. As in Lemma 1 of Zhao, Wu and Wang (2004), tedious but simple calculations show that

$$EW_k \;=\; 0, \quad \sum_k EW_k^2 = N(\psi^2 + t^2), \quad \sum_k EW_k^3 = N\,f(\psi, t),$$

$$\sum_k EW_k^4 \;\le\; 8\sum_k E(\psi^4 U_k^4 + t^4 V_k^4) \;\le\; 8N^2(\psi^4/(nq) + t^4\Delta_{1n}). \tag{6}$$

Furthermore, if we let $B_n^2 = \sum_k EW_k^2$, $\mathcal{L}_{jN} = \sum_k E|W_k|^j/B_n^j$, $j = 3, 4$, then

$$\mathcal{L}_{3N}^2 = \Big(\sum_k E|W_k|^3\Big)^2/B_n^6 \ \le\ \mathcal{L}_{4N} \ \le\ 8((nq)^{-1} + \Delta_{1n}), \tag{7}$$

and whenever $|\psi| \le (nq)^{1/4}/4$ and $|t| \le \Delta_{1n}^{-1/4}/4$,

$$s := \sqrt{\psi^2 + t^2} \quad \le \quad \mathcal{L}_{4N}^{-1/4}\big[8(\psi^4/(nq) + t^4\Delta_{1n})\big]^{1/4} \le \mathcal{L}_{4N}^{-1/4}/2. \tag{8}$$

Now, by recalling that W_k are independent r.v.s and noting that

$$h(\psi, t) = Ee^{i\sum_k W_k/\sqrt{N}} = Ee^{i\,s\sum_k W_k/B_n},$$

it follows from (7)-(8) and the classical result (see, for example, Theorem 8.6 in Bhattacharya and Ranga Rao (1976)) that, for $|\psi| \le (nq)^{1/4}/4$ and $|t| \le \Delta_{1n}^{-1/4}/4$,

$$\begin{aligned}
&\Big|h(\psi,t) - \Big(1 + \frac{i^3}{6\sqrt{N}} f(\psi,t)\Big)e^{-s^2/2}\Big| \\
&\quad = \Big|h(\psi,t) - \Big(1 + \frac{i^3 s^3}{6B_n^3}\sum_k EW_k^3\Big)e^{-s^2/2}\Big| \\
&\quad \le C(\mathcal{L}_{4N} + \mathcal{L}_{3N}^2)(s^4 + s^8)e^{-s^2/3} \\
&\quad \le C[\Delta_{1n} + (nq)^{-1}](s^4 + s^8)\exp\{-s^2/3\}.
\end{aligned}$$

This proves (5) and hence completes the proof of Lemma 1. $\square$

Lemma 2. *For $|\psi| \le (nq)^{1/4}/4$ and $|t| \le 1/4$, we have*

$$\Big|\frac{d\,h(\psi,t)}{dt} - \frac{d\,g(\psi,t)}{dt}\Big| \quad \le \quad C(1 + \psi^{12})\left((nq)^{-1} + \Delta_{1n}\right)e^{-\psi^2/4}. \tag{9}$$

Proof. We first show that if $|\psi| \le (nq)^{1/4}/4$ and $|t| \le 1/4$, then

$$\begin{aligned}
\Lambda(\psi,t) \quad &:= \quad \Big|\frac{d\,h(\psi,t)}{dt} + \Big(t + \frac{i}{6\sqrt{N}}\frac{d\,f(\psi,t)}{dt}\Big)h(\psi,t)\Big| \\
&\le \quad C(1 + \psi^4)\left((nq)^{-1} + \Delta_{1n}\right)|h(\psi,t)|.
\end{aligned} \tag{10}$$

To prove (10), define (U_k, V_k) and $W_k = \psi U_k + tV_k$ as in Lemma 1. Recall that $h(\psi,t) = E\exp\{i\sum_k W_k/\sqrt{N}\}$. It is readily seen that

$$\frac{d\,h(\psi,t)}{dt} = \frac{i\,h(\psi,t)}{\sqrt{N}}\sum_k I_{Nk}, \tag{11}$$

where $I_{Nk} = \Big[E\exp\{iW_k/\sqrt{N}\}\Big]^{-1} E\Big[V_k\exp\{iW_k/\sqrt{N}\}\Big]$. Recall $nq > 256$. It follows from (19) and (20) in Zhao, Wu and Wang (2004) that for $|\psi| \le (nq)^{1/4}/4$ and $|t| \le 1/4$,

$$\Big[E\exp\{iW_k/\sqrt{N}\}\Big]^{-1} = 1 + \theta_1 N^{-1}(\psi^2 + t^2EV_k^2), \tag{12}$$

where $|\theta_1| \le 1$ and $N^{-1}(\psi^2+t^2EV_k^2) \le 1/4$. This, together with Taylor's expansion of e^{ix}, yields that (recall $EV_k = 0$)

$$\begin{aligned}
&\left|I_{Nk} - \frac{i}{\sqrt{N}}EV_kW_k + \frac{1}{2N}EV_kW_k^2\right| \\
&\quad \le \frac{3}{N^{3/2}}E|V_k||W_k|^3 + \frac{\psi^2+t^2EV_k^2}{N}\left(\frac{1}{\sqrt{N}}|EV_kW_k| + \frac{1}{2N}|EV_kW_k^2|\right). \qquad (13)
\end{aligned}$$

As in the proof of (6), for $|t| \le 1/4$,

$$\begin{aligned}
\sum_k E|V_k||W_k|^3 &\le C(1+|\psi|^3)\sum_k E(U_k^4+V_k^4) \\
&\le C(1+|\psi|^3)N^2\left[(nq)^{-1}+\Delta_{1n}\right], \qquad (14)
\end{aligned}$$

$$\begin{aligned}
\sum_k(\psi^2+EV_k^2)|EV_kW_k| &\le C(1+|\psi|^3)\sum_k(1+EV_k^2)(EU_k^2+EV_k^2) \\
&\le C(1+|\psi|^3)\sum_k(1+EV_k^4) \\
&\le C(1+|\psi|^3)N^2[(nq)^{-1}+\Delta_{1n}], \qquad (15)
\end{aligned}$$

$$\begin{aligned}
\sum_k(\psi^2+EV_k^2)|EV_kW_k^2| &\le C(1+\psi^4)\sum_k(1+EV_k^2)\left(E|V_k|^3+E|U_k|^3\right) \\
&\le C(1+\psi^4)\sum_k\left((pq)^{-1/2}(1+EV_k^2)+E|V_k|^3+EV_k^2E|V_k|^3\right) \\
&\le C(1+\psi^4)\Big[N(pq)^{-1/2}+\sum_k\left(1+EV_k^4+\sqrt{N}EV_k^4\right)\Big] \\
&\le C(1+\psi^4)N^{5/2}\left[(nq)^{-1}+\Delta_{1n}\right], \qquad (16)
\end{aligned}$$

where, in the proof of (16), we have used the estimates: $|V_k|^3 \le 1+V_k^4$ and

$$EV_k^2E|V_k|^3 \le (EV_k^2)^{1/2}\,EV_k^4 \le \sqrt{N}EV_k^4.$$

Now (10) follows from (11), (13)-(16) and

$$\sum_k EV_kW_k = tN, \quad \sum_k EV_kW_k^2 = N(2A_1\psi t + A_2\,t^2) = \frac{N}{3}\frac{d\,f(\psi,t)}{dt}. \qquad (17)$$

We next complete the proof of Lemma 2 by virtue of (10) and Lemma 1. We first notice that, by (6), for all ψ and t,

$$\begin{aligned}
|f(\psi,t)| &\le \frac{1}{N}\sum_k E|W_k|^3 \le \frac{1}{N}\Big(\sum_k EW_k^2\sum_k EW_k^4\Big)^{1/2} \\
&\le 3\sqrt{N}(\psi^2+t^2)^{3/2}(\Delta_{1n}+(nq)^{-1})^{1/2}, \qquad (18)
\end{aligned}$$

and similarly by (17), for all ψ and t,

$$\left|\frac{d\,f(\psi,t)}{dt}\right| \le \frac{3}{N}\sum_k E|V_kW_k^2| \le 9\sqrt{N}(\psi^2+t^2)^{3/2}(\Delta_{1n}+(nq)^{-1})^{1/2}. \qquad (19)$$

It follows from (18) and Lemma 1 that for $|\psi| \le (nq)^{1/4}/4$ and $|t| \le 1/4$,

$$|h(\psi,t) - e^{-(\psi^2+t^2)/2}| \le C(\Delta_{1n} + (nq)^{-1})^{1/2} e^{-\psi^2/4}. \tag{20}$$

Therefore, by noting

$$\frac{d\,g(\psi,t)}{dt} = -tg(\psi,t) - \frac{i}{6\sqrt{N}} \frac{d\,f(\psi,t)}{dt} e^{-(\psi^2+t^2)/2},$$

simple calculations show that

$$\begin{aligned}
\Big|\frac{d\,h(\psi,t)}{dt} - \frac{d\,g(\psi,t)}{dt}\Big| &\le \Lambda(\psi,t) + t|h(\psi,t) - g(\psi,t)| \\
&\quad + \frac{1}{6\sqrt{N}} \left|\frac{d\,f(\psi,t)}{dt}\right| |h(\psi,t) - e^{-(\psi^2+t^2)/2}| \\
&\le C(1+\psi^{12})\big((nq)^{-1} + \Delta_{1n}\big) e^{-\psi^2/4},
\end{aligned}$$

where we have used (5), (10), (19) and (20). The proof of Lemma 2 is now complete. □

Lemma 3. *Assume that $|\psi| \le (nq)^{1/4}/4$. Then,*

$$|h(\psi,t)| \le C\Delta_{1n} e^{-(\psi^2+t^2)/4}, \tag{21}$$

for $\Delta_{1n}^{-1/4}/4 \le |t| \le (\Delta^*)^{-1}/16$, where

$$\Delta^* = N^{-1}\sum_k |\mu_k|^3/\sqrt{nb} + N^{-1}\sum_k E|X_k - p\mu_k|^3/(\sqrt{n}b^{3/2}).$$

Assume that $(nq)^{1/4}/4 \le |\psi| \le \pi\sqrt{nq}$. Then,

$$|h(\psi,t)| \le C(nq)^{-4}, \tag{22}$$

for all $|t| \le \delta_0(\Delta^*)^{-1}$, where δ_0 is so small that $192\delta_0^2 + 24\delta_0 \le 1 - \cos(1/16)$. If in addition $|t| \le 1/4$, then we also have

$$\Big|\frac{d\,h(\psi,t)}{dt}\Big| \le C(nq)^{-4}. \tag{23}$$

Proof. The proof of this lemma follows directly from an application of Lemmas 1-3 in Zhao, Wu and Wang(2004). The choice of δ_0 can be found in the proof of Lemma 2 in Zhao, Wu and Wang(2004). We omit the details. □

Lemma 4. *Assume that $n/N \ge 1/2$ and $\Delta_{2n} \le (nq)^{-1}/25$, where Δ_{2n} is defined as in Theorem 2. Then, for $|t| \le \frac{1}{15}\sqrt{n}b^{3/2}/\mathcal{L}_0$,*

$$|h(\psi,t)| \le \exp\{-Ct^2\}, \tag{24}$$

where $\mathcal{L}_0 = \frac{1}{N}\sum_k E|X_k|^3$ is defined as in Theorem 1.

Proof. We only need to note that the condition $\Delta_{2n} \le (nq)^{-1}/25$ implies that

$$5q\alpha_2 \le 5q\beta_4^{1/2} \le b = \frac{1}{N}\sum_k \mathrm{Var}(X_k) + q\alpha_2,$$

that is, $\frac{1}{N}\sum_k \mathrm{Var}(X_k) \ge (4/5)\, b$. Then (24) is obtained by repeating the proof of Lemma 4 in Zhao, Wu and Wang(2004). $\Box$

Lemma 5. *Assume that $n/N \ge 1/2$ and $\Delta_{2n} \le 1$. Then, for $|u| \le \frac{1}{16}\min\left\{(n/\beta_4)^{1/4}, \frac{1}{8}b\sqrt{n}/\mathcal{L}_0\right\}$,*

$$\left| E\exp\{iuS_n/\sqrt{n}\} - \exp\{-bu^2/2\}\left(1 + \frac{i^3u^3b^{3/2}}{6\sqrt{N}}A_2\right)\right| \le Cn^{-1}\,\beta_4(u^2+u^4+u^6b)\,\exp\{-0.3\,b\,u^2\}, \tag{25}$$

where $\mathcal{L}_0 = \frac{1}{N}\sum_k E|X_k|^3$ is defined as in Theorem 1.

Proof. Write, for $1 \le k \le N$,

$$f_k(u) = E\exp\{iuX_k/\sqrt{n}\}, \qquad b_k(u) = \exp\{u^2/(2n)\}f_k(u) - 1$$

and $B_j = ((-1)^{j+1}/j)\sum_{k=1}^N b_k^j(u)$ for $1 \le j \le n$. As in von Bahr (1972), we have

$$\exp\{u^2/2\}E\exp\{iuS_n/\sqrt{n}\} = \sum_{i_j\ge 0,\ 1\le j\le n}\prod_{j=1}^n \frac{(p^jB_j)^{i_j}}{i_j!}C_{N,n,\sum_{j=1}^n ji_j}, \tag{26}$$

where

$$C_{N,n,r} = \begin{cases} \binom{N-r}{n-r}\Big/\left(p^r\binom{N}{n}\right), & r \le n;\\ 0, & r > n.\end{cases}$$

In view of (28) of Zhao, Wu and Wang (2004), for $r > 0$, $C_{N,n,r} \le 1$, and for $n \ge 4$ and $r \le n$,

$$C_{N,n,r} \ge 1 - r^2/n. \tag{27}$$

To prove (25) by using (26), we need some preliminary results.

Write $\beta_{jk} = EX_k^j, j = 2,3,4$. Recall that $N^{-1}\sum_k \beta_{2k} = 1$. We have that $\beta_4 \ge 1$ and by Taylor's expansion, for $|u| \le \frac{1}{16}(n/\beta_4)^{1/4}$,

$$\exp\{u^2/(2n)\} = 1 + \frac{1}{2n}u^2 + \frac{1}{8n^2}u^4 + \frac{\theta_4}{n^3}u^6, \qquad \text{where } |\theta_4| \le 1/24,$$

$$f_k(u) = 1 + \frac{iu}{\sqrt{n}}\mu_k - \frac{u^2}{2n}\beta_{2k} - \frac{iu^3}{6n^{3/2}}\beta_{3k} + \frac{\theta_5u^4}{n^2}\beta_{4k}, \quad \text{where } |\theta_5| \le 1/24.$$

Now, by noting that $|\mu_k| \le \beta_{4k}^{1/4} \le 1 + \beta_{4k}$, $|\beta_{2k}| \le \beta_{4k}^{1/2} \le 1 + \beta_{4k}$ and $|\beta_{3k}| \le \beta_{4k}^{3/4} \le 1 + \beta_{4k}$, we obtain that, for $|u| \le \frac{1}{16}(n/\beta_4)^{1/4}$,

$$\begin{aligned} b_k(u) &= \exp\{u^2/(2n)\}f_k(u) - 1\\ &= \frac{iu}{\sqrt{n}}\mu_k + \frac{u^2}{2n}(1-\beta_{2k}) + \frac{iu^3}{6n^{3/2}}(3\mu_k - \beta_{3k}) + R_{1k}(u). \end{aligned} \tag{28}$$

where $|R_{1k}(u)| \le (1+\beta_{4k})u^4/n^2$. Furthermore, by noting that

$$\beta_{4k}^{1/4}|u|/\sqrt{n} \le (N\beta_4)^{1/4}|u|/\sqrt{n} \le 1/8$$

since $n/N \ge 1/2$ and $|u| \le \frac{1}{16}(n/\beta_4)^{1/4}$, we have

$$\begin{aligned} b_k^2(u) &= -\frac{u^2}{n}\mu_k^2 + \frac{iu^3}{n^{3/2}}\mu_k(1-\beta_{2k}) + R_{2k}(u), &(29)\\ b_k^3(u) &= -\frac{iu^3}{n^{3/2}}\mu_k^3 + R_{3k}(u), &(30)\\ |b_k^j(u)| &\le 3(1/2)^{j-4}(1+\beta_{4k})u^4/n^2, \quad \text{for } j \ge 4, &(31)\end{aligned}$$

where $|R_{2k}(u)| \le 3(1+\beta_{4k})u^4/n^2$ and $|R_{3k}(u)| \le 4(1+\beta_{4k})u^4/n^2$. Recalling that $\sum_k \mu_k = \sum_k(1-\beta_{2k}) = 0$, it follows from (28)–(31) that, for $|u| \le \frac{1}{16}(n/\beta_4)^{1/4}$,

$$pB_1 = p\sum_k b_k(u) = -\frac{iu^3}{6\sqrt{n}}\beta_3 + \theta_6\beta_4 u^4/n, \tag{32}$$

$$p^2B_2 = (-p^2/2)\sum_k b_k^2(u) = \frac{u^2p}{2}\alpha_2 + \frac{iu^3p}{2\sqrt{n}}\gamma_{12} + \theta_7\beta_4u^4/n, \tag{33}$$

$$p^3B_3 = (p^3/3)\sum_k b_k^3(u) = -\frac{ip^2u^3}{3\sqrt{n}}\alpha_3 + \theta_8\beta_4u^4/n, \tag{34}$$

$$|p^jB_j| \le \sum_k |b_k^j(u)| \le 6\beta_4u^4(1/2)^{j-4}/n, \qquad \text{for } \ j \ge 4. \tag{35}$$

where $|\theta_6| \le 2$, $|\theta_7| \le 3$ and $|\theta_8| \le 3$. By virtue of (35), it is readily seen that, for $|u| \le \frac{1}{16}(n/\beta_4)^{1/4}$,

$$\sum_{j=4}^{n}|p^jB_j| \le 12\beta_4u^4/n, \tag{36}$$

Noting that $|\alpha_3| + |\beta_3| + |\gamma_{12}| \le 3\mathcal{L}_0$ and recalling that $\Delta_{2n} = (nb^2)^{-1}\beta_4 \le 1$, it follows easily from (32)–(34) and (36) that, for $|u| \le \frac{1}{16}\min\left\{(n/\beta_4)^{1/4}, \frac{1}{8}b\sqrt{n}/\mathcal{L}_0\right\}$,

$$\begin{aligned} \sum_{j=1}^{n}|p^jB_j| &= \frac{1}{2}p\,\alpha_2\,u^2 + \theta_9\,\frac{u^3\mathcal{L}_0}{\sqrt{n}} + \theta_{10}\,\frac{u^4\beta_4}{n}\\ &= \frac{1}{2}p\,\alpha_2\,u^2 + \theta_{11}\,b\,u^2, &(37)\end{aligned}$$

where $|\theta_9| \le 3$, $|\theta_{10}| \le 20$ and $|\theta_{11}| \le 0.2$. Also, if we let $L(u) = \sum_{j=1}^3 p^jB_j - p\alpha_2u^2/2$, we have

$$|L(u)| \le 0.2bu^2, \qquad \left|L(u) + \frac{i\,u^3b^{3/2}}{6\sqrt{N}}A_2\right| \le 8\beta_4u^4/n, \tag{38}$$

where A_2 is defined as in (4). As in the proof of (6), we may obtain

$$A_2^2 = \Big(N^{-1}\sum_k EV_k^3\Big)^2 \le N^{-2}\sum_k EV_k^2\sum_k EV_k^4 \le N\Delta_{1n} \le 17N\beta_4/(nb^2).$$

This together with (38) yields, for $|u| \le \frac{1}{16}(n/\beta_4)^{1/4}$,

$$
\begin{aligned}
L^2(u) &\le \Big[\frac{|u|^3 b^{3/2}}{6\sqrt{N}}|A_2| + \frac{8}{n}\beta_4 u^4\Big]^2 \\
&\le u^6 b\beta_4/n + 128\beta_4^2 u^8/n^2 \le \beta_4(u^4 + u^6 b)/n. \qquad (39)
\end{aligned}
$$

We are now ready to prove (25) by using (26). Rewrite (26) as

$$
\exp\{u^2/2\}E\exp\{iuS_n/\sqrt{n}\} = I_1 + I_2 + I_3, \qquad (40)
$$

where

$$
\begin{aligned}
I_1 &= \sum \prod_{j=1}^{n} \frac{(p^j B_j)^{i_j}}{i_j!} C_{N,n,\sum_{j=1}^n j i_j}, \\
I_2 &= \sum_{i_j \ge 0,\ 1\le j\le 3} \prod_{j=1}^{3} \frac{(p^j B_j)^{i_j}}{i_j!} \Big(C_{N,n,\sum_{j=1}^3 j i_j} - 1\Big), \\
I_3 &= \sum_{i_j \ge 0,\ 1\le j\le 3} \prod_{j=1}^{3} \frac{(p^j B_j)^{i_j}}{i_j!},
\end{aligned}
$$

where the summation in the expression of I_1 is over all $i_j \ge 0,\ j = 1,2,3$ and $i_j > 0$ for at least one $j = 4, \cdots, n$. As in Mirakhmedov (1983), it follows from (36)-(37) that

$$
\begin{aligned}
|I_1| &\le \exp\Big\{\sum_{j=1}^{3} |p^j B_j|\Big\}\Big(\exp\Big\{\sum_{j=4}^{n} |p^j B_j|\Big\} - 1\Big) \\
&\le \sum_{j=4}^{n} |p^j B_j| \exp\Big\{\sum_{j=1}^{n} |p^j B_j|\Big\} \\
&\le Cn^{-1}\beta_4 u^4 \exp\{p\alpha_2 u^2/2 + 0.2bu^2\}.
\end{aligned}
$$

As for I_2, it follows easily from (27) and (37) that

$$
\begin{aligned}
|I_2| &\le Cn^{-1} \sum_{i_j \ge 0,\ 1\le j\le 3} \prod_{j=1}^{3} \frac{|p^j B_j|^{i_j}}{i_j!} \Big(\sum_{j=1}^{3} j i_j\Big)^2 \\
&\le Cn^{-1} \sum_{i_j \ge 0,\ 1\le j\le 3} \prod_{j=1}^{3} \frac{i_j^2 |p^j B_j|^{i_j}}{i_j!} \\
&\le Cn^{-1} \exp\Big\{\sum_{j=1}^{3} |p^j B_j|\Big\} \sum_{j=1}^{3} \big(|p^j B_j| + |p^j B_j|^2\big) \\
&\le Cn^{-1}\beta_4(u^2 + u^4)\exp\{p\alpha_2 u^2/2 + 0.2bu^2\}.
\end{aligned}
$$

We next estimate I_3. Recalling that $b = 1 - p\alpha_2$ and noting that $I_3 = e^{\sum_{j=1}^3 p^j B_j}$, we have

$$\begin{aligned}
&\Big| I_3\, e^{-u^2/2} - \Big(1 + \frac{i^3 u^3 b^{3/2}}{6\sqrt{N}} A_2\Big)\, e^{-bu^2/2} \Big| \\
&\le e^{-bu^2/2} \left| e^{L(u)} - 1 - L(u) \right| + \; e^{-bu^2/2} \left| L(u) - \frac{i^3 u^3 b^{3/2}}{6\sqrt{N}} A_2 \right| \\
&\le \big[(1/2) L^2(u) e^{|L(u)|} + 8\beta_4 u^4/n\big]\, e^{-bu^2/2} \\
&\le C\, n^{-1}\, \beta_4 (u^4 + u^6 b)\, e^{-0.3\, bu^2},
\end{aligned}$$

where $L(u) = \sum_{j=1}^3 p^j B_j - p\alpha_2 u^2/2$ and we have used (38)-(39).
Combining (40) and all above facts for I_1-I_3, we obtain

$$\begin{aligned}
&\left| E \exp\{iuS_n/\sqrt{n}\} - \exp\{-bu^2/2\} \left(1 + \frac{i^3 u^3 b^{3/2}}{6\sqrt{N}} A_2\right) \right| \\
&\quad \le \exp\{-u^2/2\}(|I_1| + |I_2|) + \Big| I_3\, e^{-u^2/2} - \Big(1 + \frac{i^3 u^3 b^{3/2}}{6\sqrt{N}} A_2\Big)\, e^{-bu^2/2} \Big| \\
&\quad \le C\, n^{-1}\, \beta_4 (u^2 + u^4 + u^6 b)\, \exp\{-0.3 bu^2\},
\end{aligned}$$

which implies (25). The proof of Lemma 5 is now completed. □

After these preliminaries, we are now ready to prove the theorems.

Proof of Theorem 1. Without loss of generality, assume that $nq > 256$ and $\Delta_{1n} < 1/16$. Write $T^{-1} = \Delta_{1n} + (nq)^{-1}$ and

$$g_n(t) = \Big(1 + \frac{i^3 t^3 A_2}{6\sqrt{N}}\Big) \exp\{-t^2/2\},$$

where A_2 is defined as in Lemma 1. We shall prove,
(i) if $|t| \le 1/4$, then

$$\big| \Psi(t) - g_n(t) \big| \le C |t|\, T^{-1}; \tag{41}$$

(ii) if $|t| \le \delta_0 (\Delta^*)^{-1}$, where δ_0 and Δ^* are defined as in Lemma 3, then

$$\big| \Psi(t) - g_n(t) \big| \le C\, T^{-1} (1 + t^8) e^{-t^2/4} + C\, (nq)^{-3}; \tag{42}$$

(iii) if $\delta_0 (\Delta^*)^{-1} \le |t| \le T$, then

$$\big| \Psi(t) - g_n(t) \big| \le C\, T^{-1}\, e^{-t^2/4} + 3\sqrt{nq}\, \exp\big\{ - nq\, \delta_N \big\}, \tag{43}$$

where δ_N is defined as in Theorem 1.
Note that $|A_2| \le \sqrt{N}/4$ by $\Delta_{1n} \le 1/16$ and the last second inequality of (39). We have $m \equiv \sup_x |G_n'(x)| \le C(1 + N^{-1/2}|A_2|) \le 2C$. So, by virtue of (41)-(43) and Esseen's smoothing

lemma, simple calculations show that

$$
\begin{aligned}
&\sup_x \Big| P(S_n/\sqrt{nb} \le x) - G_n(x) \Big| \\
&\quad \le \Big(\int_{|t|\le 1/4} + \int_{1/4\le |t| \le T_1} + \int_{T_1 \le |t| \le T} \frac{|\Psi(t) - g_n(t)|}{|t|}\, dt + Cm\, T^{-1} \\
&\quad \le C\,(\Delta_{1n} + (nq)^{-1}) + 3\sqrt{nq}\, \log(nq)\, \exp\big\{ - nq\, \delta_N \big\},
\end{aligned}
$$

where $T_1 = \min\{\delta_0(\Delta^*)^{-1}, T\}$, which implies (1) and hence Theorem 1.

We next prove (41)-(43). Throughout the proof, we write $s^2 = \psi^2 + t^2$.
Consider (42) first. Note that $g_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\psi, t) d\psi$. It is readily seen that

$$
|\Psi(t) - g_n(t)| \le II_1 + II_2 + II_3 + II_4, \tag{44}
$$

where

$$
\begin{aligned}
II_1 &= [B_n(p)]^{-1} \int_{|\psi| \le (nq)^{1/4}/4} |h(\psi, t) - g(\psi, t)|\, d\psi, \\
II_2 &= [B_n(p)]^{-1} \int_{(nq)^{1/4}/4 \le |\psi| \le \pi\sqrt{nq}} |h(\psi, t)|\, d\psi, \\
II_3 &= [B_n(p)]^{-1} \int_{|\psi| \ge (nq)^{1/4}/4} \Big(1 + \frac{|f(\psi, t)|}{6\sqrt{N}}\Big) e^{-s^2/2} d\psi, \\
II_4 &= \Big| [B_n(p)]^{-1} - (2\pi)^{-1/2} \Big| \int_{-\infty}^{\infty} \Big(1 + \frac{|f(\psi, t)|}{6\sqrt{N}}\Big) e^{-s^2/2} d\psi.
\end{aligned}
$$

To estimate $II_j, j = 1, 2, 3, 4$, we first recall that, by (18), for all ψ and t,

$$
|f(\psi, t)| \quad \le \quad 3s^3\sqrt{N}(\Delta_{1n} + (nq)^{-1})^{1/2} \le \sqrt{N} s^3, \tag{45}
$$

and by virtue of Stirling's formula,

$$
1 \le \sqrt{2\pi}/B_n(p) \le 1 + 1/nq. \tag{46}
$$

In view of (45) and (46), it is readily seen that

$$
II_3 + II_4 \le C\,(nq)^{-1}(1 + t^6)e^{-t^2/3}. \tag{47}
$$

By using (22), we have

$$
II_2 \le C\,(nq)^{-3}. \tag{48}
$$

As for II_1, if $|t| \le \min\{\Delta_{1n}^{-1/4}/4, \delta(\Delta^*)^{-1}\}$, Lemma 1 implies that

$$
II_1 \le C\,(\Delta_{1n} + (nq)^{-1})(1 + t^8)e^{-t^2/4}; \tag{49}
$$

if $\Delta_{1n}^{-1/4}/4 \le |t| \le \delta(\Delta^*)^{-1}$, then it follows from (21) and (45) that

$$
\begin{aligned}
II_1 \quad &\le \quad \int_{|\psi| \le (nq)^{1/4}/4} |h(\psi, t)| d\psi + \int_{|\psi| \le (nq)^{1/4}/4} \Big(1 + \frac{|f(\psi, t)|}{6\sqrt{N}}\Big) e^{-s^2/2} d\psi \\
&\le \quad C\,\Delta_{1n} e^{-t^2/4} + C_1(1 + |t|^3)e^{-t^2/2} \le C\,\Delta_{1n} e^{-t^2/4}.
\end{aligned} \tag{50}
$$

Taking (47)-(50) into (44), we obtain the required (42).

Secondly we prove (41). Recall that $g_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\psi, t) d\psi$. As in (44), we have

$$\left| \frac{d\,\Psi(t)}{dt} - \frac{dg_n(t)}{dt} \right| \le III_1 + III_2 + III_3 + III_4, \tag{51}$$

where

$$\begin{aligned}
III_1 &= [B_n(p)]^{-1} \int_{|\psi| \le (nq)^{1/4}/4} \left| \frac{d\,h(\psi,t)}{dt} - \frac{d\,g(\psi,t)}{dt} \right| d\psi, \\
III_2 &= [B_n(p)]^{-1} \int_{(nq)^{1/4}/4 \le |\psi| \le \pi\sqrt{nq}} \left| \frac{d\,h(\psi,t)}{dt} \right| d\psi, \\
III_3 &= [B_n(p)]^{-1} \int_{|\psi| \ge (nq)^{1/4}/4} \Big(|t| + \frac{|t||f(\psi,t)|}{6\sqrt{N}} + \frac{1}{6\sqrt{N}} \Big| \frac{d\,f(\psi,t)}{dt} \Big| \Big) e^{-s^2/2} d\psi, \\
III_4 &= \Big| [B_n(p)]^{-1} - (2\pi)^{-1/2} \Big| \int_{-\infty}^{\infty} \Big(|t| + \frac{|t||f(\psi,t)|}{6\sqrt{N}} + \frac{1}{6\sqrt{N}} \Big| \frac{d\,f(\psi,t)}{dt} \Big| \Big) e^{-s^2/2} d\psi.
\end{aligned}$$

By (18)-(19) and (46), we have that for $|t| \le 1/4$

$$III_3 + III_4 \quad \le \quad C(\Delta_{1n} + (nq)^{-1}). \tag{52}$$

By (9), (23) and (46), we have that for $|t| \le 1/4$

$$III_1 + III_2 \quad \le \quad C(\Delta_{1n} + (nq)^{-1}). \tag{53}$$

Taking these estimates into (51), we obtain for $|t| \le 1/4$,

$$\big| \Psi(t) - g_n(t) \big| \quad \le \quad |t| \sup_{|x| \le 1/4} \left| \frac{d\,\Psi(x)}{dx} - \frac{dg_n(x)}{dx} \right| \ \le \ C|t| \, (\Delta_{1n} + (nq)^{-1}),$$

which yields (41).

Finally we prove (43). We first notice that $\Delta_{1n} \ge 1/(16nb)$. Indeed, if $\alpha_2 \le 1/4$, then

$$\Delta_{1n} \ge \Big(N^{-1} \sum E(X_k - p\mu_k)^2 \Big)^2 / (nb^2) \ge (1 - 2p\alpha_2)^2/(nb^2) \ge 1/(16nb),$$

and if $\alpha_2 > 1/4$, then $\Delta_{1n} \ge (N^{-1} \sum \mu_k^2)^2/(nb) = \alpha_2^2/(nb) \ge 1/(16nb)$. This, together with the fact that

$$\Delta^* \le \frac{9}{\sqrt{n} b^{3/2}} N^{-1} \sum_k E|X_k|^3,$$

implies that if $\delta_0(\Delta^*)^{-1} \le |t| \le T$, then $\delta_0 b/(9\mathcal{L}_0) \le |t|/\sqrt{nb} \le 16\sqrt{nb}$ and hence

$$\begin{aligned}
|h(\psi,t)|^2 &\le \prod_k \Big(1 - 2pq\Big(1 - E\cos(\psi/\sqrt{nq} + tX_k/\sqrt{nb})\Big)\Big) \\
&\le \exp\Big\{-2pq\sum_k \Big(1 - E\cos(\psi/\sqrt{nq} + tX_k/\sqrt{nb})\Big)\Big\} \\
&\le \exp\Big\{-2Npq\Big(1 - \Big|(1/N)E\sum_k \exp\{i\psi/\sqrt{nq} + itX_k/\sqrt{nb}\}\Big|\Big)\Big\} \\
&\le \exp\Big\{-2Npq\Big(1 - \Big|(1/N)\sum_k E\exp\{itX_k/\sqrt{nb}\}\Big|\Big)\Big\} \\
&\le \exp\{-2nq\,\delta_N\}. \qquad (54)
\end{aligned}$$

We also note that $\Delta^* \le 2\Delta_{1n}^{1/2}$ and this together with (45) implies that, for $\delta_0(\Delta^*)^{-1} \le |t| \le T$,

$$|g_n(t)| \le \int_{-\infty}^{\infty} \Big(1 + \frac{|f(\psi,t)|}{6\sqrt{N}}\Big)e^{-s^2/2}d\psi \le C(1+|t|^3)\,e^{-t^2/2} \le C\Delta_{1n}e^{-t^2/4}. \qquad (55)$$

Combining (54) and (55) and using the estimate (46), we obtain that, for $\delta_0(\Delta^*)^{-1} \le |t| \le T$,

$$\begin{aligned}
\big|\Psi(t) - g_n(t)\big| &\le [B_n(p)]^{-1}\int_{|\psi|\le\pi\sqrt{nq}} |h(\psi,t)|\,d\psi + |g_n(t)| \\
&\le C\,\Delta_{1n}e^{-t^2/4} + 3\sqrt{nq}\,\exp\big\{-nq\,\delta_N\big\},
\end{aligned}$$

which yields (43). The proof of Theorem 1 is complete. □

Proof of Theorem 2. Without loss of generality, assume $\Delta_{2n} \le 1$. We first prove the property (2) for $n/N \ge 1/2$ and $\Delta_{2n} \le (nq)^{-1}/25$.

Write $T^* = (nb^2)/\beta_4$, $T_1^* = \frac{b^{1/2}}{16}\min\big\{(n/\beta_4)^{1/4}, \frac{1}{8}b\sqrt{n}/\mathcal{L}_0\big\}$ and $T_2^* = \frac{1}{15}\sqrt{n}b^{3/2}/\mathcal{L}_0$. As in the proof of Theorem 1, it follows from Esseen's smoothing lemma that

$$\begin{aligned}
&\sup_x \Big|P(S_n/\sqrt{nb} \le x) - G_n(x)\Big| \\
&\quad\le \Big(\int_{|t|\le T_1^*} + \int_{T_1^*\le|t|\le T_2^*} + \int_{T_2^*\le|t|\le T^*} \frac{|\Psi(t)-g_n(t)|}{|t|}\,dt + C\,\Delta_{2n} \\
&\quad= \Lambda_{1n} + \Lambda_{2n} + \Lambda_{3n} + C\,\Delta_{2n}, \quad \text{say.} \qquad (56)
\end{aligned}$$

By virtue of Lemma 5, simple calculations show that $\Lambda_{1n} \le C\,\Delta_{2n}$. Recall $\Delta_{1n} \le 17\,\Delta_{2n}$. Applying Lemma 4 and similar arguments as in the proof of (50), we obtain that $\Lambda_{2n} \le C\,\Delta_{2n}$ and also $\int_{T_2^*\le|t|\le T^*} \frac{|g_n(t)|}{|t|}\,dt \le C\Delta_{2n}$. Therefore, to prove (2), it remains to show that, for $T_2^* \le |t| \le T^*$,

$$|\Psi(t)| \le 3\sqrt{n}\,\exp\big\{-n\,\delta_{1N}\big\}. \qquad (57)$$

In fact, by using (3) in Zhao, Wu and Wang (2004), for $q > 0$,

$$\Psi(t) = E\exp\{itS_n/\sqrt{nb}\} = (\sqrt{2\pi}G_n(p))^{-1}\int_{-\pi}^{\pi} e^{-in\psi}\prod_k \big(q + pe^{i\psi + itX_j/\sqrt{nb}}\big)d\psi,$$

where $G_n(p) = \sqrt{2\pi} C_N^n p^n q^{N-n}$. This, together with the fact that $\delta_0 b/(5\mathcal{L}_0) \le |t|/\sqrt{nb} \le 16\sqrt{nb}$ whenever $T_2^* \le |t| \le T^*$, implies that

$$\begin{aligned} |\Psi(t)| &\le \sqrt{2\pi}(G_n(p))^{-1} \prod_k \left(q + p|Ee^{itX_j/\sqrt{nb}}|\right) \\ &\le \sqrt{2\pi}(G_n(p))^{-1} \exp\left\{p \sum_k \left(|Ee^{itX_j/\sqrt{nb}}| - 1\right)\right\} \\ &\le 3\sqrt{n} \exp\{-n\delta_{1N}\}, \end{aligned}$$

where we have used the inequality $\sqrt{\pi}/2 \le \sqrt{nq} G_n(p) < 1$ (see, for instance, Lemma 1 in Höglund(1978)). This proves (57) for $q > 0$. If $q = 0$, then $N = n$ hence by the independence of X_k,

$$|\Psi(t)| = \prod_k |Ee^{itX_j/\sqrt{nb}}| \le \exp\left\{\sum_k \left(|Ee^{itX_j/\sqrt{nb}}| - 1\right)\right\} \le \exp\{-n\delta_{1N}\}.$$

This implies that (57) still holds for $q = 0$. We have now completed the proof of (57) and hence (2) for $n/N \ge 1/2$ and $\Delta_{2n} \le (nq)^{-1}/25$.

Note that $\beta_4 \ge 1$, $\Delta_{1n} \le 17\,\Delta_{2n}$ and $b \ge q \ge 1/2$ if $n/N \le 1/2$. We have that $\Delta_{1n} + (nq)^{-1} \le 42\,\Delta_{2n}$, whenever $n/N \le 1/2$ or $\Delta_{2n} > (nq)^{-1}/25$. Based on this fact, by using a similar argument to that above and that in the proof of Theorem 1, we may obtain (2) for $n/N \le 1/2$ or $\Delta_{2n} > (nq)^{-1}/25$, as well. The details are omitted. The proof of Theorem 2 is now complete. □

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