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Regularizing properties for transition semigroups and semilinear parabolic equations in Banach spaces

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Abstract

We study regularizing properties for transition semigroups related to Ornstein Uhlenbeck processes with values in a Banach space E which is continuously and densely embedded in a real and separable Hilbert space H. Namely we study conditions under which the transition semigroup maps continuous and bounded functions into differentiable functions. Via a Girsanov type theorem such properties extend to perturbed Ornstein Uhlenbeck processes. We apply the results to solve in mild sense semilinear versions of Kolmogorov equations in E.

Key words: Ornstein-Uhlenbeck and perturbed Ornstein-Uhlenbeck transition semigroups; regularizing properties; parabolic equations; Banach spaces.

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1 Introduction

In this paper we study transition semigroups, related to Ornstein Uhlenbeck and perturbed Ornstein Uhlenbeck processes with values in a Banach space E, that map continuous and bounded functions into differentiable functions. The Banach space E is continuously and densely embedded in a real and separable Hilbert space H. Similar properties have been extensively studied in relation to the strong Feller property of the semigroup, see e.g. (8). For such processes we also study a more general regularizing property, introduced in the paper (20) for Hilbert space valued processes. We apply these results to prove existence and uniqueness of a mild solution for a semilinear version of a Kolmogorov equation in E.

Namely, we study E-valued processes which are solutions of the stochastic differential equation

$$\begin{cases} dX_\tau = AX_\tau d\tau + GF\left(X_\tau\right)d\tau + GdW_\tau, & \tau\in[0,T] \\ X_0 = x\in E, \end{cases} \tag{1.1}$$

where A is the generator of a semigroup of bounded linear operators e^{tA}, $t\geq 0$, which extends to a strongly continuous semigroup in H with generator denoted by A_0. W is a cylindrical Wiener process in another real and separable Hilbert space Ξ, and G is a bounded linear operator from Ξ to H. Finally, we assume that for every $\xi\in\Xi$ and for every $t>0$, $e^{tA}G\xi\in E$ and $\left\|e^{tA}G\xi\right\|_E\leq ct^{-\alpha}\left\|\xi\right\|_\Xi$, for some constant $c>0$ and $0<\alpha<\dfrac{1}{2}$. The stochastic convolution

$$W_A\left(\tau\right) = \int_0^\tau e^{(\tau-s)A}GdW_s$$

is well defined as an H valued Gaussian process when its covariance operator

$$Q_\tau = \int_0^\tau e^{sA_0}GG^* e^{sA_0^*}ds,\ \tau\geq 0,$$

is a trace class operator on H; we assume further that the stochastic convolution $W_A\left(\tau\right)$ admits an E-continuous version. The map $F:E\to\Xi$ is continuous, Gateaux differentiable and F and its derivative have polynomial growth with respect to $x\in E$. Moreover $GF:E\to E$ is dissipative. When A, F and G satisfy these assumptions, there exists a unique mild solution, that is a predictable process $\left(X_\tau^{0,x}\right)$ in E satisfying $\mathbb{P}$-a.s. for $\tau\in[0,T]$,

$$X_\tau^{0,x} = e^{\tau A}x + \int_0^\tau e^{(\tau-s)A}GF\left(X_s^{0,x}\right)ds + \int_0^\tau e^{(\tau-s)A}GdW_s.$$

Let us denote by $Z_t^{0,x}$ the Gaussian process in E associated to the case $F=0$: $Z_t^{0,x} = e^{tA}x + \int_0^t e^{(t-s)A}GdW_s$. Let us denote by R_t the transition semigroup associated to the process $Z_t^{0,x}$. First we prove that R_t maps continuous and bounded functions into Gateaux differentiable ones, if for $0<t\leq T$

$$e^{tA_0}\left(H\right)\subset Q_t^{1/2}\left(H\right).$$

Moreover, if there exists C_t, depending on t, such that the operator norm satisfies

$$\left\|Q_t^{-1/2}e^{tA_0}\right\|_{L(H,H)}\leq C_t,\ \text{for } 0<t\leq T,$$

then for every bounded and continuous function $\varphi : E \to \mathbb{R}$,

$$|\nabla R_t [\varphi](x)\, e| \leq C_t \|\varphi\|_\infty \|e\|_E, \quad 0 < t \leq T,\ x, e \in E.$$

This extends well known results in (8) for Hilbert space valued processes. By a Girsanov type theorem we are able to extend this property of the semigroup R_t, to the transition semigroup P_t, associated to the process $X_t^{0,x}$ in the case of F not equal to 0. A similar result is presented in the paper (2), for Hilbert space valued processes: in that paper the proof is achieved by means of the Malliavin calculus. We also remember that in the monograph (4) such a regularizing property for transition semigroups associated to reaction diffusion equations is achieved under weaker assumptions than ours, on the contrary in (4) no application to Kolmogorov equations is performed.

In analogy, we prove that if

$$e^{tA_0} G(\Xi) \subset Q_t^{1/2}(H)$$

and if there exists a constant C_t, depending on t, such that the operator norm satisfies

$$\left\| Q_t^{-1/2} e^{tA_0} G \right\|_{L(\Xi,H)} \leq C_t, \text{ for } 0 < t \leq T,$$

then for every bounded and continuous function $\varphi : E \to \mathbb{R}$, $R_t[\varphi]$ is Gateaux differentiable in E in the directions selected by G. More precisely, for a map $f : E \to \mathbb{R}$ the G-directional derivative ∇^G at a point $x \in E$ in direction $\xi \in \Xi$ is defined as follows, see (20):

$$\nabla^G f(x)\, \xi = \lim_{s \to 0} \frac{f(x + sG\xi) - f(x)}{s},\ s \in \mathbb{R}.$$

This definition makes sense for ξ in a dense subspace $\Xi_0 \subset \Xi$ such that $G(\Xi_0) \subset E$, for more details see section 2. We prove that for every bounded and continuous function $\varphi : E \to \mathbb{R}$ the G-derivative of $R_t[\varphi]$, satisfies, for every $0 < t \leq T$,

$$\left|\nabla^G R_t[\varphi](x)\, \xi\right| \leq C_t \|\varphi\|_\infty \|\xi\|_\Xi, \quad x \in E,\ \xi \in \Xi.$$

Again by applying the Girsanov type theorem we are able to extend this property of the semigroup R_t, to the transition semigroup P_t.

One of the main motivations is to find a unique mild solution for second order partial differential equations of parabolic type in the Banach space E. Second order partial differential equations in infinite dimensions have been extensively studied in the literature: we cite the monograph (10) as a general reference, and the papers (14) and (20), where the non linear case is treated. In that book and those papers the notion of mild solution is considered and the second order partial differential equation is studied in an infinite dimensional Hilbert space. For partial differential equations of parabolic type on an infinite dimensional Hilbert space, also the notion of viscosity solution has been studied by many authors, we refer to (5) as a general reference, and to the fundamental papers (17) and (18). We also cite (15) and (16) where the notion of viscosity solution is introduced for Hamilton Jacobi Bellman equation related respectively to boundary stochastic optimal control problems and to stochastic optimal control of the Duncan-Mortensen-Zakai equation. On of the main motivation to study Hamilton Jacobi Bellman equations is to solve a stochastic optimal control problem. In this paper we study equations on a Banach space E of the following form:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t}(t,x) = -\mathcal{A}_t u\,(t,x) + \psi\,(t,x,u\,(t,x)\,,\nabla u\,(t,x))\,, & t\in[0,T]\,,\ x\in H \\ u(T,x) = \varphi\,(x)\,, & \end{array} \right. \tag{1.2}$$

where $\mathcal{A}_t$ is formally defined by

$$\mathcal{A}_t v\,(x) = \frac{1}{2} Trace\left(GG^*\nabla^2 v\,(x)\right) + \langle Ax, \nabla v\,(x)\rangle_{E,E^*} + \langle GF\,(x)\,, \nabla v\,(x)\rangle_{E,E^*}.$$

A, F and G are the coefficients of the stochastic differential equation (1.1), ψ is a map from $[0,T]\times E\times\mathbb{R}\times E^*$ with values in $\mathbb{R}$ and $\varphi : E\to\mathbb{R}$ is bounded and continuous. By mild solution of (1.2) we mean a continuous function $u : [0,T]\times E\to\mathbb{R}$, Gateaux differentiable with respect to x for every fixed $t>0$, satisfying the integral equation

$$u(t,x) = P_{t,T}\,[\varphi]\,(x) - \int_t^T P_{t,s}\,[\psi\,(s,\cdot,u\,(s,\cdot)\,,\nabla u\,(s,\cdot))]\,(x)\,ds, \quad t\in[0,T]\,,\ x\in E.$$

In order to find a unique mild solution we use a fixed point argument, so we have to impose some Lipschitz conditions on ψ, and a regularizing property for the semigroup P_t: for every bounded and continuous function $\varphi : E\to\mathbb{R}$, we assume that $P_t\,[\varphi]\,(x)$ is Gateaux differentiable in E and for every $0<t\leq T$, $x,e\in E$

$$|\nabla P_t\,[\varphi]\,(x)\,e| \leq C_t\,\|\varphi\|_\infty\,\|e\|_E\,, \quad 0<t\leq T,$$

with $t\mapsto C_t$ in $L^1\,([0,T])$. We prove similar results for equations of the form

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t}(t,x) = -\mathcal{A}_t u\,(t,x) + \psi\left(t,x,u\,(t,x)\,,\nabla^G u\,(t,x)\right), & t\in[0,T]\,,\ x\in H \\ u(T,x) = \varphi\,(x)\,. & \end{array} \right. \tag{1.3}$$

In all the paper we will refer to equations (1.2) and (1.3) as semilinear Kolmogorov equations. Equation (1.3) has a less general structure than equation (1.2), but it can be solved in mild sense under weaker assumptions on the transition semigroup P_t: we need only to require that for every bounded and continuous function $\varphi : E\to\mathbb{R}$, $P_t\,[\varphi]$ is G-Gateaux differentiable in E and for every $t>0$, $x\in E$ and $\xi\in\Xi$

$$\left|\nabla^G P_t\,[\varphi]\,(x)\,\xi\right| \leq C_t\,\|\varphi\|_\infty\,\|\xi\|_\Xi\,, \quad 0<t\leq T,$$

with $t\mapsto C_t$ in $L^1\,([0,T])$. For what concerns already known results on partial differential equations in Banach spaces, we are not aware of any paper treating viscosity solutions for partial differential equations of parabolic type in Banach spaces; in the papers (6) and (7) viscosity solutions are presented for first order partial differential equations in Banach spaces. We also cite the paper (21), where second order partial differential equations in Banach spaces are solved in mild sense with a completely probabilistic approach: (21) generalizes the results obtained in (11), where Hamilton Jacobi Bellman equations in a Hilbert space are studied by means of backward stochastic differential equations. In (21) equations with the form of equation (1.3) are considered: φ and ψ are taken more regular than in the present paper, on the contrary no regularizing property on the transition semigroup is asked. In this paper we weaken the regularity assumptions on ψ and φ, see the discussion in section 5 for more details.

The paper is organized as follows: in section 2 we we study regularizing properties for Ornstein Uhlenbeck transition semigroups, in section 3 we present a Girsanov type theorem for a suitable perturbation of the Ornstein Uhlenbeck process, in section 4 we study regularizing properties for perturbed Ornstein Uhlenbeck processes: we want to stress the fact that to do this we cannot make a direct use of Malliavin calculus, as in (2), since we are working in Banach spaces. In section 5 we study mild solutions of semilinear Kolmogorov equations in Banach spaces and in section 6 we present some models where our results apply.

2 Regularizing properties for Ornstein Uhlenbeck semigroups

In the following, with the letter X and E we denote Banach spaces. $L(X, E)$ denotes the space of bounded linear operators from X to E, endowed with the usual operator norm. If E is a Banach space we denote by E^* its dual space. We denote by $C_b(X)$ the space of bounded and continuous functions from X to $\mathbb{R}$ endowed with the supremum norm. We denote by $C_b^1(X)$ the space of bounded and continuous functions from X to $\mathbb{R}$, with a bounded and continuous Fréchet derivative. With the letters Ξ and H we will always denote Hilbert spaces, with scalar product $\langle \cdot, \cdot \rangle$. All Hilbert spaces are assumed to be real and separable. For maps acting among topological spaces, by measurability we mean Borel measurability.

From now on, let E be a real and separable Banach space and we assume that E admits a Schauder basis. Moreover E is continuously and densely embedded in a real and separable Hilbert space H. By the Kuratowski theorem, see e.g. (22), Chapter I, Theorem 3.9, it follows that E is a Borel set in H.

We are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a filtration $\{\mathcal{F}_\tau,\ \tau \geq 0\}$ satisfying the usual conditions. For every $T > 0$, $\mathcal{H}^p([0,T], E)$ is the space of predictable processes $(Y_\tau)_{\tau\in[0,T]}$ with values in E, admitting a continuous version and such that

$$\mathbb{E} \sup_{\tau\in[0,T]} \|Y_\tau\|_E^p < \infty.$$

The Ornstein-Uhlenbeck process is the solution in E to equation

$$\begin{cases} dX_\tau = AX_\tau d\tau + GdW_\tau, & \tau \in [0,T] \\ X_0 = x \in E. \end{cases} \tag{2.1}$$

A is a linear operator in E with domain $D(A)$, W is a cylindrical Wiener process in another real and separable Hilbert space Ξ, and G is a linear operator from Ξ to H. We will need the following assumptions. We refer to (19) for the definition of sectorial operator.

Hypothesis 2.1. *We assume that either*

1. *A generates a C_0 semigroup in E; or*
2. *A is a sectorial operator in E.*

In both cases we denote by e^{tA}, $t \geq 0$, the semigroup of bounded linear operators on E generated by A and we suppose that there exists $\omega \in \mathbb{R}$ such that $\left\|e^{tA}\right\|_{L(E,E)} \leq e^{\omega t}$, for all $0 \leq t \leq T$. We assume that e^{tA}, $t \geq 0$, admits an extension to a C_0 semigroup of bounded linear operators in H, whose generator is denoted by A_0 or by A if no confusion is possible.

We have to make assumptions on G and on the stochastic convolution

$$W_A(\tau) = \int_0^\tau e^{(\tau-s)A} G dW_s.$$

Let us introduce the nonnegative symmetric operators $Q_\tau \in L(H,H)$ given by

$$Q_\tau = \int_0^\tau e^{sA_0} GG^* e^{sA_0^*} ds, \ \tau \geq 0.$$

Hypothesis 2.2. *1. $G \in L(\Xi, H)$ is such that the operators Q_τ are of trace class for every $\tau \in [0,T]$.*

2. The stochastic convolution $W_A(\tau)$ admits an E-continuous version.

It is well known that under assumption 1 alone, W_A is a well defined Gaussian process in H and Q_τ is the covariance operator of $W_A(\tau)$. Assumption 2 strengthens these properties. With hypotheses 2.1 and 2.2 there exists a unique mild solution $(X_\tau^x)_{\tau\in[0,T]}$ to equation (2.1): by definition,

$$X_\tau^x = e^{\tau A}x + W_A(\tau), \qquad \tau \in [0,T].$$

The Ornstein-Uhlenbeck semigroup is defined as

$$R_t[\phi](x) = \int_E \phi\left(e^{tA}x + y\right) \mu_t(dy),$$

where $\mu_t(dy)$ is a symmetric gaussian measure in E. Indeed, due to hypothesis 2.2, the stochastic convolution W_A is a Gaussian random variable with values in $C([0,T], E)$ with mean equal to $e^{tA}x$, see e.g. (21). Let $\mathcal{N}\left(e^{tA}x, Q_t\right)(dy)$ denote the Gaussian measure in H with mean $e^{tA}x$, and covariance operator Q_t.

Lemma 2.3. *The Gaussian measures $\mu_t(dy)$ and $\mathcal{N}(0, Q_t)(dy)$ admits the same reproducing kernel, which is given by $Q_t^{1/2}(H)$. In particular $Q_t^{1/2}(H) \subset E$.*

Proof. It is well known, see e.g. (8), that the reproducing kernel of the Gaussian measure $\mathcal{N}(0, Q_t)(dy)$ in H is given by $Q_t^{1/2}(H)$. To conclude the proof it suffices to apply proposition 2.8 in (8). By this proposition if a separable Banach space E_1 is continuously and as a Borel set embedded in another separable Banach space E_2, and if μ is a symmetric Gaussian measure on E_1 and E_2, then the reproducing kernel spaces calculated with respect to E_1 and E_2 are the same. $\square$

In order to relate regularizing properties of the semigroup R_t to the operators A and G we make the following assumptions:

Hypothesis 2.4. *Assume that for $0 < t \leq T$ we have*

$$e^{tA_0}(H) \subset Q_t^{1/2}(H). \tag{2.2}$$

We denote by C_t a constant such that

$$\left\| Q_t^{-1/2} e^{tA_0} \right\|_{L(H,H)} \leq C_t, \text{ for } 0 < t \leq T.$$

Hypothesis 2.5. *Assume that for* $0 < t \leq T$ *we have*

$$e^{tA_0} G(\Xi) \subset Q_t^{1/2}(H). \tag{2.3}$$

We denote by C_t *a constant such that*

$$\left\| Q_t^{-1/2} e^{tA_0} G \right\|_{L(\Xi,H)} \leq C_t, \textit{ for } 0 < t \leq T.$$

We note that if (2.2), respectively (2.3), holds then the operator $Q_t^{-1/2} e^{tA_0}$, respectively $Q_t^{-1/2} e^{tA_0} G$, is bounded by the closed graph theorem.

We recall that a map $f : E \to \mathbb{R}$ is Gateaux differentiable at a point $x \in E$ if f admits the directional derivative $\nabla f(x; e)$ in every directions $e \in E$ and there exists a functional, the gradient $\nabla f(x) \in E^*$, such that $\nabla f(x; e) = \nabla f(x) e$. f is Gateaux differentiable on E if it is Gateaux differentiable at every point $x \in E$. We denote by $\mathcal{G}(E)$ the class of continuous functions $f : E \to \mathbb{R}$ that are Gateaux differentiable on E and such that for every $e \in E$, $\nabla f(\cdot) e$ is continuous from E to $\mathbb{R}$.

Let $G \in L(\Xi, H)$; we recall that for a continuous function $f : H \to \mathbb{R}$ the G-directional derivative ∇^G at a point $x \in E$ in direction $\xi \in \Xi$ is defined as follows, see (20):

$$\nabla^G f(x; \xi) = \lim_{s \to 0} \frac{f(x + sG\xi) - f(x)}{s}, \ s \in \mathbb{R}.$$

A continuous function f is G-Gateaux differentiable at a point $x \in H$ if f admits the G-directional derivative $\nabla^G f(x; \xi)$ in every directions $\xi \in \Xi$ and there exists a functional, the $G-$gradient $\nabla^G f(x) \in \Xi^*$ such that $\nabla^G f(x; \xi) = \nabla^G f(x) \xi$. We want to extend this definition to continuous functions $f : E \to \mathbb{R}$, where $E \in H$ is a Banach space. In general, we can not guarantee that $G(\Xi) \subset E$. We make the following assumptions which is verified in most of the applications.

Hypothesis 2.6. *There exists a subspace* Ξ_0 *dense in* Ξ *such that* $G(\Xi_0) \subset E$.

Definition 2.7. For a map $f : E \to \mathbb{R}$ the G-directional derivative ∇^G at a point $x \in E$ in direction $\xi \in \Xi_0$ is defined as follows:

$$\nabla^G f(x; \xi) = \lim_{s \to 0} \frac{f(x + sG\xi) - f(x)}{s}, \ s \in \mathbb{R}.$$

We say that a continuous function f is G-Gateaux differentiable at a point $x \in E$ if f admits the G-directional derivative $\nabla^G f(x; \xi)$ in every directions $\xi \in \Xi_0$ and there exists a linear operator $\nabla^G f(x)$ from Ξ_0 with values in $\mathbb{R}$, such that $\nabla^G f(x; \xi) = \nabla^G f(x) \xi$ and $\left|\nabla^G f(x) \xi\right| \leq C_x \|\xi\|_\Xi$, where C_x does not depend on ξ. So the operator $\nabla^G f(x)$ can be extended to the whole Ξ, and we denote this extension again by $\nabla^G f(x)$, the G-gradient of f at x. We say that f is G-Gateaux differentiable on E if it is G-Gateaux differentiable at every point $x \in E$. We denote by $\mathcal{G}^G(E)$ the class of bounded and continuous functions $f : E \to \mathbb{R}$ that are G-Gateaux differentiable on E and such that for every $\xi \in \Xi$, $\nabla^G f(\cdot) \xi$ is continuous from E to $\mathbb{R}$.

Lemma 2.8. *Let hypotheses 2.1 and 2.2 hold true.*

1. If hypothesis 2.4 holds true, then for every $\varphi \in C_b(E)$, $P_t[\varphi] \in \mathcal{G}(E)$ and

$$|\nabla R_t[\varphi](x)e| \leq cC_t \|\varphi\|_\infty \|e\|_E, \qquad 0 < t \leq T. \tag{2.4}$$

2. If hypothesis 2.5 holds true, then for every $\varphi \in C_b(E)$, $P_t[\varphi] \in \mathcal{G}^G(E)$ and

$$\left|\nabla^G R_t[\varphi](x)\xi\right| \leq cC_t \|\varphi\|_\infty \|\xi\|_\Xi, \qquad t > 0. \tag{2.5}$$

Proof. We prove only 2, point 1 can be proved with similar arguments. We remark that when $\xi \in \Xi$, then $G\xi \in H$, but we cannot guarantee that $G\xi \in E$, anyway as a consequence of lemma 2.3, it turns out that $Q_t^{1/2}(H) \subset E$, so if inclusion (2.3) holds true, then $e^{tA_0}G(\Xi) \subset E$, so for every $\xi \in \Xi$, $e^{tA_0}G\xi \in E$ for every $t > 0$. So, for every $\xi \in \Xi$, $\nabla^G(P_t[\varphi])(x)\xi$ is well defined by the formula

$$\begin{aligned}
&\nabla^G(R_t[\varphi])(x)\xi \\
&= \lim_{s\to 0}\frac{1}{s}\mathbb{E}\left[\varphi\left(e^{tA}(x+sG\xi)+\int_0^t e^{(t-r)A}GdW_r\right)-\varphi\left(e^{tA}x+\int_0^t e^{(t-r)A}GdW_r\right)\right].
\end{aligned}$$

Let $\xi \in \Xi$ and consider

$$\begin{aligned}
&\nabla^G(R_t[\varphi])(x)\xi \\
&= \lim_{s\to 0}\frac{1}{s}\left[\mathbb{E}\varphi\left(e^{tA}(x+sG\xi)+\int_0^t e^{(t-r)A}GdW_r\right)-\varphi\left(e^{tA}x+\int_0^t e^{(t-r)A}GdW_r\right)\right] \\
&= \lim_{s\to 0}\frac{1}{s}\left[\int_E \varphi\left(y+e^{tA}G\xi\right)\mu_t^s(dy)-\int_E \varphi\left(y+e^{tA}x\right)\mu_t(dy)\right],
\end{aligned}$$

where $\mu_t^s(dy) = e^{tA}sG\xi + \mu_t(dy)$, and it is a gaussian measure on E with mean equal to $e^{tA}sG\xi$. We denote by $H(\mu_t)$ the reproducing kernel of the gaussian measure μ_t. By lemma 2.3, it coincides with $Q_t^{1/2}(H)$. Since by our assumptions $e^{tA}G\xi \in Q_t^{1/2}(H)$, the gaussian measures $\mu_t^s(dy)$ and $\mu_t(dy)$ are equivalent. Let us denote

$$d(t,\xi,s,y) := \frac{d\mu_t^s}{d\mu_t}(y)$$

the Radon-Nikodym derivative. Following (1), we denote by a_{μ_t} the mean of μ_t: for every $f \in E^*$, a_{μ_t} is defined as

$$a_{\mu_t}(f) = \int_E f(y)\,d\mu_t(y).$$

Let $E^*_{\mu_t}$ be the closure in $L^2(E,\mu_t)$ of the set $\{f - a_{\mu_t}(f) : f \in E^*\}$. Consider R_{μ_t} the covariance operator of μ_t: for every $f, g \in E^*$ it is defined as

$$R_{\mu_t}(f)(g) = \int_E (f(y) - a_{\mu_t}(f))(g(y) - a_{\mu_t}(g))\,d\mu_t(y).$$

It turns out that $h \in H(\mu_t)$ if and only if there exists $g \in E^*_{\mu_t}$ such that $h = R_{\mu_t}(g)$. By proposition 2.8 in (8), the reproducing kernel $H(\mu_t)$ coincides with $Q_t^{1/2}(H)$. So by our assumptions $se^{tA}G\xi \in H(\mu_t)$: there exists $g \in E^*_{\mu_t}$ such that $e^{tA}G\xi = R_{\mu_t}(g)$ and $se^{tA}G\xi = sR_{\mu_t}(g)$.

By the Cameron-Martin formula, see e.g. (1), corollary 2.4.3,

$$d\left(t,\xi,s,y\right)=\exp\left\{sg\left(y\right)-\frac{1}{2}s^{2}\left\|e^{tA}G\xi\right\|_{H(\mu_t)}^{2}\right\}.$$

We get

$$\begin{aligned}\nabla^{G}(R_t\left[\varphi\right])\left(x\right)\xi&=\lim_{s\to0}\int_{E}\varphi\left(y+e^{tA}x\right)\frac{\left(d\left(t,\xi,s,y\right)-1\right)}{s}\mu_t\left(dy\right)\\&=\int_{E}\varphi\left(y+e^{tA}x\right)\lim_{s\to0}\frac{\left(d\left(t,\xi,s,y\right)-1\right)}{s}\mu_t\left(dy\right),\end{aligned}$$

where in the last passage we have used dominated convergence, since φ is bounded and

$$\left|\frac{d\left(t,\xi,s,y\right)-1}{s}\right|\leq\left|g\left(y\right)-\frac{1}{2}s\left\|e^{tA}G\xi\right\|_{H(\mu_t)}^{2}\right|.$$

But $g\in E^{*}_{\mu_t}$, and so in particular $g\in L^{2}\left(E,\mu_t\right)$. So

$$\nabla^{G}(R_t\left[\varphi\right])\left(x\right)\xi=\int_{E}\varphi\left(y+e^{tA}x\right)g\left(y\right)\mu_t\left(dy\right).$$

We conclude that

$$\begin{aligned}\left|\nabla^{G}(R_t\left[\varphi\right])\left(x\right)\xi\right|&\leq\sup_{z\in E}\left|\varphi\left(z\right)\right|\int_{E}\left|g\left(y\right)\right|\mu_t\left(dy\right)\\&\leq\sup_{z\in E}\left|\varphi\left(z\right)\right|\left(\int_{E}\left|g\left(y\right)\right|^{2}\mu_t\left(dy\right)\right)^{1/2}\\&=\sup_{z\in E}\left|\varphi\left(z\right)\right|\left\|g\right\|_{L^{2}(E,\mu_t)}\\&=\sup_{z\in E}\left|\varphi\left(z\right)\right|\left\|e^{tA}G\xi\right\|_{H(\mu_t)}\\&\leq c\sup_{z\in E}\left|\varphi\left(z\right)\right|\left\|Q_t^{-1/2}e^{tA}G\right\|_{L(\Xi,H)}\left\|\xi\right\|_{\Xi}.\end{aligned}$$

In the fourth passage we have used the fact that as a map from $E^{*}_{\mu_t}$ to $H\left(\mu_t\right)$, R_{μ_t} is an isometric isomorphism, and in the last passage we have used the fact that the reproducing kernel of $H\left(\mu_t\right)$ is given by $\operatorname{Im}Q_t^{1/2}$, see lemma 2.3. We have concluded that

$$\left|\nabla(R_t\left[\varphi\right])\left(x\right)e\right|\leq c\sup_{z\in H}\left|\varphi\left(z\right)\right|\left\|Q_t^{-1/2}e^{tA}\right\|_{L(H,H)}\left\|e\right\|_{E}.$$

□

Remark 2.9. In the case of the Ornstein Uhlenbeck process, we are able to relate the assumption on the G-derivative of $R_t\left[\varphi\right]$ with properties of A and G. Also, in Hilbert spaces there are examples of Ornstein Uhlenbeck processes when it is clear that hypothesis 2.5 is less restrictive than 2.4, see (20). In particular we remember that, in the Hilbert space case, inclusion $\operatorname{Im}e^{tA}\subset\operatorname{Im}Q_t^{1/2}$ is equivalent to the strong Feller property of the Ornstein Uhlenbeck semigroup, see (8).

3 A Girsanov type theorem

We recall a result in (12) on a theorem of Girsanov type. Consider a stochastic differential equation

$$\begin{cases} dX_\tau = AX_\tau d\tau + b\left(X_\tau\right) d\tau + GdW_\tau, & \tau \in [0,T] \\ X_0 = x \in E, \end{cases} \tag{3.1}$$

with A and G satisfying hypotheses 2.1 and 2.2. Moreover we make the following assumption on b and G.

Hypothesis 3.1. *$b : E \to E$ is continuous and there exists an increasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} a(t) = \infty$ such that for every $y \in E$ and $x \in \mathcal{D}(A)$, $\langle Ax + b(x+y), x^* \rangle_{E,E^*} \leq a(\|y\|) - k\|x\|$, for some $k \geq 0$ and some $x^* \in \partial \|x\|$, the subdifferential of the norm of x. Moreover assume that G is invertible and that $\left\|G^{-1}\right\|_{L(H,\Xi)} \leq C$.*

Denote by X_τ^x the solution of equation (3.1) and by Z_τ^x the solution of equation (2.1). Then the following result holds true, see (12), theorem 1.

Theorem 3.2. *Let A, b and G satisfy hypotheses 2.1, 2.2 and 3.1. Then, for every $\varphi \in C_b(E)$,*

$$\mathbb{E}\varphi\left(X_\tau^x\right) = \mathbb{E}\left[\varphi\left(Z_\tau^x\right) \exp\left\{\int_0^\tau \left\langle G^{-1} b\left(Z_s^x\right), dW_s\right\rangle - \frac{1}{2}\int_0^\tau \left\|G^{-1} b\left(Z_s^x\right)\right\|_\Xi^2 ds\right\}\right]$$

Here we prove an analogous result for a different perturbation of the Ornstein Uhlenbeck process Z_τ^x, the solution of equation (2.1). Let us consider a stochastic differential equation

$$\begin{cases} dX_\tau = AX_\tau d\tau + f\left(X_\tau\right) d\tau + GdW_\tau, & \tau \in [0,T] \\ X_0 = x \in E. \end{cases} \tag{3.2}$$

We make the following assumptions on the non linear term f.

Hypothesis 3.3. *$f : E \longrightarrow E$ is continuous and there exists $\eta \in \mathbb{R}$ such that $A + f - \eta$ is dissipative on E. There exists $k \geq 0$ such that $\|f(x)\|_E \leq c\left(1 + \|x\|_E^k\right)$. Moreover for every $x \in E$, $f(x) \in G(\Xi)$ and we denote by $F(x) = G^{-1} f(x)$, so $F : E \longrightarrow \Xi$. We assume that as a map from E to Ξ, F is Gateaux differentiable and there exists $j \geq 0$ such that F and its derivative satisfy the following inequalities, for every $x, e \in E$:*

$$\|F(x)\|_\Xi \leq c\left(1 + \|x\|_E^j\right), \quad \|\nabla F(x) e\|_\Xi \leq c\left(1 + \|x\|_E^j\right) \|e\|_E.$$

Theorem 3.4. *Assume that hypotheses 2.1, 2.2 and 3.3 hold true. Then for every $\varphi \in C_b(E)$,*

$$\mathbb{E}\varphi\left(X_\tau^x\right) = \mathbb{E}\left[\varphi\left(Z_\tau^x\right) \exp\left\{\int_0^\tau \left\langle G^{-1} f\left(Z_s^x\right), dW_s\right\rangle - \frac{1}{2}\int_0^\tau \left\|G^{-1} f\left(Z_s^x\right)\right\|_\Xi^2 ds\right\}\right].$$

Proof. We follow (12), theorem 1, and a simple idea well known already to Girsanov, see (13). We define $\rho_T^x = \exp\left\{\int_0^T \left\langle G^{-1} f\left(Z_s^x\right), dW_s\right\rangle - \frac{1}{2}\int_0^T \left\|G^{-1} f\left(Z_s^x\right)\right\|_\Xi^2 ds\right\} = \exp V_T^x$ and the sequence of stopping times $\tau_n = \inf\left\{t > 0 : \int_0^t \left\|G^{-1} f\left(Z_s^x\right)\right\|_\Xi^2 ds > n\right\} \wedge T$. We define the probability measures $\mathbb{P}^n(d\omega) = \rho_{\tau_n}^x \mathbb{P}(d\omega)$. The Novikov condition implies that

$$W_t^n = W_t - \int_0^{t\wedge\tau_n} \left\|G^{-1} f\left(Z_s^x\right)\right\|_\Xi^2 ds$$

is a cylindrical Wiener process under the probability $\mathbb{P}^n$. We claim that $\mathbb{P}^n(\tau_n = T) \to 1$ as $n \to \infty$. In this case, in spite of the fact that Novikov condition cannot be applied directly, it is immediate that $\mathbb{E}\rho_T^x = 1$:

$$\mathbb{E}\rho_T^x \geq \mathbb{E}\left[\rho_{\tau_n}^x 1_{\{\tau_n=T\}}\right] = \mathbb{P}^n(\tau_n = T).$$

So the Theorem follows by the Girsanov theorem. We evaluate, by Markov inequality,

$$\mathbb{P}^n(\tau_n < T) = \mathbb{P}^n\left(\int_0^T \left\|G^{-1} f\left(Z_s^x\right)\right\|_\Xi^2 ds > n\right) \leq \frac{1}{n}\mathbb{E}^n \int_0^T \left\|G^{-1} f\left(Z_s^x\right)\right\|_\Xi^2 ds.$$

With respect to the probability measure $\mathbb{P}^n$, Z_τ^x is solution to the equation

$$\begin{cases} dZ_\tau^x = AZ_\tau^x d\tau + f\left(Z_\tau^x\right) 1_{[0,\tau_n]}(\tau)\, d\tau + GdW_\tau^n, & \tau \in [0,T] \\ Z_0^x = x \in E. \end{cases}$$

Let $Y_\tau^x = Z_\tau^x - W_A^n(\tau)$ and $Y_{\tau,\lambda}^x = \lambda R(\lambda, A)\, Y_\tau^x$, where $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator of A. Since $Y_{\tau,\lambda}^x \in \mathcal{D}(A)$ for every $\tau \in [0,T]$, and

$$Y_{\tau,\lambda}^x = e^{\tau A}\lambda R(\lambda, A)\, x + \int_0^\tau e^{(\tau-s)A}\lambda R(\lambda, A)\, f\left(Y_s^x + W_A^n(s)\right) 1_{[0,\tau_n]}(s)\, ds,$$

it follows that, on $[0,\tau_n]$, $Y_{\tau,\lambda}^x$ satisfies the equation

$$\begin{aligned} \frac{dY_{\tau,\lambda}^x}{d\tau} &= AY_{\tau,\lambda}^x + \lambda R(\lambda, A)\, f\left(Y_{\tau,\lambda}^x + W_A^n(\tau)\right) \\ &= AY_{\tau,\lambda}^x + f\left(Y_{\tau,\lambda}^x + W_A^n(\tau)\right) + \left[\lambda R(\lambda, A)\, f\left(Y_{\tau,\lambda}^x + W_A^n(\tau)\right) - f\left(Y_{\tau,\lambda}^x + W_A^n(\tau)\right)\right] \\ &:= AY_{\tau,\lambda}^x + f\left(Y_{\tau,\lambda}^x + W_A^n(\tau)\right) + \delta_{\tau,\lambda}. \end{aligned}$$

So, on $[0,\tau_n]$, and for some $y^* \in \partial\left\|Y_{\tau,\lambda}^x\right\|_E$, the lower derivative of $\left\|Y_{\tau,\lambda}^x\right\|_E$ satisfies the equation

$$\begin{aligned} \frac{d^-\left\|Y_{\tau,\lambda}^x\right\|_E}{d\tau} &= \left\langle AY_{\tau,\lambda}^x + f\left(Y_{\tau,\lambda}^x + W_A^n(\tau)\right) - f\left(W_A^n(\tau)\right) + f\left(W_A^n(\tau)\right) + \delta_{\tau,\lambda}, y^*\right\rangle_{E,E^*} \\ &\leq \eta\left\|Y_{\tau,\lambda}^x\right\|_E + c\left(1 + \left\|W_A^n(\tau)\right\|_E^k\right) + \left\|\delta_{\tau,\lambda}\right\|_E. \end{aligned}$$

By the Gronwall lemma

$$\left\|Y_{\tau,\lambda}^x\right\|_E \leq \|x\|_E\, e^{|\eta|T} + c\int_0^T \left(1 + \left\|W_A^n(s)\right\|_E^k\right) 1_{[0,\tau_n]}(s)\, ds + \int_0^T \left\|\delta_{s,\lambda}\right\|_E ds.$$

So, letting λ tend to ∞, we get $\|Y^x_\tau\|_E \leq \|x\|_E\, e^{|\eta| T} + c\int_0^T e^{|\eta| s}\left(1+\|W^n_A(s)\|^k_E\right) 1_{[0,\tau_n]}(s)\, ds$, and so for every $p \geq 1$

$$\begin{aligned}\mathbb{E}\sup_{\tau\in[0,T]}\|Y^x_\tau\|^p_E &\leq \|x\|^p_E\, e^{p|\eta|T} + c\mathbb{E}\left(\int_0^T \left(1+\|W_A(s)\|^k_E\right) e^{|\eta|s} ds\right)^p\\ &\leq \|x\|^p_E\, e^{p|\eta|T} + C_{T,k,p},\end{aligned}$$

where the last estimate follows from the fact that, as a process with values in $C([0,T], E)$, the stochastic convolution is a Gaussian process and so it has finite moments of every order. So the process $(Y^x_\tau)_\tau$, and consequently the process $(Z^x_\tau)_\tau$ belongs to $\mathcal{H}^p([0,T], E)$ for every $1 \leq p < \infty$. By polynomial growth assumptions on $G^{-1}f$ we get

$$\mathbb{E}^n \int_0^T \left\|G^{-1}f(Z^x_s)\right\|^2_\Xi ds \leq C,$$

with C independent of n, and consequently $\mathbb{P}^n(\tau_n = T) \to 1$ as $n \to \infty$. $\square$

4 Regularizing properties of the semigroup: from the Ornstein Uhlenbeck semigroup to the perturbed Ornstein Uhlenbeck semigroup

Let us consider the Ornstein Uhlenbeck process Z^x_τ which is a mild solution of the stochastic differential equation with values in E:

$$\begin{cases} dZ^x_\tau = AZ^x_\tau d\tau + GdW_\tau, & \tau\in[0,T]\\ Z_0 = x \in E. \end{cases}$$

In this section we assume that A and G satisfy hypotheses 2.1 and 2.2. Moreover we have to make one more assumption: as a consequence of lemma 2.3, it turns out that $Q_t^{1/2}(H) \subset E$, so if inclusion (2.2) or inclusion (2.3) hold true, then $e^{tA_0}G(\Xi) \subset E$. We have to make the following assumption on the norm of the operator $e^{tA_0}G \in L(\Xi, E)$, for every $t > 0$.

Hypothesis 4.1. *For every $\xi \in \Xi$ and for every $t > 0$ $\left\|e^{tA_0}G\xi\right\|_E \leq ct^{-\alpha}\|\xi\|_\Xi$, for some constant $c > 0$ and $0 < \alpha < \dfrac{1}{2}$.*

Let us denote by R_t the transition semigroup associated to Z^x_t, that is for every $\varphi \in C_b(E)$ and $t \in [0,T]$

$$R_t[\varphi](x) = \mathbb{E}\varphi(Z^x_t). \tag{4.1}$$

Moreover let us consider a perturbed Ornstein Uhlenbeck process X^x_τ which is a mild solution of the stochastic differential equation with values in E:

$$\begin{cases} dX^x_\tau = AX^x_\tau d\tau + f(X^x_\tau)\, d\tau + GdW_\tau, & \tau\in[0,T]\\ X_0 = x \in E. \end{cases}$$

Namely, a mild solution is an adapted and continuous E-valued process satisfying $\mathbb{P}$-a.s. the integral equation

$$X_\tau^x = e^{\tau A}x + \int_0^\tau e^{(\tau-s)A} f\left(X_s^x\right) ds + \int_0^\tau e^{(\tau-s)A} G dW_s.$$

From now on we assume that f satisfies hypothesis 3.3. From the proof of theorem 3.4, it follows that the process $\left(X_\tau^x\right)_\tau$ belongs to $\mathcal{H}^p\left([0,T], E\right)$ for every $1 \leq p < \infty$. Let us denote by P_t the transition semigroup associated to X_τ^x, that is for every $\varphi \in C_b(E)$ and $t \in [0,T]$

$$P_t[\varphi](x) = \mathbb{E}\varphi\left(X_t^x\right).$$

In this section we want to prove that if for every $\varphi \in C_b(E)$, $R_t[\varphi] \in \mathcal{G}(E)$ and $\nabla R_t[\varphi]$ satisfies inequality (2.4), then also for every $\varphi \in C_b(E)$, $P_t[\varphi] \in \mathcal{G}(E)$ and $\nabla P_t[\varphi]$ satisfies inequality (2.4). And, in analogy, if R_t is such that for every $\varphi \in C_b(E)$, $R_t[\varphi] \in \mathcal{G}^G(E)$ and $\nabla^G R_t[\varphi]$ satisfies inequality (2.5), then also P_t does. In order to prove these results we apply the Girsanov type Theorem 3.4 we have presented in the previous section: by this theorem we get

$$P_t[\varphi](x) = \mathbb{E}\left[\varphi\left(Z_t^x\right)\exp V_t^x\right], \tag{4.2}$$

and so P_t can be written in terms of the expectation of a function of the process Z^x.

Theorem 4.2. *Let hypotheses 2.1, 2.2, 3.3 and 4.1 hold true. Let R_t denote the Ornstein Uhlenbeck transition semigroup defined in (4.1) and P_t denote the perturbed Ornstein Uhlenbeck transition semigroup defined in (4.2). If for every $\varphi \in C_b(E)$, $R_t[\varphi](x) \in \mathcal{G}(E)$ and its derivative satisfies inequality (2.4), then also P_t does.*

We prove an analogous result: we show that the regularizing property expressed in (2.5) can be extended from the Ornstein Uhlenbeck semigroup to the perturbed Ornstein Uhlenbeck semigroup.

Theorem 4.3. *Let hypotheses 2.1, 2.2, 2.6, 3.3 and 4.1 hold true. Let R_t denote the Ornstein Uhlenbeck transition semigroup defined in (4.1) and P_t denote the perturbed Ornstein Uhlenbeck transition semigroup defined in (4.2). If, for every $\varphi \in C_b(E)$, $R_t[\varphi](x) \in \mathcal{G}^G(E)$ and its G-derivative satisfies inequality (2.5), then also P_t does.*

Proof. Let $\eta \in \Xi_0$. We want to prove that for every $\varphi \in C_b(E)$, $\nabla^G P_t[\varphi](x)\eta$ exists and satisfies the inequality

$$\left|\nabla^G P_t[\varphi](x)\eta\right| \leq cC_t \|\varphi\|_\infty \|\eta\|_\Xi, \quad 0 < t \leq T.$$

So, since Ξ_0 is dense in Ξ, the linear operator $\nabla^G P_t[\varphi](x)$, which is well defined on Ξ_0, admits an extension to the whole space Ξ, and we denote this extension again by $\nabla^G P_t[\varphi](x)$.

We remark that if $\eta \in \Xi_0$, $G\eta \in E$ and so the difference quotient

$$\frac{\mathbb{E}\varphi\left(X_t^{x+rG\eta}\right) - \mathbb{E}\varphi\left(X_t^x\right)}{r}$$

makes sense.

First we prove that for every $\eta \in \Xi_0$ and for every $\varphi \in C_b^1(E)$

$$\left|\nabla^G P_t[\varphi](x)\eta\right| \leq cC_t \|\varphi\|_\infty \|\eta\|_\Xi, \quad 0 < t \leq T.$$

By the definition of $\nabla^G P_t[\varphi](x)\eta$ we get

$$\begin{aligned}
&\nabla^G P_t[\varphi](x)\eta \\
&= \lim_{r\to 0} \frac{\mathbb{E}\varphi\left(X_t^{x+rG\eta}\right) - \mathbb{E}\varphi\left(X_t^x\right)}{r} \\
&= \lim_{r\to 0} \frac{\mathbb{E}\left[\varphi\left(Z_t^{x+rG\eta}\right)\exp V_t^{x+rG\eta}\right] - \mathbb{E}\left[\varphi\left(Z_t^x\right)\exp V_t^x\right]}{r} \\
&= \lim_{r\to 0} \frac{\mathbb{E}\left[\varphi\left(Z_t^{x+rG\eta}\right)\left(\exp V_t^{x+rG\eta} - \exp V_t^x\right)\right]}{r} + \lim_{r\to 0} \frac{\mathbb{E}\left[\left(\varphi\left(Z_t^{x+rG\eta}\right) - \varphi\left(Z_t^x\right)\right)\exp V_t^x\right]}{r} \\
&= \mathbb{E}\left[\varphi\left(Z_t^x\right)\exp V_t^x\left(\int_0^t \left\langle \nabla F\left(Z_s^x\right)e^{sA}G\eta, dW_s\right\rangle\right.\right. \\
&\left.\left.-\frac{1}{2}\int_0^t \left\langle F\left(Z_s^x\right)e^{sA}, \nabla F\left(Z_s^x\right)e^{sA}G\eta\right\rangle ds\right)\right] + \mathbb{E}\left[\left\langle \nabla\varphi\left(Z_t^x\right), e^{tA}G\eta\right\rangle \exp V_t^x\right] \\
&= \mathbb{E}\left[\varphi\left(X_t^x\right)\int_0^t \left\langle \nabla F\left(X_s^x\right)e^{sA}G\eta, dW_s\right\rangle\right] + \mathbb{E}\left\langle \nabla\varphi\left(X_t^x\right), e^{tA}G\eta\right\rangle.
\end{aligned}$$

Next we evaluate $\mathbb{E}\left\langle \nabla\varphi\left(X_t^x\right), e^{tA}G\eta\right\rangle$. Let $(\xi_\tau)_\tau$ be a bounded predictable process with values in Ξ. We define $X_\tau^{\varepsilon,x}$ which is the mild solution to the equation

$$\begin{cases} dX_\tau^{\varepsilon,x} = AX_\tau^{\varepsilon,x}d\tau + f\left(X_\tau^{\varepsilon,x}\right)d\tau + G\varepsilon\xi_\tau d\tau + GdW_\tau, & \tau \in [0,T] \\ X_0^\varepsilon = x, \end{cases}$$

that is

$$X_\tau^{\varepsilon,x} = e^{\tau A}x + \int_0^\tau e^{(\tau-s)A}f\left(X_s^{\varepsilon,x}\right)ds + \varepsilon\int_0^\tau e^{(\tau-s)A}G\xi_s ds + \int_0^\tau e^{(\tau-s)A}GdW_s.$$

By hypotheses 2.1, 2.2, 3.3 and 4.1 it turns out that there exists a unique mild solution; in particular by hypothesis 4.1 it turns out that $\displaystyle\int_0^\tau e^{(\tau-s)A}G\xi_s ds$, which a priori is an H-valued process, admits a version in $C([0,T],E)$.
We define the probability measure Q_ε such that

$$\frac{dQ_\varepsilon}{d\mathbb{P}} = \exp\left(\varepsilon\int_0^T \xi_\sigma dW_\sigma - \frac{\varepsilon^2}{2}\int_0^T \|\xi_\sigma\|_\Xi^2 d\sigma\right) = \rho_T^\varepsilon.$$

Since X with respect to $\mathbb{P}$ and X^ε with respect to Q_ε have the same law, it turns out that

$$\mathbb{E}\varphi\left(X_t^x\right) = \mathbb{E}\left[\varphi\left(X_t^{\varepsilon,x}\right)\rho_t^\varepsilon\right],$$

where

$$\rho_t^\varepsilon = \exp\left(\varepsilon\int_0^T \xi_\sigma dW_\sigma - \frac{\varepsilon^2}{2}\int_0^T \|\xi_\sigma\|_\Xi^2 d\sigma\right).$$

By differentiating with respect to ε, at $\varepsilon=0$, and applying the dominated convergence theorem, we get

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}_{|\varepsilon=0}\mathbb{E}\varphi\left(X_t^x\right) = \frac{d}{d\varepsilon}_{|\varepsilon=0}\mathbb{E}\left[\varphi\left(X_t^{\varepsilon,x}\right)\rho_t^{\varepsilon}\right] \\ &= \mathbb{E}\left\langle \nabla\varphi\left(X_t^x\right), \overset{.}{X}{}_t^{\xi}\right\rangle - \mathbb{E}\left[\varphi\left(X_t^x\right)\int_0^t \langle \xi_\sigma, dW_\sigma\rangle\right], \end{aligned}$$

where we have set $\overset{.}{X}{}_t^{\xi} := \dfrac{d}{d\varepsilon}_{|\varepsilon=0} X_t^{\varepsilon,x}$, $\mathbb{P}$-a.s. One can easily check that $\overset{.}{X}{}_t^{\xi}$ is the unique mild solution to the equation

$$\left\{\begin{array}{l} d\overset{.}{X}{}_\tau^{\xi} = A\overset{.}{X}{}_\tau^{\xi}d\tau + G\nabla F\left(X_\tau\right)\overset{.}{X}{}_\tau^{\xi}d\tau + G\xi_\tau d\tau, \qquad \tau\in[0,T] \\ \overset{.}{X}{}_0^{\xi} = 0, \end{array}\right. \tag{4.3}$$

that is $\overset{.}{X}{}_\tau^{\xi}$ solves the integral equation

$$\overset{.}{X}{}_\tau^{\xi} = \int_0^\tau e^{(\tau-s)A}G\nabla F\left(X_s\right)\overset{.}{X}{}_s^{\xi}ds + \int_0^\tau e^{(\tau-s)A}G\xi_s ds.$$

By hypothesis 4.1, it can be easily checked that $\overset{.}{X}{}_\tau^{\xi}$ is well defined as a process with values in the Banach space E. So for every bounded and predictable process $(\xi_\tau)_\tau$ we have proved

$$\mathbb{E}\left\langle \nabla\varphi\left(X_t^x\right), \overset{.}{X}{}_t^{\xi}\right\rangle = \mathbb{E}\left[\varphi\left(X_t^x\right)\int_0^t \langle \xi_\sigma, dW_\sigma\rangle\right]. \tag{4.4}$$

Now we want to extend this equality to predictable Ξ-valued processes $(\xi_\tau)_\tau$ such that $\mathbb{E}\displaystyle\int_0^T \|\xi_s\|_\Xi^2\, ds$ is finite. By hypothesis 4.1, also for such a process ξ, $\overset{.}{X}{}_\tau^{\xi}$ is well defined with values in E and it is the unique mild solution of equation (4.3). Moreover for such a process ξ, there exists an increasing sequence $((\xi_\tau^n)_\tau)_n$ of bounded and predictable Ξ-valued processes such that $\displaystyle\int_0^T \|\xi_s-\xi_s^n\|_\Xi^2\, ds \to 0$ a.s. We evaluate

$$\begin{aligned} &\left\|\overset{.}{X}{}_\tau^{\xi} - \overset{.}{X}{}_\tau^{\xi^n}\right\|_E \\ &\leq \int_0^\tau \left\|e^{(\tau-s)A}G\right\|_{L(\Xi,E)}\left\|\nabla F\left(X_\tau\right)\right\|_{L(E,\Xi)}\left\|\overset{.}{X}{}_s^{\xi} - \overset{.}{X}{}_s^{\xi^n}\right\|_E ds + \int_0^\tau \left\|e^{(\tau-s)A}G\left(\xi_s-\xi_s^n\right)\right\|_E ds \\ &\leq c\int_0^\tau (\tau-s)^{-\alpha}\left(1+\|X_s\|_E^k\right)\left\|\overset{.}{X}{}_s^{\xi} - \overset{.}{X}{}_s^{\xi^n}\right\|_E ds + c\int_0^\tau (\tau-s)^{-\alpha}\|\xi_s-\xi_s^n\|_\Xi\, ds \\ &\leq c\int_0^\tau (\tau-s)^{-\alpha}\left(1+\|X_s\|_E^k\right)\left\|\overset{.}{X}{}_s^{\xi} - \overset{.}{X}{}_s^{\xi^n}\right\|_E ds + c\left(\int_0^\tau (\tau-s)^{-2\alpha}ds\right)^{1/2}\left(\int_0^\tau \|\xi_s-\xi_s^n\|_\Xi^2\, ds\right)^{1/2} \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}\left\|\overset{.}{X}{}_\tau^{\xi} - \overset{.}{X}{}_\tau^{\xi^n}\right\|_E^2 &\leq \overline{c}\int_0^\tau (\tau-s)^{-2\alpha}\,\mathbb{E}\left(1+\|X_s\|_E^k\right)^2\mathbb{E}\left\|\overset{.}{X}{}_s^{\xi} - \overset{.}{X}{}_s^{\xi^n}\right\|_E^2 ds \\ &\quad + \overline{c}\left(\int_0^\tau (\tau-s)^{-2\alpha}ds\right)\mathbb{E}\int_0^\tau \|\xi_s-\xi_s^n\|_\Xi^2\, ds \end{aligned}$$

By Gronwall lemma in integral form

$$\mathbb{E}\left\|\overset{\cdot\ \xi}{X}_\tau-\overset{\cdot\ \xi^n}{X}_\tau\right\|_E^2\leq C\mathbb{E}\int_0^T\|\xi_s-\xi_s^n\|_\Xi^2\,ds\exp\left(\int_0^\tau(\tau-s)^{-2\alpha}\,\mathbb{E}\left(1+\|X_s\|_E^k\right)^2ds\right)\to 0\text{ as }n\to\infty.$$

So we deduce that

$$\mathbb{E}\left\langle\nabla\varphi\left(X_t^x\right),\overset{\cdot\ \xi^n}{X}_t\right\rangle\to\mathbb{E}\left\langle\nabla\varphi\left(X_t^x\right),\overset{\cdot\ \xi}{X}_t\right\rangle\ \text{ as }n\to\infty.$$

Moreover $\mathbb{E}\left[\varphi\left(X_t^x\right)\int_0^t\langle\xi_\sigma^n,dW_\sigma\rangle\right]\to\mathbb{E}\left[\varphi\left(X_t^x\right)\int_0^t\langle\xi_\sigma,dW_\sigma\rangle\right]$ as $n\to\infty$:

$$\begin{aligned}\mathbb{E}\left|\varphi\left(X_t^x\right)\int_0^t\langle\xi_\sigma^n,dW_\sigma\rangle-\varphi\left(X_t^x\right)\int_0^t\langle\xi_\sigma^n,dW_\sigma\rangle\right|&\leq\sup_{x\in E}|\varphi(x)|\,\mathbb{E}\left|\int_0^t\langle\xi_\sigma^n-\xi_\sigma,dW_\sigma\rangle\right|\\&\leq\sup_{x\in E}|\varphi(x)|\,\mathbb{E}\left(\int_0^t\|\xi_\sigma^n-\xi_\sigma\|_\Xi^2\,d\sigma\right)^{1/2}.\end{aligned}$$

By (4.4), for every ξ^n bounded and predictable

$$\mathbb{E}\left\langle\nabla\varphi\left(X_t^x\right),\overset{\cdot\ \xi^n}{X}_t\right\rangle=\mathbb{E}\left[\varphi\left(X_t^x\right)\int_0^t\langle\xi_\sigma^n,dW_\sigma\rangle\right].$$

Letting $n\to\infty$ we get

$$\mathbb{E}\left\langle\nabla\varphi\left(X_t^x\right),\overset{\cdot\ \xi}{X}_t\right\rangle=\mathbb{E}\left[\varphi\left(X_t^x\right)\int_0^t\langle\xi_\sigma,dW_\sigma\rangle\right].$$

for every predictable process $\xi\in L^2\left(\Omega\times[0,T],\Xi\right)$.
Now we look for a predictable process $\xi\in L^2\left(\Omega\times[0,T],\Xi\right)$ such that

$$\overset{\cdot\ \xi}{X}_t=e^{tA}G\eta.$$

Let us consider the deterministic controlled system

$$\begin{cases}\dfrac{dz_s}{ds}=Az_s+Gu_s,\\ z_0=0,\end{cases}\tag{4.5}$$

where $u\in L^2\left([0,T],\Xi\right)$. The solution of (4.5) is given by

$$z_s=\int_0^s e^{(s-r)A}Gu_r\,dr.$$

By hypothesis 4.1, the map $s\mapsto z_s$ is continuous with values in E. We claim that there exists ξ such that $\overset{\cdot\ \xi}{X}_s=z_s$, for every $s\in[0,t]$. Indeed, let us take

$$\xi_s=u_s-\nabla F\left(X_s\right)z_s.$$

For such a process ξ, we get

$$\overset{\cdot}{X}{}_s^{\xi} - z_s = \int_0^s e^{(s-r)A} G \nabla F(X_r) \left[\overset{\cdot}{X}{}_r^{\xi} - z_r \right] dr.$$

By Gronwall lemma $\overset{\cdot}{X}{}_s^{\xi} - z_s = 0$ for every $s \in [0,t]$. We are looking for $\xi \in \Xi$ such that $\overset{\cdot}{X}{}_t^{\xi} = e^{tA}G\eta$. By hypothesis 2.5, $e^{tA}G(\Xi) \subset Q_t^{1/2}(H)$ and so there exists a control $u \in L^2([0,T],\Xi)$ such that $z_t = e^{tA}G\eta$. So for such a control u, by taking $\xi_s = u_s - \nabla F(X_s) z_s$, $0 < s < t$, we get that $\overset{\cdot}{X}{}_t^{\xi} = e^{tA}G\eta$ and that

$$\mathbb{E}\left\langle \nabla\varphi(X_t^x), e^{tA}G\eta \right\rangle = \mathbb{E}\left\langle \nabla\varphi(X_t^x), \overset{\cdot}{X}{}_t^{\xi} \right\rangle = \mathbb{E}\varphi(X_t^x) \int_0^t \langle \xi_\sigma, dW_\sigma \rangle .$$

Moreover

$$\begin{aligned}
\mathbb{E}\left| \int_0^t \langle \xi_\sigma, dW_\sigma \rangle \right| &\leq \mathbb{E}\left(\int_0^t \|\xi_\sigma\|_\Xi^2 \, d\sigma \right)^{1/2} \\
&\leq \mathbb{E}\left(\int_0^t \|u_\sigma\|_\Xi^2 \, d\sigma \right)^{1/2} + \mathbb{E}\left(\int_0^t \left\|\nabla F(X_\sigma) e^{\sigma A} G\eta\right\|_\Xi^2 d\sigma \right)^{1/2}.
\end{aligned}$$

By our assumptions,

$$\begin{aligned}
\left\|\nabla F(X_\sigma) e^{\sigma A} G\eta\right\|_\Xi &\leq \|\nabla F(X_\sigma)\|_{L(E,H)} \left\|e^{\sigma A} G\eta\right\|_E \\
&\leq c\left(1 + \|X_\sigma\|_E^k\right) \left\|e^{\sigma A} G\eta\right\|_E ,
\end{aligned}$$

and so, by hypothesis 4.1, $\mathbb{E}\left(\int_0^t \left\|\nabla F(X_\sigma) e^{\sigma A} G\eta\right\|_\Xi^2 d\sigma \right)^{1/2}$, is finite and for every $\eta \in \Xi_0$ it can be estimated in terms of $\|\eta\|_\Xi$. It follows that $\mathbb{E}\left| \int_0^t \left\langle \nabla F(X_s^x) e^{sA} G\eta, dW_s \right\rangle \right|$ is finite and it can be estimated in terms of $\|\eta\|_\Xi$. So

$$\left|\nabla^G P_t[\varphi](x)\eta\right| \leq \sup_{x\in E} |\varphi(x)| \left\{ C\|\eta\|_\Xi + \left(\int_0^t \|u_\sigma\|_\Xi^2 \, d\sigma \right)^{1/2} \right\}.$$

Since the left hand side does not depend on the control u, on the right hand side we can take the infimum over all controls u steering, in the deterministic linear controlled system (4.5), the initial state 0 to $e^{tA}G\eta$ in time t. The energy to steer 0 to $e^{tA}G\eta$ in time t is given by

$$\mathcal{E}\left(t, e^{tA}G\eta\right) = \min\left\{ \left(\int_0^t \|u_s\|_\Xi^2 \, ds \right)^{1/2} : z(0) = 0,\ z(t) = e^{tA}G\eta \right\}$$

and it turns out that $\mathcal{E}\left(t, e^{tA}G\eta\right) = \left\|Q_t^{-1/2} e^{tA_0} G\eta\right\|$. So for every $\varphi \in C_b^1(E)$ and for every $\eta \in \Xi_0$

$$\left|\nabla^G P_t[\varphi](x)\eta\right| \leq C \left\|Q_t^{-1/2} e^{tA_0 G}\right\|_{L(\Xi,H)} \|\eta\|_\Xi \sup_{x\in E} |\varphi(x)| . \tag{4.6}$$

Since for every $\eta \in \Xi_0$ the norm of $\nabla^G P_t[\varphi](x)\eta$ is bounded by $\|\eta\|_\Xi$, relation (4.6) can be extended to every $\eta \in \Xi$. It remains to prove that (4.6) can be extended to every $\varphi \in C_b(E)$. We extend the result in (24) valid for Hilbert spaces: by using Schauder basis the approximation performed in that paper can be achieved also in Banach spaces. So for every $\varphi \in C_b(E)$ there exists a sequence of functions $\varphi_n \in C_b^1(E)$ such that for every $x \in E$, $\varphi_n(x) \to \varphi(x)$ as $n \to \infty$, and $\sup_{x\in E}|\varphi_n(x)| \le \sup_{x\in E}|\varphi(x)|$. So, by (4.6), we get for every $x, y \in E$

$$\begin{aligned}|P_t[\varphi_n](x) - P_t[\varphi_n](y)| &\le C\left\|Q_t^{-1/2}e^{tA_0}G\right\|_{L(\Xi,H)}\|x-y\|_E \sup_{x\in E}|\varphi_n(x)|\\ &\le C\left\|Q_t^{-1/2}e^{tA_0}G\right\|_{L(\Xi,H)}\|x-y\|_E \sup_{x\in E}|\varphi(x)|.\end{aligned}$$

Letting n tends to ∞ in the right hand side we get that

$$|P_t[\varphi](x) - P_t[\varphi](y)| \le C\left\|Q_t^{-1/2}e^{tA_0}G\right\|_{L(\Xi,H)}\|x-y\|_E \sup_{x\in E}|\varphi(x)|,$$

from which it can be deduced the strong Feller property for the semigroup P_t. We still have to prove that for every $\varphi \in C_b(E)$, $P_t[\varphi]$ is a G-Gateaux differentiable function on E. Let us consider the sequence of Frechet differentiable functions $(\varphi_n)_n$ that converges pointwise to φ in E. By previous calculations we get

$$\nabla^G P_t[\varphi_n](x)\eta = \mathbb{E}\left[\varphi_n(X_t^x)\int_0^t\left\langle \nabla F(X_s^x)e^{sA}G\eta, dW_s\right\rangle\right] + \mathbb{E}\left\langle \varphi_n(X_t^x), \int_0^t \xi_\sigma dW_\sigma\right\rangle. \quad (4.7)$$

Since $\sup_{x\in E}|\varphi_n(x)| \le \sup_{x\in E}|\varphi(x)|$, by dominated convergence $\left(\nabla^G P_t[\varphi_n](x)\right)_n$ is a Cauchy sequence in Ξ^*, indeed

$$\begin{aligned}&\nabla^G P_t[\varphi_n](x)\eta - \nabla^G P_t[\varphi_k](x)\eta\\ &= \mathbb{E}\left[(\varphi_n(X_t^x) - \varphi_k(X_t^x))\int_0^t\left\langle \nabla F(X_s^x)e^{sA}G\eta, dW_s\right\rangle\right] + \mathbb{E}\left\langle (\varphi_n(X_t^x) - \varphi_k(X_t^x)), \int_0^t \xi_\sigma dW_\sigma\right\rangle,\end{aligned}$$

and the right hand side tends to 0 uniformly with respect to $\eta \in \Xi$, $\|\eta\|_\Xi \le 1$. So there exists $H^x \in \Xi^*$ such that $\nabla P_t[\varphi_n](x) \to H^x$ in Ξ^*, as $n \to \infty$. By (4.7), for every $\eta \in \Xi$, the map $x \mapsto H^x\eta$ is continuous as a map from Ξ to $\mathbb{R}$. By the estimate (4.6) we get that

$$|H^x\eta| \le C\left\|Q_t^{-1/2}e^{tA_0}G\right\|_{L(\Xi,H)}\|\eta\|_\Xi \sup_{x\in E}|\varphi(x)|.$$

It remains to show that $P_t[\varphi]$ is G-Gateaux differentiable and that $\nabla^G P_t[\varphi](x)\eta = H^x\eta$. For every $r > 0$ and every $\eta \in \Xi$, we can write

$$P_t[\varphi_n](x + rG\eta) - P_t[\varphi_n](x) = \int_0^1 \nabla P_t[\varphi_n](x + rsG\eta)\, rG\eta ds.$$

Letting $n \to \infty$, we get

$$P_t[\varphi](x + rG\eta) - P_t[\varphi](x) = \int_0^1 H^{x+rsG\eta} rG\eta ds.$$

If we divide both sides by r and we let r tend to 0, by dominated convergence and by the continuity of $H^xG\eta$ with respect to x, we see that $P_t[\varphi]$ is G Gateaux differentiable and that $\nabla^G P_t[\varphi](x)\,G\eta = H^xG\eta$. Moreover the following estimate holds true: for every $x\in E$, $\eta\in\Xi$ there exists a constant $C>0$ such that

$$\left|\nabla^G P_t[\varphi](x)\,\eta\right| \leq C\left\|Q_t^{-1/2}e^{tA_0}G\right\|_{L(\Xi,H)}\|\eta\|_\Xi \sup_{x\in E}|\varphi(x)|.$$

□

5 Applications to Kolmogorov equations

In this section we study semilinear Kolmogorov equations in the Banach space E.

Second order partial differential equations on Hilbert spaces have been extensively studied: see e.g. the monograph (10), and the papers (14) and (20), where the non linear case is treated, and (11), where it is not made any nondegeneracy assumption on G and such equations are treated via backward stochastic differential equations, BSDEs in the following. By means of the BSDE approach, in (11) Hamilton Jacobi Bellman equations of the following form are studied

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -\mathcal{A}_t u(t,x) + \psi\left(t,x,u(t,x),\nabla^G u(t,x)\right), & t\in[0,T],\ x\in E \\ u(T,x)=\varphi(x), \end{cases} \tag{5.1}$$

When G is not invertible, this a special case of an Hamilton Jacobi Bellman equation of the following form

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -\mathcal{A}_t u(t,x) + \psi(t,x,u(t,x),\nabla u(t,x)), & t\in[0,T],\ x\in E \\ u(T,x)=\varphi(x), \end{cases} \tag{5.2}$$

Moreover, in the BSDE approach, on the Hamiltonian ψ and on the final datum φ some differentiability assumptions are required, while in the approach followed e.g. by (14), (20) and also by the present paper, the Hamiltonian ψ and the final datum φ are asked lipschitz continuous; on the contrary some regularizing properties on the transition semigroup with generator given by the second order differential operator $\mathcal{A}_t$ are needed. In the paper (21), which is a generalization of (11) to Banach spaces, the BSDE approach is used to solve second order partial differential equations in Banach spaces: also in (21) φ and ψ are taken Gateaux differentiable, and no regularizing assumptions on the transition semigroup are needed. In this paper we study Kolmogorov equations with the structure of equation 5.2 and under weaker assumptions on the transition semigroup related, with the structure of equation 5.1. One of the main motivation to study Hamilton Jacobi Bellman equations on a Banach space is to solve a stochastic optimal control problem related, which is well posed on a Banach space, for example a stochastic optimal control problem where the cost is well defined on a Banach space and/or a stochastic optimal control problem where the state evolves on a Banach space. This will be matter of a further research.

At first we study an equation of the following form

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -\mathcal{A}_t u(t,x) + \psi(t,x,u(t,x),\nabla u(t,x)), & t\in[0,T],\ x\in H \\ u(T,x)=\varphi(x), \end{cases} \tag{5.3}$$

where $G \in L(\Xi, H)$, Ξ is another separable Hilbert space and $\nabla u(t,x)$ is the gradient of u. Given $f : E \to E$ satisfying hypothesis 3.3 and the generator A of a semigroup on E, satisfying hypothesis 2.1, $\mathcal{A}_t$ is formally defined by

$$\mathcal{A}_t v(x) = \frac{1}{2} Trace\left(GG^* \nabla^2 v(x)\right) + \langle Ax, \nabla v(x)\rangle_{E,E^*} + \langle f(x), \nabla v(x)\rangle_{E,E^*},$$

and it arises as the generator of the Markov process X in E, namely of the perturbed Ornstein Uhlenbeck process $X_\tau^{t,x}$ which is a mild solution of the following stochastic differential equation with values in E:

$$\begin{cases} dX_\tau^{t,x} = AX_\tau^{t,x} d\tau + f\left(X_\tau^{t,x}\right) d\tau + G dW_\tau, & \tau \in [t,T] \\ X_t^{t,x} = x \in E. \end{cases}$$

We denote by $P_{t,\tau}$ the transition semigroup associated to X, that is for every $\varphi \in C_b(E)$,

$$P_{t,\tau}\varphi(x) = E\varphi\left(X_\tau^{t,x}\right)$$

To study equation (5.3) we also need the following assumptions on ψ and φ:

Hypothesis 5.1. *The function $\psi : [0,T] \times E \times \mathbb{R} \times E^* \to \mathbb{R}$ is Borel measurable and satisfies the following:*

1. *there exists a constant $L > 0$ such that*

$$|\psi(t,x,y_1,z_1) - \psi(t,x,y_2,z_2)| \leq L\left(|y_1 - y_2| + \|z_1 - z_2\|_{E^*}\right),$$

   for every $t \in [0,T]$, $x \in E$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in E^*$;

2. *for every $t \in [0,T]$, $\psi(t,\cdot,\cdot,\cdot)$ is continuous $E \times \mathbb{R} \times E^* \to \mathbb{R}$;*
3. *there exists $L' > 0$ such that*

$$|\psi(t,x,y,z)| \leq L'\left(1 + |y| + \|z\|_{E^*}\right),$$

   for every $t \in [0,T]$, $x \in E$, $y \in \mathbb{R}$, $z \in E^*$.

Hypothesis 5.2. *The final datum $\varphi \in C_b(E)$.*

We introduce the notion of mild solution of the non linear Kolmogorov equation (5.3). Since $\mathcal{A}_t$ is (formally) the generator of $P_{t,\tau}$, the variation of constants formula for (5.3) is:

$$u(t,x) = P_{t,T}[\varphi](x) - \int_t^T P_{t,s}\left[\psi(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\right](x)\, ds, \quad t \in [0,T],\ x \in E. \tag{5.4}$$

and we notice that this formula is meaningful if $\psi(t,\cdot,\cdot,\cdot)$, $u(t,\cdot)$, $\nabla u(t,\cdot)$ have polynomial growth, and provided they satisfy some measurability assumptions. We use this formula to give the notion of mild solution for the non linear Kolmogorov equation (5.3).

We introduce some function spaces where we seek the solution: for $\alpha \geq 0$, let $C_\alpha([0,T] \times E)$ be the linear space of continuous functions $f : [0,T) \times E \to \mathbb{R}$ with the norm

$$\|f\|_{C_\alpha} := \sup_{t\in[0,T]} \sup_{x\in E} (T-t)^\alpha |f(t,x)| < \infty.$$

$\left(C_\alpha([0,T], E), \|\cdot\|_{C_\alpha}\right)$ is a Banach space.

For $\alpha \geq 0$, we consider the linear space $C^s_\alpha([0,T] \times E, E^*)$ of the mappings $L : [0,T) \times E \to E^*$ such that for every $e \in E$, $(T-t)^\alpha L(\cdot,\cdot)e$ is bounded and continuous as a function from $[0,T) \times E$ with values in $\mathbb{R}$. The space $C^s_\alpha([0,T] \times E, E^*)$ turns out to be a Banach space if it is endowed with the norm

$$\|L\|_{C^s_\alpha(E^*)} = \sup_{t\in[0,T]} \sup_{x\in E} (T-t)^\alpha \|L(t,x)\|_{E^*}.$$

Definition 5.3. *Let $\alpha \in (0,1)$. We say that a function $u : [0,T] \times E \to \mathbb{R}$ is a mild solution of the non linear Kolmogorov equation (5.3) if the following are satisfied:*

1. *$u \in C_b([0,T] \times E)$;*
2. *$\nabla u \in C^s_\alpha([0,T] \times E, E^*)$: in particular this means that for every $t \in [0,T)$, $u(t,\cdot)$ is Gateaux differentiable;*
3. *equality (5.4) holds.*

We need the following fundamental assumption:

Hypothesis 5.4. *There exists $\alpha \in (0,1)$ such that for every $\phi \in C_b(E)$, the function $P_{t,\tau}[\phi](x)$ is Gateaux differentiable with respect to x, for every $0 \leq t < \tau \leq T$. Moreover, for every $e \in E$, the function $x \mapsto \nabla P_{t,\tau}[\phi](x)e$ is continuous and there exists a constant $c > 0$ such that for every $\phi \in C_b(H)$, for every $\xi \in \Xi$, and for $0 \leq t < \tau \leq T$,*

$$|\nabla P_{t,\tau}[\phi](x)e| \leq \frac{c}{(\tau-t)^\alpha} \|\phi\|_\infty \|e\|_E. \tag{5.5}$$

We want to stress the fact that condition (5.5) implies that the derivative blows up as τ tends to t and it is bounded with respect to x. In virtue of theorem 4.2, we know that if A and G satisfies hypotheses 2.1, 2.2, 2.4, with $C_{\tau-t} = c/(\tau-t)^\alpha$ and 4.1, then hypothesis 5.4 is satisfied.

Theorem 5.5. *Suppose that hypotheses 2.1, 2.2, 3.3, 5.1, 5.2 and 5.4 hold true. Then equation (5.3) admits a unique mild solution $u(t,x)$, in the sense of definition (5.3), satisfying, for every $e \in E$,*

$$|\nabla u(t,x)e| \leq \frac{c}{(T-t)^\alpha} \|e\|_E$$

where $\alpha \in (0,1)$ is given in hypothesis 5.4.

Proof. We give a sketch of the proof, that is similar to the proof of theorem 2.9 in (20). We define the operator $\Gamma = (\Gamma_1, \Gamma_2)$ on $C_\alpha([0,T] \times E) \times C^s_\alpha([0,T] \times E, E^*)$ endowed with the product norm.

$$\Gamma_1[u,v](t,x) = P_{t,T}[\varphi](x) - \int_t^T P_{t,s}[\psi(s,\cdot,u(s,\cdot),v(s,\cdot))](x)\,ds, \tag{5.6}$$

$$\Gamma_2\left[u,v\right](t,x)=\nabla P_{t,T}\left[\varphi\right](x)-\int_t^T\nabla P_{t,s}\left[\psi\left(s,\cdot,u\left(s,\cdot\right),v\left(s,\cdot\right)\right)\right](x)\,ds. \tag{5.7}$$

where $\alpha\in(0,1)$ is given in hypothesis 5.4. Thanks to condition (5.5) Γ is well defined on $C_\alpha\left([0,T]\times E\right)\times C^s_\alpha\left([0,T]\times E,E^*\right)$ with values in itself. We show that Γ is a contraction and so there exists a unique fixed point such that $\Gamma\left(u,v\right)=\left(u,v\right)$. Since the gradient is a closed operator, $\Gamma_2\left[u,v\right]=\nabla\Gamma_1\left[u,v\right]$. We denote by $(\overline{u},\overline{v})$ the unique fixed point of Γ, and $\overline{v}\left(t,x\right)=\nabla\overline{u}\left(t,x\right)$: $\overline{u}$ turns out to be the unique mild solution of equation (5.3). $\square$

Under less restrictive assumptions we can also study semilinear Kolmogorov equations in the Banach space E, with a more special structure. We study an equation of the following form

$$\begin{cases}\frac{\partial u}{\partial t}(t,x)=-\mathcal{A}_t u\left(t,x\right)+\psi\left(t,x,u\left(t,x\right),\nabla^G u\left(t,x\right)\right), & t\in[0,T],\ x\in E\\ u(T,x)=\varphi\left(x\right), & \end{cases} \tag{5.8}$$

where $\nabla^G u\left(t,x\right)$ is the G-gradient of u.

To study equation (5.8) we need the following assumptions on ψ:

Hypothesis 5.6. *The function $\psi:[0,T]\times E\times\mathbb{R}\times\Xi^*\to\mathbb{R}$ is Borel measurable and satisfies the following:*

1. *there exists a constant $L>0$ such that*

   $$\left|\psi\left(t,x,y_1,z_1\right)-\psi\left(t,x,y_2,z_2\right)\right|\leq L\left(|y_1-y_2|+\|z_1-z_2\|_{\Xi^*}\right),$$

   for every $t\in[0,T]$, $x\in E$, $y_1,y_2\in\mathbb{R}$, $z_1,z_2\in\Xi^*$;
2. *for every $t\in[0,T]$, $\psi\left(t,\cdot,\cdot,\cdot\right)$ is continuous $E\times\mathbb{R}\times\Xi^*\to\mathbb{R}$;*
3. *there exists $L'>0$ such that*

   $$\left|\psi\left(t,x,y,z\right)\right|\leq L'\left(1+|y|+\|z\|_{\Xi^*}\right),$$

   for every $t\in[0,T]$, $x\in E$, $y\in\mathbb{R}$, $z\in\Xi^*$.

We introduce the notion of mild solution of the non linear Kolmogorov equation (5.8): again by the variation of constants formula for (5.8) is:

$$u(t,x)=P_{t,T}\left[\varphi\right](x)-\int_t^T P_{t,s}\left[\psi\left(s,\cdot,u\left(s,\cdot\right),\nabla^G u\left(s,\cdot\right)\right)\right](x)\,ds,\quad t\in[0,T],\ x\in E. \tag{5.9}$$

and we notice that this formula is meaningful if $\psi\left(t,\cdot,\cdot,\cdot\right)$, $u(t,\cdot)$, $\nabla^G u\left(t,\cdot\right)$ have polynomial growth, and provided they satisfy some measurability assumptions.

In analogy with $C^s_\alpha\left([0,T]\times E,E^*\right)$ we introduce the space $C^s_\alpha\left([0,T]\times E,\Xi^*\right)$ of the mappings $L:[0,T)\times E\to\Xi^*$ such that for every $\xi\in\Xi$, $(T-t)^\alpha L\left(\cdot,\cdot\right)\xi$ is a bounded and continuous function from $[0,T)\times E$ to $\mathbb{R}$. The space $C^s_\alpha\left([0,T]\times E,\Xi^*\right)$ turns out to be a Banach space if it is endowed with the norm

$$\|L\|_{C^s_\alpha(\Xi^*)}=\sup_{t\in[0,T]}\sup_{x\in E}(T-t)^\alpha\left\|L\left(t,x\right)\right\|_{\Xi^*}.$$

Definition 5.7. *Let $\alpha \in (0,1)$. We say that a function $u : [0,T] \times E \to \mathbb{R}$ is a mild solution of the non linear Kolmogorov equation (5.8) if the following are satisfied:*

1. $u \in C_b([0,T] \times E)$;
2. $\nabla^G u \in C^s_\alpha([0,T] \times E, \Xi^*)$: *in particular this means that for every $t \in [0,T)$, $u(t,\cdot)$ is G-differentiable;*
3. *equality (5.9) holds.*

We need the following fundamental assumption:

Hypothesis 5.8. *There exists $\alpha \in (0,1)$ such that for every $\phi \in C_b(E)$, the function $P_{t,\tau}[\phi](x)$ is G-differentiable with respect to x, for every $0 \leq t < \tau \leq T$. Moreover, for every $\xi \in \Xi$, the function $x \mapsto \nabla^G P_{t,\tau}[\phi](x)\xi$ is continuous and there exists a constant $c > 0$ such that for every $\phi \in C_b(E)$, for every $\xi \in \Xi$, and for $0 \leq t < \tau \leq T$,*

$$\left|\nabla^G P_{t,\tau}[\phi](x)\xi\right| \leq \frac{c}{(\tau - t)^\alpha} \|\phi\|_\infty \|\xi\|_\Xi \,. \tag{5.10}$$

We want to stress the fact that condition (5.10) implies that the G-derivative blows up as τ tends to t and it is bounded with respect to x. In virtue of theorem 4.3, we know that if A and G satisfy hypotheses 2.1, 2.2, 2.5, with $C_{\tau-t} = c/(\tau-t)^\alpha$, 4.1 and 2.6, then hypothesis 5.8 is satisfied. Moreover we ask the semigroup to have a regularizing property, weaker than the one required in hypothesis 5.4.

Theorem 5.9. *Suppose that hypotheses 2.1, 2.2, 3.3, 5.2, 5.6 and 5.8 hold true. Then equation (5.8) admits a unique mild solution $u(t,x)$, in the sense of definition (5.7), and satisfying, for every $\xi \in \Xi$,*

$$\left|\nabla^G u(t,x)\xi\right| \leq \frac{c}{(T-t)^\alpha} \|\xi\|_\Xi$$

where $\alpha \in (0,1)$ is given in hypothesis 5.8.

Proof. The proof is similar to the one of theorem 5.5, and we do not give details. As in the proof of theorem 5.5, we define the operator $\Gamma = (\Gamma_1, \Gamma_2)$ on $C_\alpha([0,T] \times E) \times C^s_\alpha([0,T] \times E, \Xi^*)$ endowed with the product norm. It can be proved that Γ is a contraction on $C_\alpha([0,T] \times E) \times C^s_\alpha([0,T] \times E, \Xi^*)$ with values in itself, and its unique fixed point is the unique mild solution of equation (5.8) according to definition (5.7). □

6 Application to some specific models

In this section we collect some models where the results of the previous sections apply.

6.1 Stochastic heat equations in bounded intervals

In $(\Omega, \mathcal{F}, (\mathcal{F}_\tau)_\tau, \mathbb{P})$, we consider, for $\tau \in [0,T]$ and $\xi \in [0,1]$, the following stochastic heat equation

$$\begin{cases} \dfrac{d}{d\tau} x_\tau(\xi) = \frac{\partial^2}{\partial \xi^2} x_\tau(\xi) + \dot{W}(\tau, \xi), \\ x_0(\xi) = h(\xi), \\ x_\tau(0) = x_\tau(1) = 0 \end{cases} \tag{6.1}$$

where $\dot{W}(\tau, \xi)$ is a space-time white noise on $[0,T] \times [0,1]$, and h is a continuous function on $[0,1]$.

Equation (6.1) can be formulated in an abstract way as

$$\begin{cases} dZ_\tau = AZ_\tau d\tau + dW_\tau, \qquad \tau \in [0,T] \\ Z_0 = h, \end{cases} \tag{6.2}$$

where $(W_\tau)_\tau$ is a cylindrical Wiener process with values in $H = \Xi = L^2([0,1])$. The operator A with domain $\mathcal{D}(A)$ is defined by

$$\mathcal{D}(A) = H^2([0,1]) \cap H_0^1([0,1]), \qquad (Ay)(\xi) = \frac{\partial^2}{\partial \xi^2} y(\xi), \quad \text{for every } y \in \mathcal{D}(A).$$

Let $E = C_b([0,1])$ the space of all continuous functions on $[0,1]$ endowed with the usual norm. It turns out that E is continuously and densely embedded in H, and the restriction of A to $E = C_b([0,1])$ is given by

$$\mathcal{D}(A) = \{f \in C^2([0,1]) : f(0) = f(1) = 0\}, \qquad (Ay)(\xi) = \frac{\partial^2}{\partial \xi^2} y(\xi), \quad \text{for every } y \in \mathcal{D}(A).$$

It is well known that equation (6.1) admits a unique mild solution in H, given by

$$Z_\tau = e^{\tau A} h + \int_0^\tau e^{(\tau - s)A} dW_s,$$

since the stochastic convolution

$$W_A(\tau) = \int_0^\tau e^{(\tau - s)A} dW_s$$

is well defined as a Gaussian process with values in H. Moreover, if the initial datum $h \in E$, then the process $(Z_\tau)_\tau$ is well defined as a process with values in E, in fact the stochastic convolution $W_A(\tau)$ admits an E-continuous version, see e.g. (9), theorem 5.2.9.

So hypotheses 2.1 and 2.2 are verified for equation (6.2). Moreover hypothesis 2.4 holds true for the transition semigroup R_t associated to equation (6.2), with $C_t = \dfrac{c}{\sqrt{t}}$. Also hypothesis 4.1 holds true, with $\alpha = \dfrac{1}{4}$, since for every $f \in L^2([0,1]) = H$ and for every $t > 0$, $e^{tA} f \in C_b([0,1]) = E$ and $\left\| e^{tA} f \right\|_E \leq c t^{-1/4} \|f\|_H$.

Now let us consider a nonlinear heat equation in $[0,1]$: for $\tau \in [0,T]$ and $\xi \in [0,1]$ we consider the equation

$$\begin{cases} dx_\tau(\xi) = \frac{\partial^2}{\partial \xi^2} x_\tau(\xi) \, d\tau + g(x_\tau(\xi)) \, d\tau + W(\tau, \xi) \, d\tau, \\ x_0(\xi) = h(\xi), \end{cases} \tag{6.3}$$

where $g:\mathbb{R}\to\mathbb{R}$ is a continuous and differentiable non increasing function, and g and its derivative have polynomial growth with respect to x. We can write this equation in an abstract setting:

$$\begin{cases} dX_\tau = AX_\tau d\tau + F(X_\tau)\,d\tau + dW_\tau, & \tau\in[0,T] \\ X_0 = h, \end{cases} \tag{6.4}$$

where for every $x\in E$, $F(x)(\xi)=g(x(\xi))$. F satisfies hypothesis 3.3, see (8), example D.7. By (8), theorem 7.13, and (9), theorem 11.4.1, equation (6.4) admits a unique mild solution in E. Let us denote by P_t the transition semigroup associated to equation (6.4). By theorem 4.2, for every $\varphi\in C_b(E)$ and every $x,e\in E$, $P_t[\varphi](x)$ is Gateaux differentiable in any direction $e\in E$ and the estimate

$$|\nabla P_t[\varphi](x)e| \le \frac{C}{\sqrt{t}}\|\varphi\|_\infty \|e\|_E, \qquad 0<t\le T \tag{6.5}$$

holds true. We want to stress the fact that in the book (4) more general results about the strong Feller property for transition semigroups related to reaction diffusion equations are proved, on the contrary no applications to Kolmogorov equations in the space of continuous functions is given.

We consider the Kolmogorov equation relative to equation (6.4):

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -\mathcal{A}_t u(t,x) + \psi(t,x,u(t,x),\nabla u(t,x)), & t\in[0,T],\ x\in E \\ u(T,x)=\varphi(x), \end{cases} \tag{6.6}$$

where, at least formally,

$$\mathcal{A}_t v(x) = \frac{1}{2}Trace\left(\nabla^2 v(x)\right) + \langle Ax, \nabla v(x)\rangle_{E,E^*} + \langle F(x), \nabla v(x)\rangle_{E,E^*}.$$

By estimate (6.5) and theorem 5.5, equation (6.6) admits a unique mild solution in E, if ψ and φ satisfy respectively hypotheses 5.1 and 5.2.

6.2 Stochastic heat equations in $\mathbb{R}$

In $(\Omega,\mathcal{F},(\mathcal{F}_\tau)_\tau,\mathbb{P})$ we consider, for $\tau\in[0,T]$ and $\xi\in\mathbb{R}$:

$$\begin{cases} dx_\tau(\xi) = \frac{\partial^2}{\partial\xi^2}x_\tau(\xi)\,d\tau + W(\tau,\xi)\,d\tau, \\ x_0(\xi) = h(\xi), \end{cases} \tag{6.7}$$

Let $(e_j)_{j\ge1}$ be a complete orthonormal system in $\Xi = L^2(\mathbb{R})$. Let $\beta_j(\tau)$ be independent standard Wiener processes. We assume that $W(\tau,\xi)=\sum_{j\ge1} e_j(\xi)\beta_j(\tau)$. We consider $L^2_\rho(\mathbb{R})$, the Hilbert space of square integrable functions on $\mathbb{R}$ with respect to the (finite) measure $e^{-\rho|\xi|}d\xi$, $\rho>0$. The initial datum h belongs to $L^2_\rho(\mathbb{R})$. The choice of a weighted space is justified by the fact that we want to be able to treat constant initial functions.

In the space $H=L^2_\rho(\mathbb{R})$ equations like (6.7) can be formulated in an abstract way as

$$\begin{cases} dZ_\tau = AZ_\tau d\tau + JdW_\tau, & \tau\in[0,T] \\ Z_0 = h, \end{cases} \tag{6.8}$$

where $(W_\tau)_\tau$ is a cylindrical Wiener process with values in $\Xi = L^2(\mathbb{R})$ and J is the inclusion of Ξ in H. The operator A with domain $\mathcal{D}(A)$ is defined by

$$\mathcal{D}(A) = H^2_\rho(\mathbb{R}), \qquad (Ay)(\xi) = \frac{\partial^2}{\partial \xi^2} y(\xi), \quad \text{for every } y \in \mathcal{D}(A).$$

It is well known, see e.g. (3) or (9), chapter 11, that equation (6.7) admits a unique mild solution in H, given by

$$Z_\tau = e^{\tau A} h + \int_0^\tau e^{(\tau - s)A} J dW_s :$$

the stochastic convolution

$$W_A(\tau) = \int_0^\tau e^{(\tau - s)A} J dW_s$$

is well defined as a Gaussian process with values in H. Let $\rho > 0$: we consider the space $C_\rho(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ such that $e^{-\rho|\xi|}|f(\xi)| \to 0$ as $|\xi| \to \infty$. We consider the Banach space $E = L^2_\rho(\mathbb{R}) \cap C_{\rho/2}(\mathbb{R})$ endowed with the norm

$$\|f\|_E = \left(\int_{\mathbb{R}} e^{-\rho|\xi|} |f(\xi)|^2 d\xi \right)^{1/2} + \sup_{\xi \in \mathbb{R}} e^{-\frac{\rho}{2}|\xi|} |f(\xi)|.$$

It turns out that E is continuously and densely embedded in H, and we want to prove that if the initial datum $h \in E$, then the process $(Z_\tau)_\tau$ is well defined as a process with values in E: to this aim we have to prove that the stochastic convolution $W_A(\tau)$ admits an E-continuous version.

Lemma 6.1. *$W_A(\tau)$ admits a continuous version in $C([0,T], E)$.*

Proof. Take $\delta \in (1/4, 1/2)$: by the factorization method (see e.g. (8)), the stochastic convolution can be represented as

$$W_A(\tau)(\xi) = \frac{\sin \pi\delta}{\pi} \int_0^\tau e^{(\tau - s)A} (\tau - s)^{\delta - 1} Y_\delta(s)(\xi)\, ds$$

where

$$Y_\delta(\tau)(\xi) = \int_0^\tau (\tau - s)^{-\delta} e^{(\tau - s)A} J \sum_{j \geq 1} e_j(\xi)\, d\beta_j(s).$$

We recall that, see e.g. (3), lemma 3.1, for all $t \in [0,T]$,

$$\left\| e^{tA} \right\|_{L\left(L^2_\rho, C_{\rho/2}\right)} \leq C t^{-1/4}.$$

Following the proof of theorem 1.1 in (3),

$$\begin{aligned}
\|W_A(\tau)\|_{C_{\rho/2}(\mathbb{R})} &\leq \frac{\sin \pi\delta}{\pi} \int_0^\tau (\tau - s)^{(\delta - 1)} \left\| e^{(\tau - s)A} \right\|_{L\left(L^2_\rho, C_{\rho/2}\right)} \|Y_\delta(s)\|_{L^2_\rho(\mathbb{R})} ds \\
&\leq C \int_0^\tau (\tau - s)^{(\delta - 1 - 1/4)} \|Y_\delta(s)\|_{L^2_\rho(\mathbb{R})} ds \\
&\leq C \left(\int_0^\tau (\tau - s)^{p(\delta - 5/4)} ds \right)^{1/p} \left(\int_0^\tau \|Y_\delta(s)\|^q_{L^2_\rho(\mathbb{R})} ds \right)^{1/q} < \infty,
\end{aligned}$$

where $1 < p < \frac{1}{5/4-\delta}$, so that $p(\delta - 5/4) > -1$, and q is the conjugate exponent of p. So $W_A(\tau) \in L^\infty\left([0,T], C_{\rho/2}(\mathbb{R})\right)$. The L^∞-space can be replaced by the space of continuous functions as the space $C^\infty\left([0,T], L^2_\rho(\mathbb{R})\right)$ is dense in $L^2\left([0,T], L^2_\rho(\mathbb{R})\right)$, and this concludes the proof. □

So hypotheses 2.1 and 2.2 are verified for the coefficients of equation (6.8). Moreover hypothesis 2.5 holds true for the transition semigroup R_t associated to equation (6.8), with $C_t = \dfrac{c}{\sqrt{t}}$. Indeed, hypothesis 2.5 is equivalent to the null controllability of the deterministic linear controlled system in H

$$\begin{cases} \dfrac{d}{dt} y_t = A y_t + J u_t, \\ y_0 = x \in J(\Xi), \end{cases}$$

where $u \in L^2([0,T], \Xi)$, and to some estimate on the energy steering the initial datum x to 0 in any time $t > 0$. Namely, null controllability of this system is equivalent to the inclusion

$$e^{tA} J(\Xi) \subset Q_t^{1/2}(H).$$

As a control steering the initial state x to 0 in time t we take

$$u_s = -\frac{1}{t} e^{sA} J^{-1} x.$$

It follows

$$\int_0^t \|u_s\|^2_\Xi \, ds = \frac{1}{t^2} \int_0^t \left\| e^{sA} J^{-1} x \right\|^2_\Xi ds \le \frac{1}{t} \sup_{s \in [0,t]} \left\| e^{sA} J^{-1} x \right\|^2_\Xi = c\frac{1}{t}.$$

The energy steering $x \in J(\Xi)$ to 0 in time t is given by

$$\mathcal{E}_C(t,x) = \min \left\{ \left(\int_0^t \|u_s\|^2_\Xi \, ds \right)^{1/2} : y_0 = x, \ y_t = 0 \right\},$$

and so $\mathcal{E}_C(t,x) \le C \dfrac{1}{\sqrt{t}}$, that implies the estimate

$$\left\| Q_t^{-1/2} e^{tA} J \right\|_{L(\Xi,H)} \le C \frac{1}{\sqrt{t}}.$$

So for every $\varphi \in C_b(E)$,

$$\left| \nabla^J R_t[\varphi](x) \xi \right| \le \frac{C}{\sqrt{t}} \|\varphi\|_\infty \|\xi\|_\Xi, \qquad 0 < t \le T.$$

We remark that hypotheses 4.1 and 2.6 are satisfied, indeed for every $\xi \in \Xi$ and every $t > 0$ $e^{tA}\xi \in E$ and $\left\| e^{tA}\xi \right\|_E \le C t^{-1/4} \|\xi\|_\Xi$, and as Ξ_0 we can take $\Xi \cap C_b(\mathbb{R})$, so that $G(\Xi_0) \subset E$.

Now let us consider a nonlinear heat equation in $\mathbb{R}$: for $\tau \in [0,T]$ and $\xi \in \mathbb{R}$ we consider the equation

$$\begin{cases} dx_\tau(\xi) = \frac{\partial^2}{\partial \xi^2} x_\tau(\xi) \, d\tau + e^{-\frac{\rho}{2}|\xi|} g(x_\tau(\xi)) \, d\tau + W(\tau,\xi) \, d\tau, \\ x_0(\xi) = h(\xi), \end{cases} \tag{6.9}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous and differentiable non increasing function, and g and its derivative have linear growth with respect to x. We can write this equation in an abstract setting as

$$\begin{cases} dX_\tau = AX_\tau d\tau + f(X_\tau)\, d\tau + JdW_\tau, & \tau \in [0,T] \\ X_0 = h, \end{cases} \tag{6.10}$$

where for every $x \in E$, $f(x)(\xi) = e^{-\frac{\rho}{2}|\xi|} g(x(\xi))$. For every $x \in E$, $f(x) \in J(\Xi) = L^2(\mathbb{R})$, and satisfies hypothesis 3.3. By (8), theorem 7.13, and (9), theorem 11.4.1, equation (6.10) admits a unique mild solution in E, if the initial datum $h \in E$. Let us denote by P_t the transition semigroup associated to equation (6.10). By theorem 4.3, for every $\varphi \in C_b(E)$ and every $x \in E$ and $\xi \in \Xi$, $P_t[\varphi](x)$ is Gateaux differentiable in the direction $J\xi$ and the estimate

$$\left|\nabla^J P_t[\varphi](x)\,\xi\right| \leq \frac{C}{\sqrt{t}} \|\varphi\|_\infty \|\xi\|_\Xi, \quad 0 < t \leq T$$

holds true. This estimate allows to solve semilinear Kolmogorov equations of the form (5.8).

6.3 First order stochastic differential equations

In $(\Omega, \mathcal{F}, (\mathcal{F}_\tau)_\tau, \mathbb{P})$ let $(\beta_j(\tau))_{j=1}^n$ be real, independent, standard Wiener processes. We consider, for $\tau \in [0,T]$ and $\xi \in [0,1]$ the following stochastic equation

$$\begin{cases} dx_\tau(\xi) = \frac{\partial}{\partial \xi} x_\tau(\xi)\, d\tau + \sum_{j=1}^n e^{2\pi i k_j \xi} d\beta_{k_j}(\tau), \\ x_0(\xi) = h(\xi), \\ x_\tau(0) =_\tau (1), \end{cases} \tag{6.11}$$

where k_j are integers such that $k_i \neq k_j$ for $i \neq j$. We consider the Banach space $E = \{f \in C([0,1]) : f(0) = f(1)\}$ which is continuously and densely embedded in the Hilbert space $H = \left\{f \in L^2([0,1])\right\}$. We define $B : \mathbb{R}^n \to E$ as the map that to $(x_1, ..., x_n)$ associates the function $\sum_{j=1}^n e^{2\pi i k_j (\cdot)} x_j$. We define A_0 by

$$\mathcal{D}(A_0) = \left\{f \in H^1([0,1]) : f(0) = f(1)\right\}, \qquad (A_0 y)(\xi) = \frac{\partial}{\partial \xi} y(\xi), \text{ for every } y \in \mathcal{D}(A_0),$$

and A by

$$\mathcal{D}(A) = \left\{f \in C^1([0,1]) \bigcap E : f'(0) = f'(1)\right\}, \qquad (Ay)(\xi) = \frac{\partial}{\partial \xi} y(\xi), \text{ for every } y \in \mathcal{D}(A).$$

In the following, we write A instead of A_0.

Equation (6.11) admits the abstract formulation

$$\begin{cases} dZ_\tau = AZ_\tau d\tau + BdW_\tau, & \tau \in [0,T] \\ Z_0 = h, \end{cases} \tag{6.12}$$

where $W_\tau = (\beta_{k_1}(\tau), ..., \beta_{k_n}(\tau))$, so the space Ξ of the noise coincides with $\mathbb{R}^n$. If the initial datum $h \in H$, equation (6.12) admits a unique mild solution in H, moreover if the initial datum

$h \in E$, equation (6.12) admits a unique mild solution in E, since it is immediate to see that the stochastic convolution admits an E-continuous version.
Next we verify that

$$e^{tA_0}B(\Xi) \subset Q_t^{1/2}(H).$$

and that for some $c>0$ the operator norm satisfies

$$\left\| Q_t^{-1/2} e^{tA_0} B \right\|_{L(\Xi,H)} \leq \frac{c}{\sqrt{t}}, \text{ for } 0<t\leq T.$$

Indeed, let $f \in H$. Then $f(\xi) = \sum_{j=-\infty}^{j=+\infty} \widehat{f}(j) e^{2\pi i j\xi}$, where $\widehat{f}(j) = \displaystyle\int_0^1 f(\xi) e^{2\pi i j\xi} d\xi$. Since $A^* = -A$ with $\mathcal{D}(A^*) = \mathcal{D}(A)$, for $0 \leq t \leq T$,

$$\begin{aligned}(Q_t f)(\xi) &= \int_0^t \left(e^{sA} BB^* e^{sA^*} \right) f(\xi)\, ds \\ &= \int_0^t \sum_{j=1}^n \widehat{f}(k_j) e^{2\pi i k_j \xi} ds \\ &= t \sum_{j=1}^n \widehat{f}(k_j) e^{2\pi i k_j \xi}.\end{aligned}$$

Since for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ $e^{tA}Bx = \displaystyle\sum_{j=1}^n e^{2\pi i t k_j} x_j e^{2\pi i k_j \xi}$, it turns out that

$$e^{tA}B(\Xi) \subset Q_t^{1/2}(H)$$

and

$$Q_t^{-1/2} e^{tA} Bx = \sum_{j=1}^n \frac{e^{2\pi i t k_j}}{\sqrt{t}} x_j e^{2\pi i k_j \xi}$$

and so the operator norm satisfies

$$\left\| Q_t^{-1/2} e^{tA} B \right\|_{L(\Xi,H)} \leq \frac{1}{\sqrt{t}}, \text{ for } 0<t\leq T.$$

We remark that hypotheses 4.1 and 2.6 are satisfied, indeed $B(\Xi) \subset E$.
So if we denote by R_t the transition semigroup associated to equation (6.11) it turns out that $\varphi \in C_b(E)$ and for every $x \in E$, $R_t[\varphi](x)$ is Gateaux differentiable in the directions selected by B and for every $\xi \in \mathbb{R}^n$ the estimate

$$\left| \nabla^B R_t[\varphi](x)\xi \right| \leq \frac{1}{\sqrt{t}} \|\varphi\|_\infty \|\xi\|_{\mathbb{R}^n}, \qquad 0<t\leq T \tag{6.13}$$

holds true. We underline the fact that R_t is not strong Feller, since $Q_t^{1/2}(H)$ is a finite dimensional space, while $e^{tA}(H)$ is infinite dimensional, so $e^{tA}(H)$ cannot be contained in $Q_t^{1/2}(H)$.
We consider Kolmogorov equations with the structure of equation (5.8), where, at least formally,

$$\mathcal{A}_t v(x) = \frac{1}{2} Trace\left(BB^* \nabla^2 v(x) \right) + \langle Ax, \nabla v(x) \rangle_{E,E^*},$$

and A and B are the operators introduced in (6.12). We can solve such Kolmogorov equations by applying the results in section 5.

6.4 Stochastic wave equations

In this section we briefly show how our results can be applied to stochastic wave equations in space dimension one. In $(\Omega, \mathcal{F}, (\mathcal{F}_\tau), \mathbb{P})$ we consider, for $0 \leq \tau \leq T$ and $\xi \in [0,1]$, the following state equation:

$$\begin{cases} \frac{\partial^2}{\partial \tau^2} y(\tau, \xi) = \frac{\partial^2}{\partial \xi^2} y(\tau, \xi) + \dot{W}(\tau, \xi) \\ y(\tau, 0) = y(\tau, 1) = 0, \\ y(0, \xi) = x_0(\xi), \\ \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi). \end{cases} \tag{6.14}$$

$\dot{W}(\tau, \xi)$ is a space-time white noise on $[0,T] \times [0,1]$. This equation can be rewritten in an abstract way in the Hilbert space $H = L^2([0,1]) \oplus H^{-1}([0,1])$ in the following form:

$$\begin{cases} dX_\tau = A_0 X_\tau d\tau + G dW_\tau, \qquad \tau \in [0,T] \\ X_0 = x. \end{cases} \tag{6.15}$$

The process $\{W_\tau, \tau \geq 0\}$ is a cylindrical Wiener process with values in $\Xi = L^2([0,1])$, the operator $G : L^2([0,1]) \longrightarrow H$ is defined by $Gu = \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u$, where I is the embedding of $L^2([0,1])$ into $H^{-1}([0,1])$, and A is the wave operator, defined by

$$\mathcal{D}(A_0) = H_0^1([0,1]) \oplus L^2([0,1]), \qquad A_0 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},$$

for every $\begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A)$. The operator Λ is given by

$$\mathcal{D}(\Lambda) = H_0^1([0,1]), \qquad (\Lambda y)(\xi) = -\frac{\partial^2}{\partial \xi^2} y(\xi).$$

We underline the fact that the stochastic convolution

$$W_A(\tau) = \int_0^\tau e^{(\tau - s)A_0} G dW_s$$

is well defined in H, since the operators

$$Q_\tau = \int_0^\tau e^{sA_0} GG^* e^{sA_0^*} ds, \ 0 \leq \tau \leq T,$$

are of trace class, see e. g. (8), example 5.8.
We consider also the Hilbert space

$$H_1 = H_0^1([0,1]) \oplus L^2([0,1]).$$

On H_1 we define the operator A_1 by

$$\mathcal{D}(A_1) = H^2([0,1]) \cap H_0^1([0,1]) \oplus H_0^1([0,1]), \qquad A_1 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},$$

for every $\begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A_1)$, where the operator Λ is given by

$$\mathcal{D}(\Lambda) = H^2([0,1]) \cap H_0^1([0,1]), \qquad (\Lambda y)(\xi) = -\frac{\partial^2}{\partial \xi^2} y(\xi),$$

A_0 in H and A_1 in H_1 are the generators of the contractive group

$$e^{tA}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos\sqrt{\Lambda}t & \frac{1}{\sqrt{\Lambda}}\sin\sqrt{\Lambda}t \\ -\sqrt{\Lambda}\sin\sqrt{\Lambda}t & \cos\sqrt{\Lambda}t \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}, \quad t \in \mathbb{R}.$$

Following (21), we introduce the Banach space $E = B^s_{2,p,\{0\}}([0,1]) \oplus B^{s-1}_{2,p}([0,1])$, with $s \in (0,1)$ and $p > 2$, where $B^s_{2,p,\{0\}}([0,1])$ is the Besov space with Dirichlet boundary conditions and $B^{s-1}_{2,p}([0,1])$ is a Besov space with negative exponent. The space E can be obtained by interpolating H and H_1:

$$\begin{aligned}(H_1, H)_{s,p} &= \left(H_0^1([0,1]), L^2([0,1])\right)_{s,p} \oplus \left(L^2([0,1]), H^{-1}([0,1])\right)_{s,p} \\ &= B^s_{2,p,\{0\}}([0,1]) \oplus B^{s-1}_{2,p}([0,1]).\end{aligned}$$

For more details on real interpolation and Besov spaces see e.g. (19) and (25). The operator A with domain $\mathcal{D}(A) = B^{s+1}_{2,p,\{0\}}([0,1]) \oplus B^s_{2,p,\{0\}}([0,1])$ is the generator of a group in E, and moreover since A is dissipative both in H_1 and in H, it turns out that A is dissipative in E. Equation (6.14) can be written as an evolution equation in the Banach space E:

$$\begin{cases} dX_\tau = AX_\tau d\tau + GdW_\tau, & \tau \in [0,T] \\ X_0 = x, \end{cases} \tag{6.16}$$

where $G : L^2([0,1]) \longrightarrow E$ is defined by $Gu = \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u$, and I is the embedding of $L^2([0,1])$ in $B^{s-1}_{2,p}([0,1])$. We note that if $s - 1/2 > 0$, then $B^s_{2,p,\{0\}}([0,1])$ is contained in $C([0,1])$, see (25), theorem 4.6.1. Moreover E is continuously and densely embedded in the Hilbert space H, and the stochastic convolution admits a version in $C([0,T], E)$, for p sufficiently large, see (21). So if the initial datum belongs to E, then the solution X of equation (6.16) evolves in E. We remark that hypotheses 2.6 and 4.1 are satisfied, indeed $G(\Xi) \subset E$.

By (20), section 6.1, it turns out that for every $0 < t \leq T$

$$e^{tA_0}G(\Xi) \subset Q_t^{1/2}(H)$$

and the operator norm satisfies

$$\left\| Q_t^{-1/2} e^{tA} G \right\|_{L(\Xi,H)} \leq \frac{c}{\sqrt{t}}, \text{ for } 0 < t \leq T.$$

So if we denote by R_t the transition semigroup associated to equation (6.16) it turns out that $\varphi \in C_b(E)$ and for every $x \in E$, $R_t[\varphi](x) \in \mathcal{G}^G(E)$ and for every $\xi \in \Xi$ the following estimate holds true:

$$\left|\nabla^G R_t[\varphi](x)\xi\right| \leq \frac{1}{\sqrt{t}} \|\varphi\|_\infty \|\xi\|_\Xi, \quad 0 < t \leq T. \tag{6.17}$$

We consider Kolmogorov equations with the structure of equation (5.8), where, at least formally,

$$\mathcal{A}_t v(x) = \frac{1}{2} Trace\left(GG^* \nabla^2 v(x)\right) + \langle Ax, \nabla v(x)\rangle_{E,E^*},$$

and A and G are the operators introduced in (6.16. We can solve such Kolmogorov equations by applying the results in section 5.

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