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Distributions of Sojourn Time, Maximum and Minimum for Pseudo-Processes Governed by Higher-Order Heat-Type Equations

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Abstract: The higher-order heat-type equation $\partial u/\partial t = \pm\partial^n u/\partial x^n$ has been investigated by many authors. With this equation is associated a pseudo-process $(X_t)_{t\geqslant 0}$ which is governed by a signed measure. In the even-order case, Krylov, [9], proved that the classical arc-sine law of Paul Lévy for standard Brownian motion holds for the pseudo-process $(X_t)_{t\geqslant 0}$, that is, if T_t is the sojourn time of $(X_t)_{t\geqslant 0}$ in the half line $(0,+\infty)$ up to time t, then $\mathbb{P}(T_t \in ds) = \frac{ds}{\pi\sqrt{s(t-s)}}$, $0 < s < t$. Orsingher, [13], and next Hochberg and Orsingher, [7], obtained a counterpart to that law in the odd cases $n = 3, 5, 7$. Actually Hochberg and Orsingher proposed a more or less explicit expression for that new law in the odd-order general case and conjectured a quite simple formula for it. The distribution of T_t subject to some conditioning has also been studied by Nikitin & Orsingher, [11], in the cases $n = 3, 4$. In this paper, we prove that the conjecture of Hochberg and Orsingher is true and we extend the results of Nikitin & Orsingher for any integer n. We also investigate the distributions of maximal and minimal functionals of $(X_t)_{t\geqslant 0}$, as well as the distribution of the last time before becoming definitively negative up to time t.

Key words and phrases: Arc-sine law, sojourn time, maximum and minimum, higher-order heat-type equations, Vandermonde systems and determinants.

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1 Introduction

Several authors have considered higher-order heat-type equations

$$\frac{\partial u}{\partial t} = \kappa_n \frac{\partial^n u}{\partial x^n} \tag{1}$$

for integral n and $\kappa_n = \pm 1$. More precisely, they choose $\kappa_n = (-1)^{p+1}$ for $n = 2p$ and $\kappa_n = \pm 1$ for $n = 2p+1$. They associated with Eq. (1) a pseudo Markov process $(X_t)_{t\geqslant 0}$ governed by a signed measure which is not a probability measure and studied

- some analytical properties of the sample paths of that pseudo-process;
- the distribution of the sojourn time spent by that pseudo-process on the positive half-line up to time t: $T_t = \text{meas}\{s \in [0,t] : X_s > 0\} = \int_0^t 1\!\!1_{\{X_s>0\}}\, ds$;
- the distribution of the maximum $M_t = \max_{0\leqslant s\leqslant t} X_s$ together with that of the minimum $m_t = \min_{0\leqslant s\leqslant t} X_s$ with possible conditioning on some values of X_t;
- the joint distribution of M_t and X_t.

The first remarkable result is due to Krylov, [9], who showed that the well-known arc-sine law of Paul Lévy for standard Brownian motion holds for T_t for all even integers n:

$$\mathbb{P}(T_t \in ds)/ds = \frac{1\!\!1_{(0,t)}(s)}{\pi\sqrt{s(t-s)}}$$

which is also characterized by the double Laplace transform

$$\int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}(e^{-\mu T_t})\, dt = \frac{1}{\sqrt{\lambda(\lambda+\mu)}}.$$

Hochberg, [6], studied the pseudo-process $(X_t)_{t\geqslant 0}$ and derived many analytical properties: especially, he defined a stochastic integral and proposed an Itô formula, he obtained a formula for the distribution of M_t in the case $n = 4$ with an extension to the even-order case. Noteworthy, the sample paths do not seem to be continuous in the case $n = 4$. Later, Orsingher, [13], obtained some explicit distributions for the random variable T_t as well as the random variables M_t and m_t in the case $n = 3$. For example, when $\kappa_n = +1$, he found for the double Laplace transform of T_t the simple expression

$$\int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}(e^{-\mu T_t})\, dt = \frac{1}{\sqrt[3]{\lambda^2(\lambda+\mu)}}.$$

Hochberg and Orsingher proposed a more or less explicit formula for the double Laplace transform of T_t when n is an odd integer (Theorem 5.1 of [7]). They simplified their formula in the particular cases $n = 5$ and $n = 7$ and found the following simple expressions when $\kappa_n = +1$ for instance:

$$\int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}(e^{-\mu T_t})\, dt = \begin{cases} \dfrac{1}{\sqrt[5]{\lambda^2(\lambda+\mu)^3}} & \text{for } n = 5, \\[2ex] \dfrac{1}{\sqrt[7]{\lambda^4(\lambda+\mu)^3}} & \text{for } n = 7. \end{cases}$$

In view of these relations, Hochberg and Orsingher guessed that the following formula should hold for any odd integer $n = 2p + 1$ when $\kappa_n = +1$ (conjecture (5.17) of [7]):

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}(e^{-\mu T_t})\,dt = \begin{cases} \dfrac{1}{\sqrt[n]{\lambda^{p+1}(\lambda+\mu)^p}} & \text{if } p \text{ is odd,} \\ \dfrac{1}{\sqrt[n]{\lambda^{p}(\lambda+\mu)^{p+1}}} & \text{if } p \text{ is even.} \end{cases} \tag{2}$$

In this paper, we prove that their conjecture is true (see Theorem 7 and Corollary 9 below). The way we use consists of solving a partial differential equation coming from the Feynman-Kac formula. This leads to a Vandermonde system for which we produce an explicit solution. As a result, it is straightforward to invert (2) as in [7] (formulae (3.13) to (3.16)) and to derive then the density of T_t which is the counterpart to the arc-sine law:

$$\mathbb{P}(T_t \in ds)/ds = \begin{cases} \dfrac{k_n \mathbb{1}_{(0,t)}(s)}{\sqrt[n]{s^p(t-s)^{p+1}}} & \text{if } \kappa_n = (-1)^p, \\ \dfrac{k_n \mathbb{1}_{(0,t)}(s)}{\sqrt[n]{s^{p+1}(t-s)^{p}}} & \text{if } \kappa_n = (-1)^{p+1}. \end{cases}$$

with

$$k_n = \frac{1}{\pi}\sin\frac{p}{n}\pi = \frac{1}{\pi}\sin\frac{p+1}{n}\pi.$$

Nikitin and Orsingher, [11], studied the law of T_t conditioned on the events $X_t = 0$, $X_t > 0$ or $X_t < 0$ in the cases $n = 3$ and $n = 4$ and obtained the uniform law for first conditioning as well as some Beta distributions for the others. For instance,

$$\mathbb{P}(T_t \in ds | X_t > 0)/ds = \begin{cases} \dfrac{3^{3/2}}{4\pi t}\left(\dfrac{s}{t-s}\right)^{2/3} \mathbb{1}_{(0,t)}(s) & \text{for } n = 3 \text{ and } \kappa_n = +1, \\ \dfrac{2}{\pi t}\sqrt{\dfrac{s}{t-s}}\,\mathbb{1}_{(0,t)}(s) & \text{for } n = 4. \end{cases} \tag{3}$$

Through a way similar to that used in the unconditioned case, we have also found the uniform law for conditioning on $X_t = 0$ (see Theorem 13) and we have obtained an extension to (3) (Theorem 14). Beghin et al., [1, 2], studied the laws of M_t and m_t conditioned on the event $X_t = 0$ as well as coupling M_t with X_t in the same cases. In particular, in the unconditioned case,

$$\mathbb{P}(M_t \leqslant a) \;=\; \mathbb{P}(X_t \leqslant a) - \mathbb{P}(X_t > a) + \frac{1}{\Gamma(\frac{2}{3})}\int_0^t \frac{\partial p}{\partial x}(s;a)\,\frac{ds}{\sqrt[3]{t-s}} \quad \text{if } n = 3, \tag{4}$$

$$\mathbb{P}(M_t \leqslant a) \;=\; \mathbb{P}(X_t \leqslant a) - \mathbb{P}(X_t > a) + \frac{2}{\sqrt{\pi}}\int_0^t \frac{\partial p}{\partial x}(s;a)\,\frac{ds}{\sqrt{t-s}} \quad \text{if } n = 4, \tag{5}$$

where $p(t;x)$ is the fundamental solution of Eq. (1) in the cases $n = 3, 4$ and $\kappa_n = -1$. Actually, formula (4) is due to Orsingher, [13]. Concerning formula (5), the constant before the integral in [2] should be corrected into $2/\sqrt{\pi}$. We shall also study the distributions of M_t and m_t in the general case by relying them on that of T_t. Specifically, the Laplace transform of M_t, $\mathbb{E}(e^{-\mu M_t})$ say, may be acquired from that of T_t by letting μ tend to $+\infty$.

Finally, let us mention that Eq. (1) has been studied especially in the case $n = 4$ (biharmonic operator) under other points of view by Funaki, [4], and next Hochberg and Orsingher, [8], in relation with compound processes, Nishioka, [12], who adopts a distributional approach for first hitting time and place of a half-line, Motoo, [10], who study stochastic integration, Benachour et al., [3], who provide other probabilistic interpretations. See also the references therein.

The paper is organized as follows: in Section 2, we write down the general notations. In Section 3, we exhibit several properties for the fundamental solution of Eq. 1 (heat-type kernel $p(t;x)$) that will be useful. Section 4 is devoted to some duality notion for the pseudo-process $(X_t)_{t\geqslant 0}$. The subsequent sections 5 and 6 deal with the distributions of T_t subject to certain conditioning. Section 7 concerns the distributions of the maximal and minimal functionals and Section 8 concerns the distribution of the last time before becoming definitively negative up to time t:

$$O_t = \sup\{s \in [0,t] : X_s > 0\}$$

with the convention $\sup(\emptyset) = 0$.

2 Notations

We first introduce the dual operators

$$\mathcal{D}_x = \kappa_n \frac{\partial^n}{\partial x^n} \qquad \text{and} \qquad \mathcal{D}_y^* = (-1)^n \kappa_n \frac{\partial^n}{\partial y^n}.$$

Let $p(t;z)$ be the fundamental solution of Eq. (1). The function p is characterized by its Fourier transform

$$\int_{-\infty}^{+\infty} e^{iuz} p(t;z)\,dz = e^{\kappa_n t(-iu)^n}. \tag{6}$$

Put

$$p(t;x,y) = p(t;x-y) = \mathbb{P}_x(X_t \in dy)/dy.$$

The foregoing relation defines a pseudo Markov process $(X_t)_{t\geqslant 0}$ which is governed by a signed measure, this one being of infinite total variation. The function $(t;x,y) \longmapsto p(t;x,y)$ is the fundamental solution to the backward and forward equations

$$\mathcal{D}_x u = \mathcal{D}_y^* u = \frac{\partial u}{\partial t}.$$

Some properties of p are exhibited in the next section and we refer to [6] for several analytical properties about the pseudo-process $(X_t)_{t\geqslant 0}$ in the even-order case.

Set now

$$\begin{aligned}
u(t,\mu;x) &= \mathbb{E}_x(e^{-\mu T_t}),\\
v(t,\mu;x,y) &= \mathbb{E}_x(e^{-\mu T_t}, X_t \in dy)/dy,\\
w_M(t;x,a) &= \mathbb{P}_x(M_t \leqslant a),\\
w_m(t;x,a) &= \mathbb{P}_x(m_t \geqslant a),\\
z(t,\mu;x) &= \mathbb{E}_x(e^{-\mu O_t})
\end{aligned}$$

together with their Laplace transforms:

$$U(\lambda,\mu;x) = \int_0^{+\infty} e^{-\lambda t}\, u(t,\mu;x)\,dt,$$

$$\begin{aligned} V(\lambda,\mu;x,y) &= \int_0^{+\infty} e^{-\lambda t}\, v(t,\mu;x,y)\,dt,\\ W_M(\lambda;x,a) &= \int_0^{+\infty} e^{-\lambda t}\, w_M(t;x,a)\,dt,\\ W_m(\lambda;x,a) &= \int_0^{+\infty} e^{-\lambda t}\, w_m(t;x,a)\,dt,\\ Z(\lambda,\mu;x) &= \int_0^{+\infty} e^{-\lambda t}\, z(t,\mu;x)\,dt. \end{aligned}$$

The functions u and v are solutions of some partial differential equations. We recall that by the Feynman-Kac formula, see e.g. [6], if f and g are any suitable functions, the function φ defined as

$$\varphi(t;x) = \mathbb{E}_x[e^{-\int_0^t f(X_s)\,ds} g(X_t)]$$

where the "expectation" must be understood as $\lim_{m\to\infty} \mathbb{E}_x[e^{-\frac{t}{m}\sum_{k=0}^m f(X_{kt/m})} g(X_t)]$ solves the partial differential equation

$$\frac{\partial\varphi}{\partial t} = \kappa_n \frac{\partial^n \varphi}{\partial x^n} - f\varphi \tag{7}$$

together with the initial condition $\varphi(0;x) = g(x)$ and then the Laplace transform ϕ of φ,

$$\phi(\lambda;x) = \int_0^{+\infty} e^{-\lambda t}\, \varphi(t;x)\,dt,$$

solves the ordinary differential equation

$$\kappa_n \frac{d^n\phi}{dx^n} = (f+\lambda)\phi - g. \tag{8}$$

For solving Eq. (8) that will be of concern to us, the related solutions will be linear combinations of exponential functions and we shall have to consider the n^{th} roots of κ_n (θ_j for $0 \leqslant j \leqslant n-1$ say) and distinguish the indices j such that $\Re\theta_j < 0$ and $\Re\theta_j > 0$ (one never has $\Re\theta_j = 0$). So, let us introduce the following set of indices

$$\begin{aligned} I &= \{j \in \{0,\dots,n-1\} : \Re\theta_j < 0\},\\ J &= \{j \in \{0,\dots,n-1\} : \Re\theta_j > 0\}. \end{aligned}$$

We have

$$I \cup J = \{0,\dots,n-1\} \quad \text{and} \quad I \cap J = \emptyset.$$

If $n = 2p$, then $\kappa_n = (-1)^{p+1}$, $\theta_j = e^{i[(2j+p+1)\pi/n]}$,

$$I = \{0,\dots,p-1\} \quad \text{and} \quad J = \{p,\dots,2p-1\}.$$

The numbers of elements of the sets I and J are

$$\#I = \#J = p.$$

If $n = 2p+1$, then

- for $\kappa_n = +1$: $\theta_j = e^{i[2j\pi/n]}$ and

$$I = \left\{\frac{p}{2}+1,\dots,\frac{3p}{2}\right\} \quad \text{and } J = \left\{0,\dots,\frac{p}{2}\right\} \cup \left\{\frac{3p}{2}+1,\dots,2p\right\} \quad \text{if } p \text{ is even},$$

$$I = \left\{\frac{p+1}{2},\dots,\frac{3p+1}{2}\right\} \text{ and } J = \left\{0,\dots,\frac{p-1}{2}\right\} \cup \left\{\frac{3p+3}{2},\dots,2p\right\} \quad \text{if } p \text{ is odd}.$$

The numbers of elements of the sets I and J are

$$\begin{array}{llll} \#I = p & \text{and} & \#J = p+1 & \text{if } p \text{ is even,} \\ \#I = p+1 & \text{and} & \#J = p & \text{if } p \text{ is odd.} \end{array}$$

- for $\kappa_n = -1$: $\theta_j = e^{i[(2j+1)\pi/n]}$ and

$$\begin{array}{lll} I = \left\{\frac{p}{2}, \dots, \frac{3p}{2}\right\} & \text{and } J = \left\{0, \dots, \frac{p}{2} - 1\right\} \cup \left\{\frac{3p}{2} + 1, \dots, 2p\right\} & \text{if } p \text{ is even,} \\ I = \left\{\frac{p+1}{2}, \dots, \frac{3p-1}{2}\right\} & \text{and } J = \left\{0, \dots, \frac{p-1}{2}\right\} \cup \left\{\frac{3p+1}{2}, \dots, 2p\right\} & \text{if } p \text{ is odd.} \end{array}$$

The numbers of elements of the sets I and J are

$$\begin{array}{llll} \#I = p+1 & \text{and} & \#J = p & \text{if } p \text{ is even,} \\ \#I = p & \text{and} & \#J = p+1 & \text{if } p \text{ is odd.} \end{array}$$

We finally set $\omega_j = e^{i[2j\pi/n]}$. In all cases one has $\theta_j/\theta_k = \omega_{j-k}$.

3 Some properties of the heat-type kernel $p(t;z)$

In this section, we write down some properties for the kernel $p(t;z)$ that will be used further.

3.1 Integral representations

Inverting the Fourier transform (6), we obtain the following complex integral representation for p:

$$p(t;z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-izu+\kappa_n t(-iu)^n} \, du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{izu+\kappa_n t(iu)^n} \, du \tag{9}$$

from which we also deduce some real integral representations:

- if $n = 2p$ and $\kappa_n = (-1)^{p+1}$,

$$p(t;z) = \frac{1}{\pi} \int_0^{+\infty} \cos(zu) e^{-tu^n} \, du;$$

- if $n = 2p+1$ and $\kappa_n = \pm 1$,

$$p(t;z) = \begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^{+\infty} \cos(zu + tu^n) \, du \stackrel{\text{def}}{=} p^+(t;z) & \text{for } \kappa_n = (-1)^p, \\ \dfrac{1}{\pi} \displaystyle\int_0^{+\infty} \cos(zu - tu^n) \, du \stackrel{\text{def}}{=} p^-(t;z) & \text{for } \kappa_n = (-1)^{p+1}. \end{cases}$$

We obviouly have $p^{\pm}(t;z) = p^{\mp}(t;-z)$.

3.2 Value at $z = 0$

Substituting $z = 0$ in the above expressions immediately gives the value of $p(t;0)$ below. This will be used in the evaluation of the distribution of the random variable T_t conditioned on $X_t = 0$.

Proposition 1 *We have*

$$p(t;0) = \begin{cases} \dfrac{\Gamma(\frac{1}{n})}{n\pi\sqrt[n]{t}} & \text{if } n \text{ is even,} \\[2ex] \dfrac{\Gamma(\frac{1}{n})\cos\frac{\pi}{2n}}{n\pi\sqrt[n]{t}} & \text{if } n \text{ is odd.} \end{cases} \tag{10}$$

3.3 Asymptotics

It is possible to determine some asymptotics for $p(t;z)$ as $z/\sqrt[n]{t} \to \pm\infty$ (which include the cases $z \to \pm\infty$ and $t \to 0^+$) by using the Laplace's method (the steepest descent). Since the computations are rather intricate, we postpone them to Appendix A and only report the results below.

Proposition 2 *Set $\zeta = z/\sqrt[n]{nt}$. We have as $z/\sqrt[n]{t} \to +\infty$:*

- *when $n = 2p$:*
  - *for even p:*

$$\begin{aligned} p(t;z) \;\sim\;& \frac{2\,\zeta^{-\frac{n-2}{2(n-1)}}}{\sqrt{2(n-1)\pi}\,\sqrt[n]{nt}} \sum_{k=0}^{(p/2)-1} e^{-(1-\frac{1}{n})\sin(\frac{2k+\frac{1}{2}}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}} \\ & \times \cos\left((1-\frac{1}{n})\cos(\frac{2k+\frac{1}{2}}{n-1}\pi)\,\zeta^{\frac{n}{n-1}} + \frac{2k+\frac{1}{2}}{2(n-1)}\pi - \frac{\pi}{4}\right); \end{aligned}$$

  - *for odd p:*

$$\begin{aligned} p(t;z) \;\sim\;& \frac{\zeta^{-\frac{n-2}{2(n-1)}}}{\sqrt{2(n-1)\pi}\,\sqrt[n]{nt}} \left[2 \sum_{k=0}^{(p-3)/2} e^{-(1-\frac{1}{n})\sin(\frac{2k+\frac{1}{2}}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}}\right. \\ & \times \cos\left((1-\frac{1}{n})\cos(\frac{2k+\frac{1}{2}}{n-1}\pi)\,\zeta^{\frac{n}{n-1}} + \frac{2k+\frac{1}{2}}{2(n-1)}\pi - \frac{\pi}{4}\right) \\ & \left. - e^{-(1-\frac{1}{n})\zeta^{\frac{n}{n-1}}}\right]; \end{aligned}$$

- *when $n = 2p+1$:*
  - *if $\kappa_n = (-1)^p$,*
    - *for even p:*

$$\begin{aligned} p(t;z) \;\sim\;& \frac{2\,\zeta^{-\frac{n-2}{2(n-1)}}}{\sqrt{2(n-1)\pi}\,\sqrt[n]{nt}} \sum_{k=0}^{(p/2)-1} e^{-(1-\frac{1}{n})\sin(\frac{2k+1}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}} \\ & \times \cos\left((1-\frac{1}{n})\cos(\frac{2k+1}{n-1}\pi)\,\zeta^{\frac{n}{n-1}} + \frac{2k+1}{2(n-1)}\pi - \frac{\pi}{4}\right); \end{aligned}$$

* *for odd p:*

$$p(t;z) \sim \frac{\zeta^{-\frac{n-2}{2(n-1)}}}{\sqrt{2(n-1)\pi}\,\sqrt[n]{nt}}\left[2\sum_{k=0}^{(p-3)/2} e^{-(1-\frac{1}{n})\sin(\frac{2k+1}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}} \times\cos\left((1-\frac{1}{n})\cos(\frac{2k+1}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}+\frac{2k+1}{2(n-1)}\pi-\frac{\pi}{4}\right) - e^{-(1-\frac{1}{n})\zeta^{\frac{n}{n-1}}}\right];$$

notice that the integral $\int_0^{+\infty} p(t;z)\,dz$ *is absolutely convergent;*

- *if* $\kappa_n = (-1)^{p+1}$,
  * *for even p:*

$$p(t;z) \sim \frac{\zeta^{-\frac{n-2}{2(n-1)}}}{\sqrt{2(n-1)\pi}\,\sqrt[n]{nt}}\left[2\sum_{k=0}^{(p/2)-1} e^{-(1-\frac{1}{n})\sin(\frac{2k}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}} \times\cos\left((1-\frac{1}{n})\cos(\frac{2k}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}+\frac{k}{n-1}\pi-\frac{\pi}{4}\right) - e^{-(1-\frac{1}{n})\zeta^{\frac{n}{n-1}}}\right];$$

  * *for odd p:*

$$p(t;z) \sim \frac{2\,\zeta^{-\frac{n-2}{2(n-1)}}}{\sqrt{2(n-1)\pi}\,\sqrt[n]{nt}}\sum_{k=0}^{(p-1)/2} e^{-(1-\frac{1}{n})\sin(\frac{2k}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}} \times\cos\left((1-\frac{1}{n})\cos(\frac{2k}{n-1}\pi)\,\zeta^{\frac{n}{n-1}}+\frac{k}{n-1}\pi-\frac{\pi}{4}\right);$$

notice that the integral $\int_0^{+\infty} p(t;z)\,dz$ *is semi-convergent; this is due to the presence of the term* $\cos((1-\frac{1}{n})\,\zeta^{\frac{n}{n-1}}-\frac{\pi}{4})$ *corresponding to the index* $k=0$.

Asymptotics for the case $z/\sqrt[n]{t} \to -\infty$ can be immediately deduced from the previous ones by reminding that

- if n is even, the function p is symmetric;
- if n is odd, we have $p^{\pm}(t;z) = p^{\mp}(t;-z)$.

3.4 Value of the spatial integral

Conditioning on $X_t > 0$ requires the evaluation of the integral $\int_0^{+\infty} p(t;z)\,dz$. Its value is supplied by the following proposition.

Proposition 3 *The value of the integral of the function $z \longmapsto p(t;z)$ on $(0,+\infty)$ is given by*

$$\int_0^{+\infty} p(t;z)\,dz = \begin{cases} \dfrac{1}{2} & \text{if } n \text{ is even,} \\ \dfrac{1}{2}\left(1-\dfrac{1}{n}\right) & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^p, \\ \dfrac{1}{2}\left(1+\dfrac{1}{n}\right) & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^{p+1}. \end{cases} \tag{11}$$

PROOF. 1. If n is even, the function $z \mapsto p(t;z)$ is symmetric and then its integral on $(0,+\infty)$ is the half of the one on $(-\infty,+\infty)$, that is to say $\frac{1}{2}$.

2. If n is odd ($n = 2p+1$), due to $\int_{-\infty}^{+\infty} p(t;z)\,dz = 1$ we only have to evaluate one of both integrals $\int_0^{+\infty} p(t;z)\,dz$ and $\int_{-\infty}^0 p(t;z)\,dz$. Actually, as it has been said in Proposition 2 one is absolutely convergent whereas the other is semi-convergent. More precisely,

- if $\kappa_n = (-1)^p$, $\int_0^{+\infty} p(t;z)\,dz$ is absolutely convergent;
- if $\kappa_n = (-1)^{p+1}$, $\int_{-\infty}^0 p(t;z)\,dz$ is absolutely convergent.

So, as $\lambda \to 0^+$, the dominated convergence theorem applies to the integral $\int_0^{+\infty} e^{-\lambda z} p(t;z)\,dz$ in the case $\kappa_n = (-1)^p$ and to $\int_{-\infty}^0 e^{\lambda z} p(t;z)\,dz$ in the case $\kappa_n = (-1)^{p+1}$. For instance, in the first case,

$$\int_0^{+\infty} e^{-\lambda z} p(t;z)\,dz \underset{\lambda\to 0^+}{\longrightarrow} \int_0^{+\infty} p(t;z)\,dz.$$

Now let us compute the integral $\int_0^{+\infty} e^{-\lambda z} p(t;z)\,dz$ for $\kappa_n = (-1)^p$ and $\lambda > 0$. By (9) we have

$$\int_0^{+\infty} e^{-\lambda z} p(t;z)\,dz = \Re\left[\frac{1}{\pi}\int_0^{+\infty} e^{-\lambda z}\,dz \int_0^{+\infty} e^{i(zu+tu^n)}\,du\right]$$

Integrating over an angular contour, we see that the integral of the function $u \longmapsto e^{i(zu+tu^n)}$ on the half-line $(0,+\infty)$ coincides with the one on the half-line of polar angle $\frac{\pi}{2n}$: putting $\varpi = e^{i\frac{\pi}{2n}}$,

$$\int_0^{+\infty} e^{i(zu+tu^n)}\,du = \varpi \int_0^{+\infty} e^{i\varpi zu - tu^n}\,du.$$

Therefore,

$$\begin{aligned} \int_0^{+\infty} e^{-\lambda z} p(t;z)\,dz &= \Re\left[\frac{1}{\pi}\varpi \int_0^{+\infty} e^{-tu^n}\,du \int_0^{+\infty} e^{-(\lambda - i\varpi u)z}\,dz\right] \\ &= \frac{1}{\pi}\int_0^{+\infty} \Re\left(\frac{\varpi}{\lambda - i\varpi u}\right) e^{-tu^n}\,du \\ &= \frac{\lambda\cos\frac{\pi}{2n}}{\pi}\int_0^{+\infty} \frac{e^{-tu^n}}{u^2 + 2\lambda u \sin\frac{\pi}{2n} + \lambda^2}\,du \\ &= \frac{\cos\frac{\pi}{2n}}{\pi}\int_0^{+\infty} \frac{e^{-\lambda^n t u^n}}{u^2 + 2u\sin\frac{\pi}{2n} + 1}\,du. \end{aligned}$$

By dominated convergence,

$$\int_0^{+\infty} e^{-\lambda z} p(t;z)\,dz \underset{\lambda\to 0+}{\longrightarrow} \frac{\cos\frac{\pi}{2n}}{\pi}\int_0^{+\infty}\frac{du}{u^2+2u\sin\frac{\pi}{2n}+1}.$$

Finally, we use the elementary identity ([5, 2.172, p. 81]), for $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$,

$$\int_0^{+\infty}\frac{du}{u^2+2u\sin\alpha+1}=\frac{1}{\cos\alpha}\Big[\frac{\pi}{2}-\alpha\Big],$$

and the result follows by choosing $\alpha=\frac{\pi}{2n}$. □

3.5 Value of the temporal Laplace transform

It will be seen that the distribution of the maximum and minimum functionals may be expressible by means of the successive derivatives $\frac{\partial^j p}{\partial z^j}(t;z)$, $0\leqslant j\leqslant n-1$. So, we give below some expressions for their Laplace transform.

Proposition 4 *1. The Laplace transform of the function $t\longmapsto p(t;z)$ is given by*

$$\Phi(\lambda;z)\overset{\text{def}}{=}\int_0^{+\infty}e^{-\lambda t}\,p(t;z)\,dt=\begin{cases}-\dfrac{1}{n\lambda^{1-\frac{1}{n}}}\displaystyle\sum_{k\in I}\theta_k e^{\theta_k\delta z} & \text{if } z\geqslant 0,\\ \dfrac{1}{n\lambda^{1-\frac{1}{n}}}\displaystyle\sum_{k\in J}\theta_k e^{\theta_k\delta z} & \text{if } z\leqslant 0,\end{cases}\tag{12}$$

where $\delta=\sqrt[n]{\lambda}$.
2. The successive partial derivatives of Φ are given, for $j\leqslant n-1$ and $z\neq 0$, by

$$\frac{\partial^j\Phi}{\partial z^j}(\lambda;z)=\int_0^{+\infty}e^{-\lambda t}\,\frac{\partial^j p}{\partial z^j}(t;z)\,dt=\begin{cases}-\dfrac{1}{n\lambda^{1-\frac{j+1}{n}}}\displaystyle\sum_{k\in I}\theta_k^{j+1}e^{\theta_k\delta z} & \text{if } z>0,\\ \dfrac{1}{n\lambda^{1-\frac{j+1}{n}}}\displaystyle\sum_{k\in J}\theta_k^{j+1}e^{\theta_k\delta z} & \text{if } z<0,\end{cases}\tag{13}$$

and for $j=n$, by

$$\frac{\partial^n\Phi}{\partial z^n}(\lambda;z)=\int_0^{+\infty}e^{-\lambda t}\,\frac{\partial^n p}{\partial z^n}(t;z)\,dt=\kappa_n[\lambda\Phi(\lambda;z)-\delta_0(z)].\tag{14}$$

Moreover, the derivatives of Φ up to order $n-2$ are continuous at 0.
3. The Laplace transform of the function $t\longmapsto\mathbb{P}(X_t\leqslant x)-\mathbb{P}(X_t>x)$ is

$$\begin{aligned}\Psi(\lambda;x)&\overset{\text{def}}{=}\int_0^{+\infty}e^{-\lambda t}\,[\mathbb{P}(X_t\leqslant x)-\mathbb{P}(X_t>x)]\,dt\\&=\begin{cases}\dfrac{1}{\lambda}\Big[1-\dfrac{2}{n}\displaystyle\sum_{k\in J}e^{-\theta_k\delta x}\Big] & \text{if } x\geqslant 0,\\ \dfrac{1}{\lambda}\Big[\dfrac{2}{n}\displaystyle\sum_{k\in I}e^{-\theta_k\delta x}-1\Big] & \text{if } x\leqslant 0.\end{cases}\end{aligned}\tag{15}$$

Proof. 1. Using (9), it is possible to show that

$$\int_0^{+\infty}e^{-\lambda t}\,p(t;z)\,dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{izu}}{\lambda-\kappa_n(iu)^n}\,du=\frac{1}{2i\pi}\int_{-i\infty}^{+i\infty}\frac{e^{zu}}{\lambda-\kappa_n u^n}\,du;$$

the details are recorded in Appendix B. Now, assume $z \geqslant 0$. Let us integrate on a large half-circle lying in the half-plane $\Re u \leqslant 0$, with center at 0 and diameter on the imaginary axis. By residus, we get

$$\int_0^{+\infty} e^{-\lambda t}\, p(t; z)\, dt = \sum_{k\in I} \mathrm{Res}\left(\frac{e^{zu}}{\lambda - \kappa_n u^n}\right)_{u=\theta_k\delta} = -\sum_{k\in J} \frac{1}{n\kappa_n(\theta_k\delta)^{n-1}}\, e^{\theta_k\delta z}$$

which proves (12) by remarking that $\kappa_n\theta_k^{n-1} = \frac{1}{\theta_k}$. The case $z \leqslant 0$ can be treated in a similar way.

2. Differentiating several times the right-hand side of (12) leads to the right-hand sides of (13) and (14). Nevertheless, the proof of $\frac{\partial^j \Phi}{\partial z^j}(\lambda; z) = \int_0^{+\infty} e^{-\lambda t}\, \frac{\partial^j p}{\partial z^j}(t; z)\, dt$ for $j \leqslant n-1$ is very intricate, so we postpone it to the Appendix B. The continuity of the derivatives of Φ up to order $n-2$ immediately comes from the equality $\sum_{k=0}^{n-1} \theta_k^{j+1} = 0$ for $j \leqslant n-2$.

3. Assume e.g. $x \geqslant 0$. Integrating (12) with respect to x gives now

$$\begin{aligned}
\int_0^{+\infty} e^{-\lambda t}\, \mathbb{P}(X_t \leqslant x)\, dt &= \int_{-\infty}^{x} \Phi(\lambda; -z)\, dz \\
&= \int_{-\infty}^{0} -\frac{1}{n\lambda}\sum_{k\in I} \theta_k\delta\, e^{-\theta_k\delta z}\, dz + \int_0^x \frac{1}{n\lambda}\sum_{k\in J} \theta_k\delta\, e^{-\theta_k\delta z}\, dz \\
&= \frac{1}{n\lambda}\left[\sum_{k\in I} 1 + \sum_{k\in J}(1 - e^{-\theta_k\delta x})\right] \\
&= \frac{1}{\lambda}\left[1 - \frac{1}{n}\sum_{k\in J} e^{-\theta_k\delta x}\right].
\end{aligned}$$

Similarly,

$$\int_0^{+\infty} e^{-\lambda t}\, \mathbb{P}(X_t > x)\, dt = \frac{1}{n\lambda}\sum_{k\in J} e^{-\theta_k\delta x}$$

which completes the proof of (15) in the case $x \geqslant 0$. The other case is quite analogous. □

4 A note on duality

In this part, we introduce the dual pseudo-process $(X_t^*)_{t\geqslant 0}$ defined by its transition functions as

$$\begin{aligned}
p^*(t; x, y) &= \mathbb{P}_x^*(X_t^* \in dy)/dy \\
&= \mathbb{P}(X_t^* \in dy | X_0^* = x)/dy \\
&= p(t; y, x).
\end{aligned}$$

The duality seems to have been implicitly employed by Krylov, [9], in the even-order case (for which the process $(X_t)_{t\geqslant 0}$ is symmetric, see below). On the other hand Nikitin & Orsingher, [11], have used a conditioned Feynman-Kac formula in studying the sojourn time T_t subject to some conditioning in the cases $n = 3, 4$. Actually, duality may avoid its use and we shall make use of the ordinary Feynman-Kac formula together with duality for studying conditioned T_t. Due to the equality

$p(t;x,y)=p(t;x-y)$ we plainly have $p^*(t;x,y)=p(t;-x,-y)$. This observation allows us to view the pseudo-process $(X_t^*)_{t\geqslant 0}$ as $(-X_t)_{t\geqslant 0}$. Notice that in the case of even order the function $z\longmapsto p(t;z)$ is symmetric and then the pseudo-process $(X_t)_{t\geqslant 0}$ is also symmetric which entails that $(X_t^*)_{t\geqslant 0}$ and $(X_t)_{t\geqslant 0}$ may be seen as the same processes.

In the odd-order case, we call $(X_t^+)_{t\geqslant 0}$ the process $(X_t)_{t\geqslant 0}$ associated with the constant $\kappa_n=+1$ and $(X_t^-)_{t\geqslant 0}$ the one associated with $\kappa_n=-1$. We clearly have the following connections between $(X_t^+)_{t\geqslant 0}$, $(X_t^-)_{t\geqslant 0}$ and their dual processes.

Proposition 5 *If n is odd, the dual pseudo-process of $(X_t^+)_{t\geqslant 0}$ (resp. $(X_t^-)_{t\geqslant 0}$) may be viewed as $(X_t^-)_{t\geqslant 0}$ (resp. $(X_t^+)_{t\geqslant 0}$).*

As in the classical theory of duality for Markov processes, it may be easily seen that the following relationship between duality and time inversion holds:

$$\begin{aligned}
[(X_s)_{0\leqslant s\leqslant t}|X_0=x,X_t=y] &= [(X_{t-s}^*)_{0\leqslant s\leqslant t}|X_0^*=y,X_t^*=x]\\
&= [(-X_{t-s})_{0\leqslant s\leqslant t}|X_0=-y,X_t=-x]
\end{aligned}$$

which also implies

$$[(X_s)_{0\leqslant s\leqslant t},X_t\in dy|X_0=x]/dy = [(-X_{t-s})_{0\leqslant s\leqslant t},X_t\in -dx|X_0=-y]/dx. \quad (16)$$

For being quite rigourous, the first (for instance) of the foregoing identities should be understood, for any suitable functionals F, as

$$\begin{aligned}
&\lim_{m\to\infty}\mathbb{E}[F((X_{kt/m})_{1\leqslant k\leqslant m-1})|X_0=x,X_t=y]\\
&\qquad = \lim_{m\to\infty}\mathbb{E}[F((-X_{t-kt/m})_{1\leqslant k\leqslant m-1})|X_0=-y,X_t=-x]
\end{aligned}$$

and this is sufficient for what we are doing in the present work.

5 The law of T_t

The way we use for deriving the Laplace transform of the distribution of T_t consists of solving Eq. (7). This approach has already been adopted by Krylov, [9], for an even integer n and by Hochberg and Orsingher, [7], for an odd integer n. Actually, the approach of Krylov was slightly different. Indeed, he wrote

$$\begin{aligned}
U(\lambda,\mu;x) &= \int_0^{+\infty} e^{-\lambda t}\,u(t,\mu;x)\,dt\\
&= \int_0^{+\infty} e^{-\lambda t}\,dt\int_{-\infty}^{+\infty} v(t,\mu;x,y)\,dy\\
&= \int_{-\infty}^{+\infty} V(\lambda,\mu;x,y)\,dy.
\end{aligned}$$

and subsumed that, in the case where n is even, $V(\lambda,\mu;x,y)=V(\lambda,\mu;y,x)$. In fact, this is the consequence of self-duality for the pseudo-process $(X_t)_{t\geqslant 0}$, that is the latter coincides with its dual (see Section 4). Actually, we shall see in the proof of Theorem 14 that $V(\lambda,\mu;x,y)=V(\lambda+\mu,-\mu;-y,-x)$. Next, he solved the distributional differential equation

$$\mathcal{D}_y V=(\lambda+\mu\mathbb{1}_{(0,+\infty)})V-\delta_x, \quad x \text{ fixed}$$

in the case $x = 0$ and integrated the solution with respect to y. This finally leads him to the classical arc-sine law.

Here, we imitate their arguments and unify their results for any integer n without distinction. Ultimately, we achieve the computations carried out by Hochberg and Orsingher, [7], for odd integers and prove their conjecture (5.17)–(5.18).

The function u solves Eq. (7) with the choices $f = \mu 1\!\!1_{(0,+\infty)}$, $g = 1$ and is bounded over the real line. Put $\gamma = \sqrt[n]{\lambda + \mu}$ and $\delta = \sqrt[n]{\lambda}$.

According to Eq. (8), the corresponding ϕ-function is U and has the form

$$U(\lambda, \mu; x) = \begin{cases} \displaystyle\sum_{k \in I} c_k e^{\theta_k \gamma x} + \frac{1}{\lambda + \mu} & \text{if } x > 0, \\ \displaystyle\sum_{k \in J} d_k e^{\theta_k \delta x} + \frac{1}{\lambda} & \text{if } x < 0, \end{cases} \tag{17}$$

where c_k and d_k are some constants that will be determined by using some regularity conditions. Indeed, by integrating (8) several times on the interval $[-\varepsilon, \varepsilon]$ and letting ε tend to 0, it is easily seen that the function U admits $n - 1$ derivatives at 0, that is

$$\forall \ell \in \{0, \dots, n-1\}, \quad \frac{\partial^\ell U}{\partial x^\ell}(\lambda; 0^+) = \frac{\partial^\ell U}{\partial x^\ell}(\lambda; 0^-)$$

and then

$$\begin{cases} \displaystyle\sum_{k \in I} c_k + \frac{1}{\lambda + \mu} & = \displaystyle\sum_{k \in J} d_k + \frac{1}{\lambda}, \\ \displaystyle\sum_{k \in I} (\theta_k \gamma)^\ell c_k & = \displaystyle\sum_{k \in J} (\theta_k \delta)^\ell d_k \quad \text{for } 1 \leqslant \ell \leqslant n-1. \end{cases}$$

Put $x_k = \begin{cases} c_k & \text{if } k \in I \\ -d_k & \text{if } k \in J \end{cases}$ and $\alpha_k = \begin{cases} \theta_k \gamma & \text{if } k \in I \\ \theta_k \delta & \text{if } k \in J \end{cases}$. The system takes the form

$$\sum_{k=0}^{n-1} \alpha_k^\ell x_k = \begin{cases} \dfrac{\mu}{\lambda(\lambda + \mu)} & \text{if } \ell = 0, \\ 0 & \text{if } 1 \leqslant \ell \leqslant n-1. \end{cases}$$

This is a Vandermonde system, the solution of which being

$$x_k = \frac{\mu}{\lambda(\lambda + \mu)} \prod_{\substack{j=0 \\ j \neq k}}^{n-1} \frac{\alpha_j}{\alpha_j - \alpha_k}.$$

The value of the foregoing product is given in the Lemma below.

Lemma 6 *We have*

$$\prod_{\substack{j=0 \\ j \neq k}}^{n-1} \frac{\alpha_j}{\alpha_j - \alpha_k} = \begin{cases} \dfrac{(\gamma/\delta) - 1}{(\gamma/\delta)^n - 1} \displaystyle\prod_{i \in I \setminus \{k\}} \frac{\theta_i - \theta_k (\gamma/\delta)}{\theta_i - \theta_k} & \textit{if } k \in I, \\ \dfrac{1 - (\delta/\gamma)}{1 - (\delta/\gamma)^n} \displaystyle\prod_{j \in J \setminus \{k\}} \frac{\theta_j - \theta_k (\delta/\gamma)}{\theta_j - \theta_k} & \textit{if } k \in J. \end{cases}$$

PROOF. By excising the set of indices $\{0,\ldots,n-1\}$ into I and J, we get

$$\prod_{\substack{j=0\\j\neq k}}^{n-1}\frac{\alpha_j}{\alpha_j-\alpha_k} = \begin{cases} \displaystyle\prod_{i\in I\setminus\{k\}}\frac{\theta_i\gamma}{\theta_i\gamma-\theta_k\gamma}\prod_{j\in J}\frac{\theta_j\delta}{\theta_j\delta-\theta_k\gamma} & \text{if } k\in I\\ \displaystyle\prod_{j\in J\setminus\{k\}}\frac{\theta_j\delta}{\theta_j\delta-\theta_k\delta}\prod_{i\in I}\frac{\theta_i\gamma}{\theta_i\gamma-\theta_k\delta} & \text{if } k\in J \end{cases}$$

$$= \begin{cases} \displaystyle\prod_{i\in I\setminus\{k\}}(1-\omega_{k-i})^{-1}\prod_{j\in J}(1-\omega_{k-j}\frac{\gamma}{\delta})^{-1} & \text{if } k\in I,\\ \displaystyle\prod_{j\in J\setminus\{k\}}(1-\omega_{k-j})^{-1}\prod_{i\in I}(1-\omega_{k-i}\frac{\delta}{\gamma})^{-1} & \text{if } k\in J. \end{cases}$$

Now, we decompose the polynomial $1-x^n$ into elementary factors as follows:

$$\begin{aligned} 1-x^n &= \prod_{j=0}^{n-1}(1-\omega_j x) = \prod_{j=0}^{n-1}(1-\omega_{k-j}x)\\ &= \begin{cases} \displaystyle(1-x)\prod_{i\in I\setminus\{k\}}(1-\omega_{k-i}x)\prod_{j\in J}(1-\omega_{k-j}x) & \text{if } k\in I\\ \displaystyle(1-x)\prod_{j\in J\setminus\{k\}}(1-\omega_{k-j}x)\prod_{i\in I}(1-\omega_{k-i}x) & \text{if } k\in J \end{cases} \end{aligned}$$

and then

$$\begin{aligned} \prod_{j\in J}(1-\omega_{k-j}x)^{-1} &= \frac{1-x}{1-x^n}\prod_{i\in I\setminus\{k\}}(1-\omega_{k-i}x) \quad \text{if } k\in I,\\ \prod_{i\in I}(1-\omega_{k-i}x)^{-1} &= \frac{1-x}{1-x^n}\prod_{j\in J\setminus\{k\}}(1-\omega_{k-j}x) \quad \text{if } k\in J. \end{aligned}$$

Putting $x=\gamma/\delta$ or $x=\delta/\gamma$ yields the result. □

Therefore, since $(\gamma/\delta)^n-1=\mu/\lambda$, the constants c_k and d_k in (17) are given by

$$\begin{aligned} c_k &= \frac{(\gamma/\delta)-1}{\lambda+\mu}\prod_{i\in I\setminus\{k\}}\frac{\theta_i-\theta_k(\gamma/\delta)}{\theta_i-\theta_k} \qquad \text{for } k\in I,\\ d_k &= -\frac{1-(\delta/\gamma)}{\lambda}\prod_{j\in J\setminus\{k\}}\frac{\theta_j-\theta_k(\delta/\gamma)}{\theta_j-\theta_k} \qquad \text{for } k\in J, \end{aligned}$$

and we obtain

$$U(\lambda,\mu;x) = \begin{cases} \displaystyle\frac{1}{\lambda+\mu}\left[1+\left(\left(\frac{\lambda+\mu}{\lambda}\right)^{1/n}-1\right)\right.\\ \displaystyle\left.\times\sum_{k\in I}\prod_{i\in I\setminus\{k\}}\frac{\theta_i-\theta_k(\gamma/\delta)}{\theta_i-\theta_k}\,e^{\theta_k\gamma x}\right] & \text{if } x\geqslant 0,\\ \displaystyle\frac{1}{\lambda}\left[1-\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{1/n}\right)\right.\\ \displaystyle\left.\times\sum_{k\in J}\prod_{j\in J\setminus\{k\}}\frac{\theta_j-\theta_k(\delta/\gamma)}{\theta_j-\theta_k}\,e^{\theta_k\delta x}\right] & \text{if } x\leqslant 0. \end{cases} \tag{18}$$

In particular, for $x=0$, we get an intermediate expression for $U(\lambda,\mu;0)$:

$$U(\lambda,\mu;0)=\frac{1}{\lambda+\mu}\left[1+\left(\left(\frac{\lambda+\mu}{\lambda}\right)^{1/n}-1\right)\sum_{k\in I}\prod_{i\in I\setminus\{k\}}\frac{\theta_i-\theta_k(\gamma/\delta)}{\theta_i-\theta_k}\right]. \tag{19}$$

Formula (19) has been obtained by Hochberg & Orsingher, [7], in the odd-order case. We now state the following theorem.

Theorem 7 *The Laplace transform of $\mathbb{E}_0(e^{-\mu T_t})$ is given by*

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_0(e^{-\mu T_t})\,dt &= \frac{1}{\sqrt[n]{\lambda^{\#I}(\lambda+\mu)^{\#J}}} \\ &= \begin{cases}\dfrac{1}{\sqrt{\lambda(\lambda+\mu)}} & \text{if } n \text{ is even},\\[2mm] \dfrac{1}{\sqrt[n]{\lambda^{p}(\lambda+\mu)^{p+1}}} & \text{if } n=2p+1 \text{ and } \kappa_n=(-1)^p,\\[2mm] \dfrac{1}{\sqrt[n]{\lambda^{p+1}(\lambda+\mu)^{p}}} & \text{if } n=2p+1 \text{ and } \kappa_n=(-1)^{p+1}.\end{cases}\end{aligned} \tag{20}$$

Rewriting formula (20) for $U(\lambda,\mu;0)$ as

$$U(\lambda,\mu;0)=\frac{1}{\lambda+\mu}\left(\frac{\gamma}{\delta}\right)^{\#I}=\frac{1}{\lambda}\left(\frac{\delta}{\gamma}\right)^{\#J},$$

we see that we have to prove the following identity.

Lemma 8 *We have for any real number $x\neq 1$,*

$$\sum_{k\in I}\prod_{i\in I\setminus\{k\}}\frac{\theta_i x-\theta_k}{\theta_i-\theta_k}=\sum_{k\in I}\prod_{i\in I\setminus\{k\}}\frac{\theta_i-\theta_k x}{\theta_i-\theta_k}=\frac{x^{\#I}-1}{x-1}.$$

PROOF. First, observe that if the first and third terms coincide, then the second one and the third one also coincide as it can be seen by replacing x by $1/x$. Set now

$$Q(x)=\frac{x^{\#I}-1}{x-1}=\sum_{k=0}^{\#I-1}x^k$$

and

$$P(x)=\sum_{k\in I}\prod_{i\in I\setminus\{k\}}P_{i,k}(x)\quad\text{with}\quad P_{i,k}(x)=\frac{\theta_i x-\theta_k}{\theta_i-\theta_k}.$$

The functions P and Q are polynomials of degree not higher than $\#I$. Therefore, proving the identity $P=Q$ is equivalent to checking the equalities between their successive derivatives at 1 for instance:

$$\forall \ell\in\{0,\dots,\#I-1\},\ P^{(\ell)}(1)=Q^{(\ell)}(1). \tag{21}$$

Plainly,

$$Q^{(\ell)}(1)=\ell!\sum_{k=0}^{\#I-1}\binom{k}{\ell}=\ell!\binom{\#I}{\ell+1}.$$

The evaluation of $P^{(\ell)}(1)$ is more intricate. We require Leibniz rule for a product of functions P_j:

$$(\prod_{j=1}^{q} P_j)^{(\ell)} = \sum_{\substack{\ell_1+\cdots+\ell_q=\ell \\ \ell_1,\ldots,\ell_q\geqslant 0}} \frac{\ell!}{\ell_1!\ldots\ell_q!} P_1^{(\ell_1)} \cdots P_q^{(\ell_q)}.$$

In the case where all the functions P_j are polynomials of degree one, we get

$$\begin{aligned}
(\prod_{j=1}^{q} P_j)^{(\ell)} &= \ell! \prod_{j=1}^{q} P_j \sum_{\substack{\ell_1+\cdots+\ell_q=\ell \\ \ell_1,\ldots,\ell_q\in\{0,1\}}} \frac{P_1^{(\ell_1)}}{P_1} \cdots \frac{P_q^{(\ell_q)}}{P_q} \\
&= \ell! \prod_{j=1}^{q} P_j \sum_{1\leqslant j_1<\cdots<j_\ell\leqslant q} \frac{P'_{j_1}}{P_{j_1}} \cdots \frac{P'_{j_\ell}}{P_{j_\ell}} \\
&= \prod_{j=1}^{q} P_j \sum_{\substack{1\leqslant j_1,\ldots,j_\ell\leqslant q \\ j_1,\ldots,j_\ell \text{ differents}}} \frac{P'_{j_1}}{P_{j_1}} \cdots \frac{P'_{j_\ell}}{P_{j_\ell}}.
\end{aligned}$$

In the second equality, we have used

$$\frac{P_j^{(\ell_i)}}{P_j} = \begin{cases} 1 & \text{if } \ell_i = 0, \\ \dfrac{P'_j}{P_j} & \text{if } \ell_i = 1, \\ 0 & \text{if } \ell_i \geqslant 2, \end{cases}$$

and introduced the ordered indices $1 \leqslant j_1 < \cdots < j_\ell \leqslant q$ for which $\ell_{j_1} = \cdots = \ell_{j_\ell} = 1$. The last equality comes obviously from the fact that the $\ell!$ permutations of $j_1, \ldots, j_\ell$ yield the same result for the quantity $\frac{P'_{j_1}}{P_{j_1}} \cdots \frac{P'_{j_\ell}}{P_{j_\ell}}$. Thus, in our case, since the polynomials $P_{i,k}$ have degree one and are such that $P_{i,k}(1) = 1$ and $P'_{i,k}(1) = \frac{\theta_i}{\theta_i-\theta_k}$, we obtain

$$\begin{aligned}
P^{(\ell)}(1) &= \sum_{k\in I} \sum_{\substack{i_1,\ldots,i_\ell\in I\setminus\{k\} \\ i_1,\ldots,i_\ell \text{ differents}}} \frac{\theta_{i_1}}{\theta_{i_1}-\theta_k} \cdots \frac{\theta_{i_\ell}}{\theta_{i_\ell}-\theta_k} \\
&= \sum_{\substack{i_1,\ldots,i_\ell,k\in I \\ i_1,\ldots,i_\ell,k \text{ differents}}} \prod_{j=1}^{\ell} \frac{\theta_{i_j}}{\theta_{i_j}-\theta_k} \\
&= \sum_{\substack{i_0,\ldots,i_\ell\in I \\ i_0,\ldots,i_\ell \text{ differents}}} \prod_{\substack{j=0 \\ j\neq 0}}^{\ell} \frac{\theta_{i_j}}{\theta_{i_j}-\theta_{i_0}}
\end{aligned}$$

where we put $k = i_0$ in the last equality. Next, observing that the last expression is invariant by permutation,

$$\begin{aligned}
P^{(\ell)}(1) &= \frac{1}{\ell+1} \sum_{k=0}^{\ell} \sum_{\substack{i_0,\ldots,i_\ell\in I \\ i_0,\ldots,i_\ell \text{ differents}}} \prod_{\substack{j=0 \\ j\neq k}}^{\ell} \frac{\theta_{i_j}}{\theta_{i_j}-\theta_{i_k}} \\
&= \frac{1}{\ell+1} \sum_{\substack{i_0,\ldots,i_\ell\in I \\ i_0,\ldots,i_\ell \text{ differents}}} \left(\sum_{k=0}^{\ell} \prod_{\substack{j=0 \\ j\neq k}}^{\ell} \frac{\theta_{i_j}}{\theta_{i_j}-\theta_{i_k}} \right).
\end{aligned}$$

Now, we remark that the family $\left(\prod_{\substack{j=0\\j\neq k}}^{\ell}\frac{\theta_{i_j}}{\theta_{i_j}-\theta_{i_k}}\right)_{0\leqslant k\leqslant \ell}$ solves the Vandermonde system $\sum_{j=0}^{\ell}\theta_{i_j}^k x_j=\delta_{k0}$, $0\leqslant k\leqslant \ell$. Hence, the sum within braces corresponds to the equation numbered by $i=0$, so it equals 1 and then

$$P^{(\ell)}(1)=\frac{1}{\ell+1}\#\{(i_0,\dots,i_\ell)\in I\,:\,i_0,\dots,i_\ell\text{ are differents}\}=\ell!\left(\begin{array}{c}\#I\\\ell+1\end{array}\right).$$

□

By inverting the Laplace transform (20) as it is done in [7], one gets the density of T_t.

Corollary 9 *The distribution of T_t is a Beta law with parameters $\#I/n$ and $\#J/n$:*

$$\mathbb{P}_0(T_t\in ds)/ds=\frac{k_n\mathbb{1}_{(0,t)}(s)}{\sqrt[n]{s^{\#I}(t-s)^{\#J}}}$$

with

$$k_n=\frac{1}{\pi}\sin\left(\frac{\#I}{n}\pi\right)=\begin{cases}\dfrac{1}{\pi} & \text{if } n \text{ is even},\\[2ex] \dfrac{1}{\pi}\sin\dfrac{p\pi}{n} & \text{if } n=2p+1.\end{cases}$$

In particular, for any even integer n, $\#I=\#J=\frac{n}{2}$ and then the distribution of T_t is the arc-sine law:

$$\mathbb{P}_0(T_t\in ds)/ds=\frac{\mathbb{1}_{(0,t)}(s)}{\pi\sqrt{s(t-s)}}.$$

Remark. The distribution function of T_t is expressible by means of hypergeometric function by integrating its density. Indeed, by the change of variables $\sigma=s\tau$ and [5, formula 9.111, p. 1066], we get

$$\begin{aligned}\int_0^s\frac{d\sigma}{\sigma^\alpha(t-\sigma)^{1-\alpha}} &= \left(\frac{s}{t}\right)^{1-\alpha}\int_0^1\tau^{-\alpha}\,(1-\frac{s}{t}\,\tau)^{\alpha-1}\,d\tau\\ &= \frac{1}{1-\alpha}\left(\frac{s}{t}\right)^{1-\alpha}{}_2F_1(1-\alpha,1-\alpha;2-\alpha;\frac{s}{t}).\end{aligned}$$

Hence, for $\alpha=\frac{\#I}{n}$,

$$\mathbb{P}_0(T_t\leqslant s)/ds=\frac{nk_n}{\#J}\left(\frac{s}{t}\right)^{\#J/n}{}_2F_1\Big(\frac{\#J}{n},\frac{\#J}{n};\frac{\#J}{n}+1;\frac{s}{t}\Big).\tag{22}$$

6 Conditioned laws of T_t

By choosing $f=\mu\mathbb{1}_{(0,+\infty)}$ and $g=\delta_y$ in Eq. (7), v solves the partial differential equation

$$\frac{\partial v}{\partial t}=\mathcal{D}_x v-\mu\mathbb{1}_{(0,+\infty)}v$$

together with

$$v(0,\mu;x,y)=\delta_y(x).$$

In the last equation, the symbol δ_y denotes the Dirac measure at y. It can be viewed as the weak limit as $\varepsilon\to 0^+$ of the family of functions $\partial_{y,\varepsilon}: x\mapsto(\varepsilon-|x-y|)^+/\varepsilon^2$ which satisfy $\int_{y-\varepsilon}^{y+\varepsilon}\partial_{y,\varepsilon}(x)\,dx=1$. This approach may be invoked for justifying the computations related to δ_y that will be done in the sequel of the section. For the sake of simplicity, we shall perform them formally.

According to (8), the Laplace transform V of v solves the distributional differential equation

$$\mathcal{D}_x V=(\lambda+\mu 1\!\!1_{(0,+\infty)})V-\delta_y. \tag{23}$$

Put

$$\tilde V(x)=V(\lambda,\mu;x,0).$$

We have

$$\mathcal{D}_x\tilde V(x)=\begin{cases}(\lambda+\mu)\tilde V(x) & \text{if } x>0,\\ \lambda\tilde V(x) & \text{if } x<0.\end{cases}$$

The bounded solution of the previous system has the form

$$\tilde V(x)=\begin{cases}\displaystyle\sum_{k\in I}c'_k e^{\theta_k\gamma x} & \text{if } \ x>0,\\ \displaystyle\sum_{k\in J}d'_k e^{\theta_k\delta x} & \text{if } \ x<0,\end{cases}$$

where c'_k and d'_k are some constants that will be determined by using some regularity conditions at 0. Indeed, by integrating Eq. (23) on $(-\varepsilon,\varepsilon)$ as in [11],

$$\int_{-\varepsilon}^{\varepsilon}\mathcal{D}_x\tilde V(x)\,dx=\kappa_n\Big[\frac{\partial^{n-1}\tilde V}{\partial x^{n-1}}(\varepsilon)-\frac{\partial^{n-1}\tilde V}{\partial x^{n-1}}(-\varepsilon)\Big]=\int_{-\varepsilon}^{\varepsilon}(\lambda+\mu 1\!\!1_{(0,+\infty)})(x)\tilde V(x)\,dx-1$$

and letting ε tend to zero (recall that $\kappa_n=\pm1$),

$$\frac{\partial^{n-1}\tilde V}{\partial x^{n-1}}(0^+)-\frac{\partial^{n-1}\tilde V}{\partial x^{n-1}}(0^-)=-\kappa_n.$$

Similarly, integrating several times on a neighbourhood of 0 yields

$$\frac{\partial^{\ell}\tilde V}{\partial x^{\ell}}(0^+)=\frac{\partial^{\ell}\tilde V}{\partial x^{\ell}}(0^-)\quad\text{for } 0\leqslant\ell\leqslant n-2.$$

Hence, we derive the following conditions for the constants c'_k and d'_k:

$$\begin{cases}\displaystyle\sum_{k\in I}c'_k(\theta_k\gamma)^{\ell} &=\displaystyle\sum_{k\in J}d'_k(\theta_k\delta)^{\ell}\quad\text{for }\ 0\leqslant\ell\leqslant n-2,\\ \displaystyle\sum_{k\in I}c'_k(\theta_k\gamma)^{n-1} &=\displaystyle\sum_{k\in J}d'_k(\theta_k\delta)^{n-1}-\kappa_n\end{cases}$$

which, by putting $x'_k=\begin{cases}c'_k & \text{if } k\in I\\ -d'_k & \text{if } k\in J\end{cases}$ and $\alpha_k=\begin{cases}\theta_k\gamma & \text{if } k\in I\\ \theta_k\delta & \text{if } k\in J\end{cases}$, lead to the Vandermonde system

$$\sum_{k=0}^{n-1}\alpha_k^{\ell}x'_k=\begin{cases}0 & \text{for }\ 0\leqslant\ell\leqslant n-2,\\ -\kappa_n & \text{for }\ \ell=n-1.\end{cases}$$

Its solution is given by

$$x'_k = -\kappa_n \prod_{\substack{j=0\\ j\neq k}}^{n-1} \frac{1}{\alpha_k - \alpha_j} = (-1)^n \frac{\kappa_n \alpha_k}{\prod_{j=0}^{n-1} \alpha_j} \times \prod_{\substack{j=0\\ j\neq k}}^{n-1} \frac{\alpha_j}{\alpha_j - \alpha_k}.$$

We evidently have

$$\prod_{j=0}^{n-1} \alpha_j = \gamma^{\#I} \delta^{\#J} \prod_{j=0}^{n-1} \theta_j \quad \text{and} \quad \prod_{j=0}^{n-1} \theta_j = (-1)^{n-1} \kappa_n$$

and then

$$x'_k = -\frac{\alpha_k}{\gamma^{\#I} \delta^{\#J}} \prod_{\substack{j=0\\ j\neq k}}^{n-1} \frac{\alpha_j}{\alpha_j - \alpha_k}$$

where the foregoing product is given by Lemma 6. Therefore, the constant c'_k for instance can be evaluated as

$$\begin{aligned} c'_k &= -\frac{\theta_k((\gamma/\delta)-1)}{\gamma^{\#I-1}\delta^{\#J}((\gamma/\delta)^n-1)} \prod_{i\in I\setminus\{k\}} \frac{\theta_i - \theta_k(\gamma/\delta)}{\theta_i - \theta_k} \\ &= -\theta_k \left(\frac{\delta}{\gamma}\right)^{\#I-1} \frac{\gamma-\delta}{\gamma^n - \delta^n} \prod_{i\in I\setminus\{k\}} \frac{\theta_i - \theta_k(\gamma/\delta)}{\theta_i - \theta_k} \end{aligned}$$

since $n - \#I = \#J$. Hence,

$$c'_k = -\theta_k \frac{\gamma-\delta}{\mu} \prod_{i\in I\setminus\{k\}} \frac{\theta_i(\delta/\gamma) - \theta_k}{\theta_i - \theta_k} \quad \text{for } k \in I.$$

Similarly,

$$d'_k = \theta_k \frac{\gamma-\delta}{\mu} \prod_{j\in J\setminus\{k\}} \frac{\theta_j(\gamma/\delta) - \theta_k}{\theta_j - \theta_k} \quad \text{for } k \in J.$$

As a result, we have obtained the expression below for $V(\lambda,\mu;x,0)$.

Proposition 10 *The Laplace transform of $\mathbb{E}_x(e^{-\mu T_t}, X_t \in dy)/dy$ at $y=0$ is given by*

$$\int_0^{+\infty} e^{-\lambda t} \left[\mathbb{E}_x(e^{-\mu T_t}, X_t \in dy)/dy\right]_{y=0} dt$$
$$= \begin{cases} -\dfrac{1}{\mu}(\gamma-\delta) \displaystyle\sum_{k\in I} \theta_k \Big(\prod_{i\in I\setminus\{k\}} \frac{\theta_i(\delta/\gamma)-\theta_k}{\theta_i-\theta_k} \Big) e^{\theta_k \gamma x} & \textit{if } x \geqslant 0, \\ \dfrac{1}{\mu}(\gamma-\delta) \displaystyle\sum_{k\in J} \theta_k \Big(\prod_{j\in J\setminus\{k\}} \frac{\theta_j(\gamma/\delta)-\theta_k}{\theta_j-\theta_k} \Big) e^{\theta_k \delta x} & \textit{if } x \leqslant 0. \end{cases} \tag{24}$$

In particular, for $x=0$:

$$\begin{aligned} V(\lambda,\mu;0,0) &= \sum_{k\in I} c'_k = \sum_{k\in J} d'_k \\ &= -\frac{\gamma-\delta}{\mu} \sum_{k\in I} \theta_k \prod_{i\in I\setminus\{k\}} \frac{\theta_i(\gamma/\delta)-\theta_k}{\theta_i-\theta_k}. \end{aligned}$$

The above sum can be simplified. Its value is written out in the lemma below.

Lemma 11 *We have for any real number x*

$$\sum_{k\in I}\theta_k\prod_{i\in I\setminus\{k\}}\frac{\theta_i x-\theta_k}{\theta_i-\theta_k}=\sum_{k\in I}\theta_k$$

with

$$\sum_{k\in I}\theta_k=-\sum_{k\in J}\theta_k=\begin{cases}-\dfrac{1}{\sin\frac{\pi}{n}} & \textit{if } n \textit{ is even,}\\ -\dfrac{1}{2\sin\frac{\pi}{2n}} & \textit{if } n \textit{ is odd.}\end{cases}$$

Proof. 1. The proof is similar to that of Lemma 8. Set

$$\tilde{P}(x)=\sum_{k\in I}\theta_k\prod_{i\in I\setminus\{k\}}P_{i,k}(x)$$

where the polynomial $P_{i,k}$ has been defined in the proof of Lemma 8. We first have

$$\tilde{P}(1)=\sum_{k\in I}\theta_k.$$

Second, we evaluate the successive derivatives at 1 of the polynomial $\tilde{P}$:

$$\begin{aligned}\forall \ell\in\{0,\dots,\#I-1\},\ \tilde{P}^{(\ell)}(1) &= \sum_{k\in I}\theta_k\sum_{\substack{i_1,\dots,i_\ell\in I\setminus\{k\}\\ i_1,\dots,i_\ell\text{ differents}}}\frac{\theta_{i_1}}{\theta_{i_1}-\theta_k}\cdots\frac{\theta_{i_\ell}}{\theta_{i_\ell}-\theta_k}\\ &= \sum_{\substack{i_1,\dots,i_\ell,k\in I\\ i_1,\dots,i_\ell,k\text{ differents}}}\theta_k\prod_{j=1}^{\ell}\frac{\theta_{i_j}}{\theta_{i_j}-\theta_k}\\ &= \sum_{\substack{i_0,\dots,i_\ell\in I\\ i_0,\dots,i_\ell\text{ differents}}}\theta_{i_0}\prod_{\substack{j=0\\ j\neq 0}}^{\ell}\frac{\theta_{i_j}}{\theta_{i_j}-\theta_{i_0}}\end{aligned}$$

where we put $k=i_0$ in the last equality. Next, observing that the last expression is invariant by permutation,

$$\begin{aligned}\tilde{P}^{(\ell)}(1) &= \frac{1}{\ell+1}\sum_{k=0}^{\ell}\sum_{\substack{i_0,\dots,i_\ell\in I\\ i_0,\dots,i_\ell\text{ differents}}}\theta_{i_k}\prod_{\substack{j=0\\ j\neq k}}^{\ell}\frac{\theta_{i_j}}{\theta_{i_j}-\theta_{i_k}}\\ &= \frac{1}{\ell+1}\sum_{\substack{i_0,\dots,i_\ell\in I\\ i_0,\dots,i_\ell\text{ differents}}}\left(\sum_{k=0}^{\ell}\theta_{i_k}\prod_{\substack{j=0\\ j\neq k}}^{\ell}\frac{\theta_{i_j}}{\theta_{i_j}-\theta_{i_k}}\right).\end{aligned}$$

As in the proof of Lemma 8, we invoke an argument related to a Vandermonde system to assert that the sum within braces equals 0 and then

$$\tilde{P}^{(\ell)}(1)=0$$

which entails that

$$\tilde{P}(x)=\tilde{P}(1)=\sum_{k\in I}\theta_k.$$

2. Now, we have to evaluate the sum $\sum_{k\in I}\theta_k$. For this, we use the elementary equality

$$\sum_{k=a}^{b}\omega_k = e^{i[(a+b)\pi/n]}\,\frac{\sin\frac{(b-a+1)\pi}{n}}{\sin\frac{\pi}{n}}. \tag{25}$$

For $n=2p$ we have $\theta_k=\omega_{k+(p+1)/2}=\omega_{(p+1)/2}\,\omega_k$ and by choosing $a=0$ and $b=p-1$ in formula (25) we get

$$\sum_{k=0}^{p-1}\theta_k=\omega_{(p+1)/2}\,e^{i[(p-1)\pi/n]}\,\frac{\sin\frac{p\pi}{n}}{\sin\frac{\pi}{n}}=-\frac{1}{\sin\frac{\pi}{n}}.$$

For $n=2p+1$,

- if $\kappa_n=+1$: $\theta_k=\omega_k$ and we choose

$$\begin{aligned} a&=\frac{p}{2} &&\text{and}\quad b=\frac{3p}{2} &&\text{if } p \text{ is even,}\\ a&=\frac{p+1}{2} &&\text{and}\quad b=\frac{3p-1}{2} &&\text{if } p \text{ is odd;}\end{aligned}$$

- if $\kappa_n=-1$: $\theta_k=\omega_{k+1/2}$ and we choose

$$\begin{aligned} a&=\frac{p}{2}+1 &&\text{and}\quad b=\frac{3p}{2} &&\text{if } p \text{ is even,}\\ a&=\frac{p+1}{2} &&\text{and}\quad b=\frac{3p+1}{2} &&\text{if } p \text{ is odd.}\end{aligned}$$

In both cases, we have, noticing that $\sin\frac{p\pi}{n}=\sin\frac{(p+1)\pi}{n}=\cos\frac{\pi}{2n}$ and that $\sin\frac{\pi}{n}=2\sin\frac{\pi}{2n}\,\cos\frac{\pi}{2n}$,

$$\sum_{k\in I}\theta_k=\omega_{n/2}\,\frac{\sin\frac{p\pi}{n}}{\sin\frac{\pi}{n}}=-\frac{1}{2\sin\frac{\pi}{2n}}.$$

The proof of Lemma 11 is complete. □

We can write out an explicit expression for the function $V(\lambda,\mu;0,0)$.

Proposition 12 *The Laplace transform of $\mathbb{E}_0(e^{-\mu T_t},X_t\in dy)/dy|_{y=0}$ is given by*

$$\int_0^{+\infty} e^{-\lambda t}\,[\mathbb{E}_0(e^{-\mu T_t},X_t\in dy)/dy]_{y=0}\,dt=\frac{1}{l_n\mu}\big[\sqrt[n]{\lambda+\mu}-\sqrt[n]{\lambda}\,\big] \tag{26}$$

with

$$l_n=\begin{cases}\sin\dfrac{\pi}{n} & \text{if } n \text{ is even,}\\[2mm] 2\sin\dfrac{\pi}{2n} & \text{if } n \text{ is odd.}\end{cases}$$

Remark. Nikitin & Orsingher obtained the previous formula in the cases $n=3$ and $n=4$ by solving differential equations with respect to the variable y related to the operator $\mathcal{D}_y^*$. Actually, their equations are associated with the dual pseudo-process $(X_t^*)_{t\geqslant 0}$ and this connection has been explained in Section 4.

We now state the important consequence.

Theorem 13 *We have*

$$\mathbb{P}_0(T_t \in ds | X_t = 0)/ds = \frac{1}{t}\,\mathbb{1}_{(0,t)}(s),$$

that is, T_t is uniformly distributed on $[0,t]$.

PROOF. Using the elementary identity (see e.g. [5, formula 3.434.1, p. 378])

$$\int_0^{+\infty} (e^{-as} - e^{-bs})\,\frac{ds}{s^{\nu+1}} = \frac{\Gamma(1-\nu)}{\nu}\,(b^\nu - a^\nu)$$

it is easy to invert the Laplace transform (26) for deriving

$$v(t,\mu;0,0) = \frac{1}{l_n n \Gamma(1-\frac{1}{n})}\,\frac{1-e^{-\mu t}}{\mu\, t^{1+\frac{1}{n}}}.$$

Therefore, due to (10), we find

$$\mathbb{E}_0(e^{-\mu T_t} | X_t = 0) = \frac{v(t,\mu;0,0)}{p(t;0)} = \frac{1-e^{-\mu t}}{\mu t}.$$

from which we immediately deduce through another inversion the uniform distribution. □

Theorem 14 *We have*

$$\mathbb{P}_0(T_t \in ds | X_t > 0)/ds = \frac{k_n^+}{t}\left(\frac{s}{t-s}\right)^{\#J/n} \mathbb{1}_{(0,t)}(s)$$

with

$$k_n^+ = \frac{\sin\frac{\#J\pi}{n}}{\pi\int_0^{+\infty} p(t;y)\,dy} = \begin{cases} \dfrac{2}{\pi} & \text{if } n \text{ is even,} \\[2ex] \dfrac{2\sin\frac{p\pi}{n}}{\pi(1-\frac{1}{n})} & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^p, \\[2ex] \dfrac{2\sin\frac{p\pi}{n}}{\pi(1+\frac{1}{n})} & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^{p+1}. \end{cases}$$

Likewise,

$$\mathbb{P}_0(T_t \in ds | X_t < 0)/ds = \frac{k_n^-}{t}\left(\frac{t-s}{s}\right)^{\#I/n} \mathbb{1}_{(0,t)}(s)$$

with

$$k_n^- = \frac{\sin\frac{\#I\pi}{n}}{\pi\int_{-\infty}^{0} p(t;y)\,dy} = \begin{cases} \dfrac{2}{\pi} & \text{if } n \text{ is even,} \\[2ex] \dfrac{2\sin\frac{p\pi}{n}}{\pi(1+\frac{1}{n})} & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^p, \\[2ex] \dfrac{2\sin\frac{p\pi}{n}}{\pi(1-\frac{1}{n})} & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^{p+1}. \end{cases}$$

Proof. 1. We did not get any explicit expression for $v(t,\mu;0,y)$ yet. Nevertheless, we got someone for $v(t,\mu;x,0)$. Actually, both quantities are closely linked through duality. We refer to Section 4 and especially to (16) for obtaining

$$\begin{aligned} v(t,\mu;x,y) &= \mathbb{E}_y^*[e^{-\mu\int_0^t \mathbb{1}_{\{X_{t-s}^*>0\}}\,ds}, X_t^* \in dx]/dx \\ &= \mathbb{E}_{-y}[e^{-\mu\int_0^t \mathbb{1}_{\{X_s<0\}}\,ds}, X_t \in -dx]/dx. \end{aligned}$$

As a result, observing that $\int_0^t \mathbb{1}_{\{X_s\leqslant 0\}}\,ds = t - \int_0^t \mathbb{1}_{\{X_s>0\}}\,ds$,

$$v(t,\mu;x,y) = e^{-\mu t}\,v(t,-\mu;-y,-x)$$

and then

$$V(\lambda,\mu;x,y) = V(\lambda+\mu,-\mu;-y,-x). \tag{27}$$

On the right-hand side of the last relation, the inherent parameters γ and δ must be exchanged. Next, we have

$$\mathbb{E}_0(e^{-\mu T_t}|X_t>0) = \frac{\int_0^{+\infty} v(t,\mu;0,y)\,dy}{\int_0^{+\infty} p(t;y)\,dy}. \tag{28}$$

Invoking (27) together with (24), the Laplace transform of the numerator of the fraction can be computed as follows:

$$\begin{aligned} \int_0^{+\infty} V(\lambda,\mu;0,y)\,dy &= \int_{-\infty}^0 V(\lambda+\mu,-\mu;y,0)\,dy \\ &= \sum_{k\in J}\frac{d_k'}{\theta_k\gamma} = \frac{\gamma-\delta}{\mu\gamma}\sum_{k\in J}\Big(\prod_{j\in J\setminus\{k\}}\frac{\theta_j(\delta/\gamma)-\theta_k}{\theta_j-\theta_k}\Big). \end{aligned}$$

Due to Lemma 8, the value of the sum in the right-hand side is $\dfrac{1-(\delta/\gamma)^{\#J}}{1-(\delta/\gamma)}$ and so

$$\int_0^{+\infty} V(\lambda,\mu;0,y)\,dy = \frac{1}{\mu}\Big[1-\Big(\frac{\lambda}{\lambda+\mu}\Big)^{\#J/n}\Big]. \tag{29}$$

Likewise,

$$\int_{-\infty}^0 V(\lambda,\mu;0,y)\,dy = -\sum_{k\in I}\frac{c_k'}{\theta_k\delta} = \frac{1}{\mu}\Big[\Big(\frac{\lambda+\mu}{\lambda}\Big)^{\#I/n}-1\Big]. \tag{30}$$

2. Let us now invert the Laplace transform (29). Set

$$\varphi_\alpha(s) = s^{\alpha-1}\,e^{-\mu s} \quad\text{and}\quad \psi_\beta(s) = s^{\alpha-1}(1-e^{-\mu s}).$$

The Laplace transforms of φ_α and ψ_β are respectively given, for any positive real numbers α and β, by

$$\begin{aligned} \mathcal{L}\varphi_\alpha(\lambda) &= \int_0^{+\infty} e^{-\lambda s}\,\varphi_\alpha(s)\,ds &= \frac{\Gamma(\alpha)}{(\lambda+\mu)^\alpha}, \\ \mathcal{L}\psi_\beta(\lambda) &= \int_0^{+\infty} e^{-\lambda s}\,\psi_\beta(s)\,ds &= \Gamma(\beta)\Big[\frac{1}{\lambda^\beta}-\frac{1}{(\lambda+\mu)^\beta}\Big]. \end{aligned}$$

Actually, we can observe that the second displayed equality is also valid for any real number $\beta\in(-1,0)$ as it can been easily checked by writing

$$e^{-\lambda s}-e^{-(\lambda+\mu)s} = \int_\lambda^{\lambda+\mu} s\,e^{-su}\,du.$$

Consequently, for $\alpha \in (0,1)$,

$$\begin{aligned}
\frac{1}{(\lambda+\mu)^\alpha} &= \frac{1}{\Gamma(\alpha)}\,\mathcal{L}\varphi_\alpha(\lambda),\\
(\lambda+\mu)^\alpha - \lambda^\alpha &= -\frac{1}{\Gamma(-\alpha)}\,\mathcal{L}\psi_{-\alpha}(\lambda).
\end{aligned}$$

Thus, by convolution,

$$1-\left(\frac{\lambda}{\lambda+\mu}\right)^\alpha = \mathcal{L}(\varphi_\alpha\star\psi_{-\alpha})(\lambda)$$

with

$$\begin{aligned}
(\varphi_\alpha\star\psi_{-\alpha})(t) &= -\frac{1}{\Gamma(-\alpha)\Gamma(\alpha)}\int_0^t \frac{e^{-\mu s}-e^{-\mu t}}{s^{1-\alpha}(t-s)^{1+\alpha}}\,ds\\
&= \frac{\alpha\sin\alpha\pi}{\pi}\int_0^t \frac{ds}{s^{1-\alpha}(t-s)^{1+\alpha}}\int_s^t \mu\,e^{-\mu u}\,du\\
&= \frac{\alpha\sin\alpha\pi}{\pi}\int_0^t \mu\,e^{-\mu u}\,du\int_0^u \left(\frac{s}{t-s}\right)^\alpha \frac{ds}{s(t-s)}.
\end{aligned}$$

Using the change of variables $v=\left(\frac{s}{t-s}\right)^\alpha$, it comes

$$\int_0^u \left(\frac{s}{t-s}\right)^\alpha \frac{ds}{s(t-s)} = \frac{1}{\alpha t}\left(\frac{s}{t-s}\right)^\alpha.$$

Finally, choosing $\alpha=\#J/n$ yields

$$\int_0^{+\infty} v(t,\mu;0,y)\,dy = \frac{\sin\frac{\#J\pi}{n}}{\pi t}\int_0^t e^{-\mu s}\left(\frac{s}{t-s}\right)^{\#J/n} ds$$

and the result for positive conditioning is proved thanks to (11) and (28). Inverting the Laplace transform (30) can be carried out in a similar way by replacing φ_α by the function defined as $\varphi_\alpha(s)=s^{\alpha-1}$. $\square$

Remark. As in the unconditioned case, the distribution function of $(T_t|X_t>0)$ for instance is expressible by means of hypergeometric function by integrating its density. In effect, by the change of variables $\sigma=s\tau$ and [5, formula 9.111, p. 1066], we obtain

$$\begin{aligned}
\int_0^s \frac{1}{t}\left(\frac{\sigma}{t-\sigma}\right)^\alpha d\sigma &= \left(\frac{s}{t}\right)^{\alpha+1}\int_0^1 \tau^\alpha\,(1-\frac{s}{t}\,\tau)^{-\alpha}\,d\tau\\
&= \frac{1}{\alpha+1}\left(\frac{s}{t}\right)^{\alpha+1} {}_2F_1(\alpha,\alpha+1;\alpha+2;\frac{s}{t}).
\end{aligned}$$

Hence, for $\alpha=\frac{\#J}{n}$,

$$\mathbb{P}_0(T_t\leqslant s|X_t>0) = \frac{k_n^+}{1+\#J/n}\left(\frac{s}{t}\right)^{1+\#J/n} {}_2F_1\Big(\frac{\#J}{n},\frac{\#J}{n}+1;\frac{\#J}{n}+2;\frac{s}{t}\Big).$$

In an analogous manner, we can show that

$$\mathbb{P}_0(T_t\leqslant s|X_t<0) = \frac{n\,k_n^+}{\#J}\left(\frac{s}{t}\right)^{\#J/n} {}_2F_1\Big(\frac{\#J}{n}-1,\frac{\#J}{n};\frac{\#J}{n}+1;\frac{s}{t}\Big).$$

As a check, we notice that

$$\begin{aligned}
\mathbb{P}_0(T_t \leqslant s) &= \mathbb{P}_0(X_t>0)\,\mathbb{P}_0(T_t\leqslant s|X_t>0)+\mathbb{P}_0(X_t<0)\,\mathbb{P}_0(T_t\leqslant s|X_t<0)\\
&= \frac{\sin\frac{\#J\pi}{n}}{\pi}\left[\frac{1}{1+\#J/n}\left(\frac{s}{t}\right)^{1+\#J/n}{}_2F_1\Big(\frac{\#J}{n},\frac{\#J}{n}+1;\frac{\#J}{n}+2;\frac{s}{t}\Big)\right.\\
&\quad\left.+\frac{n}{\#J}\left(\frac{s}{t}\right)^{\#J/n}{}_2F_1\Big(\frac{\#J}{n}-1,\frac{\#J}{n};\frac{\#J}{n}+1;\frac{s}{t}\Big)\right]\\
&= \frac{n\,k_n}{\#J(1+\#J/n)}\left(\frac{s}{t}\right)^{\#J/n}\\
&\quad\times\left[\frac{\#J}{n}\,{}_2F_1\Big(\frac{\#J}{n},\frac{\#J}{n}+1;\frac{\#J}{n}+2;\frac{s}{t}\Big)\right.\\
&\quad\left.+\Big(1+\frac{\#J}{n}\Big){}_2F_1\Big(\frac{\#J}{n}-1,\frac{\#J}{n};\frac{\#J}{n}+1;\frac{s}{t}\Big)\right],
\end{aligned}$$

and we retrieve (22) by using [5, formula 9.137.11, p. 1071]. On the other hand, in the case of even order, we notice by successively making use of formulae 9.121.1 on p. 1067, 9.121.26 on p. 1068 and 9.137.4 on p. 1071 of [5] that

$$\begin{aligned}
\frac{s}{t}\,{}_2F_1\Big(\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{s}{t}\Big) &= \frac{3}{2}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{s}{t}\Big)-\frac{3}{2}\,{}_2F_1\Big(-\frac{1}{2},\frac{3}{2};\frac{3}{2};\frac{s}{t}\Big)\\
&= \frac{3}{2}\left[\frac{\arcsin\sqrt{s/t}}{\sqrt{s/t}}-\sqrt{1-\frac{s}{t}}\right]
\end{aligned}$$

and we retrieve formula (3.15) of Nikitin & Orsingher, [11]:

$$\mathbb{P}_0(T_t\leqslant s|X_t>0)=\frac{2}{\pi}\left[\arcsin\sqrt{\frac{s}{t}}-\frac{1}{t}\sqrt{s(t-s)}\right].$$

7 The distributions of M_t and m_t

The functionals M_t and T_t are related according to the equality

$$\begin{aligned}
w_M(t;x,a) &= \mathbb{P}_x(M_t\leqslant a)\\
&= \lim_{\mu\to+\infty}\mathbb{E}_x\left[e^{-\mu\int_0^t \mathbb{1}_{\{X_s>a\}}\,ds}\right]\\
&= \lim_{\mu\to+\infty} u(t,\mu;x-a),
\end{aligned}$$

the quantity $\mathbb{P}_x(M_t\leqslant a)$ being understood as $\lim_{m\to\infty}\mathbb{P}(\max_{0\leqslant k\leqslant m}X_{kt/m}\leqslant a)$. Indeed, decomposing on the events $\{M_t\leqslant a\}$ and $\{M_t>a\}$,

$$\lim_{\mu\to+\infty}\mathbb{E}_x\left[e^{-\mu\int_0^t \mathbb{1}_{\{X_s>a\}}\,ds}\right]=\lim_{\mu\to+\infty}\mathbb{E}_x\left[\mathbb{1}_{\{M_t\leqslant a\}}+\mathbb{1}_{\{M_t>a\}}e^{-\mu\int_0^t \mathbb{1}_{\{X_s>a\}}\,ds}\right].$$

The first part of the "expectation" on the right-hand side of the previous equality is $\mathbb{P}_x(M_t\leqslant a)$ and the second one tends to 0. Evidently, for $x\geqslant a$, $w_M(t;x,a)=0$ and for $x\leqslant a$ the Laplace transform of w is given by

$$W_M(\lambda;x,a)=\lim_{\mu\to+\infty}U(\lambda,\mu;x-a).$$

Since we have $\delta/\gamma\to 0$ as $\mu\to+\infty$, $e^{\theta_k\gamma(x-a)}\to 0$ for any $k\in I$ and it is then easy to derive from (18) a simple expression for the Laplace transform of the function

$t \longmapsto \mathbb{P}_x(M_t \leqslant a)$. It is provided by the proposition below where the distribution of m_t is considered as well.

Theorem 15 *The Laplace transforms of the functions $t \longmapsto \mathbb{P}_x(M_t \leqslant a)$ and $t \longmapsto \mathbb{P}_x(m_t \geqslant a)$ are respectively given by*

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x(M_t \leqslant a)\,dt = W_M(\lambda;x,a) = \frac{1}{\lambda}\Big[1-\sum_{k\in J} A_k\, e^{\theta_k\delta(x-a)}\Big] \quad \textit{for } x \leqslant a,$$
$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x(m_t \geqslant a)\,dt = W_m(\lambda;x,a) = \frac{1}{\lambda}\Big[1-\sum_{k\in I} B_k\, e^{\theta_k\delta(x-a)}\Big] \quad \textit{for } x \geqslant a,$$

with

$$A_k = \prod_{j\in J\setminus\{k\}} \frac{\theta_j}{\theta_j-\theta_k} \quad \textit{and} \quad B_k = \prod_{j\in I\setminus\{k\}} \frac{\theta_j}{\theta_j-\theta_k}.$$

Remarks. 1. Since $\sum_{k\in J} A_k = \sum_{k\in I} B_k = 1$ as it can be seen by using a Vandermonde system, we observe that $\mathbb{P}_a(M_t \leqslant a) = \mathbb{P}_a(m_t \geqslant a) = 0$.

2. The functions $x \longmapsto W_M(\lambda;x,a)$ and $x \longmapsto W_m(\lambda;x,a)$ are the bounded solutions of the respective boundary value problems $\mathcal{D}_x W = \lambda W - 1$ on $(-\infty, a)$ with $\frac{d^kW}{dx^k}(\lambda;a^-,a) = 0$ for $0 \leqslant k \leqslant \#J-1$, and $\mathcal{D}_x W = \lambda W - 1$ on $(a,+\infty)$ with $\frac{d^kW}{dx^k}(\lambda;a^+,a) = 0$ for $0 \leqslant k \leqslant \#I-1$.

The expressions for W_M and W_m given in Theorem 15 involve complex numbers and we wish to convert them into real forms. So, we have to gather together the conjugate terms therein. Since the computations are elementary but cumbersome, we carry out them in the case $n = 2p$ and we only report the results corresponding to the case $n = 2p+1$. We shall adopt the convention $\prod_{j=a}^{b} = 1$ if $a > b$. Set

$$a_k = \prod_{j=0}^{k} \cos\frac{j\pi}{n} \quad \text{and} \quad b_k = \prod_{j=0}^{k} \cos\frac{2j+1}{2n}\,\pi.$$

Since $1-\omega_j = 2\,e^{i(\frac{j}{n}-\frac12)\pi}\sin\frac{j\pi}{n}$ and

$$\prod_{j\in\{0,\dots,n-1\}\setminus\{k\}} \Big(1-\frac{\theta_k}{\theta_j}\Big) = \prod_{j=1}^{n-1}(1-\omega_j) = \lim_{x\to 1}\frac{1-x^n}{1-x} = n,$$

we can rewrite A_k as

$$A_k = \frac{1}{n}\prod_{i\in I}(1-\omega_{k-i}) = \frac{2^{\#I}}{n}\, e^{i\pi[(\frac{k}{n}-\frac12)\#I-\frac1n\sum_{i\in I} i]} \prod_{i\in I}\sin\frac{(k-i)\pi}{n}.$$

- For $n = 2p$:

$$A_k = 2^p e^{i[(2k-3p+1)\pi/4]} \prod_{j=k-p+1}^{k} \sin\frac{j\pi}{n}.$$

We notice that, under the action of the symmetry between indices $k \in J \longmapsto 3p-1-k \in J$, the numbers θ_k and A_k are mapped into their conjugates, namely $\theta_{3p-1-k} = \overline{\theta_k}$ and $A_{3p-1-k} = \overline{A_k}$, and then

– if $p = 2q$:

$$
\begin{aligned}
W_M(\lambda; x, a) &= \frac{1}{\lambda}\Big[1 - \frac{1}{n}\Big(\sum_{k=2q}^{3q-1} + \sum_{k=3q}^{4q-1}\Big) A_k\, e^{\theta_k \delta(x-a)}\Big] \\
&= \frac{1}{\lambda}\Big[1 - \frac{2}{n}\sum_{k=3q}^{4q-1} \Re\big(A_k\, e^{\theta_k \delta(x-a)}\big)\Big] \\
&= \frac{1}{\lambda}\Big[1 - \frac{2}{n}\sum_{k=0}^{q-1} \Re\big(A_{k+3q}\, e^{\theta_{k+3q} \delta(x-a)}\big)\Big]
\end{aligned}
$$

where $\theta_{k+3q} = \omega_{k+1/2}$ and $A_{k+3q} = 2^p\, e^{i[(2k+1)\pi/4]} \prod_{j=k+q+1}^{k+3q} \sin\frac{j\pi}{n}$. The product in A_{k+3q} may be symplified into

$$
\prod_{j=k-q+1}^{k+q} \sin\frac{(j+p)\pi}{n} = \prod_{j=0}^{q+k} \cos\frac{j\pi}{n} \prod_{j=0}^{q-1-k} \cos\frac{j\pi}{n} = a_{q+k}\, a_{q-1-k}.
$$

Consequently,

$$
\begin{aligned}
W_M(\lambda; x, a) &= \frac{1}{\lambda}\Big[1 - \frac{2^{p+1}}{n}\sum_{k=0}^{q-1} a_{q+k}\, a_{q-1-k}\, e^{\delta(x-a)\cos\frac{2k+1}{n}\pi} \\
&\qquad \cos\Big(\delta(x-a)\sin\frac{2k+1}{n}\pi + \frac{2k+1}{4}\pi\Big)\Big]
\end{aligned}
$$

– if $p = 2q + 1$:

$$
\begin{aligned}
W_M(\lambda; x, a) &= \frac{1}{\lambda}\Big[1 - \frac{1}{n}\Big(\sum_{k=2q+1}^{3q} + \sum_{k\in\{3q+1\}} + \sum_{k=3q+2}^{4q+1}\Big) A_k\, e^{\theta_k \delta(x-a)}\Big] \\
&= \frac{1}{\lambda}\Big[1 - \frac{2}{n}\sum_{k=3q+2}^{4q+1} \Re\big(A_k\, e^{\theta_k \delta(x-a)}\big) - \frac{A_{3q+1}}{n}\, e^{\theta_{3q+1}\delta(x-a)}\Big] \\
&= \frac{1}{\lambda}\Big[1 - \frac{2}{n}\sum_{k=1}^{q} \Re\big(A_{k+3q+1}\, e^{\theta_{k+3q+1} \delta(x-a)}\big) \\
&\qquad - \frac{A_{3q+1}}{n}\, e^{\theta_{3q+1}\delta(x-a)}\Big]
\end{aligned}
$$

where $\theta_{k+3q+1} = \omega_k$ and $A_{k+3q+1} = 2^p\, e^{i[k\pi/2]} \prod_{j=k+q+1}^{k+3q+1} \sin\frac{j\pi}{n}$. The product in A_{k+3q+1} can be rewritten as

$$
\prod_{j=k-q}^{k+q} \sin\frac{(j+p)\pi}{n} = \prod_{j=0}^{q+k} \cos\frac{j\pi}{n} \prod_{j=0}^{q-k} \cos\frac{j\pi}{n} = a_{q+k}\, a_{q-k}.
$$

In particular, for $k = 0$ we obtain $\theta_{3q+1} = 1$ and $A_{3q+1} = a_q^2$. Thus,

$$
\begin{aligned}
W_M(\lambda; x, a) &= \frac{1}{\lambda}\Big[1 - \frac{2^{p+1}}{n}\sum_{k=0}^{q-1} a_{q+k}\, a_{q-k}\, e^{\delta(x-a)\cos\frac{2k}{n}\pi} \\
&\qquad \cos\Big(\delta(x-a)\sin\frac{2k}{n}\pi + \frac{k}{2}\pi\Big) - \frac{2^p}{n}\, a_q^2\, e^{\delta(x-a)}\Big]
\end{aligned}
$$

- For $n = 2p + 1$, we shall use the notation $W_M^{\pm}$ for W_M, the $\pm$ signs referring to the cases $\kappa_n = \pm 1$.

– If $\kappa_n=+1$, thanks to the symmetry $k\in J\setminus\{0\}\longmapsto n-k\in J\setminus\{0\}$ for which $\theta_{n-k}=\overline{\theta_k}$ and $A_{n-k}=\overline{A_k}$, we derive

  * if $p=2q$:

$$\begin{aligned}W_M^+(\lambda;x,a) &= \frac{1}{\lambda}\bigg[1-\frac{2^p}{n}\,b_{q-1}^2\,e^{\delta(x-a)}\\ &\quad -\frac{2^{p+1}}{n}\sum_{k=1}^{q} b_{q+k-1}\,b_{q-1-k}\,e^{\delta(x-a)\cos\frac{2k}{n}\pi}\\ &\quad \times\cos\Big(\delta(x-a)\sin\frac{2k}{n}\pi+\frac{kp}{n}\pi\Big)\bigg];\end{aligned}$$

  * if $p=2q+1$:

$$\begin{aligned}W_M^+(\lambda;x,a) &= \frac{1}{\lambda}\bigg[1-\frac{2^{p+1}}{n}\,b_{q}^2\,e^{\delta(x-a)}\\ &\quad -\frac{2^{p+2}}{n}\sum_{k=1}^{q} b_{q+k}\,b_{q-k}\,e^{\delta(x-a)\cos\frac{2k}{n}\pi}\\ &\quad \times\cos\Big(\delta(x-a)\sin\frac{2k}{n}\pi+\frac{k(p+1)}{n}\pi\Big)\bigg].\end{aligned}$$

– If $\kappa_n=-1$, thanks to the symmetry $k\in J\setminus\{0\}\longmapsto n-1-k\in J\setminus\{0\}$ for which $\theta_{n-1-k}=\overline{\theta_k}$ and $A_{n-1-k}=\overline{A_k}$, we derive

  * if $p=2q$:

$$\begin{aligned}W_M^-(\lambda;x,a) &= \frac{1}{\lambda}\bigg[1-\frac{2^{p+2}}{n}\sum_{k=0}^{q-1} b_{q+k}\,b_{q-1-k}\,e^{\delta(x-a)\cos\frac{2k+1}{n}\pi}\\ &\quad \cos\Big(\delta(x-a)\sin\frac{2k+1}{n}\pi+\frac{(2k+1)(2p+1)}{2n}\pi\Big)\bigg];\end{aligned}$$

  * if $p=2q+1$:

$$\begin{aligned}W_M^-(\lambda;x,a) &= \frac{1}{\lambda}\bigg[1-\frac{2^{p+1}}{n}\sum_{k=0}^{q} b_{q+k}\,b_{q-1-k}\,e^{\delta(x-a)\cos\frac{2k+1}{n}\pi}\\ &\quad \cos\Big(\delta(x-a)\sin\frac{2k+1}{n}\pi+\frac{(2k+1)p}{2n}\pi\Big)\bigg].\end{aligned}$$

Concerning the minimum functional, since $m_t=-\max_{0\leqslant s\leqslant t}(-X_s)$ and $(-X_t)_{t\geqslant 0}$ is equivalent to the dual process $(X_t^*)_{t\geqslant 0}$, we can replace m_t by the dual maximum functional, M_t^* say, and then

$$\mathbb{P}_x(m_t\geqslant a)=\mathbb{P}_0(m_t\geqslant a-x)=\mathbb{P}_0^*(M_t^*\leqslant x-a)=\mathbb{P}_a^*(M_t^*\leqslant x).$$

So, reminding the results of Section 4,

- if n is even, $\mathbb{P}_x(m_t\geqslant a)=\mathbb{P}_a(M_t\leqslant x)$;
- if n is odd, $\mathbb{P}_x(m_t^+\geqslant a)=\mathbb{P}_a(M_t^-\leqslant x)$ and $\mathbb{P}_x(m_t^-\geqslant a)=\mathbb{P}_a(M_t^+\leqslant x)$ where the plus and minus signs in $m_t^{\pm}$ and $M_t^{\pm}$ correspond to the choices $\kappa_n=\pm 1$.

Examples.

- For $n=2$ (Brownian motion),

$$W_M(\lambda;x,a)=\frac{1}{\lambda}[1-e^{\sqrt{\lambda}(x-a)}]$$

and

$$W_m(\lambda;x,a)=W_M(\lambda;a,x).$$

- For $n=3$ (Orsingher, [13]),

$$\begin{aligned}
W_M^+(\lambda;x,a) &= \frac{1}{\lambda}[1-e^{\sqrt[3]{\lambda}(x-a)}],\\
W_M^-(\lambda;x,a) &= \frac{1}{\lambda}\Big[1-e^{\sqrt[3]{\lambda}(x-a)/2}\Big(\cos\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(x-a)-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(x-a)\Big)\Big]
\end{aligned}$$

and

$$W_m^+(\lambda;x,a)=W_M^-(\lambda;a,x),\quad W_m^-(\lambda;x,a)=W_M^+(\lambda;a,x).$$

- For $n=4$ (Hochberg, [6], and Beghin et al., [2]),

$$W_M(\lambda;x,a)=\frac{1}{\lambda}\Big[1-e^{\sqrt[4]{\lambda}(x-a)/\sqrt{2}}\Big(\cos\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}(x-a)-\sin\frac{1}{\sqrt{2}}\sqrt[4]{\lambda}(x-a)\Big)\Big]$$

and

$$W_m(\lambda;x,a)=W_M(\lambda;a,x).$$

- For $n=5$,

$$\begin{aligned}
W_M^+(\lambda;x,a) &= \frac{1}{\lambda}\Big[1-\frac{1+\sqrt{5}}{2\sqrt{5}}\,e^{\sqrt[5]{\lambda}(x-a)}\\
&\quad -\frac{2}{\sqrt{5}}\,e^{\sqrt[5]{\lambda}(x-a)\cos(2\pi/5)}\cos\left(\sqrt[5]{\lambda}(x-a)\sin\frac{2\pi}{5}+\frac{2\pi}{5}\right)\Big],\\
W_M^-(\lambda;x,a) &= \frac{1}{\lambda}\Big[1-e^{\sqrt[5]{\lambda}(x-a)\cos(\pi/5)}\Big(\cos\left(\sqrt[5]{\lambda}(x-a)\sin\frac{\pi}{5}\right)-\\
&\quad -\frac{3+\sqrt{5}}{\sqrt{5}}\,\sin\left(\sqrt[5]{\lambda}(x-a)\sin\frac{\pi}{5}\right)\Big)\Big]
\end{aligned}$$

and

$$W_m^+(\lambda;x,a)=W_M^-(\lambda;a,x),\quad W_m^-(\lambda;x,a)=W_M^+(\lambda;a,x).$$

Remarks. 1. The problem we just studied can be treated in the case of even-order through another approach based on Spitzer's identity as it has been observed by Hochberg, [6]. In effect, let $(Y_j)_{j\geqslant 1}$ be a sequence of independent random variables having the same distribution which is assumed to be symmetric, and let $S_0=0$ and $S_j=\sum_{k=1}^j Y_k$ for $j\geqslant 1$. Put

$$\varphi_j(\mu)=\mathbb{E}[e^{-\mu\max_{0\leqslant k\leqslant j}S_k}]\quad\text{and}\quad\psi_j(\mu)=\mathbb{E}[e^{-\mu(S_j\vee 0)}].$$

Then

$$\sum_{j=0}^{\infty}\varphi_j(\mu)\,z^j=\exp\left[\sum_{j=1}^{\infty}\psi_j(\mu)\frac{z^j}{j}\right]$$

and also

$$(1-z)\sum_{j=0}^{\infty}\varphi_j(\mu)\,z^j=\exp\left[-\sum_{j=1}^{\infty}[1-\psi_j(\mu)]\frac{z^j}{j}\right].$$

Actually, Spitzer's identity remains valid also in the case of signed measures provided the total measure is one, so it applies here. Put

$$M_{N,t} = \max_{0 \leqslant k \leqslant \lfloor Nt \rfloor} X_{k/N}.$$

Let us recall that an expectation of the form $\mathbb{E}[F(M_t)]$ should be interpretated as $\lim_{N\to\infty} \mathbb{E}[F(M_{N,t})]$ along this work. The function $t \longmapsto M_{N,t}$ is piecewise constant since for $t \in [\frac{j}{N}, \frac{j+1}{N})$, $M_{N,t} = M_{N,j/N}$. Therefore, applying Spitzer's identity to $Y_k = X_{k/N} - X_{k/N-1}$ successively gives

$$\begin{aligned}
\int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_0(e^{-\mu M_{N,t}})\, dt &= \sum_{j=0}^{\infty} \int_{j/N}^{(j+1)/N} e^{-\lambda t}\, \mathbb{E}_0(e^{-\mu M_{N,j/N}})\, dt \\
&= \frac{1-e^{-\lambda/N}}{\lambda} \sum_{j=0}^{\infty} \varphi_j(\mu)\, e^{-\lambda j/N} \\
&= \frac{1}{\lambda} \exp\left[\sum_{j=1}^{\infty} [1-\psi_j(\mu)] \frac{e^{-\lambda j/N}}{j}\right].
\end{aligned}$$

Letting N tend to infinity yields

$$\begin{aligned}
\int_0^{+\infty} e^{-\mu\xi}\, W_M(\lambda;0,\xi)\, d\xi &= \int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_0(e^{-\mu M_t})\, dt \\
&= \frac{1}{\lambda} \exp\Big[-\int_0^{+\infty} e^{-\lambda t}\, [1-\mathbb{E}_0(e^{-\mu(X_t\vee 0)})]\, \frac{dt}{t}\Big] \\
&= \frac{1}{\lambda} \exp\Big[-\int_\lambda^{+\infty} ds \int_0^{+\infty} e^{-st}\, [1-\mathbb{E}_0(e^{-\mu(X_t\vee 0)})]\, dt\Big].
\end{aligned}$$

Approaching $1-\mathbb{E}_0(e^{-\mu(X_t\vee 0)})$ by $\mathbb{E}_0(e^{-\varepsilon(X_t\vee 0)}) - \mathbb{E}_0(e^{-\mu(X_t\vee 0)})$, this term being evaluated with the aid of (9), introducing the null integral $\displaystyle\int_{-\infty}^{+\infty} \frac{dz}{(z+i\varepsilon)(z+i\mu)}$ for avoiding some problems of divergence and letting $\varepsilon \to 0^+$, we get:

$$1-\mathbb{E}_0(e^{-\mu(X_t\vee 0)}) = \frac{\mu}{2\pi} \int_{-\infty}^{+\infty} \frac{1-e^{-tz^n}}{z(z+i\mu)}\, dz$$

and then

$$\begin{aligned}
\int_0^{+\infty} e^{-st}\, [1-\mathbb{E}_0(e^{-\mu(X_t\vee 0)})]\, dt &= \frac{\mu}{2\pi} \int_{-\infty}^{+\infty} \frac{dz}{z(z+i\mu)} \int_0^{+\infty} [e^{-st} - e^{-(z^n+s)t}]\, dt \\
&= \frac{\mu}{2\pi s} \int_{-\infty}^{+\infty} \frac{z^{n-1}}{(z^n+s)(z+i\mu)}\, dz.
\end{aligned}$$

By residus, if $n = 2p$,

$$\int_{-\infty}^{+\infty} \frac{z^{n-1}}{(z^n+s)(z+i\mu)}\, dz = \frac{2\pi}{n} \sum_{k\in J} \frac{1}{\theta_k s^{1/n} + \mu}.$$

Therefore

$$\begin{aligned}
\int_0^{+\infty} e^{-\lambda t}\, [1-\mathbb{E}_0(e^{-\mu(X_t\vee 0)})]\, \frac{dt}{t} &= \sum_{k\in J} \int_{\sqrt[n]{\lambda}}^{+\infty} \frac{\mu\, d\sigma}{\sigma(\theta_k\sigma+\mu)} \\
&= \sum_{k\in J} \ln \frac{\theta_k \sqrt[n]{\lambda} + \mu}{\theta_k \sqrt[n]{\lambda}}
\end{aligned}$$

which implies

$$\int_0^{+\infty} e^{-\mu\xi}\, W_M(\lambda;0,\xi)\,d\xi = \frac{1}{\sqrt{\lambda}} \prod_{k\in J}\theta_k \prod_{k\in J}(\mu+\theta_k\delta)^{-1}.$$

Decomposing the last product into partial fractions, we obtain

$$\prod_{k\in J}(\mu+\theta_k\delta)^{-1} = \frac{\delta}{\sqrt{\lambda}} \sum_{k\in J} \frac{1}{\prod_{j\in J\setminus\{k\}}(\theta_j-\theta_k)}\,\frac{1}{\mu+\theta_k\delta}$$

and then

$$\prod_{k\in J}\theta_k \prod_{k\in J}(\mu+\theta_k\delta)^{-1} = \frac{\delta}{\sqrt{\lambda}} \sum_{k\in J} \frac{A_k\theta_k}{\mu+\theta_k\delta}.$$

Finally, observing that

$$\frac{1}{\mu+\theta_k\delta} = \int_0^{+\infty} e^{-(\mu+\theta_k\delta)\xi}\,d\xi,$$

we can invert the Laplace transform and deduce

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_0(M_t\in d\xi)\,dt = \frac{1}{\lambda}\sum_{k\in J} A_k\theta_k\delta\, e^{-\theta_k\delta\xi}.$$

This is in good accordance with Theorem 15 in the even-order case.

2. Let us introduce the first times τ_a^+ and τ_a^- the pseudo-process $(X_t)_{t\geqslant 0}$ becomes greater or less than a:

$$\tau_a^\pm = \inf\{s\geqslant 0 : X_s \gtrless a\}$$

with the convention $\inf\emptyset = +\infty$. Plainly, the variables $\tau_a^\pm$ are related to the maximal and minimal functionals according as

$$\tau_a^+ < t \Longleftrightarrow \max_{s\in[0,t)} X_s > a \quad \text{and} \quad \tau_a^- < t \Longleftrightarrow \min_{s\in[0,t)} X_s < a.$$

On the other hand, it may be easily seen that the variables $\max_{s\in[0,t)} X_s$ and $\max_{s\in[0,t]} X_s$ have the same distributions and the same holds for the minimal functionals. Hence,

$$\begin{aligned}
\mathbb{P}_x(\tau_a^+ < t) &= \mathbb{P}_x(M_t > a) \quad \text{if } x\leqslant a,\\
\mathbb{P}_x(\tau_a^- < t) &= \mathbb{P}_x(m_t < a) \quad \text{if } x\geqslant a,
\end{aligned}$$

and then

$$\mathbb{E}_x(e^{-\lambda\tau_a^+}) = \lambda\int_0^{+\infty} e^{-\lambda t}\,\mathbb{P}_x(\tau_a^+ < t)\,dt = 1-\lambda W_M(\lambda;x,a)$$

that is

$$\mathbb{E}_x(e^{-\lambda\tau_a^+}) = \begin{cases} \displaystyle\sum_{k\in J} A_k\, e^{\theta_k\delta(x-a)} & \text{if } x\leqslant a,\\ 1 & \text{if } x\geqslant a. \end{cases}$$

Similarly,

$$\mathbb{E}_x(e^{-\lambda\tau_a^-}) = \begin{cases} \displaystyle\sum_{k\in I} B_k\, e^{\theta_k\delta(x-a)} & \text{if } x\geqslant a,\\ 1 & \text{if } x\leqslant a. \end{cases}$$

3. We can provide a proof that the paths are not continuous. In fact, suppose for the time being the paths of the pseudo-process $(X_t)_{t\geqslant 0}$ are continuous. We would then have $X_{\tau_0^\pm}=0$. Since

$$T_t=\begin{cases}\tau_0^-+T_{t-\tau_0^-}\circ\theta_{\tau_0^-} & \text{if } x>0 \text{ and } \tau_0^-\leqslant t,\\ T_{t-\tau_0^+}\circ\theta_{\tau_0^+} & \text{if } x<0 \text{ and } \tau_0^+\leqslant t,\end{cases}$$

where $(\theta_t)_{t\geqslant 0}$ stands for the shift operators family defined as $X_t\circ\theta_s=X_{s+t}$, by the strong Markov property we would get for $x>0$:

$$\mathbb{E}_x(e^{-\mu T_t})=\mathbb{E}_x\Big[1\!\!1_{\{\tau_0^-\leqslant t\}}\,e^{-\mu\tau_0^-}\,\mathbb{E}_0(e^{-\mu T_{t-\tau_0^-}})\Big]+e^{-\mu t}\,\mathbb{P}_x(\tau_0^->t)$$

and then

$$\begin{aligned}U(\lambda,\mu;x) &= \mathbb{E}_x\Big[e^{-\mu\tau_0^-}\int_{\tau_0^-}^{+\infty}e^{-\lambda t}\,\mathbb{E}_0(e^{-\mu T_{t-\tau_0^-}})\,dt\Big]+\int_0^{+\infty}e^{-(\lambda+\mu)t}\,\mathbb{P}_x(\tau_0^->t)\,dt\\ &= \mathbb{E}_x(e^{-(\lambda+\mu)\tau_0^-})\,U(\lambda,\mu;0)+\frac{1}{\lambda+\mu}\,[1-\mathbb{E}_x(e^{-(\lambda+\mu)\tau_0^-})]\\ &= \frac{1}{\lambda+\mu}+\Big[U(\lambda,\mu;0)-\frac{1}{\lambda+\mu}\Big]\,\mathbb{E}_x(e^{-(\lambda+\mu)\tau_0^-})\\ &= \frac{1}{\lambda+\mu}\Big[1+\Big(\Big(\frac{\gamma}{\delta}\Big)^{\#I}-1\Big)\sum_{k\in I}B_k\,e^{\theta_k\gamma x}\Big].\end{aligned}\tag{31}$$

We would deduce in the same way, for $x<0$:

$$U(\lambda,\mu;x)=\frac{1}{\lambda}\Big[1-\Big(1-\Big(\frac{\delta}{\gamma}\Big)^{\#J}\Big)\sum_{k\in J}A_k\,e^{\theta_k\gamma x}\Big].\tag{32}$$

Let us now rewrite (18) as follows:

$$U(\lambda,\mu;x)=\begin{cases}\dfrac{1}{\lambda+\mu}\Big[1-\displaystyle\sum_{k\in I}\prod_{i\in I}(1-\omega_{k-i}\,\frac{\gamma}{\delta})\,e^{\theta_k\gamma x}\Big] & \text{if } x>0,\\ \dfrac{1}{\lambda}\Big[1-\displaystyle\sum_{k\in J}\prod_{j\in J}(1-\omega_{k-j}\,\frac{\delta}{\gamma})\,e^{\theta_k\gamma x}\Big] & \text{if } x<0.\end{cases}\tag{33}$$

Comparing (31) and (32) to (33), due to the linear independence of the families $(x\longmapsto e^{\theta_k\gamma x})_{k\in I}$ and $(x\longmapsto e^{\theta_k\delta x})_{k\in J}$, we would have

$$\forall k\in I,\quad \prod_{i\in I}(1-\omega_{k-i}x)=1-x^{\#I}\quad\text{and}\quad \forall k\in J,\quad \prod_{j\in J}(1-\omega_{k-j}x)=1-x^{\#J}$$

The above identities are valid only in the Brownian case $n=2$ (for which we know that the paths are continuous!) and the discontinuity of the paths ensues whenever $n\geqslant 3$.

We now inverse the Laplace transforms lying in Theorem 15; we obtain the following representation.

Theorem 16 *The distribution of the maximum functional M_t can be expressed by means of the successive derivatives of the kernel $p(t;z)$ as follows:*

$$\mathbb{P}_x\{M_t > a\} = \sum_{j=0}^{\#J-1} \frac{\tilde{\alpha}_j}{\Gamma(\frac{j+1}{n})} \int_0^t \frac{\partial^j}{\partial x^j}[p(s;x-a)] \frac{ds}{(t-s)^{1-\frac{j+1}{n}}} \tag{34}$$

where the $\tilde{\alpha}_j$'s are the solution of the Vandermonde system

$$\sum_{j=0}^{\#J-1} \theta_k^j \, \tilde{\alpha}_j = \frac{n\, A_k}{\theta_k}, \quad k \in J. \tag{35}$$

Moreover the $\tilde{\alpha}_j$'s are real numbers.

PROOF. Invoking Proposition 4 together with the identity $\int_0^{+\infty} e^{-\lambda t} \frac{dt}{t^{1-\alpha}} = \frac{\Gamma(\alpha)}{\lambda^\alpha}$, we see that the Laplace transform of the right-hand side of (34) is given by

$$\begin{aligned}
\sum_{j=0}^{\#J-1} \frac{\tilde{\alpha}_j}{\lambda^{\frac{j+1}{n}}} \frac{\partial^j}{\partial x^j}[\Phi(\lambda; x-a)] &= \frac{1}{n\lambda} \sum_{k \in J} \Big(\sum_{j=0}^{\#J-1} \theta_k^{j+1} \tilde{\alpha}_j \Big) e^{\theta_k \delta(x-a)} \\
&= \frac{1}{\lambda} \sum_{k \in J} A_k \, e^{\theta_k \delta(x-a)}
\end{aligned}$$

and, by virtue of Theorem 15, the last term equals $\frac{1}{\lambda} - W(\lambda; x, a)$ which is exactly the Laplace transform of the left-hand side of (34).

Last, we have to check that the $\tilde{\alpha}_j$'s are real numbers. This is the result of the fact (that we have already seen) that the θ_k's and the A_k's, $k \in J$, are either real numbers or two by two conjugate complex numbers. More precisely, there is a permutation $s : J \to J$ such that $\theta_{s(k)} = \overline{\theta_k}$ and $A_{s(k)} = \overline{A_k}$. Then, taking the conjugate equations of system (35), we see that the $\overline{\tilde{\alpha}_j}$'s solve the system

$$\sum_{j=0}^{\#J-1} \theta_{s(k)}^j \overline{\tilde{\alpha}_j} = \frac{n\, A_{s(k)}}{\theta_{s(k)}}, \quad k \in J$$

which coincides with (35). By unicity of the solution, we conclude that $\overline{\tilde{\alpha}_j} = \tilde{\alpha}_j$ for all j and hence $\tilde{\alpha}_j \in \mathbb{R}$. □

We can also provide a relationship between the quantities $\mathbb{P}_x\{M_t \leqslant a\}$ and $\mathbb{P}_x\{X_t \leqslant a\} - \mathbb{P}_x\{X_t > a\}$. Indeed, the Laplace transform of the latter, $\Psi(\lambda; a-x)$, is displayed in Proposition 4 and, as in the foregoing proof, we can write

$$\frac{1}{\lambda} - \Psi(\lambda; a-x) = \frac{2}{n\lambda} \sum_{k \in J} e^{\theta_k \delta(x-a)} = \sum_{j=0}^{\#J-1} \frac{\tilde{\beta}_j}{\lambda^{\frac{j+1}{n}}} \frac{\partial^j}{\partial^j x}[\Phi(\lambda; x-a)]$$

where the $\tilde{\beta}_j$'s are the solution of the Vandermonde system

$$\sum_{j=0}^{\#J-1} \theta_k^j \, \tilde{\beta}_j = \frac{2}{\theta_k}, \quad k \in J. \tag{36}$$

It may be easily seen as previously that the $\tilde{\beta}_j$'s are real. Consequently, we obtain the following result.

Theorem 17 *The quantities $\mathbb{P}_x\{M_t \leqslant a\}$ and $\mathbb{P}_x\{X_t \leqslant a\} - \mathbb{P}_x\{X_t > a\}$ are related together according to*

$$\begin{aligned}\mathbb{P}_x\{M_t \leqslant a\} &= \mathbb{P}_x\{X_t \leqslant a\} - \mathbb{P}_x\{X_t > a\} \\ &\quad + \sum_{j=0}^{\#J-1} \frac{\tilde{\beta}_j - \tilde{\alpha}_j}{\Gamma(\frac{j+1}{n})} \int_0^t \frac{\partial^j}{\partial x^j}[p(s;x-a)] \frac{ds}{(t-s)^{1-\frac{j+1}{n}}}. \end{aligned} \tag{37}$$

Corollary 18 *The pseudo-process $(X_t)_{t\geqslant 0}$ satisfies the reflection principle, i.e. $\mathbb{P}_x\{M_t > a\} = 2\,\mathbb{P}_x\{X_t > a\}$, only in the Brownian case $n = 2$.*

Proof. Since the exponentials $e^{\theta_k \delta(x-a)}$, $k \in J$, are independent, taking the Laplace transform of (37), for the reflection principle being sastisfied, we must have

$$\sum_{j=0}^{\#J-1} \theta_k^j (\tilde{\beta}_j - \tilde{\alpha}_j) = 0, \quad k \in J,$$

which entails $A_k = \frac{2}{n}$ for all $k \in J$. This can occur in the sole case $n = 2$ since some A_k are not real for $n > 2$. □

Examples.

- For $n = 3$:
  - if $\kappa_3 = +1$: $J = \{0\}$, $A_0 = \theta_0 = 1$. The solution of systems (35) and (36) are $\tilde{\alpha}_0 = 3$ and $\tilde{\beta}_0 = 2$ and then

    $$\mathbb{P}_x\{M_t > a\} = \frac{3}{\Gamma(\frac{1}{3})} \int_0^t p(s;x-a)\,\frac{ds}{(t-s)^{2/3}}$$

    and

    $$\mathbb{P}_x\{M_t \leqslant a\} = \mathbb{P}_x\{X_t \leqslant a\} - \mathbb{P}_x\{X_t > a\} - \frac{1}{\Gamma(\frac{1}{3})} \int_0^t p(s;x-a)\,\frac{ds}{(t-s)^{2/3}};$$

  - if $\kappa_3 = -1$: $J = \{0, 2\}$, $A_0 = \frac{1}{2\sqrt{3}}(\sqrt{3} + i)$, $\theta_0 = \frac{1}{2}(1 + i\sqrt{3})$ and $A_2 = \overline{A_0}$, $\theta_2 = \overline{\theta_0}$. Systems (35) and (36) are

    $$\left\{\begin{array}{rcl} \tilde{\alpha}_0 + \theta_0\,\tilde{\alpha}_1 &=& \frac{1}{2}(3 - i\sqrt{3}) \\ \tilde{\alpha}_0 + \theta_2\,\tilde{\alpha}_1 &=& \frac{1}{2}(3 + i\sqrt{3}) \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} \tilde{\beta}_0 + \theta_0\,\tilde{\beta}_1 &=& 1 - i\sqrt{3} \\ \tilde{\beta}_0 + \theta_2\,\tilde{\beta}_1 &=& 1 + i\sqrt{3} \end{array}\right.$$

    whose solutions are

    $$\tilde{\alpha}_0 = 2,\ \tilde{\alpha}_1 = -1 \quad \text{and} \quad \tilde{\beta}_0 = 2,\ \tilde{\beta}_1 = -2.$$

    Thus,

    $$\mathbb{P}_x\{M_t > a\} = \frac{2}{\Gamma(\frac{1}{3})} \int_0^t p(s;x-a)\,\frac{ds}{(t-s)^{2/3}} - \frac{1}{\Gamma(\frac{2}{3})} \int_0^t \frac{\partial}{\partial x}[p(s;x-a)]\,\frac{ds}{\sqrt[3]{t-s}}$$

    and

    $$\mathbb{P}_x\{M_t \leqslant a\} = \mathbb{P}_x\{X_t \leqslant a\} - \mathbb{P}_x\{X_t > a\} - \frac{1}{\Gamma(\frac{2}{3})} \int_0^t \frac{\partial}{\partial x}[p(s;x-a)]\,\frac{ds}{\sqrt[3]{t-s}}$$

    which agrees with formula (4) of Orsingher.

- For $n=4$: $J=\{2,3\}$, $A_2=\frac{1}{2}(1-i)$, $\theta_2=\frac{1}{\sqrt{2}}(1-i)$ and $A_3=\overline{A_2}$, $\theta_3=\overline{\theta_2}$. Systems (35) and (36) are

$$\left\{\begin{array}{lcl}\tilde{\alpha}_0+\theta_2\,\tilde{\alpha}_1 &=& 2\sqrt{2}\\ \tilde{\alpha}_0+\theta_3\,\tilde{\alpha}_1 &=& 2\sqrt{2}\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{lcl}\tilde{\beta}_0+\theta_2\,\tilde{\beta}_1 &=& \sqrt{2}\,(1+i)\\ \tilde{\beta}_0+\theta_3\,\tilde{\beta}_1 &=& \sqrt{2}\,(1-i)\end{array}\right.$$

whose solutions are

$$\tilde{\alpha}_0=2\sqrt{2},\ \tilde{\alpha}_1=0 \quad\text{and}\quad \tilde{\beta}_0=2\sqrt{2},\ \tilde{\beta}_1=-2.$$

Thus,

$$\mathbb{P}_x\{M_t>a\}=\frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\int_0^t p(s;x-a)\,\frac{ds}{(t-s)^{3/4}}$$

and

$$\mathbb{P}_x\{M_t\leqslant a\}=\mathbb{P}_x\{X_t\leqslant a\}-\mathbb{P}_x\{X_t>a\}-\frac{2}{\sqrt{\pi}}\int_0^t \frac{\partial}{\partial x}[p(s;x-a)]\,\frac{ds}{\sqrt{t-s}}.$$

This is formula (5) of Beghin et al., [2], where the constant before the integral therein should be corrected into $2/\sqrt{\pi}$.

8 The distribution of O_t

The time O_t is related to the maximal functional as follows: for $s<t$,

$$O_t\leqslant s \Longleftrightarrow \max_{u\in(s,t]} X_u\leqslant 0.$$

On the other hand, it may be easily seen that the functionals $\max_{u\in(s,t]} X_u$ and $\max_{u\in[s,t]} X_u$ have the same distributions. Then, for $s<t$,

$$\begin{aligned}\mathbb{P}_x(O_t\leqslant s) &= \mathbb{P}_x(\max_{u\in[0,t-s]} X_{u+s}\leqslant 0)\\ &= \mathbb{E}_x[1\!\!1_{\{X_s<0\}}\mathbb{P}_{X_s}(M_{t-s}\leqslant 0)]\\ &= \int_{-\infty}^0 p(s;x,y)\,w_M(t-s;y,0)\,dy,\end{aligned}$$

and for $s\geqslant t$, plainly $\mathbb{P}_x(O_t\leqslant s)=1$. Therefore, we have

$$\begin{aligned}z(t,\mu;x) &= \mathbb{E}_x(e^{-\mu O_t})\\ &= \mu\int_0^{+\infty} e^{-\mu s}\,\mathbb{P}_x(O_t\leqslant s)\,ds\\ &= \mu\int_0^t e^{-\mu s}\,ds\int_{-\infty}^0 p(s;x,y)\,w_M(t-s;y,0)\,dy+\mu\int_t^{+\infty} e^{-\mu s}\,ds\\ &= \mu\int_{-\infty}^0 dy\int_0^t e^{-\mu s}\,p(s;x,y)\,w_M(t-s;y,0)\,ds+e^{-\mu t}.\end{aligned}$$

Observing that the integral with respect to the variable s is a convolution, we get for the Laplace transform of the function z:

$$\begin{aligned}Z(\lambda,\mu;x) &= \mu\int_{-\infty}^0 dy\left[\int_0^{+\infty} e^{-(\lambda+\mu)t}\,p(t;x,y)\,dt\times\int_0^{+\infty} e^{-\lambda t}\,w_M(t;y,0)\,dt\right]\\ &\quad +\frac{1}{\lambda+\mu}\\ &= \mu\int_{-\infty}^0 \Phi(\lambda+\mu;x,y)\,W_M(\lambda;y,0)\,dy+\frac{1}{\lambda+\mu}.\end{aligned}$$

This becomes, by (12) and Theorem 15, for $x \geqslant 0$:

$$\begin{aligned}
Z(\lambda,\mu;x) &= -\frac{\mu}{n\lambda(\lambda+\mu)}\int_{-\infty}^{0}\sum_{i\in I}\theta_i\gamma\, e^{-\theta_i\gamma(y-x)}\Big(1-\sum_{j\in J}A_j\, e^{\theta_j\delta y}\Big)\,dy+\frac{1}{\lambda+\mu}\\
&= \frac{\mu}{n\lambda(\lambda+\mu)}\Big[-\sum_{i\in I}\theta_i\gamma\int_{-\infty}^{0} e^{-\theta_i\gamma(y-x)}\,dy\\
&\quad+\sum_{(i,j)\in I\times J}A_j\theta_i\gamma e^{\theta_i\gamma x}\int_{-\infty}^{0} e^{(\theta_j\delta-\theta_i\gamma)y)}\,dy\Big]+\frac{1}{\lambda+\mu}\\
&= \frac{\mu}{n\lambda(\lambda+\mu)}\Big[\sum_{i\in I}\Big(1-\sum_{j\in J}A_j\,\frac{\theta_i\gamma}{\theta_i\gamma-\theta_j\delta}\Big)e^{\theta_i\gamma x}\Big]+\frac{1}{\lambda+\mu} \qquad (38)
\end{aligned}$$

and for $x \leqslant 0$:

$$\begin{aligned}
Z(\lambda,\mu;x) &= -\frac{\mu}{n\lambda(\lambda+\mu)}\Big[\int_{-\infty}^{x}\sum_{i\in I}\theta_i\gamma\, e^{-\theta_i\gamma(y-x)}\Big(1-\sum_{j\in J}A_j\, e^{\theta_j\delta y}\Big)\,dy\\
&\quad-\int_{x}^{0}\sum_{i\in J}\theta_i\gamma\, e^{-\theta_i\gamma(y-x)}\Big(1-\sum_{j\in J}A_j\, e^{\theta_j\delta y}\Big)\,dy+\frac{1}{\lambda+\mu}\\
&= \frac{\mu}{n\lambda(\lambda+\mu)}\Big[-\sum_{i\in I}\theta_i\gamma\int_{-\infty}^{x} e^{-\theta_i\gamma(y-x)}\,dy+\sum_{i\in J}\theta_i\gamma\int_{x}^{0} e^{-\theta_i\gamma(y-x)}\,dy\\
&\quad+\sum_{(i,j)\in I\times J}A_j\theta_i\gamma e^{\theta_i\gamma x}\int_{-\infty}^{x} e^{(\theta_j\delta-\theta_i\gamma)y)}\,dy\\
&\quad-\sum_{(i,j)\in J\times J}A_j\theta_i\gamma e^{\theta_i\gamma x}\int_{x}^{0} e^{(\theta_j\delta-\theta_i\gamma)y)}\,dy\Big]+\frac{1}{\lambda+\mu}\\
&= \frac{1}{\lambda}-\frac{\mu}{n\lambda(\lambda+\mu)}\Big[\sum_{i\in J}\Big(1-\sum_{j\in J}A_j\,\frac{\theta_i\gamma}{\theta_i\gamma-\theta_j\delta}\Big)e^{\theta_i\gamma x}\\
&\quad-\sum_{j\in J}\Big(\sum_{i=0}^{n-1}\frac{\theta_i\gamma}{\theta_i\gamma-\theta_j\delta}\Big)A_j\, e^{\theta_j\delta x}\Big]. \qquad (39)
\end{aligned}$$

Looking at formulae (38) and (39), we see that we have to evaluate the sums $\sum_{j\in J}A_j\,\frac{\theta_i\gamma}{\theta_i\gamma-\theta_j\delta}$ and $\sum_{i=0}^{n-1}\frac{\theta_i\gamma}{\theta_i\gamma-\theta_j\delta}$ which are respectively of the form $\sum_{j\in J}\frac{A_j}{1-a\theta_j}$ with $a=\frac{\delta}{\theta_i\gamma}$ and $\sum_{i=0}^{n-1}\frac{1}{1-\omega_{j-i}(\delta/\gamma)}$. The latter is easy to calculate, upon invoking an expansion into partial fractions:

$$\sum_{i=0}^{n-1}\frac{1}{1-\omega_{j-i}(\delta/\gamma)}=\sum_{i=0}^{n-1}\frac{1}{1-\omega_i(\delta/\gamma)}=\frac{n}{1-(\delta/\gamma)^n}=n\,\frac{\lambda+\mu}{\mu}.$$

For computing the former, we state an intermediate result which will be useful.

Lemma 19 *We have for $n \geqslant r \geqslant 2$*

$$\begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_r \\ \theta_1^2 & \dots & \theta_r^2 \\ \vdots & & \vdots \\ \theta_1^{r-2} & \dots & \theta_r^{r-2} \\ \theta_1^{n-1} & \dots & \theta_r^{n-1} \end{vmatrix} = \prod_{1 \leqslant k < l \leqslant r} (\theta_l - \theta_k) \sum_{\substack{i_1, \dots, i_r \geqslant 0 \\ i_1 + \dots + i_r = n-r}} \theta_1^{i_1} \dots \theta_r^{i_r}. \tag{40}$$

PROOF. Set

$$\Delta_{r,n} = \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_r \\ \theta_1^2 & \dots & \theta_r^2 \\ \vdots & & \vdots \\ \theta_1^{r-2} & \dots & \theta_r^{r-2} \\ \theta_1^{n-1} & \dots & \theta_r^{n-1} \end{vmatrix}.$$

Substracting the $(i-1)^{\text{th}}$ row multiplied by θ_r to the i^{th} row for $2 \leqslant i \leqslant r-1$, and the first row multiplied by θ_r^{n-1} to the last row yields

$$\begin{aligned}
\Delta_{r,n} &= (-1)^{r+1} \begin{vmatrix} \theta_1 - \theta_r & \dots & \theta_{r-1} - \theta_r \\ (\theta_1 - \theta_r)\theta_1 & \dots & (\theta_{r-1} - \theta_r)\theta_{r-1} \\ (\theta_1 - \theta_r)\theta_1^2 & \dots & (\theta_{r-1} - \theta_r)\theta_{r-1}^2 \\ \vdots & & \vdots \\ (\theta_1 - \theta_r)\theta_1^{r-3} & \dots & (\theta_{r-1} - \theta_r)\theta_{r-1}^{r-3} \\ \theta_1^{n-1} - \theta_r^{n-1} & \dots & \theta_{r-1}^{n-1} - \theta_r^{n-1} \end{vmatrix} \\
&= \prod_{k=1}^{r-1} (\theta_r - \theta_k) \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{r-1} \\ \vdots & & \vdots \\ \theta_1^{r-3} & \dots & \theta_{r-1}^{r-3} \\ \sum_{m=0}^{n-2} \theta_1^m \theta_r^{n-2-m} & \dots & \sum_{m=0}^{n-2} \theta_{r-1}^m \theta_r^{n-2-m} \end{vmatrix} \\
&= \prod_{k=1}^{r-1} (\theta_r - \theta_k) \sum_{m=r-2}^{n-2} \theta_r^{n-2-m} \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{r-1} \\ \vdots & & \vdots \\ \theta_1^{r-3} & \dots & \theta_{r-1}^{r-3} \\ \theta_1^m & \dots & \theta_{r-1}^m \end{vmatrix}.
\end{aligned}$$

In the last sum, we only retained the indices $m \geqslant r-2$ since the determinants therein corresponding to $0 \leqslant m \leqslant r-3$ vanish. So we just obtained the recursion

$$\Delta_{r,n} = \prod_{k=1}^{r-1} (\theta_r - \theta_k) \sum_{m=r-1}^{n-1} \theta_r^{n-m-1} \Delta_{r-1,m}.$$

It is easily seen that (40) is valid for $r = 2$. Suppose now the result (40) is true for $\Delta_{r-1,m}$, $m \geqslant r-1$; here, the recursion hypothesis is related to the index r. We get

$$\Delta_{r,n} = \prod_{k=1}^{r-1} (\theta_r - \theta_k) \prod_{1 \leqslant k < l \leqslant r-1} (\theta_l - \theta_k)$$

$$\times \sum_{m=r-1}^{n-1} \theta_r^{n-m-1} \sum_{\substack{i_1,\dots,i_{r-1}\geqslant 0\\ i_1+\dots+i_{r-1}=m-r+1}} \theta_1^{i_1}\dots\theta_{r-1}^{i_{r-1}}$$
$$= \prod_{1\leqslant k<l\leqslant r} (\theta_l-\theta_k) \sum_{\substack{i_1,\dots,i_r\geqslant 0\\ i_1+\dots+i_r=n-r}} \theta_1^{i_1}\dots\theta_r^{i_r}.$$

Indeed, in the last sum, we put $i_r = n-m-1$ and (40) is also valid for $\Delta_{r,n}$, $n \geqslant r$. □

Lemma 20 *Recall that* $A_k = \prod_{j\in J} \frac{\theta_k}{\theta_k-\theta_j}$ *for* $k \in J$*. We have*

$$\sum_{j\in J} \frac{A_j}{1-a\theta_j} = 1-(-1)^{\#J} \prod_{j\in J} \frac{a\theta_j}{1-a\theta_j}. \tag{41}$$

Proof. Let us expand $\frac{1}{1-a\theta_j}$ for $|a| < \min_{j\in J} \frac{1}{\theta_j}$ as a Taylor series. This yields

$$\sum_{j\in J} \frac{A_j}{1-a\theta_j} = \sum_{n=0}^{\infty} \Big[\sum_{j\in J} A_j\theta_j^n\Big] a^n. \tag{42}$$

It is convenient for the sequel to relabelling J as the set $\{1,2,\dots,r\}$ with $r = \#J$. Reminding that the A_j's are solutions of the Vandermonde system $\sum_{j\in J} A_j\theta_j^n = \delta_{0n}$, $0 \leqslant n \leqslant r-1$, we rewrite A_j as $A_j = \frac{\Delta_j}{\Delta}$ where

$$\Delta = \Delta_{r,r} = \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_r \\ \theta_1^2 & \dots & \theta_r^2 \\ \vdots & & \vdots \\ \theta_1^{r-1} & \dots & \theta_r^{r-1} \end{vmatrix} = \prod_{1\leqslant k<l\leqslant r} (\theta_l-\theta_k)$$

and

$$\begin{aligned}
\Delta_j &= \begin{vmatrix} 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{j-1} & 0 & \theta_{j+1} & \dots & \theta_r \\ \theta_1^2 & \dots & \theta_{j-1}^2 & 0 & \theta_{j+1}^2 & \dots & \theta_r^2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{r-1} & \dots & \theta_{j-1}^{r-1} & 0 & \theta_{j+1}^{r-1} & \dots & \theta_r^{r-1} \end{vmatrix} \\
&= (-1)^{j+1} \begin{vmatrix} \theta_1 & \dots & \theta_{j-1} & \theta_{j+1} & \dots & \theta_r \\ \theta_1^2 & \dots & \theta_{j-1}^2 & \theta_{j+1}^2 & \dots & \theta_r^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ \theta_1^{r-1} & \dots & \theta_{j-1}^{r-1} & \theta_{j+1}^{r-1} & \dots & \theta_r^{r-1} \end{vmatrix} \\
&= (-1)^{j+1} \prod_{k\in J\setminus\{j\}} \theta_k \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{j-1} & \theta_{j+1} & \dots & \theta_r \\ \vdots & & \vdots & \vdots & & \vdots \\ \theta_1^{r-2} & \dots & \theta_{j-1}^{r-2} & \theta_{j+1}^{r-2} & \dots & \theta_r^{r-2} \end{vmatrix}.
\end{aligned}$$

Then, for $n \geqslant r$,

$$\begin{aligned}\sum_{j\in J} A_j\theta_j^n &= \frac{1}{\Delta}\prod_{k\in J}\theta_k\sum_{j\in J}(-1)^{j+1}\theta_j^{n-1}\begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1\\ \theta_1 & \dots & \theta_{j-1} & \theta_{j+1} & \dots & \theta_r\\ \vdots & & \vdots & \vdots & & \vdots\\ \theta_1^{r-2} & \dots & \theta_{j-1}^{r-2} & \theta_{j+1}^{r-2} & \dots & \theta_r^{r-2}\end{vmatrix}\\ &= (-1)^{\#J+1}\,\frac{\Delta_{r,n}}{\Delta}\prod_{k\in J}\theta_k.\end{aligned}$$

In the last equality we have used the expansion of the determinant $\Delta_{r,n}$ with respect to its last row. By (40) we get for $n \geqslant r$

$$\sum_{j\in J} A_j\theta_j^n = (-1)^{\#J+1}\prod_{k\in J}\theta_k\sum_{\substack{i_1,\dots,i_r\geqslant 0\\ i_1+\dots+i_r=n-r}}\theta_1^{i_1}\dots\theta_r^{i_r}$$

and then, by (42), since $\sum\limits_{j\in J} A_j\theta_j^n$ equals 1 for $n=0$ and vanishes for $1\leqslant n\leqslant r-1$,

$$\begin{aligned}\sum_{j\in J}\frac{A_j}{1-a\theta_j} &= 1+\sum_{n=r}^{\infty}\Big[\sum_{j\in J}A_j\theta_j^n\Big]a^n\\ &= 1-(-1)^{\#J}\prod_{j\in J}\theta_j\sum_{\substack{n\geqslant r,i_1,\dots,i_r\geqslant 0\\ i_1+\dots+i_r=n-r}}a^n\theta_1^{i_1}\dots\theta_r^{i_r}.\end{aligned}$$

Replacing n by $i_1+\dots+i_r+r$ in the sum of the last equality, we obtain

$$\begin{aligned}\sum_{j\in J}\frac{A_j}{1-a\theta_j} &= 1-(-a)^{\#J}\prod_{j\in J}\theta_j\sum_{i_1,\dots,i_r\geqslant 0}(a\theta_1)^{i_1}\dots(a\theta_r)^{i_r}\\ &= 1-(-1)^{\#J}\prod_{j\in J}\frac{a\theta_j}{1-a\theta_j}.\end{aligned}$$

□

We can now pursue the calculation of $Z(\lambda,\mu;x)$. We have by (41)

$$\begin{aligned}\sum_{j\in J}A_j\,\frac{\theta_i\gamma}{\theta_i\gamma-\theta_j\delta} &= 1-(-1)^{\#J}\prod_{j\in J}\frac{\theta_j\delta}{\theta_i\gamma-\theta_j\delta}\\ &= 1-\prod_{j\in J}\frac{1}{1-\omega_{i-j}(\gamma/\delta)}.\end{aligned}$$

Since

$$-\prod_{j=0}^{n-1}(1-\omega_{i-j}\,\frac{\gamma}{\delta}) = \Big(\frac{\gamma}{\delta}\Big)^n-1=\frac{\mu}{\lambda},$$

this becomes

$$\sum_{j\in J}A_j\,\frac{\theta_i\gamma}{\theta_i\gamma-\theta_j\delta} = 1+\frac{\lambda}{\mu}\prod_{j\in I}(1-\omega_{i-j}\,\frac{\gamma}{\delta}). \tag{43}$$

As a result, putting (43) in (38) and (39), we obtain

$$Z(\lambda,\mu;x)=\begin{cases}\dfrac{1}{\lambda+\mu}\Big[1-\dfrac{1}{n}\displaystyle\sum_{i\in I}\prod_{j\in I}(1-\omega_{i-j}\,\frac{\gamma}{\delta})\,e^{\theta_i\gamma x}\Big] & \text{if } x\geqslant 0,\\[2ex] \dfrac{1}{n(\lambda+\mu)}\displaystyle\sum_{j\in J}\prod_{i\in I}(1-\omega_{j-i}\,\frac{\gamma}{\delta})\,e^{\theta_j\gamma x}+\frac{1}{\lambda}\Big[1-\sum_{j\in J}A_j\,e^{\theta_j\delta x}\Big] & \text{if } x\leqslant 0.\end{cases}$$

REMARK. Differentiating n times the foregoing relation, it may be easily seen that $Z(\lambda,\mu;x)$ solves the differential equation

$$\mathcal{D}_x Z(\lambda,\mu;x)=\begin{cases}(\lambda+\mu)Z(\lambda,\mu;x)-1-\dfrac{\mu}{\lambda}\Big[1-\displaystyle\sum_{j\in J}A_j\,e^{\theta_j\delta x}\Big] & \text{if } x>0,\\ (\lambda+\mu)Z(\lambda,\mu;x)-1 & \text{if } x<0.\end{cases}$$

Moreover, the function $Z(\lambda,\mu;x)$ is C^n at 0. Indeed, we have

$$\begin{aligned}
&Z(\lambda,\mu;0^+)-Z(\lambda,\mu;0^-)\\
&\qquad=\frac{1}{\lambda+\mu}-\frac{1}{\lambda}\Big[1-\sum_{j\in J}A_j\Big]-\frac{1}{n(\lambda+\mu)}\sum_{j=0}^{n-1}\Big[\prod_{i\in I}(1-\omega_{j-i}\,\frac{\gamma}{\delta})\Big]\\
&\qquad=\frac{1}{n(\lambda+\mu)}\Big[n-\sum_{j=0}^{n-1}\prod_{i\in I}(1-\omega_{j-i}\,\frac{\gamma}{\delta})\Big]
\end{aligned}$$

and for $1\leqslant\ell\leqslant n-1$,

$$\begin{aligned}
&\frac{d^\ell Z}{dx^\ell}(\lambda,\mu;0^+)-\frac{d^\ell Z}{dx^\ell}(\lambda,\mu;0^-)\\
&\qquad=\frac{1}{\lambda}\sum_{j\in J}A_j(\theta_j\delta)^\ell-\frac{1}{n(\lambda+\mu)}\sum_{j=0}^{n-1}\Big[\prod_{i\in I}(1-\omega_{j-i}\,\frac{\gamma}{\delta})\Big](\theta_j\gamma)^\ell\\
&\qquad=\frac{1}{n}\,\gamma^{\ell-n}\Big[n\Big(\frac{\gamma}{\delta}\Big)^{n-\ell}\sum_{j\in J}A_j\theta_j^\ell-\sum_{j=0}^{n-1}\theta_j^\ell\prod_{i\in I}(1-\omega_{j-i}\,\frac{\gamma}{\delta})\Big]
\end{aligned}$$

We know that the value of $\sum_{j\in J}A_j\theta_j^\ell$ is 1 for $\ell=0$ and 0 for $1\leqslant\ell\leqslant\#J-1$. On the other hand, expanding the product $\prod_{i\in I}(1-\omega_{j-i}\,x)$, we obtain for the coefficient of x^m within $\sum_{j=0}^{n-1}\theta_j^\ell\prod_{i\in I}(1-\omega_{j-i}\,x)$:

$$\begin{aligned}
(-1)^m\sum_{j=0}^{n-1}\theta_j^\ell\sum_{\substack{i_1,\dots,i_m\in I\\ i_1<\dots<i_m}}\omega_{j-i_1}\dots\omega_{j-i_m}&=(-1)^m\sum_{j=0}^{n-1}\theta_j^\ell\,\omega_1^{mj}\sum_{\substack{i_1,\dots,i_m\in I\\ i_1<\dots<i_m}}\omega_1^{-(i_1+\dots+i_m)}\\
&=(-1)^m\theta_0^\ell\sum_{j=0}^{n-1}\omega_1^{(m+\ell)j}\times\sum_{\substack{i_1,\dots,i_m\in I\\ i_1<\dots<i_m}}\omega_1^{-(i_1+\dots+i_m)}
\end{aligned}$$

where in the last equality we used $\theta_j=\theta_0\omega_j$. The sum $\sum_{j=0}^{n-1}\omega_1^{(m+\ell)j}$ equals 0 if $m\neq n-\ell$ and n if $m=n-\ell$. So, for $0\leqslant\ell\leqslant\#J-1$,

$$\frac{d^\ell Z}{dx^\ell}(\lambda,\mu;0^+)-\frac{d^\ell Z}{dx^\ell}(\lambda,\mu;0^-)=0.$$

It may be shown that the jump at 0 for the derivatives of order ℓ such that $\#J\leqslant\ell\leqslant n$ also vanish but we omit the intricate proof. We only point out that the following formulae are needed: with the aid of (40),

$$\sum_{j\in J}A_j\theta_j^\ell=(-1)^{r-1}\prod_{j\in J}\theta_j\sum_{\substack{k_1,\dots,k_r\geqslant 0\\ k_1+\dots+k_r=\ell-r}}\theta_{j_1}^{k_1}\dots\theta_{j_r}^{k_r}$$

where $r=\#J$, $J=\{j_1,\dots,j_r\}$ and, putting $\omega_1=\omega$,

$$\sum_{0\leqslant k_1<\dots<k_s\leqslant t-1}\omega^{k_1+\dots+k_s}=\omega^{s(s-1)/2}\,\frac{(\omega^t-1)(\omega^{t-1}-1)\dots(\omega^{t-s+1}-1)}{(\omega^s-1)(\omega^{s-1}-1)\dots(\omega-1)}.$$

In the particular case $x = 0$, we obtain the following expression for $Z(\lambda, \mu; 0)$.

Theorem 21 *The Laplace transform of* $\mathbb{E}_0(e^{-\mu O_t})$ *is given by*

$$\int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_0(e^{-\mu O_t})\, dt = \frac{1}{\lambda+\mu}\Big[1 - \frac{1}{n} \sum_{i\in I} \prod_{j\in I} (1 - \omega_{i-j}\, \frac{\gamma}{\delta})\Big].$$

Remark. The discontinuity of the paths can be seen once more. If the paths were continuous, we would have, as in the remark just before Theorem 16,

$$\mathbb{E}_x(e^{-\mu O_t}) = \begin{cases} \mathbb{E}_x\Big[\mathbb{1}_{\{\tau_0^- \leqslant t\}}\, e^{-\mu\tau_0^-}\, \mathbb{E}_0(e^{-\mu O_{t-\tau_0^-}})\Big] + e^{-\mu t}\, \mathbb{P}_x(\tau_0^- > t) & \text{if } x > 0, \\ \mathbb{E}_x\Big[\mathbb{1}_{\{\tau_0^+ \leqslant t\}}\, e^{-\mu\tau_0^+}\, \mathbb{E}_0(e^{-\mu O_{t-\tau_0^+}})\Big] + \mathbb{P}_x(\tau_0^+ > t) & \text{if } x < 0, \end{cases}$$

and then

$$Z(\lambda, \mu; x) = \begin{cases} \dfrac{1}{\lambda+\mu}\Big[1 - (1-(\lambda+\mu)Z(\lambda,\mu;0)) \displaystyle\sum_{i\in I} B_i\, e^{\theta_i \gamma x}\Big] & \text{if } x > 0, \\ \dfrac{1}{\lambda}\Big[1 - \displaystyle\sum_{j\in J} A_j\, e^{\theta_j \delta x}\Big] + Z(\lambda,\mu;0) \displaystyle\sum_{j\in J} A_j\, e^{\theta_j \gamma x} & \text{if } x < 0. \end{cases}$$

This would entails

$$\begin{aligned} \forall k \in J, \quad \prod_{i\in I} (1-\omega_{k-i} x) &= A_k \sum_{j\in J} \prod_{i\in I} (1-\omega_{j-i} x), \\ \forall k \in I, \quad \prod_{i\in I} (1-\omega_{k-i} x) &= B_k \sum_{j\in I} \prod_{i\in I} (1-\omega_{j-i} x). \end{aligned}$$

The above identities are valid only in the case $n = 2$.

Examples.

- For $n = 2$,

$$Z(\lambda,\mu;0) = \frac{1}{2}\Big[\frac{1}{\lambda+\mu} + \frac{1}{\sqrt{\lambda(\lambda+\mu)}}\Big].$$

- For $n = 3$,

$$\begin{aligned} Z^+(\lambda,\mu;0) &= \frac{1}{3}\Big[\frac{1}{\lambda+\mu} + \frac{1}{\sqrt[3]{\lambda(\lambda+\mu)^2}} + \frac{1}{\sqrt[3]{\lambda^2(\lambda+\mu)}}\Big], \\ Z^-(\lambda,\mu;0) &= \frac{1}{3}\Big[\frac{2}{\lambda+\mu} + \frac{1}{\sqrt[3]{\lambda(\lambda+\mu)^2}}\Big]. \end{aligned}$$

- For $n = 4$,

$$Z(\lambda,\mu;0) = \frac{1}{2}\Big[\frac{1}{\lambda+\mu} + \frac{1}{\sqrt[4]{\lambda(\lambda+\mu)^3}}\Big].$$

- For $n = 5$,

$$\begin{aligned} Z^+(\lambda,\mu;0) &= \frac{1}{5}\Big[\frac{3}{\lambda+\mu} + \frac{3+\sqrt{5}}{2\sqrt[5]{\lambda(\lambda+\mu)^4}} + \frac{1-\sqrt{5}}{2\sqrt[5]{\lambda^2(\lambda+\mu)^3}}\Big], \\ Z^-(\lambda,\mu;0) &= \frac{1}{5}\Big[\frac{2}{\lambda+\mu} + \frac{3+\sqrt{5}}{2\sqrt[5]{\lambda(\lambda+\mu)^4}} + \frac{1}{\sqrt[5]{\lambda^2(\lambda+\mu)^3}} + \frac{1-\sqrt{5}}{2\sqrt[5]{\lambda^3(\lambda+\mu)^2}}\Big]. \end{aligned}$$

- For $n = 6$,

$$Z(\lambda,\mu;0) = \frac{1}{6}\Big[\frac{3}{\lambda+\mu} + \frac{4}{\sqrt[6]{\lambda(\lambda+\mu)^5}} - \frac{1}{\sqrt{\lambda(\lambda+\mu)}}\Big].$$

Appendix

A Asymptotics for p

Let us prove Proposition 2. Suppose for instance $z > 0$. We first rewrite (9) using the change of variables $u = \zeta^{\frac{1}{n-1}} v$ as

$$p(t;z) = \frac{\zeta^{\frac{1}{n-1}}}{2\pi \sqrt[n]{nt}} q(\zeta^{\frac{n}{n-1}}), \quad \zeta = \frac{z}{\sqrt[n]{nt}}$$

with

$$q(\zeta) = \int_{-\infty}^{+\infty} e^{\zeta h(v)}\, dv, \quad h(v) = iv + \kappa_n \frac{(iv)^n}{n}.$$

More precisely,

- if $n = 2p$ and $\kappa_n = (-1)^{p+1}$,

$$q(\zeta) = \int_{-\infty}^{+\infty} e^{\zeta h(u)}\, du, \quad h(u) = iu - \frac{u^n}{n};$$

- if $n = 2p+1$,

$$q(\zeta) = \int_{-\infty}^{+\infty} e^{\zeta h^{\pm}(u)}\, du, \quad h^{\pm}(u) = i(u \pm \frac{u^n}{n}),$$

  the $\pm$ sign in h referring to the respective cases $\kappa_n = (-1)^p$ and $\kappa_n = (-1)^{p+1}$.

We will apply to q the method of the steepest descent for describing its asymptotic behaviour as $\zeta \to +\infty$. The principle of that method consists of deforming the integration path into another one passing trough some critical points, ϑ_k, $k \in K$ say, for the function h, at which the real part of h is maximal. For this, we must consider the level curves $\Im(h(u)) = \Im(h(\vartheta_k))$, $k \in K$. For a fixed k, these curves are made of several branches and two branches pass through the critical points ϑ_k (*saddle points*). One of the latter is included in the region $\Re(h(u)) \leqslant \Re(h(\vartheta_k))$ (it has the *steepest descent*) and the other is included in the region $\Re(h(u)) \geqslant \Re(h(\vartheta_k))$ (it has the *steepest ascent*). We shall only retain the branch of steepest descent which we call L_k since the function $\Re(h)$ attains its maximum on it at the sole point ϑ_k. Finally, we deform the real line $(-\infty, +\infty)$ into the union of the lines of steepest descent L_k, $k \in K$ with the aid of Cauchy's theorem, and we replace the function h within the integral on L_k by its Taylor expansion of order two at ϑ_k, as well as the path L_k by a small arc l_k, or the tangent, D_k say, to l_k at ϑ_k (the integral along L_k and D_k are then asymptotically equivalent), the evaluation of the integral of the Taylor expansion on D_k being now easy to carry out.

- Critical points: they are solutions to the equation $h'(u) = 0$, that is to say $u^{n-1} = -\kappa_n (-i)^{n-1}$. They are given by

$$\vartheta_k = \begin{cases} e^{i\frac{2k+1/2}{n-1}\pi} & \text{if } n = 2p, \\ e^{i\frac{2k+1}{n-1}\pi} & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^p, \\ e^{i\frac{2k}{n-1}\pi} & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^{p+1}. \end{cases}$$

We plainly have $h(\vartheta_k) = (1-\frac{1}{n})\,i\vartheta_k$ and $h''(\vartheta_k) = -\frac{n-1}{2\vartheta_k}\,i$. Notice that for $n = 2p+1$ and $\kappa_n = (-1)^{p+1}$, there are real critical points. These latter will make divergent the integral of $|q|$ on $(0,+\infty)$ whereas in the other cases the integral of q on $(0,+\infty)$ turns out to be absolutely convergent.

- Taylor expansion of order two: it is given by

$$\begin{aligned} h(u) &= h(\vartheta_k) + h'(\vartheta_k)(u-\vartheta_k) + \frac{1}{2}h''(\vartheta_k)(u-\vartheta_k)^2 + o((u-\vartheta_k)^2) \\ &= (1-\frac{1}{n})\,i\vartheta_k - \frac{n-1}{2\vartheta_k}\,i(u-\vartheta_k)^2 + o((u-\vartheta_k)^2). \end{aligned}$$

  Setting $u = \vartheta_k + \rho e^{i\varphi}$ and $h''(\vartheta_k) = \rho_k e^{i\varphi_k}$, where $\rho_k = |h''(\vartheta_k)|$ and $\varphi_k = \arg(h''(\vartheta_k))$, this can be rewritten as

$$h(u) = (1-\frac{1}{n})\,i\vartheta_k - \frac{1}{2}\rho_k\rho^2 e^{i(2\varphi+\varphi_k)} + o(\rho^2).$$

- Level curves: it is easy to see that the curve $\Im(h(u)) = \Im(h(\vartheta_k))$ admits the asymptotes whose equation is $\Im((iu)^n) = 0$, which are the lines of polar angles $\frac{k}{n}\,\pi$, $k \in \{0,\dots,2n-1\}$ if n is even and $\frac{2k+1}{2n}\,\pi$, $k \in \{0,\dots,2n-1\}$ if n is odd. On the other hand, this curve has n branches, and both tangents at the saddle point ϑ_k admit $-\frac{1}{2}\varphi_k$ (steepest ascent) and $\frac{1}{2}(\pi-\varphi_k)$ (steepest descent) as polar angles. We used MAPLE to draw the level curves in some particular cases; they are represented in Figures 1, 3, 5 below. We call L_k the branch passing through ϑ_k and having the steepest descent. We have also represented both perpendicular tangents at the critical point on the figures.

- Choice of a new path of integration: the modulus of the function $u \longmapsto e^{\zeta h(u)}$ over a circle of large radius is given by

$$\begin{aligned} |e^{\zeta h(Re^{i\theta})}| &= e^{-R\zeta\sin\theta + \kappa_n\frac{R^n}{n}\zeta\cos(n\theta+n\frac{\pi}{2})} \\ &= \begin{cases} e^{-R\zeta\sin\theta - \frac{R^n}{n}\zeta\cos(n\theta)} & \text{if } n = 2p, \\ e^{-R\zeta\sin\theta - \frac{R^n}{n}\zeta\sin(n\theta)} & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^p, \\ e^{-R\zeta\sin\theta + \frac{R^n}{n}\zeta\sin(n\theta)} & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^{p+1}, \end{cases} \end{aligned}$$

  which is of order $o(\frac{1}{R})$ in the respective angular sectors

$$\theta \in \begin{cases} \displaystyle\bigcup_{j=0}^{n-1}\Big(\frac{2j-\frac{1}{2}}{n}\,\pi, \frac{2j+\frac{1}{2}}{n}\,\pi\Big) & \text{if } n = 2p, \\ \displaystyle\bigcup_{j=0}^{n-1}\Big(\frac{2j}{n}\,\pi, \frac{2j+1}{n}\,\pi\Big) & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^p, \\ \displaystyle\bigcup_{j=0}^{n-1}\Big(\frac{2j-1}{n}\,\pi, \frac{2j}{n}\,\pi\Big) & \text{if } n = 2p+1 \text{ and } \kappa_n = (-1)^{p+1}. \end{cases}$$

  For each k we must choose the branch L_k lying in one of the above unions of asymptotical sectors:

  - if $n = 2p$: both extremities (near infinity) of L_k $(0 \leqslant k \leqslant p-1)$ lie at infinity on the asymptotes of polar angles $\frac{2k}{n}\,\pi$ and $\frac{2k+2}{n}\,\pi$ and both tails lie within the respective sectors $(\frac{4k}{2n}\,\pi, \frac{4k+1}{2n}\,\pi)$ and $(\frac{4k+3}{2n}\,\pi, \frac{4k+4}{2n}\,\pi)$;
  - if $n = 2p+1$:

* when $\kappa_n=(-1)^p$, both extremities of L_k $(0\leqslant k\leqslant p-1)$ lie at infinity on the asymptotes of polar angles $\frac{4k+1}{2n}\,\pi$ and $\frac{4k+5}{2n}\,\pi$ and both tails lie within the sectors $(\frac{4k+1}{2n}\,\pi,\frac{4k+2}{2n}\,\pi)$ and $(\frac{4k+4}{2n}\,\pi,\frac{4k+5}{2n}\,\pi)$;
* when $\kappa_n=(-1)^{p+1}$, both extremities of L_k $(0\leqslant k\leqslant p)$ lie at infinity on the asymptotes of polar angles $\frac{4k-1}{2n}\,\pi$ and $\frac{4k+3}{2n}\,\pi$ and both tails lie within the sectors $(\frac{4k-1}{2n}\,\pi,\frac{4k}{2n}\,\pi)$ and $(\frac{4k+2}{2n}\,\pi,\frac{4k+3}{2n}\,\pi)$.

The deformation of the real line is now possible by using Cauchy's theorem: we set $L=\bigcup_{k\in K}L_k$ where

$$K=\begin{cases}\{0,\dots,p-1\} & \text{if } n=2p,\\ \{0,\dots,p-1\} & \text{if } n=2p+1 \text{ and } \kappa_n=(-1)^p,\\ \{0,\dots,p\} & \text{if } n=2p+1 \text{ and } \kappa_n=(-1)^{p+1}.\end{cases}$$

The integration paths are represented in Figures 2, 4, 6 below; the forbidden sectors are shaded therein. More precisely, the path L goes

- if $n=2p$: from $-\infty$ to $+\infty$ passing successively through ϑ_{p-1}, $e^{i\frac{n-2}{n}\pi}\infty$, ϑ_{p-2}, $e^{i\frac{n-4}{n}\pi}\infty$, …, $e^{i\frac{4}{n}\pi}\infty$, ϑ_2, $e^{i\frac{2}{n}\pi}\infty$, ϑ_1 (see Fig. 4);
- if $n=2p+1$:
  * when $\kappa_n=(-1)^p$, from $e^{i\frac{4p+1}{2n}\pi}\infty$ to $e^{i\frac{1}{2n}\pi}\infty$ passing successively through ϑ_{p-1}, $e^{i\frac{4p-3}{2n}\pi}\infty$, ϑ_{p-2}, $e^{i\frac{4p-7}{2n}\pi}\infty$, …, $e^{i\frac{9}{2n}\pi}\infty$, ϑ_2, $e^{i\frac{5}{2n}\pi}\infty$, ϑ_1 (see Fig. 6);
  * when $\kappa_n=(-1)^{p+1}$, from $e^{i\frac{4p+3}{2n}\pi}\infty$ to $e^{-i\frac{1}{2n}\pi}\infty$ passing successively through ϑ_p, $e^{i\frac{4p-1}{2n}\pi}\infty$, ϑ_{p-1}, $e^{i\frac{4p-5}{2n}\pi}\infty$, …, $e^{i\frac{7}{2n}\pi}\infty$, ϑ_1, $e^{i\frac{3}{2n}\pi}\infty$, ϑ_0 (see Fig. 2).

We have

$$q(\zeta)=\int_L e^{\zeta h(u)}\,du=\sum_{k\in K}\int_{L_k}e^{\zeta h(u)}\,du. \tag{44}$$

- Asymptotic expansion: we have

$$\begin{aligned}\int_{L_k}e^{\zeta h(u)}\,du &\sim e^{(1-\frac{1}{n})\,i\vartheta_k\zeta}\int_{l_k}e^{-\frac{n-1}{2\vartheta_k}\,i\zeta(u-\vartheta_k)^2}\,du\\ &\sim e^{(1-\frac{1}{n})\,i\vartheta_k\zeta}\int_{D_k}e^{-\frac{n-1}{2\vartheta_k}\,i\zeta(u-\vartheta_k)^2}\,du\end{aligned}$$

where l_k is a small arc included in L_k containing ϑ_k and D_k is the tangent of L_k (and l_k) at ϑ_k; the polar angle of D_k is $\frac{3\pi}{4}+\frac{1}{2}\arg\vartheta_k$. Finally, we plainly have

$$\int_{D_k}e^{-\frac{n-1}{2\vartheta_k}\,i\zeta(u-\vartheta_k)^2}\,du=-i\sqrt{\frac{2\pi}{n-1}}\,\frac{1}{\sqrt{\zeta}}\,(i\vartheta_k)^{1/2}$$

where $(i\vartheta_k)^{1/2}$ is the square root whose argument lies in the angular sector $(\frac{\pi}{4},\frac{3\pi}{4})$.

As a result, we have obtained an asymptotical expansion for q which is displayed in the following Proposition.

Proposition 22 *As $\zeta \to +\infty$,*

$$q(\zeta) \sim -i\sqrt{\frac{2\pi}{n-1}}\,\frac{1}{\sqrt{\zeta}}\sum_{k\in K}(i\vartheta_k)^{1/2}\,e^{(1-\frac{1}{n})\,i\vartheta_k\zeta}.$$

Proposition 2 immediately ensues.

B The derivatives of p and Φ

Proposition 23 *We have*

$$\frac{\partial^j p}{\partial z^j}(t;z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(iu)^j\,e^{izu+\kappa_n t(iu)^n}\,du \quad \text{if } j \leqslant n-2, \tag{45}$$

$$\frac{\partial^{n-1} p}{\partial z^{n-1}}(t;z) = -\kappa_n\,\frac{z}{nt}\,p(t;z). \tag{46}$$

Proof. Performing an integration by parts gives

$$\begin{aligned}
2\pi\,p(t;z) &= \int_{-\infty}^{+\infty} e^{izu+\kappa_n t(iu)^n}\,du \\
&= \int_{-1}^{1} e^{izu+\kappa_n t(iu)^n}\,du + \frac{\kappa_n}{nit}\int_{|u|>1}\frac{e^{izu}}{(iu)^{n-1}}\,d(e^{\kappa_n t(iu)^n}) \\
&= \int_{-1}^{1} e^{izu+\kappa_n t(iu)^n}\,du - \frac{\kappa_n}{nit}\Bigg\{\Big[\frac{1}{(iu)^{n-1}}\,e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1} \\
&\qquad +i\int_{|u|>1}\Big(\frac{z}{(iu)^{n-1}} - \frac{n-1}{(iu)^n}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\Bigg\}.
\end{aligned}$$

We can now calculate the j^{th} derivative without difficulty:

$$\begin{aligned}
2\pi\,\frac{\partial^j p}{\partial z^j}(t;z) &= \int_{-1}^{1}(iu)^j\,e^{izu+\kappa_n t(iu)^n}\,du - \frac{\kappa_n}{nit}\Bigg\{\Big[\frac{1}{(iu)^{n-j-1}}\,e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1} \\
&\qquad +i\int_{|u|>1}\Big(\frac{z}{(iu)^{n-j-1}} - \frac{(n-j-1)}{(iu)^{n-j}}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\Bigg\}
\end{aligned} \tag{47}$$

which is clearly valid for $j \leqslant n-3$ since the last integral is absolutely convergent. Next, a backward integration by parts yields

$$\begin{aligned}
2\pi\,\frac{\partial^j p}{\partial z^j}(t;z) &= \int_{-1}^{1}(iu)^j\,e^{izu+\kappa_n t(iu)^n}\,du + \frac{\kappa_n}{nit}\int_{|u|>1}\frac{1}{(iu)^{n-j-1}}\,e^{izu}\,d(e^{\kappa_n t(iu)^n}) \\
&= \int_{-\infty}^{+\infty}(iu)^j\,e^{izu+\kappa_n t(iu)^n}\,du.
\end{aligned}$$

This proves (45) for $j \leqslant n-3$. For $j = n-3$ formula (47) reads

$$2\pi\,\frac{\partial^{n-3} p}{\partial z^{n-3}}(t;z) = \int_{-1}^{1}(iu)^{n-3}\,e^{izu+\kappa_n t(iu)^n}\,du$$

$$
-\frac{\kappa_n}{nit}\left\{\Big[\frac{1}{(iu)^2}\,e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1}\right.
$$

$$
\left.+i\int_{|u|>1}\Big(\frac{z}{(iu)^2}-\frac{2}{(iu)^3}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\right\}.
$$

The last integral can not directly be differentiated, so it needs to be transformed through another integration by parts:

$$
\begin{aligned}
&\int_{|u|>1}\Big(\frac{z}{(iu)^2}-\frac{2}{(iu)^3}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\\
&\quad = -\frac{\kappa_n}{nit}\left\{\Big[\Big(\frac{z}{(iu)^{n+1}}-\frac{2}{(iu)^{n+2}}\Big)\,e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1}\right.\\
&\qquad \left.+i\int_{|u|>1}\Big(\frac{z^2}{(iu)^{n+1}}-(n+3)\,\frac{z}{(iu)^{n+2}}+(n+2)\,\frac{2}{(iu)^{n+3}}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\right\}.
\end{aligned}
$$

The new integral can be differentiated without problem and (47) holds also for $j=n-2$. Next, a backward integration by parts proves that (45) is valid for $j=n-2$.

Applying (47) to $j=n-2$ gives

$$
\begin{aligned}
2\pi\,\frac{\partial^{n-2}p}{\partial z^{n-2}}(t;z) &= \int_{-1}^{1}(iu)^{n-2}\,e^{izu+\kappa_n t(iu)^n}\,du-\frac{\kappa_n}{nit}\left\{\Big[\frac{1}{iu}\,e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1}\right.\\
&\quad \left.+i\int_{|u|>1}\Big(\frac{z}{iu}-\frac{1}{(iu)^2}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\right\}. \qquad (48)
\end{aligned}
$$

But

$$
\begin{aligned}
&\int_{|u|>1}\Big(\frac{z}{iu}-\frac{1}{(iu)^2}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\\
&\quad = -\frac{\kappa_n}{nit}\left\{\Big[\Big(\frac{z}{(iu)^{n}}-\frac{1}{(iu)^{n+1}}\Big)\,e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1}\right.\\
&\qquad \left.+i\int_{|u|>1}\Big(\frac{z^2}{(iu)^{n}}-(n+1)\,\frac{z}{(iu)^{n+1}}+(n+1)\,\frac{1}{(iu)^{n+2}}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\right\},
\end{aligned}
$$

the derivative of this integral is then

$$
\begin{aligned}
&-\frac{\kappa_n}{nit}\left\{\Big[\frac{z}{(iu)^{n-1}}\,e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1}+i\int_{|u|>1}\Big(\frac{z^2}{(iu)^{n-1}}-(n-1)\,\frac{z}{(iu)^{n}}\Big)\,e^{izu+\kappa_n t(iu)^n}\,du\right\}\\
&\quad = -\frac{\kappa_n z}{nit}\left\{\Big[\frac{1}{(iu)^{n-1}}\,e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1}+\int_{|u|>1}e^{\kappa_n t(iu)^n}\,d\Big(\frac{e^{izu}}{(iu)^{n-1}}\Big)\right\}\\
&\quad = \frac{\kappa_n z}{nit}\int_{|u|>1}\frac{e^{izu}}{(iu)^{n-1}}\,d(e^{\kappa_n t(iu)^n})\\
&\quad = z\int_{|u|>1}e^{izu+\kappa_n t(iu)^n}\,du.
\end{aligned}
$$

Hence, the derivative of (48) is

$$
2\pi\,\frac{\partial^{n-1}p}{\partial z^{n-1}}(t;z) = \int_{-1}^{1}(iu)^{n-1}\,e^{izu+\kappa_n t(iu)^n}\,du-\frac{\kappa_n}{nit}\left\{\Big[e^{izu+\kappa_n t(iu)^n}\Big]_{u=-1}^{u=1}\right.
$$

$$+iz\int_{|u|>1} e^{izu+\kappa_n t(iu)^n}\,du\Bigg\}.$$

We finally observe that

$$\int_{-1}^{1}(iu)^{n-1}\,e^{izu+\kappa_n t(iu)^n}\,du=\frac{\kappa_n}{nit}\left\{\left[e^{izu+\kappa_n t(iu)^n}\right]_{u=-1}^{u=1}-iz\int_{-1}^{1}e^{izu+\kappa_n t(iu)^n}\,du\right\}$$

and therefore

$$2\pi\,\frac{\partial^{n-1}p}{\partial z^{n-1}}(t;z)=-\kappa_n\,\frac{z}{nt}\int_{-\infty}^{+\infty}e^{izu+\kappa_n t(iu)^n}\,du=-\kappa_n\,\frac{z}{nt}\,p(t;z)$$

which ends up the proof of Proposition 23. □

Lemma 24 *We have*

$$\Phi(\lambda;z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{izu}}{\lambda-\kappa_n(iu)^n}\,du. \tag{49}$$

Proof. We should have, at least formally,

$$\begin{aligned}\Phi(\lambda;z)&=\frac{1}{2\pi}\int_0^{+\infty}e^{-\lambda t}\,dt\int_{-\infty}^{+\infty}e^{izu+\kappa_n t(iu)^n}\,du\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{izu}\,du\int_0^{+\infty}e^{-(\lambda-\kappa_n(iu)^n)t}\,dt\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{izu}}{\lambda-\kappa_n(iu)^n}\,du.\end{aligned}$$

However the double integral is not absolutely convergent and Fubini's theorem can not directly apply. So we rewrite p as follows:

$$\begin{aligned}2\pi\,p(t;z)&=\int_{-\infty}^{+\infty}e^{izu+\kappa_n t(iu)^n}\,du\\&=\int_{|u|\leqslant(z/t)^{\frac{1}{n-1}}}e^{izu+\kappa_n t(iu)^n}\,du+\int_{|u|>(z/t)^{\frac{1}{n-1}}}e^{izu+\kappa_n t(iu)^n}\,du.\end{aligned}$$

Performing an integration by parts on the second integral of the last equality yields

$$\begin{aligned}\int_{|u|>(z/t)^{\frac{1}{n-1}}}e^{izu+\kappa_n t(iu)^n}\,du&=\frac{\kappa_n}{nit}\int_{|u|>(z/t)^{\frac{1}{n-1}}}\frac{e^{izu}}{(iu)^{n-1}}\,d(e^{\kappa_n t(iu)^n})\\&=-\frac{\kappa_n}{nit}\Bigg\{\left[\frac{1}{(iu)^{n-1}}\,e^{izu+\kappa_n t(iu)^n}\right]_{u=-(z/t)^{\frac{1}{n-1}}}^{u=(z/t)^{\frac{1}{n-1}}}\\&\qquad+\int_{|u|>(z/t)^{\frac{1}{n-1}}}e^{\kappa_n t(iu)^n}\,d\Big(\frac{e^{izu}}{(iu)^{n-1}}\Big)\Bigg\}.\end{aligned}$$

Therefore

$$2\pi\,\Phi(\lambda;z)=I_1-\frac{\kappa_n}{ni}\,(I_2+I_3) \tag{50}$$

where we put

$$
\begin{aligned}
I_1 &= \int_0^{+\infty} e^{-\lambda t}\,dt \int_{|u|\leqslant (z/t)^{\frac{1}{n-1}}} e^{izu+\kappa_n t(iu)^n}\,du,\\
I_2 &= \int_0^{+\infty} \left[\frac{1}{(iu)^{n-1}}\,e^{izu+\kappa_n t(iu)^n}\right]_{u=-(z/t)^{\frac{1}{n-1}}}^{u=(z/t)^{\frac{1}{n-1}}} e^{-\lambda t}\,\frac{dt}{t},\\
I_3 &= \int_0^{+\infty} e^{-\lambda t}\,\frac{dt}{t} \int_{|u|>(z/t)^{\frac{1}{n-1}}} e^{\kappa_n t(iu)^n}\,d\Big(\frac{e^{izu}}{(iu)^{n-1}}\Big).
\end{aligned}
$$

Now, Fubini's theorem applies to both double integrals I_1 and I_3 and then

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{+\infty} e^{izu}\,du \int_0^{z/|u|^{n-1}} e^{-(\lambda-\kappa_n(iu)^n)t}\,dt,\\
I_3 &= \int_{-\infty}^{+\infty} d\Big(\frac{e^{izu}}{(iu)^{n-1}}\Big) \int_{z/|u|^{n-1}}^{+\infty} e^{-(\lambda-\kappa_n(iu)^n)t}\,\frac{dt}{t}.
\end{aligned}
$$

Another integration by parts leads to

$$
\begin{aligned}
I_3 &= \left[\frac{e^{izu}}{(iu)^{n-1}} \int_{z/|u|^{n-1}}^{+\infty} e^{-(\lambda-\kappa_n(iu)^n)t}\,\frac{dt}{t}\right]_{u=-\infty}^{u=+\infty}\\
&\quad -\int_{-\infty}^{+\infty} e^{izu}\left[n\kappa_n i \int_{z/|u|^{n-1}}^{+\infty} e^{-(\lambda-\kappa_n(iu)^n)t}\,dt\right.\\
&\quad \left. -\frac{(n-1)i}{(iu)^n}\,e^{-(\lambda-\kappa_n(iu)^n)\,z/|u|^{n-1}}\right] du\\
&= -(n\kappa_n i)\,I_4 - I_5 \qquad (51)
\end{aligned}
$$

where

$$
\begin{aligned}
I_4 &= \int_{-\infty}^{+\infty} e^{izu} \int_{z/|u|^{n-1}}^{+\infty} e^{-(\lambda-\kappa_n(iu)^n)t}\,dt,\\
I_5 &= (n-1)i \int_{-\infty}^{+\infty} e^{izu-(\lambda-\kappa_n(iu)^n)\,z/|u|^{n-1}}\,\frac{du}{(iu)^n}.
\end{aligned}
$$

Using the change of variables $t = z/|u|^{n-1}$ it is easily seen that $I_5 = I_2$ and then, by (50) and (51),

$$
2\pi\,\Phi(\lambda;z) = I_1 + I_4 = \int_{-\infty}^{+\infty} \frac{e^{izu}}{\lambda-\kappa_n(iu)^n}\,du.
$$

□

Lemma 25 *We have*

$$
\frac{\partial^j\Phi}{\partial z^j}(\lambda;z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{(iu)^j\,e^{izu}}{\lambda-\kappa_n(iu)^n}\,du \qquad (52)
$$

for $j \leqslant n-2$ or [$j = n-1$ and $z \neq 0$].

Proof. Differentiating (49) leads plainly to (52) for $j \leqslant n-2$ since the integral therein is absolutely convergent. The $(n-1)^{\text{th}}$ derivative is more difficult to obtain. We invoke once again some integrations by parts. For $z \neq 0$,

$$
2\pi\,\frac{\partial^{n-2}\Phi}{\partial z^{n-2}}(\lambda;z)
$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \frac{(iu)^{n-2}\, e^{izu}}{\lambda - \kappa_n (iu)^n}\, du \\
&= \frac{1}{iz} \int_{-\infty}^{+\infty} \frac{(iu)^{n-2}}{\lambda - \kappa_n (iu)^n}\, d(e^{izu}) \\
&= -\frac{1}{iz} \int_{-\infty}^{+\infty} e^{izu}\, d\Big(\frac{(iu)^{n-2}}{\lambda - \kappa_n (iu)^n}\Big) \\
&= -\frac{1}{z} \int_{-\infty}^{+\infty} e^{izu} \Big((n-2)\, \frac{(iu)^{n-3}}{\lambda - \kappa_n (iu)^n} + n\kappa_n\, \frac{(iu)^{2n-3}}{[\lambda - \kappa_n (iu)^n]^2}\Big)\, du.
\end{aligned}
$$

That last integral can now be differentiated according as

$$
\begin{aligned}
&2\pi\, \frac{\partial^{n-1}\Phi}{\partial z^{n-1}}(\lambda; z) \\
&\quad= \frac{1}{z^2} \int_{-\infty}^{+\infty} e^{izu} \Big((n-2)\, \frac{(iu)^{n-3}}{\lambda - \kappa_n (iu)^n} + n\kappa_n\, \frac{(iu)^{2n-3}}{[\lambda - \kappa_n (iu)^n]^2}\Big)\, du \\
&\qquad -\frac{1}{z} \int_{-\infty}^{+\infty} e^{izu} \Big((n-2)\, \frac{(iu)^{n-2}}{\lambda - \kappa_n (iu)^n} + n\kappa_n\, \frac{(iu)^{2n-2}}{[\lambda - \kappa_n (iu)^n]^2}\Big)\, du \\
&\quad= \frac{1}{iz^2} \int_{-\infty}^{+\infty} e^{izu}\, d\Big(\frac{(iu)^{n-2}}{\lambda - \kappa_n (iu)^n}\Big) \\
&\qquad -\frac{1}{z} \int_{-\infty}^{+\infty} e^{izu} \Big((n-2)\, \frac{(iu)^{n-2}}{\lambda - \kappa_n (iu)^n} + n\kappa_n\, \frac{(iu)^{2n-2}}{[\lambda - \kappa_n (iu)^n]^2}\Big)\, du \\
&\quad= -\frac{1}{z} \int_{-\infty}^{+\infty} e^{izu}\, \frac{(iu)^{n-2}}{\lambda - \kappa_n (iu)^n}\, du \\
&\qquad -\frac{1}{z} \int_{-\infty}^{+\infty} e^{izu} \Big((n-2)\, \frac{(iu)^{n-2}}{\lambda - \kappa_n (iu)^n} + n\kappa_n\, \frac{(iu)^{2n-2}}{[\lambda - \kappa_n (iu)^n]^2}\Big)\, du \\
&\quad= -\frac{1}{z} \int_{-\infty}^{+\infty} e^{izu} \Big((n-1)\, \frac{(iu)^{n-2}}{\lambda - \kappa_n (iu)^n} + n\kappa_n\, \frac{(iu)^{2n-2}}{[\lambda - \kappa_n (iu)^n]^2}\Big)\, du \\
&\quad= -\frac{1}{iz} \int_{-\infty}^{+\infty} e^{izu}\, d\Big(\frac{(iu)^{n-1}}{\lambda - \kappa_n (iu)^n}\Big) \\
&\quad= \int_{-\infty}^{+\infty} e^{izu}\, \frac{(iu)^{n-1}}{\lambda - \kappa_n (iu)^n}\, du.
\end{aligned}
$$

Hence formula (52) is also valid in the case $j = n-1$ for $z \neq 0$. □

Proposition 26 *For $j \leqslant n$:*

$$
\frac{\partial^j \Phi}{\partial z^j}(\lambda; z) = \int_0^{+\infty} e^{-\lambda t}\, \frac{\partial^j p}{\partial z^j}(t; z)\, dt. \tag{53}
$$

Proof. By (52), for $j \leqslant n-2$,

$$
\frac{\partial^j \Phi}{\partial z^j}(\lambda; z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (iu)^j\, e^{izu}\, du \int_0^{+\infty} e^{-(\lambda - \kappa_n (iu)^n)t}\, dt.
$$

It may be shown using a method similar to that of deriving (49) that, for $j \leqslant n-2$,

$$
\frac{\partial^j \Phi}{\partial z^j}(\lambda; z) = \frac{1}{2\pi} \int_0^{+\infty} e^{-\lambda t}\, dt \int_{-\infty}^{+\infty} (iu)^j\, e^{izu + \kappa_n t (iu)^n}\, du
$$

which proves (53) thanks to (45). Actually, formula holds for $j \leqslant n$ as it may be seen by using several integrations by parts. $\square$

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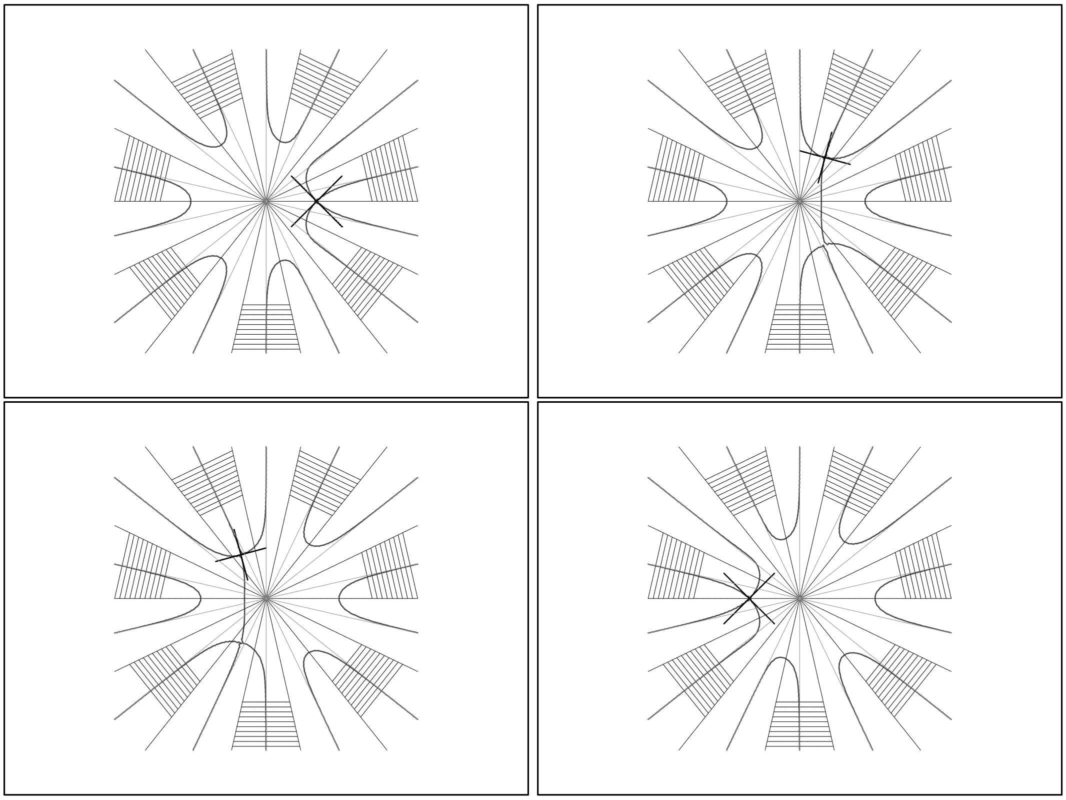

Figure 1: The level curves $\Im(h^+(u)) = \Im(h^+(\vartheta_k))$ for $n = 7$

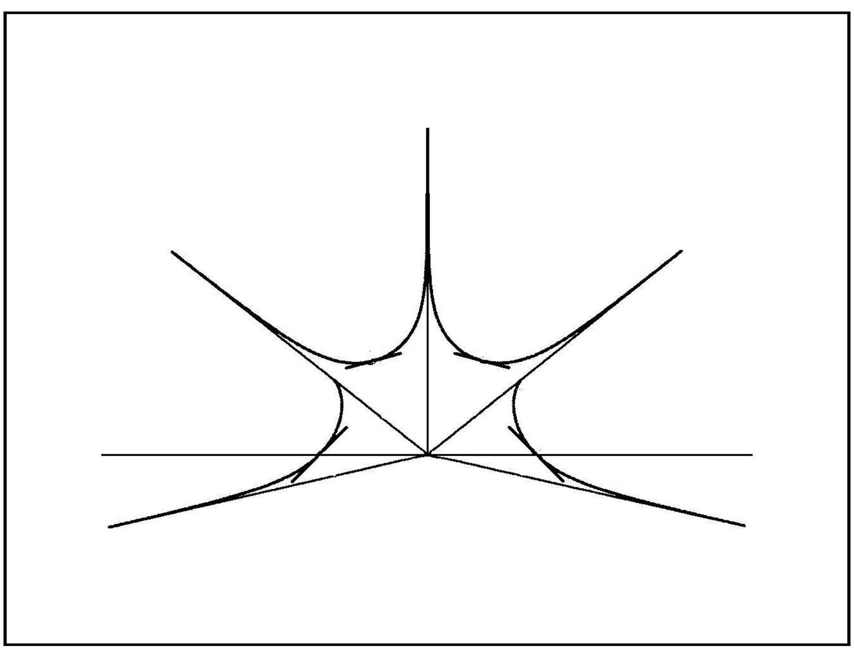

Figure 2: Path of integration for $n = 7$ and $\kappa_n = +1$

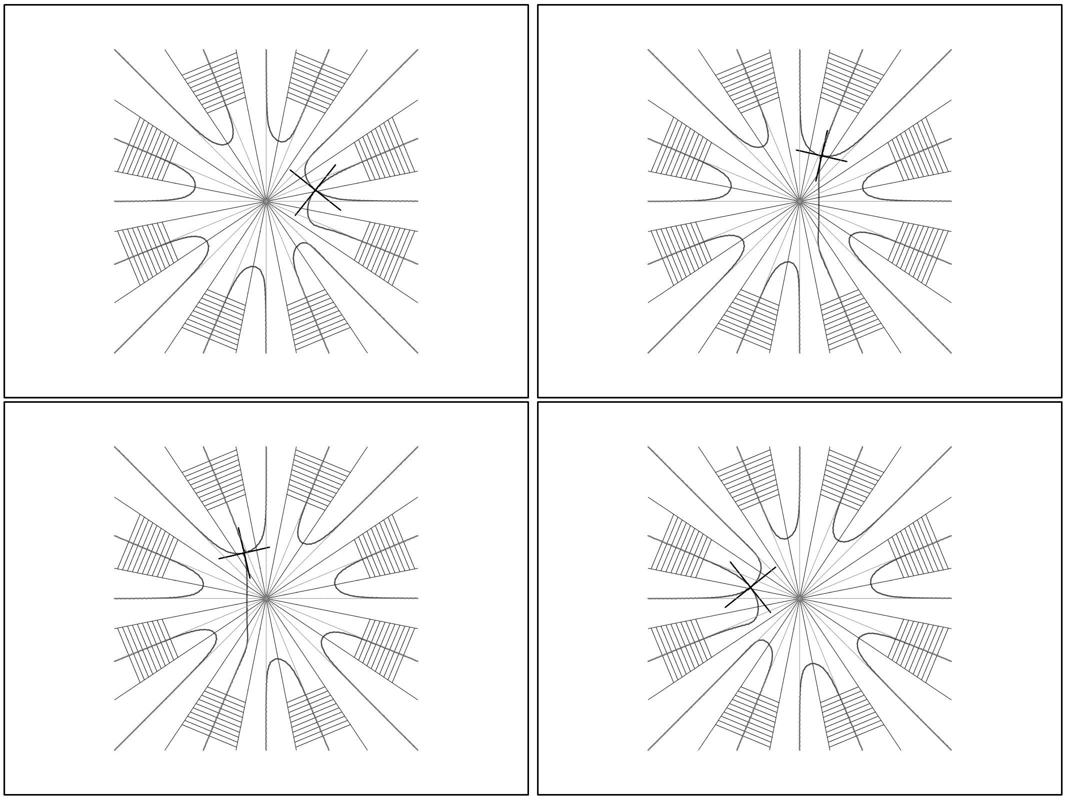

Figure 3: The level curves $\Im(h(u)) = \Im(h(\vartheta_k))$ for $n = 8$

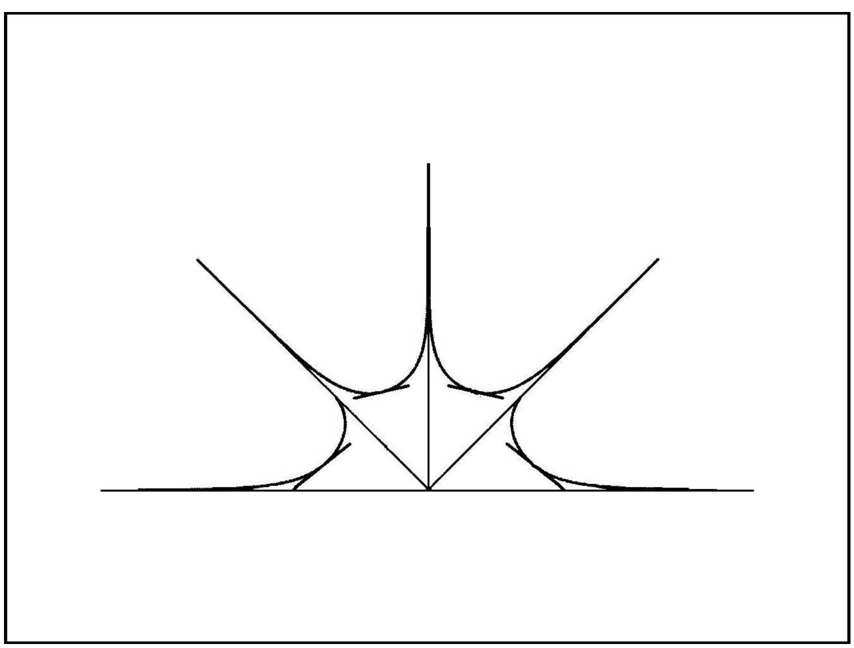

Figure 4: Path of integration for $n = 8$

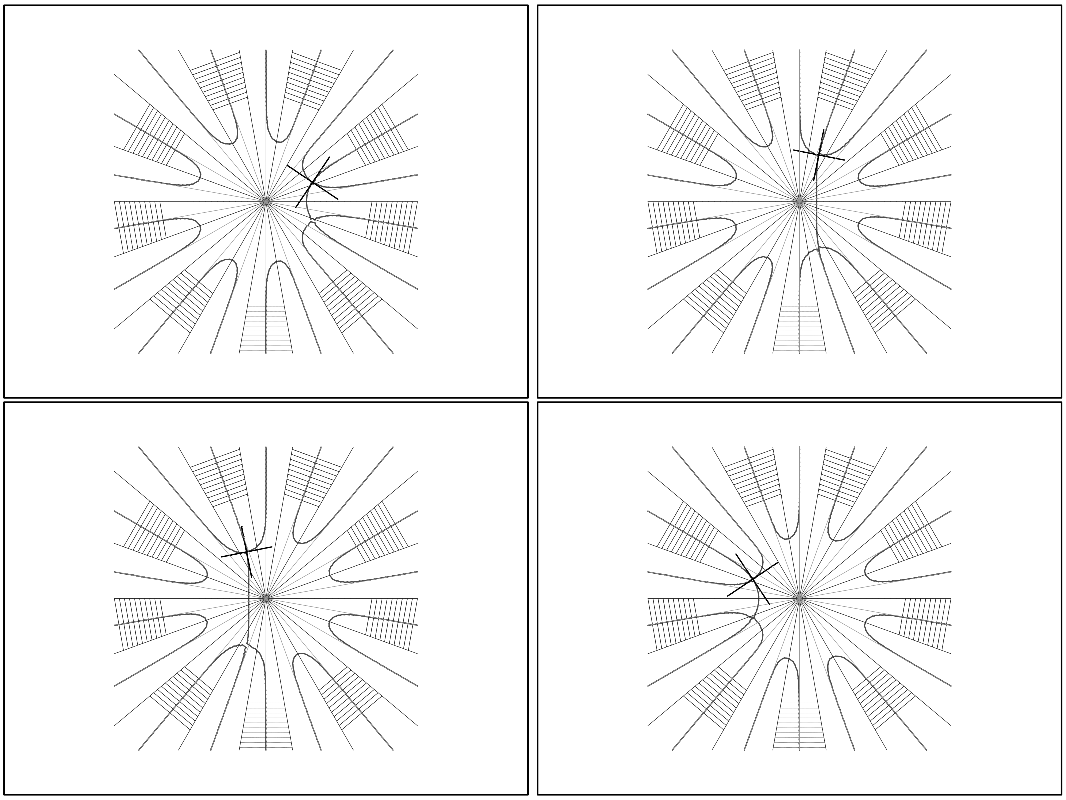

Figure 5: The level curves $\Im(h^{-}(u)) = \Im(h^{-}(\vartheta_k))$ for $n = 9$

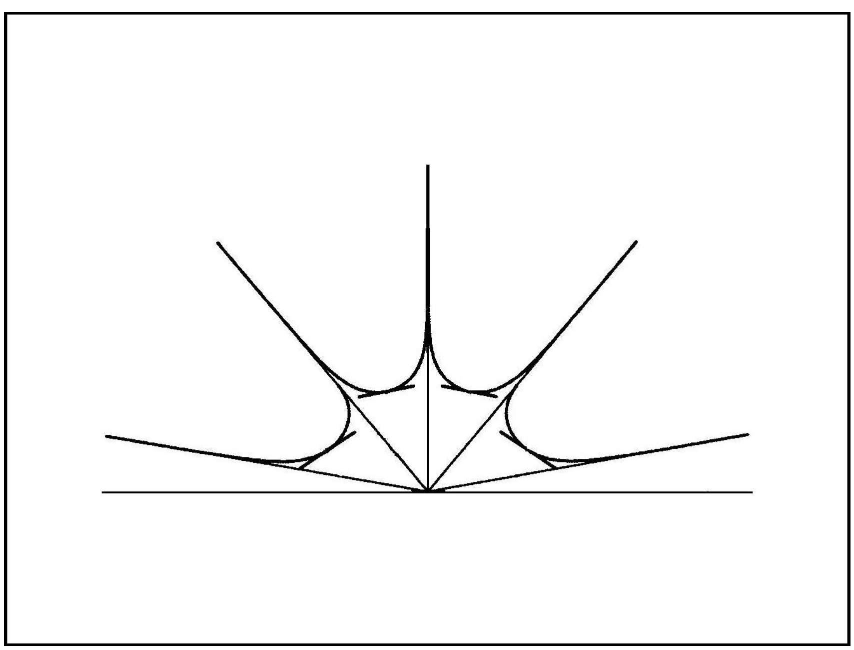

Figure 6: Path of integration for $n = 9$ and $\kappa_n = +1$