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Three upsilon transforms related to tempered stable distributions

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Abstract

We discuss the properties of three upsilon transforms, which are related to the class of p-tempered α-stable (TS^p_α) distributions. In particular, we characterize their domains and show how they can be represented as compositions of each other. Further, we show that if $-\infty < \beta < \alpha < 2$ and $0 < q < p < \infty$ then they can be used to transform the Lévy measures of TS^p_β distributions into those of TS^q_α.

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1 Introduction

Over the past decade there has been considerable interest in the study of transforms of Lévy measures, especially upsilon transforms, which are closely related to stochastic integration with respect to Lévy processes. Although upsilon transforms were formally defined in [2], the concept goes back, at least, to [7]. In this paper, we study the properties of three upsilon transforms, which are related to tempered stable distributions.

Let $\mathfrak{M}_{\sigma f}$ be the collection of all σ-finite Borel measures on $\mathbb{R}^d$ such that every $M \in \mathfrak{M}_{\sigma f}$ satisfies $M(\{0\}) = 0$, and let ρ be a nonzero σ-finite Borel measure on $(0, \infty)$. A mapping $\Upsilon_\rho : \mathfrak{M}_{\sigma f} \mapsto \mathfrak{M}_{\sigma f}$ is called an upsilon transform with dilation measure ρ if, for any $M \in \mathfrak{M}_{\sigma f}$, we have

$$[\Upsilon_\rho M](B) = \int_0^\infty M(s^{-1}B)\rho(\mathrm{d}s), \quad B \in \mathfrak{B}(\mathbb{R}^d), \tag{1.1}$$

where $\mathfrak{B}(\mathbb{R}^d)$ refers to the Borel sets in $\mathbb{R}^d$.

We are particularly interested in the case when $\Upsilon_\rho M$ is a Lévy measure. Recall that a Borel measure $M \in \mathfrak{M}_{\sigma f}$ is called a Lévy measure if

$$M(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} \left(1 \wedge |x|^2\right) M(\mathrm{d}x) < \infty. \tag{1.2}$$

Theorem 3.2 in [2] tells us that if $\Upsilon_\rho M$ is a Lévy measure then, necessarily, M is a Lévy measure as well. However, $\Upsilon_\rho M$ need not be a Lévy measure even if M is. We write $\mathfrak{D}(\Upsilon_\rho)$ to denote the collection of all Lévy measures for which $\Upsilon_\rho M$ remains a

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Lévy measure. This is called the domain of Υ_ρ. Further, we write $\mathfrak{R}(\Upsilon_\rho)$ to denote the collection of all Lévy measures M' for which there exists an $M \in \mathfrak{D}(\Upsilon_\rho)$ with $M' = \Upsilon_\rho M$. This is called the range of Υ_ρ.

A probabilistic interpretation of Υ_ρ is given in [2]. Specifically, for an upsilon transform Υ_ρ let $\eta_\rho(t) = \rho([t,\infty))$ for $t > 0$. Assume that $\eta_\rho(t) < \infty$ for each $t > 0$ and let η_ρ^* be the inverse of η_ρ in the sense $\eta_\rho^*(t) = \inf\{s > 0 : \eta_\rho(s) \le t\}$ for $t > 0$. Let $\{X_t : t \ge 0\}$ be a Lévy process such that the distribution of X_1 has Lévy measure M. Define (if possible) the stochastic integral

$$Y = \int_0^{\eta_\rho(0)} \eta_\rho^*(t) \mathrm{d}X_t$$

in the sense of [17]. If the integral exists then $M \in \mathfrak{D}(\Upsilon_\rho)$ and the distribution of Y is infinitely divisible with Lévy measure $\Upsilon_\rho M$. However, even if $M \in \mathfrak{D}(\Upsilon_\rho)$ this does not guarantee that the integral exists since we must be careful with the Gaussian part and the shift. See Theorem 3.5 in [17] for the exact conditions under which the stochastic integral exists.

In this paper we focus on upsilon transforms with the following dilation measures:

1. For $\alpha \in \mathbb{R}$ and $p > 0$ let

$$\psi_{\alpha,p}(\mathrm{d}s) = s^{-\alpha-1} e^{-s^p} 1_{s>0} \mathrm{d}s. \tag{1.3}$$

2. For $-\infty < \beta < \alpha < \infty$ and $p > 0$ let

$$\tau_{\beta\to\alpha,p}(\mathrm{d}s) = \frac{1}{K_{\alpha,\beta,p}} s^{-\alpha-1}(1-s^p)^{\frac{\alpha-\beta}{p}-1} 1_{0<s<1} \mathrm{d}s, \tag{1.4}$$

where

$$K_{\alpha,\beta,p} = \int_0^\infty u^{\alpha-\beta-1} e^{-u^p} \mathrm{d}u = p^{-1}\Gamma\left(\frac{\alpha-\beta}{p}\right).$$

3. For $0 < q < p < \infty$ and $\alpha \in \mathbb{R}$ let

$$\pi_{\alpha,p\to q}(\mathrm{d}s) = p f_{q/p}(s^{-p}) s^{-\alpha-p-1} 1_{s>0} \mathrm{d}s, \tag{1.5}$$

where, for $r \in (0,1)$, f_r is the density of a fully right skewed r-stable distribution with Laplace transform

$$\int_0^\infty e^{-tx} f_r(x) \mathrm{d}x = e^{-t^r}. \tag{1.6}$$

For simplicity of notation we write

$$\Psi_{\alpha,p} = \Upsilon_{\psi_{\alpha,p}},\ \mathfrak{T}_{\beta\to\alpha,p} = \Upsilon_{\tau_{\beta\to\alpha,p}},\ \text{and } \mathfrak{P}_{\alpha,p\to q} = \Upsilon_{\pi_{\alpha,p\to q}}.$$

The transform $\Psi_{\alpha,p}$ was first introduced in [10] and then further studied in [11] and [12]. Several important subclasses were studied in [16], [1], and [8]. The transform, $\mathfrak{T}_{\alpha\to\beta,p}$ was discussed in Section 4 of [12], and the case where $p = 1$ was considered in [16], [18], and [9]. The transform $\mathfrak{P}_{\alpha,p\to q}$ is essentially new, although it appears, implicitly, in [4]. Further, a related transform is studied in [3].

The transform $\Psi_{\alpha,p}$ is closely related to the class of tempered stable distributions, which is a class of models that is obtained by modifying the tails of infinite variance stable distributions to make them lighter. These models were first introduced in [14]. The more general class of p-tempered α-stable distributions (TS_α^p), where $p > 0$ and $\alpha < 2$, was introduced in [4] as a class of infinitely divisible distributions with no Gaussian part and a Lévy measure of the form $\Psi_{\alpha,p}M$, where $M \in \mathfrak{D}(\Psi_{\alpha,p})$. If we allow these distributions to have a Gaussian part then we get the class of models studied in [10]. This, in turn, contains important subclasses including the Thorin class, the Goldie-Steutel-Bondesson class, the class of type M distributions, and the class of generalized type G distributions. For more about tempered stable distributions and their use in a variety of application areas see [13], [5], [6], and the references therein.

The relationship between tempered stable distributions and the transforms $\mathfrak{T}_{\beta\to\alpha,p}$ and $\mathfrak{P}_{\alpha,p\to q}$ will become apparent from studying the relationships among the transforms. Several such relationships are known. Specifically, if $-\infty < \gamma < \beta < \alpha < \infty$ then Theorem 3.1 in [16] (see also [9]) implies that

$$\Psi_{\alpha,1} = \mathfrak{T}_{\beta\to\alpha,1}\Psi_{\beta,1}$$

and Theorem 4.7 in [18] implies that

$$\mathfrak{T}_{\gamma\to\alpha,1} = \mathfrak{T}_{\beta\to\alpha,1}\mathfrak{T}_{\gamma\to\beta,1}.$$

We will show that these relations hold with 1 replaced by any $p > 0$. Further, we show that if $0 < r < q < p < \infty$ and $\alpha \in \mathbb{R}$ then

$$\Psi_{\alpha,q} = \mathfrak{P}_{\alpha,p\to q}\Psi_{\alpha,p}$$

and

$$\mathfrak{P}_{\alpha,p\to r} = \mathfrak{P}_{\alpha,q\to r}\mathfrak{P}_{\alpha,p\to q}.$$

Putting these together implies that if $-\infty < \beta < \alpha < \infty$ and $0 < q < p < \infty$ then

$$\Psi_{\alpha,q} = \mathfrak{P}_{\alpha,p\to q}\mathfrak{T}_{\beta\to\alpha,p}\Psi_{\beta,p} = \mathfrak{T}_{\beta\to\alpha,q}\mathfrak{P}_{\beta,p\to q}\Psi_{\beta,p}.$$

Thus we can transform $\Psi_{\beta,p}$ into $\Psi_{\alpha,q}$ by using the other two transforms. In the context of tempered stable distributions this means that we can transform the Lévy measures of TS^p_β distributions into those of TS^q_α.

2 Main Results

We begin by characterizing the domains of the transforms of interest. Toward this end we introduce some notation. For $\alpha \in [0,2]$ let $\mathfrak{M}^\alpha$ be the class of Borel measures on $\mathbb{R}^d$ such that $M \in \mathfrak{M}^\alpha$ if and only if

$$M(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} \left(|x|^2 \wedge |x|^\alpha\right) M(\mathrm{d}x) < \infty. \tag{2.1}$$

Note that if $0 < \alpha_1 < \alpha_2 < 2$ then $\mathfrak{M}^2 \subsetneq \mathfrak{M}^{\alpha_2} \subsetneq \mathfrak{M}^{\alpha_1} \subsetneq \mathfrak{M}^0$, and that $\mathfrak{M}^0$ is the class of all Lévy measures on $\mathbb{R}^d$. Let $\mathfrak{M}^{\log}$ be the subclass of $\mathfrak{M}^0$ such that $M \in \mathfrak{M}^{\log}$ satisfies $\int_{|x|\le 1} |x|^2 M(\mathrm{d}x) + \int_{|x|>1} \log|x| M(\mathrm{d}x) < \infty$.

Theorem 2.1. *1. For $-\infty < \beta < \alpha < \infty$ and $p > 0$ we have*

$$\mathfrak{D}(\mathfrak{T}_{\beta\to\alpha,p}) = \mathfrak{D}(\Psi_{\alpha,p}) = \begin{cases} \mathfrak{M}^0 & \text{if } \alpha < 0 \\ \mathfrak{M}^{\log} & \text{if } \alpha = 0 \\ \mathfrak{M}^\alpha & \text{if } \alpha \in (0,2) \\ \{0\} & \text{if } \alpha \ge 2 \end{cases}.$$

2. For $\alpha \in \mathbb{R}$ and $0 < q < p < \infty$ we have

$$\mathfrak{D}(\mathfrak{P}_{\alpha,p\to q}) = \begin{cases} \mathfrak{M}^0 & \text{if } \alpha - q < 0 \\ \mathfrak{M}^{\log} & \text{if } \alpha - q = 0 \\ \mathfrak{M}^{\alpha-q} & \text{if } \alpha - q \in (0,2) \\ \{0\} & \text{if } \alpha - q \ge 2 \end{cases}.$$

The proof follows from a general result and is given in Section 4. We note that $\mathfrak{D}(\Psi_{\alpha,p})$ was already fully characterized in [10]. Henceforth, we assume that $\alpha < 2$ in the case of $\mathfrak{T}_{\beta\to\alpha,p}$ and $\Psi_{\alpha,p}$ and that $\alpha < 2+q$ in the case of $\mathfrak{P}_{\alpha,p\to q}$. Of course, our results will, trivially, remain true for the other cases.

We now turn to the composition of transforms. Let $\Upsilon_{\rho_1}, \Upsilon_{\rho_2}$ be two upsilon transforms and define the composition $\Upsilon_{\rho_2}\Upsilon_{\rho_1}$ on the domain

$$\mathfrak{D}(\Upsilon_{\rho_2}\Upsilon_{\rho_1}) = \{M \in \mathfrak{D}(\Upsilon_{\rho_1}) : \Upsilon_{\rho_1}M \in \mathfrak{D}(\Upsilon_{\rho_2})\}.$$

Proposition 4.1 in [2] tells us that $\mathfrak{D}(\Upsilon_{\rho_2}\Upsilon_{\rho_1}) = \mathfrak{D}(\Upsilon_{\rho_1}\Upsilon_{\rho_2})$ and that

$$\Upsilon_{\rho_2}\Upsilon_{\rho_1} = \Upsilon_{\rho_1}\Upsilon_{\rho_2}.$$

Thus compositions of upsilon transforms commute. To give a better understanding of the domains of compositions we give the following.

Lemma 2.2. *If $-\infty < \gamma < \beta < \alpha < \infty$ and $0 < r < q < p < \infty$ then*

$$\mathfrak{R}(\mathfrak{T}_{\beta\to\alpha,p}) \subset \mathfrak{D}(\mathfrak{T}_{\gamma\to\beta,p}) = \mathfrak{D}(\Psi_{\beta,p})$$

and

$$\mathfrak{R}(\Psi_{\alpha,p}), \mathfrak{R}(\mathfrak{P}_{\alpha,q\to r}), \mathfrak{R}(\mathfrak{T}_{\beta\to\alpha,p}) \subset \mathfrak{D}(\mathfrak{P}_{\alpha,p\to q}).$$

The proof follows from a general result and is given in Section 4. We now state our main result.

Theorem 2.3. *1. If $-\infty < \beta < \alpha < 2$ and $p > 0$ then*

$$\Psi_{\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\Psi_{\beta,p}.$$

2. If $-\infty < \gamma < \beta < \alpha < 2$ and $p > 0$ then

$$\mathfrak{T}_{\gamma\to\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\mathfrak{T}_{\gamma\to\beta,p}.$$

3. If $\alpha < 2$ and $0 < q < p < \infty$ then

$$\Psi_{\alpha,q} = \mathfrak{P}_{\alpha,p\to q}\Psi_{\alpha,p}.$$

4. If $0 < r < q < p < \infty$ and $-\infty < \alpha < 2 + r$ then

$$\mathfrak{P}_{\alpha,p\to r} = \mathfrak{P}_{\alpha,q\to r}\mathfrak{P}_{\alpha,p\to q}.$$

The proof is given in Section 4. In all cases equality of domains is part of the result. Further, in the above, all compositions commute. Before proceeding, we recall a result from [10].

Proposition 2.4. *If $-\infty < \alpha < 2$ and $p > 0$ then the transform $\Psi_{\alpha,p}$ is one-to-one.*

Combining this with Theorem 2.3 will give the following.

Corollary 2.5. *If $-\infty < \beta < \alpha < 2$ and $p > 0$ then the transform $\mathfrak{T}_{\beta\to\alpha,p}$ is one-to-one. If $0 < q < p$ and $\alpha < 0$ then the transform $\mathfrak{P}_{\alpha,p\to q}$ is one-to-one.*

For $\mathfrak{T}_{\beta\to\alpha,p}$ a different proof was given in [12]. For $\mathfrak{P}_{\alpha,p\to q}$, the case where $\alpha \geq 0$ is more complicated and will be dealt with in a future work.

Proof. We begin with Part 1. Let $M, M' \in \mathfrak{D}(\mathfrak{T}_{\beta\to\alpha,p}) = \mathfrak{D}(\Psi_{\alpha,p})$. If $\mathfrak{T}_{\beta\to\alpha,p}M = \mathfrak{T}_{\beta\to\alpha,p}M'$ then $\Psi_{\beta,p}\mathfrak{T}_{\beta\to\alpha,p}M = \Psi_{\beta,p}\mathfrak{T}_{\beta\to\alpha,p}M'$ and hence by commutativity and Theorem 2.3 we have $\Psi_{\alpha,p}M = \Psi_{\alpha,p}M'$. From here Proposition 2.4 implies that $M = M'$ and hence $\mathfrak{T}_{\beta\to\alpha,p}$ is one-to-one. The proof of Part 2 is similar. We just need to note that, in this case, $\mathfrak{D}(\Psi_{\alpha,q}) = \mathfrak{M}^0$. $\square$

We now interpret Theorem 2.3 in the context of tempered stable distributions. For $\alpha < 2$ and $p > 0$ let LTS_α^p be the class of Lévy measures of p-tempered α-stable distributions, and note that $LTS_\alpha^p = \mathfrak{R}(\Psi_{\alpha,p})$. For $-\infty < \beta < \alpha < 2$ and $0 < q < p < \infty$ let $\mathfrak{T}_{\beta\to\alpha,p}^{TS}$ and $\mathfrak{P}_{\alpha,p\to q}^{TS}$ be the restrictions of $\mathfrak{T}_{\beta\to\alpha,p}$ and $\mathfrak{P}_{\alpha,p\to q}$ to the domains $LTS_\beta^p \cap \mathfrak{D}(\mathfrak{T}_{\beta\to\alpha,p})$ and LTS_α^p respectively. Note that, by Lemma 2.2, $LTS_\alpha^p \subset \mathfrak{D}(\mathfrak{P}_{\alpha,p\to q})$.

Corollary 2.6. *For $-\infty < \beta < \alpha < 2$ and $p > 0$ the mapping $\mathfrak{T}_{\beta\to\alpha,p}^{TS}$ is a bijection from $LTS_\beta^p \cap \mathfrak{D}(\mathfrak{T}_{\beta\to\alpha,p})$ onto LTS_α^p. For $0 < q < p$ and $\alpha < 2$ the mapping $\mathfrak{P}_{\alpha,p\to q}^{TS}$ is a bijection from LTS_α^p onto LTS_α^q.*

Proof. The result is immediate from Theorem 2.3 and Corollary 2.5, except in the case of $\mathfrak{P}_{\alpha,p\to q}^{TS}$ with $\alpha \in [0,2)$. In this case we can show that $\mathfrak{P}_{\alpha,p\to q}^{TS}$ is one-to-one by arguments similar to the proof of Corollary 2.5. $\square$

3 Probabilistic Interpretation

In this section we interpret Theorem 2.3 in terms of stochastic integration. First, recall that every infinitely divisible distribution μ on $\mathbb{R}^d$ has a characteristic function of the form $\hat{\mu}(z) = \exp\{C_\mu(z)\}$ for $z \in \mathbb{R}^d$, where

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle b, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{|x|\le 1}\right) M(\mathrm{d}x),$$

A is a symmetric nonnegative-definite $d\times d$ matrix called the Gaussian part, $b \in \mathbb{R}^d$ is called the shift, and M is a Lévy measure. The measure μ is uniquely identified by the Lévy triplet (A, M, b) and we write $\mu = ID(A, M, b)$. Further, we write $\{X_t^{(\mu)} : t \ge 0\}$ to denote a Lévy process with $X_1 \sim \mu$. A Lévy measure M on $\mathbb{R}^d$ is called symmetric if $M(B) = M(-B)$ for every $B \in \mathfrak{B}(\mathbb{R}^d)$.

We say that a dilation measure ρ satisfies Assumption A if $\rho(\mathrm{d}t) = r(t)\mathrm{d}t$ for some Borel function r for which there exists a $b \in (0,\infty]$ with $r(t) > 0$ for $t \in (0, b)$ and $r(t) = 0$ otherwise. For any dilation measure ρ satisfying Assumption A let $\eta_\rho(t) = \rho([t,\infty))$ for $t \in (0, b)$ and note that $\eta_\rho : (0, b) \mapsto (0, \eta_\rho(0))$ is differentiable, strictly decreasing, and invertible. Let $\eta_\rho^*(t)$ be its inverse function.

Fix a dilation measure ρ satisfying Assumption A. For any $\mu = ID(0, M, 0)$, where $M \in \mathfrak{D}(\Upsilon_\rho)$ is a *symmetric* Lévy measure, Theorem 3.5 in [17] implies that the stochastic integral

$$\int_0^{\eta_\rho(0)} \eta_\rho^*(t)\mathrm{d}X_t^{(\mu)}$$

exists. For such μ define the transform

$$\Upsilon^\rho(\mu) = \mathcal{L}\left(\int_0^{\eta_\rho(0)} \eta_\rho^*(t)\mathrm{d}X_t^{(\mu)}\right),$$

where $\mathcal{L}(X)$ is the law of X. Theorem 3.10 in [17] implies that $\Upsilon^\rho(\mu) = ID(0, \Upsilon_\rho M, 0)$.

We can now give a probabilistic interpretation of Theorem 2.3. We only present it for the first part of the theorem since the rest are similar. Fix $-\infty < \beta < \alpha < 2$, $p > 0$, and let M be a symmetric Lévy measure with $M \in \mathfrak{D}(\Psi_{\alpha,p})$. If $\mu = ID(0, M, 0)$ then the first part of Theorem 2.3 implies

$$\begin{aligned}\int_0^{\eta_{\psi_{\alpha,p}}(0)} \eta_{\psi_{\alpha,p}}^*(t)\mathrm{d}X_t^{(\mu)} &\overset{d}{=} \int_0^{\eta_{\tau_{\beta\to\alpha,p}}(0)} \eta_{\tau_{\beta\to\alpha,p}}^*(t)\mathrm{d}X_t^{(\Upsilon^{\psi_{\beta,p}}\mu)}\\ &\overset{d}{=} \int_0^{\eta_{\psi_{\beta,p}}(0)} \eta_{\psi_{\beta,p}}^*(t)\mathrm{d}X_t^{(\Upsilon^{\tau_{\beta\to\alpha,p}}\mu)}.\end{aligned}$$

In this section we focused on the case where $\mu = ID(0, M, 0)$ and M is a symmetric Lévy measure. This was done for simplicity. In the general case, the conditions for the existence of the stochastic integral and the form of its Lévy triplet are a bit more complicated, see Theorems 3.5 and 3.10 of [17]. Never-the-less, one can obtain more general results analogous to those given in this section.

4 Proofs

In this section we prove our main results. First, recall that, for $r \in (0, 1)$, f_r is the probability density of a fully right-skewed r-stable distribution with Laplace transform given by (1.6). Recall further that the class of all stable distributions on $\mathbb{R}$ is a parametric family with four parameters. While there are many ways to parametrize this family, a common parametrization is given in Definition 14.16 of [15]. We note that, in this parametrization, f_r is the density of a stable distribution with parameters $\left(r, 1, 0, \cos \frac{\pi r}{2}\right)$. We now give some properties of this density.

Lemma 4.1. *1. If $r \in (0, 1)$ then there is a $K > 0$ depending on r such that*

$$f_r(x) \sim K x^{-r-1} \text{ as } x \to \infty.$$

2. If $r \in (0, 1)$ and $\beta \in (-\infty, r)$ then

$$\int_0^\infty s^\beta f_r(s) \mathrm{d}s < \infty.$$

3. If $r, p \in (0, 1)$ then

$$f_{rp}(u) = \int_0^\infty f_r(u y^{-1/r}) y^{-1/r} f_p(y) \mathrm{d}y.$$

Proof. Part 1 follows from (14.37) in [15]. When $\beta < 0$ Part 2 follows from Theorem 5.4.1 in [19] and when $\beta \in [0, r)$ it follows from Part 1. Now, let $X \sim f_r$ and $Y \sim f_p$ be independent random variables. The fact that

$$\mathrm{E}\left[e^{-tY^{1/r}X}\right] = \mathrm{E}\left[\mathrm{E}\left[e^{-tY^{1/r}X} | Y\right]\right] = \mathrm{E}\left[e^{-t^r Y}\right] = e^{-t^{rp}}$$

implies that $Y^{1/r} X \sim f_{rp}$. From here Part 3 follows by representing the density of $Y^{1/r} X$ in terms of the densities of X and Y. $\square$

Lemma 4.2. *Assume that $\rho(\mathrm{d}s) = g(s) 1_{s>0} \mathrm{d}s$ and that there exist $\delta \in (0, 1)$, $\alpha \in \mathbb{R}$, and $0 < a < b < \infty$ such that $a < s^{\alpha+1} g(s) < b$ for all $s \in (0, \delta)$. When $\alpha < 2$ assume also that $\int_0^\infty s^2 g(s) \mathrm{d}s < \infty$. In this case*

$$\mathfrak{D}(\Upsilon_\rho) = \begin{cases} \mathfrak{M}^0 & \text{if } \alpha < 0 \\ \mathfrak{M}^{\log} & \text{if } \alpha = 0 \\ \mathfrak{M}^\alpha & \text{if } \alpha \in (0, 2) \\ \{0\} & \text{if } \alpha \geq 2 \end{cases}.$$

Further, when $\alpha \in (0, 2)$ we have $\mathfrak{R}(\Upsilon_\rho) \subset \mathfrak{M}^\beta$ for every $\beta \in [0, \alpha)$.

We note that a related result is given in Theorem 4.1 of [18].

Proof. Fix $M \in \mathfrak{M}^0$. We need to characterize when

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge 1\right) [\Upsilon_\rho M](\mathrm{d}x) < \infty.$$

First assume $\alpha \geq 2$. If $M \neq 0$ then there exists a $\delta' \in (0, \delta)$ such that $M(|x| \leq 1/\delta') > 0$ and

$$\begin{aligned}\int_{|x|\leq 1} |x|^2 [\Upsilon_\rho M](\mathrm{d}x) &= \int_{\mathbb{R}^d} |x|^2 \int_0^{1/|x|} s^2 g(s)\mathrm{d}s M(\mathrm{d}x) \\ &\geq \int_{|x|\leq 1/\delta'} |x|^2 \int_0^{\delta'} s^2 g(s)\mathrm{d}s M(\mathrm{d}x) \\ &\geq a \int_{|x|\leq 1/\delta'} |x|^2 \int_0^{\delta'} s^{1-\alpha}\mathrm{d}s M(\mathrm{d}x) = \infty.\end{aligned}$$

Now assume that $\alpha < 2$. We have

$$\begin{aligned}\int_{|x|>1} [\Upsilon_\rho M](\mathrm{d}x) &= \int_{|x|\leq 1/\delta} \int_{1/|x|}^{\infty} g(s)\mathrm{d}s M(\mathrm{d}x) \\ &\quad + \int_{|x|>1/\delta} \int_{1/|x|}^{\delta} g(s)\mathrm{d}s M(\mathrm{d}x) \\ &\quad + \int_{|x|>1/\delta} \int_{\delta}^{\infty} g(s)\mathrm{d}s M(\mathrm{d}x) =: I_1 + I_2 + I_3.\end{aligned}$$

Since $M \in \mathfrak{M}^0$ and for any $c > 0$ we have $\int_c^\infty g(s)\mathrm{d}s \leq c^{-2}\int_0^\infty s^2 g(s)\mathrm{d}s < \infty$ we have $I_3 < \infty$ and

$$I_1 \leq \int_{|x|\leq 1/\delta} |x|^2 M(\mathrm{d}x) \int_0^\infty s^2 g(s)\mathrm{d}s < \infty.$$

Now note that

$$a \int_{|x|>1/\delta} \int_{1/|x|}^{\delta} s^{-1-\alpha}\mathrm{d}s M(\mathrm{d}x) \leq I_2 \leq b \int_{|x|>1/\delta} \int_{1/|x|}^{\delta} s^{-1-\alpha}\mathrm{d}s M(\mathrm{d}x).$$

From here the fact that $\int_{1/|x|}^{\delta} s^{-1-\alpha}\mathrm{d}s = (|x|^\alpha - \delta^{-\alpha})/\alpha$ when $\alpha \neq 0$ and it equals $\log|x\delta|$ when $\alpha = 0$ gives the necessity of our conditions. Now note that

$$\begin{aligned}&\int_{|x|\leq 1} |x|^2 [\Upsilon_\rho M](\mathrm{d}x) = \int_{\mathbb{R}^d} |x|^2 \int_0^{1/|x|} s^2 g(s)\mathrm{d}s M(\mathrm{d}x) \\ &\leq \int_{|x|\leq 1/\delta} |x|^2 M(\mathrm{d}x) \int_0^\infty s^2 g(s)\mathrm{d}s + b \int_{|x|>1/\delta} |x|^2 \int_0^{1/|x|} s^{1-\alpha}\mathrm{d}s M(\mathrm{d}x) \\ &= \int_{|x|\leq 1/\delta} |x|^2 M(\mathrm{d}x) \int_0^\infty s^2 g(s)\mathrm{d}s + \frac{b}{2-\alpha} \int_{|x|>1/\delta} |x|^\alpha M(\mathrm{d}x).\end{aligned}$$

This gives sufficiency of the conditions and completes the proof of the first part.

Now assume that $\alpha \in (0, 2)$ and $\beta \in [0, \alpha)$. It suffices to show that for any $M \in \mathfrak{M}^\alpha$

$$\int_{|x|>1} |x|^\beta [\Upsilon_\rho M](\mathrm{d}x) = \int_{\mathbb{R}^d} |x|^\beta \int_{|x|^{-1}}^\infty s^\beta g(s)\mathrm{d}s M(\mathrm{d}x) < \infty.$$

Observing that

$$\int_{|x|\leq 1/\delta} |x|^\beta \int_{|x|^{-1}}^\infty s^\beta g(s)\mathrm{d}s M(\mathrm{d}x) \leq \int_{|x|\leq 1/\delta} |x|^2 M(\mathrm{d}x) \int_0^\infty s^2 g(s)\mathrm{d}s < \infty,$$

$$\int_{|x|>1/\delta} |x|^\beta \int_\delta^\infty s^\beta g(s)\mathrm{d}s M(\mathrm{d}x) < \infty,$$

and

$$\begin{aligned}\int_{|x|>1/\delta}|x|^{\beta}\int_{|x|^{-1}}^{\delta}s^{\beta}g(s)\mathrm{d}sM(\mathrm{d}x) &\le b\int_{|x|>1/\delta}|x|^{\beta}\int_{|x|^{-1}}^{\infty}s^{\beta-\alpha-1}\mathrm{d}sM(\mathrm{d}x)\\ &= \frac{b}{\alpha-\beta}\int_{|x|>1/\delta}|x|^{\alpha}M(\mathrm{d}x)<\infty\end{aligned}$$

gives the result. □

We can now prove Theorem 2.1 and Lemma 2.2.

Proof of Theorem 2.1 and Lemma 2.2. Both results follow easily from Lemma 4.2, we just need to check that the assumptions hold. We only verify this for $\mathfrak{P}_{\alpha,p\to q}$ as it is immediate for the other cases. In this case we have $g(s)=pf_{q/p}(s^{-p})s^{-\alpha-p-1}$. Lemma 4.1 implies that

$$\int_0^\infty s^2 g(s)\mathrm{d}s = p\int_0^\infty f_{q/p}(s^{-p})s^{1-\alpha-p}\mathrm{d}s=\int_0^\infty f_{q/p}(v)v^{-(2-\alpha)/p}\mathrm{d}v<\infty$$

and that $g(s)\sim pKs^{q-\alpha-1}$ as $s\downarrow 0$. From here the result follows. □

Proof of Theorem 2.3. In all cases, equality of the domains follows from Theorem 2.1 and Lemma 2.2. We now turn to proving the equalities. We begin with Part 1. Fix $M\in\mathfrak{D}(\mathfrak{T}_{\beta\to\alpha,p}\Psi_{\beta,p})$ and let $M'=\mathfrak{T}_{\beta\to\alpha,p}\Psi_{\beta,p}M$. For any $B\in\mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned}M'(B) &= K^{-1}_{\alpha,\beta,p}\int_0^1[\Psi_{\beta,p}M](u^{-1}B)u^{-\alpha-1}\left(1-u^p\right)^{\frac{\alpha-\beta}{p}-1}\mathrm{d}u\\ &= K^{-1}_{\alpha,\beta,p}\int_0^\infty\int_0^1 M((ut)^{-1}B)t^{-1-\beta}e^{-t^p}u^{-\alpha-1}\left(1-u^p\right)^{\frac{\alpha-\beta-p}{p}}\mathrm{d}u\mathrm{d}t\\ &= K^{-1}_{\alpha,\beta,p}\int_0^\infty\int_0^t M(v^{-1}B)t^{\alpha-\beta-1}e^{-t^p}v^{-\alpha-1}\left(1-\frac{v^p}{t^p}\right)^{\frac{\alpha-\beta-p}{p}}\mathrm{d}v\mathrm{d}t\\ &= K^{-1}_{\alpha,\beta,p}\int_0^\infty\int_v^\infty M(v^{-1}B)t^{p-1}e^{-t^p}v^{-\alpha-1}\left(t^p-v^p\right)^{\frac{\alpha-\beta-p}{p}}\mathrm{d}t\mathrm{d}v\\ &= K^{-1}_{\alpha,\beta,p}\int_0^\infty M(v^{-1}B)e^{-v^p}v^{-\alpha-1}\mathrm{d}v\int_0^\infty e^{-s^p}s^{\alpha-\beta-1}\mathrm{d}s\\ &= \int_0^\infty M(v^{-1}B)e^{-v^p}v^{-\alpha-1}\mathrm{d}v=[\Psi_{\alpha,p}M](B),\end{aligned}$$

where the third line follows by the substitution $v=ut$ and the fifth by the substitution $s^p=t^p-v^p$

We now show Part 2. Note that by the well-known relationship between beta and gamma functions

$$\begin{aligned}\int_0^1 w^{\alpha-\beta-1}\left(1-w^p\right)^{\frac{\beta-\gamma}{p}-1}\mathrm{d}w &= p^{-1}\int_0^1 w^{\frac{\alpha-\beta}{p}-1}\left(1-w\right)^{\frac{\beta-\gamma}{p}-1}\mathrm{d}w\\ &= p^{-1}\frac{\Gamma\left(\frac{\alpha-\beta}{p}\right)\Gamma\left(\frac{\beta-\gamma}{p}\right)}{\Gamma\left(\frac{\alpha-\gamma}{p}\right)}=\frac{K_{\alpha,\beta,p}K_{\beta,\gamma,p}}{K_{\alpha,\gamma,p}}.\end{aligned}$$

For simplicity of notation let $A=K^{-1}_{\alpha,\beta,p}K^{-1}_{\beta,\gamma,p}$ and note that

$$K^{-1}_{\alpha,\gamma,p}=A\int_0^1 w^{\alpha-\beta-1}\left(1-w^p\right)^{\frac{\beta-\gamma}{p}-1}\mathrm{d}w.$$

Fix $M \in \mathfrak{D}(\mathfrak{T}_{\beta\to\alpha,p}\mathfrak{T}_{\gamma\to\beta,p})$ and let $M' = \mathfrak{T}_{\beta\to\alpha,p}\mathfrak{T}_{\gamma\to\beta,p}M$. For $B \in \mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned}
M'(B) &= K^{-1}_{\alpha,\beta,p}\int_0^1 [\mathfrak{T}_{\gamma\to\beta,p}M](u^{-1}B)u^{-\alpha-1}\left(1-u^p\right)^{\frac{\alpha-\beta}{p}-1}\mathrm{d}u\\
&= A\int_0^1\int_0^1 M((ut)^{-1}B)u^{-\alpha-1}\left(1-u^p\right)^{\frac{\alpha-\beta}{p}-1}\mathrm{d}ut^{-\beta-1}\left(1-t^p\right)^{\frac{\beta-\gamma}{p}-1}\mathrm{d}t\\
&= A\int_0^1\int_0^t M(v^{-1}B)v^{-\alpha-1}\left(1-\frac{v^p}{t^p}\right)^{\frac{\alpha-\beta}{p}-1}\mathrm{d}vt^{\alpha-\beta-1}\left(1-t^p\right)^{\frac{\beta-\gamma}{p}-1}\mathrm{d}t\\
&= A\int_0^1 M(v^{-1}B)v^{-\alpha-1}\int_v^1\left(t^p-v^p\right)^{\frac{\alpha-\beta}{p}-1}\left(1-t^p\right)^{\frac{\beta-\gamma}{p}-1}t^{p-1}\mathrm{d}t\mathrm{d}v\\
&= A\int_0^1 M(v^{-1}B)v^{-\alpha-1}(1-v^p)^{\frac{\alpha-\gamma}{p}-1}\mathrm{d}v\int_0^1 w^{\alpha-\beta-1}\left(1-w^p\right)^{\frac{\beta-\gamma}{p}-1}\mathrm{d}w\\
&= K^{-1}_{\alpha,\gamma,p}\int_0^1 M(v^{-1}B)v^{-\alpha-1}(1-v^p)^{\frac{\alpha-\gamma}{p}-1}\mathrm{d}v = [\mathfrak{T}_{\gamma\to\alpha,p}M](B),
\end{aligned}$$

where the third line follows by the substitution $v = ut$ and the fifth by the substitution $w^p = (t^p - v^p)/(1-v^p)$, which implies $1 - t^p = (1-w^p)(1-v^p)$.

To show Part 3, fix $M \in \mathfrak{D}(\mathfrak{P}_{\alpha,p\to q}\Psi_{\alpha,p})$ and note that for $B \in \mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned}
[\mathfrak{P}_{\alpha,p\to q}\Psi_{\alpha,p}M](B) &= p\int_0^\infty f_{q/p}(s^{-p})s^{-\alpha-p-1}[\Psi_{\alpha,p}M](s^{-1}B)\mathrm{d}s\\
&= \int_0^\infty f_{q/p}(s)s^{\alpha/p}[\Psi_{\alpha,p}M](s^{1/p}B)\mathrm{d}s\\
&= \int_0^\infty f_{q/p}(s)s^{\alpha/p}\int_0^\infty M(s^{1/p}t^{-1}B)t^{-1-\alpha}e^{-t^p}\mathrm{d}t\mathrm{d}s\\
&= \int_0^\infty M(v^{-1}B)v^{-1-\alpha}\int_0^\infty e^{-v^p s}f_{q/p}(s)\mathrm{d}s\mathrm{d}v\\
&= \int_0^\infty M(v^{-1}B)v^{-1-\alpha}e^{-v^q}\mathrm{d}v = [\Psi_{\alpha,q}M](B),
\end{aligned}$$

where the fourth line follows by the substitution $v = s^{-1/p}t$.

For Part 4, fix $M \in \mathfrak{D}(\mathfrak{P}_{\alpha,q\to r}\mathfrak{P}_{\alpha,p\to q})$ and let $M' = \mathfrak{P}_{\alpha,q\to r}\mathfrak{P}_{\alpha,p\to q}M$. For any $B \in \mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned}
M'(B) &= \int_0^\infty f_{r/q}(t)t^{\alpha/q}[\mathfrak{P}_{\alpha,p\to q}M](t^{1/q}B)\mathrm{d}t\\
&= \int_0^\infty f_{q/p}(s)s^{\alpha/p}\int_0^\infty f_{r/q}(t)t^{\alpha/q}M(t^{1/q}s^{1/p}B)\mathrm{d}t\mathrm{d}s\\
&= \int_0^\infty M(u^{1/p}B)u^{\alpha/p}\int_0^\infty f_{q/p}(ut^{-p/q})f_{r/q}(t)t^{-p/q}\mathrm{d}t\mathrm{d}u\\
&= \int_0^\infty M(u^{1/p}B)u^{\alpha/p}f_{r/p}(u)\mathrm{d}u = [\mathfrak{P}_{\alpha,p\to r}M](B),
\end{aligned}$$

where the third line follows by the substitution $u = st^{p/q}$ and the fourth by Lemma 4.1. □

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