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Concentration of random polytopes around the expected convex hull

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Abstract

We provide a streamlined proof and improved estimates for the weak multivariate Gnedenko law of large numbers on concentration of random polytopes within the space of convex bodies (in a fixed or a high dimensional setting), as well as a corresponding strong law of large numbers.

Keywords: random polytope; law of large numbers ; log-concave; expected convex hull; floating body.
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1 Introduction

Let $d \in \mathbb{N}$ and let μ be a probability measure on $\mathbb{R}^d$ with a log-concave density $f = d\mu/dx$, i.e. $-\log f$ is a convex extended real valued function. Let $n \geq d+1$ and let $(X_i)_1^n$ denote an i.i.d. sequence of random vectors with common distribution μ. The convex hull

$$P_n = \mathrm{conv}\{X_i\}_1^n \tag{1.1}$$

is a random polytope and, as such, is a random element w.p.1 of the space $\mathcal{K}_d$ of all convex bodies in $\mathbb{R}^d$ (compact convex sets with non-empty interior). There are various metrics and metric-like functions on $\mathcal{K}_d$, such as the Hausdorff distance $d_{\mathcal{H}}$ and the Banach-Mazur distance δ^{BM} (for origin symmetric bodies). We refer the reader to [28] for general background on convex bodies, and to [18] specifically for metric, and other, structures on $\mathcal{K}_d$.

It was shown in [13] that if $n \geq c\exp(\exp(5d))$, then with probability at least $1 - 3^{d+3}(\log n)^{-1000}$, there exists $x \in \mathbb{R}^n$ and

$$\lambda \leq 1 + c'd^2 \frac{\log\log n}{\log n}$$

such that

$$\lambda^{-1}(F_{1/n} - x) + x \subseteq P_n \subseteq \lambda(F_{1/n} - x) + x \tag{1.2}$$

where $c, c' > 0$ are universal constants and $F_{1/n}$ is the floating body defined by

$$F_\delta = \cap\{\mathfrak{H} : \mu(\mathfrak{H}) \geq 1 - \delta\} \tag{1.3}$$

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where the intersection runs through the collection of all closed half-spaces $\mathfrak{H}$ of μ-mass at least $1-\delta$ ($\delta < e^{-1}$). The body $F_{1/n}$ was defined in [29] (see also [5]) in the case of Lebesgue measure on a convex body and has often been used to model random polytopes, see for example [4, 33]. This follows an earlier type of floating body defined in [10].

Being log-concave, the density f decays at least as quickly as an exponential function. A bound on the decay rate of f translates to a bound on the Hausdorff distance $d_{\mathcal{H}}(P_n, F_{1/n})$. For example if the tails of μ are sub-Gaussian (with universally bounded constants), then $\operatorname{diam}(F_{1/n}) \leq c(\log n)^{1/2}$ and (1.2) translates to

$$d_{\mathcal{H}}(P_n, F_{1/n}) \leq c' d^2 \frac{\log\log n}{\sqrt{\log n}} \tag{1.4}$$

where $c, c' > 0$ are universal constants. This is an embodiment of the concentration of measure phenomenon: the polytope P_n, as a random element of the metric space $(\mathcal{K}_d, d_{\mathcal{H}})$, is concentrated around $F_{1/n}$.

In the case $d = 1$, P_n reduces to the interval

$$[\min\{X_i\}_1^n, \max\{X_i\}_1^n]$$

and we see that the above mentioned result generalizes a theorem of Gnedenko [15] on concentration of the maximum and minimum of a large i.i.d. sample (under rapid decay of the tails of μ). Other multivariate analogs of Gnedenko's law of large numbers are included in [14] for the multivariate normal distribution, [17] for Gaussian measures on infinite dimensional spaces, [8, 11, 12] for regularly varying distributions, and [19, 22] for more general distributions.

The proof of (1.2) was complicated by the fact that there is no convenient expression for the support function of the floating body,

$$h_{F_{1/n}}(\theta) = \max_{x \in F_{1/n}} \langle \theta, x \rangle$$

In this paper we study concentration of P_n around the expected convex hull

$$\mathbb{E}P_n = \{x \in \mathbb{R}^n : \forall \theta \in S^{d-1}, \langle \theta, x \rangle \leq \mathbb{E} \max_{1 \leq i \leq n} \langle \theta, X_i \rangle\} \tag{1.5}$$

which is easily seen to be a convex body with support function

$$h_{\mathbb{E}P_n}(\theta) = \mathbb{E} \max_{1 \leq i \leq n} \langle \theta, X_i \rangle \tag{1.6}$$

Using the expected convex hull leads to a streamlined proof of (1.2). The notion of the expectation of a random convex body follows the theory of integrals of set valued functions, see for example [1, 3, 9, 20, 23] and the references therein. It was used in [2] for the purpose of a Kolmogorov strong law of large numbers and has appeared as an approximant to floating bodies in bounded domains [6], as well as in other contexts e.g. [16, 24, 30, 31, 32, 34].

In the original paper [13] we were mainly interested in a quantitative dependence on n. Although our bounds included dependence on dimension, the required sample size was very large. Theorem 2.1 includes improved bounds on the required sample size and is more in the spirit of the high dimensional theory. The quantitative dependence that we achieve is essentially the same as that in Dvoretzky's theorem, see for example [26]. This result should also be compared to the main result in [7].

To make the present exposition brief, we refer the reader to [13] for a more detailed discussion.

2 Main results

Theorem 2.1. *Let* $d \in \mathbb{N}$ *and let* μ *be a log-concave probability measure on* $\mathbb{R}^d$ *with center of mass at the origin and non-singular covariance matrix. Consider any* $\varepsilon \in (0, 1/2)$ *and let* $n \geq \exp(7d\varepsilon^{-1}\log\varepsilon^{-1})$. *Let* $(X_i)_1^n$ *be an i.i.d. sample from* μ, $P_n = \mathrm{conv}\{X_i\}_1^n$, *and let* $\mathbb{E}P_n$ *denote the expected convex hull as defined by (1.5). With probability at least* $1 - 3n^{-\varepsilon/4}$,

$$(1-\varepsilon)\mathbb{E}P_n \subseteq P_n \subseteq (1+\varepsilon)\mathbb{E}P_n \tag{2.1}$$

Setting $\varepsilon = (4q + 32d)(\log\log n)/\log n$ (where $q \geq 1$ can be chosen arbitrarily), we see that whenever $n \geq 8(q + 8d)\exp(8q + 64d)$, (2.1) holds with probability at least $1 - 3(\log n)^{-q-8d}$. Theorem 2.1 above therefore implies Theorem 1 in [13] with improved estimates. The following Theorem, which is similar to the main result in [6], is a consequence of Lemma 5.1.

Theorem 2.2. *Let* $d \in \mathbb{N}$ *and let* μ *be a log-concave probability measure on* $\mathbb{R}^d$ *with center of mass at the origin and non-singular covariance matrix. Let* $\mathbb{E}P_n$ *denote the expected convex hull as defined by (1.5), and let* $F_{1/n}$ *denote the floating body defined by (1.3). Then provided* $n \geq 12$,

$$(1 - 3/\log n)\mathbb{E}P_n \subseteq F_{1/n} \subseteq (1 + 1/\log n)\mathbb{E}P_n$$

Theorem 2.3. *Let* $d \in \mathbb{N}$ *and let* μ *be a log-concave probability measure on* $\mathbb{R}^d$ *with center of mass at the origin and non-singular covariance matrix. Let* $(X_i)_1^\infty$ *be an i.i.d. sample from* μ, *and let* $(P_n)_{d+1}^\infty$ *and* $(\mathbb{E}P_n)_{d+1}^\infty$ *be the random polytopes and expected convex hulls defined by (1.1) and (1.5) respectively. Then with probability* 1, *there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$,

$$\left(1 - \frac{3\log\log n}{\log n}\right)\mathbb{E}P_n \subseteq P_n \subseteq \left(1 + \frac{8\log\log n}{\log n}\right)\mathbb{E}P_n \tag{2.2}$$

3 Approximation in the Hausdorff distance

Let us comment briefly on how the main results of this paper give rise to estimates in terms of the Hausdorff distance $d_{\mathcal{H}}$. Let $\mathbb{E}\mu$ and $\mathrm{Cov}(\mu)$ denote (respectively) the center of mass and covariance matrix of μ. If we assume, in addition to log-concavity, that $\mathbb{E}\mu = 0$ and $\mathrm{Cov}(\mu) = I_d$, in which case μ is called isotropic, then it is easy to show that the diameter of both $\mathbb{E}P_n$ and $F_{1/n}$ are bounded above by $c\log n$, where $c > 0$ is a universal constant. In addition, it follows from the definition of the Hausdorff distance $d_{\mathcal{H}}$ that for any two convex bodies $A, B \subset \mathbb{R}^d$ with $0 \in \mathrm{int}(A) \cap \mathrm{int}(B)$, $d_{\mathcal{H}}(A, B) \leq \mathrm{diam}(B)\inf\{\varepsilon > 0 : (1+\varepsilon)^{-1}A \subseteq B \subseteq (1+\varepsilon)A\}$. Using these bounds, Theorems 2.1, 2.2 and 2.3 may be written in terms of $d_{\mathcal{H}}$.

For certain distributions the estimate $c\log n$ on the diameter may be substantially improved. Consider the case where the density function $f = d\mu/dx$ has the form

$$f(x) = \left(\frac{p}{2\Gamma(p^{-1})}\right)^d \exp\left(-\sum_{i=1}^n |x_i|^p\right)$$

for $1 \leq p < \infty$. In this case $\mathbb{E}\mu = 0$ and $\mathrm{Cov}(\mu) = (\Gamma(3/p)/\Gamma(1/p))^{1/2} I_d$ (see the proof of Lemma 2, part 4 with $q = 2$ in [27]). It follows from Theorem 2.2 above and Theorem 3 in [13] that for $n > n_0(d, p)$, $\mathrm{diam}\,(\mathbb{E}P_n) \leq (2.01)\max\{d^{1/2-1/p}, 1\}(\log n)^{1/p}$. By Theorem 2.3, with probability 1 there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d_{\mathcal{H}}(P_n, \mathbb{E}P_n) \leq \frac{17(\log\log n)}{(\log n)^{1-1/p}}\max\{d^{1/2-1/p}, 1\}$$

As another example, when the tails of μ are sub-Gaussian then with probability 1 there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d_{\mathcal{H}}(P_n, \mathbb{E}P_n) \leq \frac{c \log\log n}{(\log n)^{1/2}}$$

which can be compared to (1.4).

4 Notation

If J is the cumulative distribution function associated to a probability measure μ on $\mathbb{R}$, then the generalized inverse $J^{-1} : (0,1) \to \mathbb{R}$ is defined as

$$J^{-1}(t) = \sup\{x \in \mathbb{R} : J(x) < t\} = \inf\{x \in \mathbb{R} : J(x) \geq t\}$$

If μ has a log-concave density function then $J(J^{-1}(t)) = t$ for all $t \in (0,1)$ and $J^{-1}(J(x)) = x$ for all x in the support of μ. If $(Y_i)_1^n$ is an i.i.d. sample from μ, then $Y_{(n)} = \max_{1 \leq i \leq n} Y_i$ denotes the n^{th} order statistic.

If $K \subset \mathbb{R}^d$ is a convex body then the function

$$h_K(x) = \max_{y \in K} \langle x, y \rangle$$

is known as the support function of K. If $0 \in \mathrm{int}(K)$ then the Minkowski functional is defined as

$$\|x\|_K = \min\{\lambda \geq 1 : x \in \lambda K\}$$

and the support function is the Minkowski functional of the polar body

$$K^\circ = \{y \in \mathbb{R}^d : \forall x \in K, \langle x, y \rangle \leq 1\}$$

i.e. $h_K(\cdot) = \|\cdot\|_{K^\circ}$. In the case when K is centrally symmetric, i.e. $K = -K$, then $h_K(\cdot)$ and $\|\cdot\|_K$ are norms.

5 Proofs

The following lemma is a natural extension of Lemma 7 in [13].

Lemma 5.1. *Let μ be a probability measure on $\mathbb{R}$ with mean 0 and log-concave density $f = d\mu/dx$. Let $n \geq 12$ and let $(Y_i)_1^n$ be an i.i.d. sample from μ. Then for all $t > 0$,*

$$\mathbb{P}\{Y_{(n)} \leq (1+t)\mathbb{E}Y_{(n)}\} \geq 1 - n^{-t/2} \tag{5.1}$$

$$\mathbb{P}\{Y_{(n)} \geq (1-t)\mathbb{E}Y_{(n)}\} \geq 1 - \exp(-n^{t/2}/3) \tag{5.2}$$

Proof. Let J be the common distribution function of each Y_i. Let f_n and J_n denote the density and distribution function of $Y_{(n)}$,

$$\begin{aligned} J_n(t) &= J(t)^n \\ f_n(t) &= \frac{d}{dt} J_n(t) = nJ(t)^{n-1} f(t) \end{aligned}$$

Since f is log-concave, so is J (see for example Theorem 5.1 in [21] or Lemma 5 in [13]). The product of log-concave functions is certainly log-concave, and therefore so is f_n. By a standard result, see for example Lemma 5.4 in [21], $J_n^{-1}(e^{-1}) \leq \mathbb{E}Y_{(n)} \leq J_n^{-1}(1 - e^{-1})$. Just as the left tail J is log-concave, so is the right tail $1 - J$, and the function $u(t) = -\log(1 - J(t))$ is convex. This implies that,

$$\frac{u(J^{-1}(1 - n^{-t/2}/n)) - u(J^{-1}(1 - 1/n))}{J^{-1}(1 - n^{-t/2}/n) - J^{-1}(1 - 1/n)} \geq \frac{u(J^{-1}(1 - 1/n)) - u(0)}{J^{-1}(1 - 1/n)}$$

which translates to

$$\frac{J^{-1}(1-n^{-t/2}/n)-J^{-1}(1-1/n)}{J^{-1}(1-1/n)} \le \frac{t\log n}{2(\log n-1)} \le t$$

Now,

$$\mathbb{P}\{Y_{(n)} \le J^{-1}(1-n^{-t/2}/n)\} = (1-n^{-t/2}/n)^n \ge 1-n^{-t/2}$$

By definition of J_n, $J_n(J^{-1}(1-1/n)) = (1-1/n)^n < e^{-1}$, so $\mathbb{E}Y_{(n)} \ge J_n^{-1}(e^{-1}) \ge J^{-1}(1-1/n)$ and (5.1) follows. Again by convexity of u,

$$\frac{u(J^{-1}(1-9/(20n)))-u(J^{-1}(1-9n^{t/2-1}/20))}{J^{-1}(1-9/(20n))-J^{-1}(1-9n^{t/2-1}/20)} \ge \frac{u(J^{-1}(1-9/(20n)))-u(0)}{J^{-1}(1-9/(20n))}$$

which translates to

$$\frac{J^{-1}(1-9/(20n))-J^{-1}(1-9n^{t/2-1}/20)}{J^{-1}(1-9/(20n))} \le \frac{(t/2)\log n}{\log n-1+\log(20/9)} \le t$$

Now,

$$\mathbb{P}\{Y_{(n)} \le J^{-1}(1-9n^{t/2-1}/20)\} = (1-9n^{t/2-1}/20)^n \le \exp(-9n^{t/2}/20)$$

As before, $J_n(J^{-1}(1-9/(20n))) = (1-9/(20n))^n > 1-e^{-1}$, so $\mathbb{E}Y_{(n)} \le J_n^{-1}(1-e^{-1}) < J^{-1}(1-9/(20n))$ and (5.2) follows. □

Proof of Theorem 2.2. Since $J^{-1}(1-1/n) = J_n^{-1}((1-1/n)^n)$, where $J_n(x) = \mathbb{P}\{Y_{(n)} \le x\}$, $\mathbb{P}\{Y_{(n)} \le J^{-1}(1-1/n)\} \ge 1/3$ and by inequality (5.2) of Lemma 5.1, this can only be true if $J^{-1}(1-1/n) \ge (1-(\log 18)/\log n)\mathbb{E}Y_{(n)}$. By similar reasoning, $\mathbb{P}\{Y_{(n)} > J^{-1}(1-1/n)\} \ge 1-e^{-1}$, which by inequality (5.1) of Lemma 5.1 implies that $J^{-1}(1-1/n) \le (1+1/\log n)\mathbb{E}Y_{(n)}$. The result now follows from the definitions of $F_{1/n}$ and $\mathbb{E}P_n$, see (1.3) and (1.6). □

The following lemma appears in Lemmas 4.10 and 4.11 in [25] under the assumption that K is centrally symmetric. We sketch the proof to show that it can also be used in the non-symmetric case.

Lemma 5.2. *Let $K \subset \mathbb{R}^d$ be any convex body with $0 \in \mathrm{int}(K)$ and $0 < \varepsilon < 1/2$. Then there exists a set $\mathcal{N} \subset \partial K$ with $|\mathcal{N}| \le (3/\varepsilon)^d$ such that for all $\theta \in \partial K$ there exist sequences $(\omega_i)_0^\infty \subseteq \mathcal{N}$ and $(\varepsilon_i)_1^\infty \subseteq [0,\infty)$ such that $0 \le \varepsilon_i \le \varepsilon^i$ for all i and*

$$\theta = \omega_0 + \sum_{i=1}^{\infty} \varepsilon_i \omega_i$$

Proof. Consider a subset $\mathcal{N} \subset \partial K$, minimal with respect to set inclusion, with the following property: for all $z \in \partial K$ there exists $\omega \in \mathcal{N}$ such that $\|z-\omega\|_K \le \varepsilon$. Such a set can easily be constructed recursively, and we shall refer to $\mathcal{N}$ as an ε-net. Note that since K may be non-symmetric, we may have $\|z-\omega\|_K \ne \|\omega-z\|_K$ and order becomes important. By the standard volumetric argument $|\mathcal{N}| \le (3/\varepsilon)^d$. By the defining property of $\mathcal{N}$, for all $x \in \mathbb{R}^d$ there exists $\omega \in \mathcal{N}$ such that

$$\|x - \|x\|_K\, \omega\|_K \le \varepsilon\, \|x\|_K \tag{5.3}$$

Now consider $\theta \in \partial K$. By (5.3) there exists $\omega_0 \in \mathcal{N}$ such that $\|\theta-\omega_0\|_K \le \varepsilon$. By applying (5.3) again, there exists $\omega_1 \in \mathcal{N}$ such that $\|\theta-\omega_0-\|\theta-\omega_0\|_K\,\omega_1\|_K \le \varepsilon\,\|\theta-\omega_0\|_K \le \varepsilon^2$. Iterating this procedure defines a sequence $(\omega_i)_0^\infty$ such that for all $N \in \mathbb{N}$,

$$\left\|\theta-\omega_0-\sum_{i=1}^{N}\varepsilon_i\omega_i\right\|_K \le \varepsilon^{N+1}$$

where $\varepsilon_i = \left\|\theta-\omega_0-\sum_{i=1}^{i-1}\varepsilon_i\omega_i\right\|_K \le \varepsilon^i$. □

Proof of Theorem 2.1. Set $\delta = 3n^{-\varepsilon/(4d)}$ and let $\mathcal{N} \subset \partial((\mathbb{E}P_n)^\circ)$ be a δ-net as in Lemma 5.2. By the bounds imposed on n, $\delta \le \varepsilon/5 < 1/10$. From the union bound and Lemma 5.1, the following event occurs with probability at least $1 - (3/\delta)^d 3n^{-\varepsilon/2} \ge 1 - 3n^{-\varepsilon/4}$: for all $\omega \in \mathcal{N}$,

$$(1-\varepsilon/2)\,\|\omega\|_{(\mathbb{E}P_n)^\circ} \le \|\omega\|_{P_n{}^\circ} \le (1+\varepsilon/2)\,\|\omega\|_{(\mathbb{E}P_n)^\circ} \tag{5.4}$$

For any $\theta \in \partial((\mathbb{E}P_n)^\circ)$, write $\theta = \omega_0 + \sum_1^\infty \delta_i\omega_i$, with $\omega_i \in \mathcal{N}$ and $0 \le \delta_i \le \delta^i$ for all i. By the triangle inequality and (5.4),

$$\|\theta\|_{P_n{}^\circ} \le (1+\varepsilon/2)\sum_{i=0}^{\infty}\delta^i \le (1+2\delta)(1+\varepsilon/2) \le 1+\varepsilon$$

and

$$\|\theta\|_{P_n{}^\circ} \ge \|\omega_0\|_{P_n{}^\circ} - \sum_{1}^{\infty}\delta^i\,\|\omega_i\|_{P_n{}^\circ} \ge 1-\varepsilon/2-(1+\varepsilon/2)\delta(1-\delta)^{-1} \ge 1-\varepsilon$$

and the result follows. □

Proof of Theorem 2.3. Here d and μ are fixed, and we treat $n \to \infty$ as a variable. In particular we may assume that $n > n_0$, where n_0 is suitably large. From comparing successive terms in the binomial theorem and using the fact that $n^{-k}\binom{n}{k}$ is a decreasing function of k, for all $\delta \in (0, 1/2)$

$$(1-2\delta/n)^n = (1-\delta) - \delta + \binom{n}{2}\left(\frac{2\delta}{n}\right)^2 + \sum_{k=3}^{n}(-1)^k\binom{n}{k}\left(\frac{2\delta}{n}\right)^k \le 1-\delta$$

By Theorem 2.1, $1 - 3n^{-\varepsilon/4} \le \mathbb{P}\{P_n \subseteq (1+\varepsilon)\mathbb{E}P_n\} = (\mathbb{P}\{X_1 \in (1+\varepsilon)\mathbb{E}P_n\})^n$, and it follows that $\mu((1+\varepsilon)\mathbb{E}P_n) \ge (1-3n^{-\varepsilon/4})^{1/n} \ge 1 - 6n^{-1-\varepsilon/4}$ (provided $3n^{-\varepsilon/4} < 1/2$). Setting $\varepsilon = 8(\log\log n)/\log n$ yields

$$\sum_{n=12}^{\infty}\mathbb{P}\{X_n \notin (1+\varepsilon)\mathbb{E}P_n\} \le 6\sum_{n=12}^{\infty} n^{-1-\varepsilon/4} = 6\sum_{n=12}^{\infty}\frac{1}{n(\log n)^2} < \infty$$

Therefore, by the Borel-Cantelli lemma, with probability 1 there exists $N^{(1)} \in \mathbb{N}$ such that for all $n \ge N^{(1)}$,

$$P_n \subseteq (1 + 8(\log\log n)/\log n)\mathbb{E}P_n \tag{5.5}$$

For each $n \in \mathbb{N}$, let E_n be the event that (5.5) holds. Consider any sufficiently large (deterministic) $n \in \mathbb{N}$. Set $\varepsilon = 3(\log\log n)/\log n$ and $\delta = 3\exp(-n^{-\varepsilon/2}/(6d))$. Let $\mathcal{N} \subset \partial((\mathbb{E}P_n)^\circ)$ be a δ-net as in Lemma 5.2. As before, $\delta \le \varepsilon/10 \le 1/20$. By the union bound and Lemma 5.1, the following event, to be denoted F_n, occurs with probability at least $1 - (3/\delta)^d\exp(-n^{\varepsilon/2}/3) \ge 1 - \exp(-n^{\varepsilon/2}/6) \ge 1 - n^{-2}$: for all $\omega \in \mathcal{N}$,

$$(1-\varepsilon/2)\,\|\omega\|_{(\mathbb{E}P_n)^\circ} \le \|\omega\|_{P_n{}^\circ}$$

The Borel-Cantelli lemma again implies that with probability 1 there exists $N^{(2)} \in \mathbb{N}$ such that F_n occurs for all $n \ge N^{(2)}$. For all $n \ge \max\{N^{(1)}, N^{(2)}\}$, $E_n \cap F_n$ occurs w.p.1, and expressing an arbitrary $\theta \in \partial((\mathbb{E}P_n)^\circ)$ as $\theta = \omega_0 + \sum_1^\infty \delta_i\omega_i$ as in Lemma 5.2 and using the triangle inequality,

$$\|\theta\|_{P_n{}^\circ} \ge \|\omega_0\|_{P_n{}^\circ} - \sum_{1}^{\infty}\delta^i\,\|\omega_i\|_{P_n{}^\circ} \ge 1-\varepsilon/2-2\delta(1-\delta)^{-1} \ge 1-\varepsilon$$

which implies (2.2). □

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