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STOCHASTIC FLOWS OF DIFFEOMORPHISMS FOR ONE-DIMENSIONAL SDE WITH DISCONTINUOUS DRIFT

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Abstract
The existence of a stochastic flow of class $C^{1,\alpha}$, for $\alpha < \frac{1}{2}$, for a 1-dimensional SDE will be proved under mild conditions on the regularity of the drift. The diffusion coefficient is assumed constant for simplicity, while the drift is an autonomous BV function with distributional derivative bounded from above or from below. To reach this result the continuity of the local time with respect to the initial datum will also be proved.

1 Introduction

The problem of the existence and smoothness of the stochastic flow under conditions of low regularity of the coefficients has been much studied. Apart from the intrinsic interest of the problem, there is an interest due to the range of possible applications of these results to PDE theory. For example, in [3] the uniqueness of the stochastic linear transport equation with Hölder continuous drift was proved, through new results about stochastic flows of class $C^{1,\alpha}$. Because of the greater regularity of stochastic flows compared to deterministic flows, there are cases in which a PDE admits infinitely many solutions in the deterministic case, but it becomes well posed if it is perturbed by a stochastic noise. In addition in [2] it is proved that in some cases, through a zero-noise limit, it is possible to find a criterion to select one particular solution.
We consider an equation of the form

$$\begin{cases} dX_t^x = b(X_t^x)dt + dW_t \\ X^x(0) = x \end{cases} \tag{1}$$

A complete proof of existence of the stochastic flows of class $C^{1,\alpha}$ is known only when b is Hölder continuous and bounded, see [3]. In the 1-dimensional case, an important example that deals with discontinuous b has been studied in [5]. Moreover there are preliminary results in [6]. The class of bounded variation (BV) fields b emerges from these works as a natural candidate for the flow property, although only a few partial properties have been proved. Moreover, BV fields are the most general class considered also in the deterministic literature, see [1]: in any dimension,

when $b \in BV$ and the negative part of the distributional divergence of b is bounded, a generalized notion of flow exists and is unique.
The aim of this work is to give a complete proof of existence of the stochastic flows of class $C^{1,\alpha}$, for $\alpha < \frac{1}{2}$, in dimension one, when $b \in BV$ and the positive or the negative part of the distributional derivative of b is bounded. This result, although restricted to the 1-dimensional case, is stronger than the deterministic one both because we accept a bound on b' from any side, and because we construct a flow of class $C^{1,\alpha}$, not only a generalized flow.
The partial results of the paper [6] suggest the problem whether $b \in BV$ is sufficient. We cannot reach this result without a one-side control on b'. The fact that a similar assumption is imposed in [1] is maybe an indication that it is not possible to avoid it.

2 Flow of homeomorphisms and known results

All results contained in this paper will be proved under the following hypothesis:

1. $b \in BV(\mathbb{R})$ and $b = b^1 - b^2$, with b^1 and b^2 increasing and bounded functions
2. $b^1 \in W^{1,\infty}$ or $b^2 \in W^{1,\infty}$

We will first suppose $b^1 \in W^{1,\infty}$. Under this hypothesis we will prove that the local time of the stochastic differential equations (SDE) solutions is Hölder continuous with respect to the initial data. Thanks to this result we will prove the existence of the stochastic flow of class $C^{1,\alpha}$, for $\alpha < \frac{1}{2}$. At the end of the paper, using standard facts on the backward equations, we will show that the results proved hold if we replace the hypothesis $b^1 \in W^{1,\infty}$ with the hypothesis $b^2 \in W^{1,\infty}$.
A result about one-dimension stochastic flows, under the hypothesis $b \in BV$, was given in [6]. There it was first proved the non-coalescence property through an elegant proof different from the one proposed in these notes. Then, the flow continuity was proved, and this property, together with the continuity of the flow of the backward equation, implies the homeomorphic property of the flow. However the proof of the continuity of the flow appears to be incomplete. Indeed, in order to apply Kolmogorov's lemma, the following inequality is shown:

$$E\left[\sup_{0\le s\le t}(X^x_{s\wedge\tau} - X^y_{s\wedge\tau})^2\right] \le n(x-y)^2$$

where τ is a stopping time depending on x and y. This doesn't appear sufficient to apply Kolmogorov's lemma.
Note that, as a standard consequence of the pathwise uniqueness we have that $\forall h > 0$, a.s., $X^{x+h}_t \ge X^x_t$. Moreover, assuming $b^1 \in W^{1,\infty}$, we have

$$(X^{x+h}_t - X^x_t) - (X^{x+h}_s - X^x_s) = \int_s^t b(X^{x+h}_r) - b(X^x_r)dr \le \int_s^t \|Db^1\|_\infty (X^{x+h}_r - X^x_r)dr$$

From this inequality, using Gronwall's lemma we obtain:

$$(X^{x+h}_t - X^x_t) \le e^{(t-s)\|Db^1\|_\infty}(X^{x+h}_s - X^x_s)$$

This fact, together with the proof of the non-coalescence property, contained in [6], is sufficient to prove the existence of a stochastic flow of homeomorphisms. Therefore the proof of the homeomorphic property contained in [6] can be corrected easily under our hypothesis 1 and 2.

However we are interested in stronger results about smoothness of the flow. In particular we are interested in the smoothness of the inverse flow, which is a basic ingredient, for instance, in the analysis of stochastic transport equations. While the homeomorphic property implies only the continuity of the inverse flow, we will prove that the inverse flow is of class $C^{1,\alpha}$ for $\alpha < \frac{1}{2}$.

Notation

Throughout the paper we will assume given a stochastic basis with a 1-dimensional Brownian motion $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathscr{F}, P, W_t)$. Moreover, for each $0 \leq s < t$ we denote by $\mathscr{F}_{s,t}$ the completed σ-algebra generated by $W_u - W_r$ for $s \leq r \leq u \leq t$. We will use the following notation: $L = \|b\|_\infty$, $K = \|Db^1\|_\infty$, $b^* = b^1 + b^2$, $L^* = \|b^*\|_\infty$.

We will denote by X_t^x the unique solution of the stochastic differential equation (1).

3 Local-time continuity with respect to the initial data

Definition 3.1. *Let X_t^x be the unique solution of equation (1), and let $a \in \mathbb{R}$. We will denote by $L_t^a(X^x)$ its local time at a, i.e. the continuous and increasing process such that*

$$|X_t^x - a| = |x - a| + \int_0^t \mathrm{sgn}(X_s^x - a) dX_s^x + L_t^a(X^x)$$

Further details about local time can be found in [8].

Remark 1. Recall the following inequality which is used, for example, to prove the continuity with respect to (a, t) of the local time: Let $X_t = X_0 + A_t + M_t$ be a continuous semimartingale, where M_t is a continuous local martingale, vanishing in 0 and A_t is a continuous process with bounded variation, vanishing in 0. Suppose that $\sup_{t\leq T} |M_t| \vee \sup_{t\leq T} |A_t| \leq K$. Then it holds:

$$E\left[\left|\int_0^t \mathbb{1}_{\{a<X_s\leq b\}} d\langle M\rangle_s\right|^p\right] \leq C_{p,K}|a - b|^p$$

Theorem 3.2. *There exists a modification of $L_t^a(X^x)$ which is jointly continuous in (a, t, x), and it is Hölder continuous in (a, x), of order α, for $\alpha < \frac{1}{2}$.*

Proof. Define an increasing sequence of stopping times as follows:

$$T_n := \inf\{t \geq 0 : |W_t| \vee tL \geq n\}$$

We have $T_n \uparrow \infty$ a.s. Denote by X_{T_n} the unique stopped solution of equation (1). It is sufficient to prove the theorem for $L_{t\wedge T_n}^a(X^x)$. Note that $\forall t \geq 0$ $|X_{t\wedge T_n} - x| \leq 2n$, and X_{T_n} satisfies the hypothesis of remark 1. By definition of $L_t^a(X^x)$ it follows:

$$L_{t\wedge T_n}^a(X^x) = |X_{t\wedge T_n}^x - a| - |x - a| - \int_0^{t\wedge T_n} \mathrm{sgn}(X_s^x - a) b(X_s^x) ds - \int_0^{t\wedge T_n} \mathrm{sgn}(X_s^x - a) dW_s$$

Thanks to the inequality $|X_t^x - X_t^y| \leq e^{Kt}|x - y|$, which holds a.s. the first term on the right hand admits a modification jointly continuous in (a, t, x), and lipschitz continuous in (a, x). In particular it is Hölder continuous in (a, x), of order α, for $\alpha < \frac{1}{2}$. The second term is obviously continuous in (a, t, x) and Hölder continuous in (a, x), of order α, for $\alpha < \frac{1}{2}$. We now prove that the third one admits a modification jointly continuous in (a, t, x), and Hölder continuous in (a, x),

of order α, for $\alpha < \frac{1}{2}$. We will apply Kolmogorov's lemma to the space $C([0,\infty))$, endowed with the sup norm. It will be useful to apply remark 1.
Let $(a,x) \in \mathbb{R}^2$ and $(b,y) \in \mathbb{R}^2$. We will consider only the case $b > a$ and $y > x$. In the other cases the following estimates are similar. It holds:

$$E\left[\sup_{t\geq 0}\left|\int_0^{t\wedge T_n} \operatorname{sgn}(X_s^x - a)b(X_s^x) - \operatorname{sgn}(X_s^y - b)b(X_s^y)ds\right|^p\right]$$

$$\leq C_pE\left[\sup_{t\geq 0}\left|\int_0^{t\wedge T_n} \operatorname{sgn}(X_s^x - a)b(X_s^x) - \operatorname{sgn}(X_s^x - b)b(X_s^x)ds\right|^p\right]$$

$$+C_pE\left[\sup_{t\geq 0}\left|\int_0^{t\wedge T_n} \operatorname{sgn}(X_s^x - b)b(X_s^x) - \operatorname{sgn}(X_s^y - b)b(X_s^y)ds\right|^p\right]$$

$$\leq C_pE\left[\left|2L\int_0^{T_n} \mathbb{1}_{a\leq X_s^x<b}ds\right|^p\right] + C_pE\left[\left|\int_0^{T_n} |b(X_s^x)(\operatorname{sgn}(X_s^x - b) - \operatorname{sgn}(X_s^y - b))|ds\right|^p\right]$$

$$+C_pE\left[\left|\int_0^{T_n} |\operatorname{sgn}(X_s^y - b)(b(X_s^x) - b(X_s^y))|ds\right|^p\right] \leq C_pE\left[\left|2L\int_0^{T_n} \mathbb{1}_{a\leq X_s^x<b}ds\right|^p\right]$$

$$+C_pE\left[\left|2L\int_0^{T_n} \mathbb{1}_{X_s^x\leq b<X_s^y}ds\right|^p\right] + C_pE\left[\left|\int_0^{T_n} |b(X_s^x) - b(X_s^y)|ds\right|^p\right]$$

$$\leq C_pE\left[\left|2L\int_0^{T_n} \mathbb{1}_{a\leq X_{s\wedge T_n}^x<b}ds\right|^p\right] + C_pE\left[\left|2L\int_0^{T_n} \mathbb{1}_{b-e^{KT}(y-x)<X_{s\wedge T_n}^x\leq b}ds\right|^p\right]$$

$$+C_pE\left[\left|\int_0^{T_n} b^*(X_s^y) - b^*(X_s^x)ds\right|^p\right]$$

$$\leq C_{p,n}L^p|a-b|^p + C_{p,n}L^p|x-y|^p + C_pE\left[\left|\int_0^{T_n} b^*(X_s^y) - b^*(X_s^x)ds\right|^p\right]$$

We need to estimate the last term to apply Kolmogorov's lemma: define $h = e^{KT}(y-x)$; let f be such that $f^{''}(r) = b^*(hr+h) - b^*(hr)$, and $f^{'}(r) = -L^* + \int_{-\infty}^r f^{''}(s)ds$. Note that $f^{''}(r) \geq 0\ \forall r$, and that $\int_{-\infty}^{+\infty} f^{''}(s)ds = \lim_{r\to+\infty} b^*(r) - b^*(-r) \leq 2L^*$. So we have $|f^{'}(r)| \leq L^*\ \forall r \in \mathbb{R}$. Using Itô formula and the boundness of $f^{'}$ we will obtain the L^p-boundness of $\frac{1}{h}\int_0^{T_n} f^{''}\left(\frac{X_s^x}{h}\right)ds$. Indeed we have

$$\frac{1}{2}\left|\int_0^{T_n} b^*(X_s^y) - b^*(X_s^x)ds\right| \leq \frac{1}{2}\int_0^{T_n} b^*(X_s^x + h) - b^*(X_s^x)ds = \frac{1}{2}\int_0^{T_n} f^{''}\left(\frac{X_s^x}{h}\right)ds$$

$$\leq h^2\left|f\left(\frac{X_{T_n}^x}{h}\right) - f\left(\frac{x}{h}\right)\right| + h\left|\int_0^{T_n} f^{'}\left(\frac{X_s^x}{h}\right)b(X_s^x)ds\right| + h\left|\int_0^{T_n} f^{'}\left(\frac{X_s^x}{h}\right)dW_s\right|$$

$$\leq L^* h \left| X_{T_n} - x \right| + L^* h \int_0^{T_n} |b(X_s^x)| ds + h \left| \int_0^{T_n} f^{'} \left(\frac{X_s^x}{h} \right) dW_s \right|$$

$$\leq L^* h(2n+n) + h \left| \int_0^{T_n} f^{'} \left(\frac{X_s^x}{h} \right) dW_s \right|$$

Thus we have:

$$E\left[\left| \int_0^{T_n} b^*(X_s^y) - b^*(X_s^x) ds \right|^p \right] \leq h^p C_{n,p}^{'} \left[1 + E\left[\left| \int_0^{T_n} \left(f^{'} \left(\frac{X_s^x}{h} \right) \right)^2 ds \right|^{\frac{p}{2}} \right] \right] \leq h^p C_{n,p}^{''}$$

Therefore, thanks to Kolmogorov's lemma we have proved that

$$\int_0^{t\wedge T_n} \operatorname{sgn}(X_s^x - a) b(X_s^x) ds$$

admits a modification jointly continuous in (a,t,x), and Hölder continuous in (a,x), of order α, for $\alpha < \frac{1}{2}$.

To complete the proof we have to show that $\int_0^{t\wedge T_n} \operatorname{sgn}(X_s^x - a) dW_s$ admits a modification jointly continuous in (a,t,x), and Hölder continuous in (a,x), of order α, for $\alpha < \frac{1}{2}$. We have:

$$E\left[\sup_{t\geq 0} \left| \int_0^{t\wedge T_n} \operatorname{sgn}(X_s^x - a) - \operatorname{sgn}(X_s^y - b) dWs \right|^p \right]$$

$$\leq C_p E\left[\left(\int_0^{T_n} |\operatorname{sgn}(X_s^x - a) - \operatorname{sgn}(X_s^y - b)|^2 ds \right)^{\frac{p}{2}} \right]$$

$$\leq C_p E\left[\left(\int_0^{T_n} |\operatorname{sgn}(X_s^x - a) - \operatorname{sgn}(X_s^x - b)|^2 ds \right)^{\frac{p}{2}} \right]$$

$$+ C_p E\left[\left(\int_0^{T_n} |\operatorname{sgn}(X_s^x - b) - \operatorname{sgn}(X_s^y - b)|^2 ds \right)^{\frac{p}{2}} \right]$$

$$\leq C_p E\left[\left(\int_0^{T_n} \mathbb{1}_{a \leq X_s^x < b} ds \right)^{\frac{p}{2}} \right] + C_p E\left[\left(\int_0^{T_n} \mathbb{1}_{b - e^{KT}(y-x) < X_s^x \leq b} ds \right)^{\frac{p}{2}} \right]$$

$$\leq C_{p,n} |a-b|^{\frac{p}{2}} + C_{p,n} |x-y|^{\frac{p}{2}}$$

This inequalities and Kolmogorov's lemma prove that $\int_0^{t\wedge T_n} \operatorname{sgn}(X_s^x - a) dW_s$ admits a modification jointly continuous in (a,t,x), and Hölder continuous in (a,x), of order α, for $\alpha < \frac{1}{2}$. The proof is complete. □

From now on we will consider only the continuous version of $L_t^a(X^x)$.

Corollary 3.3. *Let $T \geq 0$ and $x \leq y$. Then the process*

$$s \to \sup_{t\in[0,T]} \sup_{u\in[x,y]} \sup_{a\in\mathbb{R}} L^a_{t+s}(X^u) - L^a_t(X^u) \tag{2}$$

is continuous.

Proof. Note that $\forall s \leq t$, and $\forall u \in \mathbb{R}$ it holds $|X^u_s - u| \leq Lt + \sup_{s\leq t}|W_s|$.
Thus $a \notin [u - Lt - \sup_{s\leq t}|W_s|, u + Lt + \sup_{s\leq t}|W_s|]$ implies $L^a_t(X^u) = 0$. Denote by A_s the random compact set $[x - L(T+s) - \sup_{r\leq T+s}|W_r|, y + L(T+s) + \sup_{r\leq T+s}|W_r|]$. Note that a.s. $s < r$ implies $A_s \subset A_r$. Thus $\forall s \leq r$ it holds:

$$\sup_{t\in[0,T]} \sup_{u\in[x,y]} \sup_{a\in\mathbb{R}} L^a_{t+s}(X^u) - L^a_t(X^u) = \sup_{t\in[0,T]} \sup_{u\in[x,y]} \sup_{a\in A_r} L^a_{t+s}(X^u) - L^a_t(X^u)$$

Thanks to this equality and to the compactness of $[0,T]\times[x,y]\times A_r$, the continuity of the process (2) is proved on the interval $[0,r]$. Because of the arbitrariness of r the claim is proved. □

4 Existence of the stochastic flow of diffeomorphisms

We will now prove the non-coalescence property of the solutions of equation (1). This result has been already proved in [6] under more general hypothesis. However, for the sake of completeness we will give a proof based on the continuity of $L^a_t(X^x)$. The following lemma, which appears in [8], and in [7] with a complete proof, will be useful.

Lemma 4.1. *Let X be a continuous semimartingale, and denote by $\langle X\rangle_t$, its quadratic variation. Let $f : \mathbb{R}^+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a bounded measurable function. Then a.s., $\forall t \geq 0$*

$$\int_0^t f(s, X_s, \cdot)d\langle X\rangle_t = \int_{\mathbb{R}} da \int_0^t f(s, a, \cdot)dL^a_s(X)$$

Proposition 4.2. $\forall x \in \mathbb{R}$, $\forall h > 0$, *and* $T \geq 0$, *a.s.* $X^{x+h}_T - X^x_T > 0$.

Proof. Fix $x \in \mathbb{R}$, $h > 0$ and $T \geq 0$. From corollary 3.3, it follows that the process $s \to \sup_{t\in[0,T]} \sup_{a\in\mathbb{R}} L^a_{t+s}(X^x) - L^a_t(X^x)$, is continuous and vanishing in 0. Therefore $\forall \varepsilon \in (0,1)$ a.s. exists $s_{\varepsilon,x}(\omega) > 0$ such that $s \in [0, s_{\varepsilon,x}(\omega)]$ implies $L^a_{t+s}(X^x) - L^a_t(X^x) < \frac{\varepsilon}{2L^*(1+\varepsilon)}$ $\forall a \in \mathbb{R}$ and $t \in [0,T]$. So, a.s. it holds $\forall t \in [0,T]$:

$$\inf_{r\in[t,t+s_{\varepsilon,x}(\omega)]} (X^{x+h}_r - X^x_r) - (X^{x+h}_t - X^x_t) \geq \int_t^{t+s_{\varepsilon,x}(\omega)} b^*(X^x_u) - b^*(X^x_u + (X^{x+h}_t - X^x_t))du$$

$$= \int_{\mathbb{R}} \left[b^*(a) - b^*\left(a + \left(X^{x+h}_t - X^x_t\right)\right)\right] \times \left[L^a_{t+s_{\varepsilon,x}(\omega)}(X^x) - L^a_t(X^x)\right] da$$

$$\geq -\|L^{\cdot}_{t+s_{\varepsilon,x}(\omega)}(X^x) - L^{\cdot}_t(X^x)\|_\infty \times \|b^*(\cdot) - b^*\left(\cdot + \left(X^{x+h}_t - X^x_t\right)\right)\|_1$$

$$\geq -\left(\sup_{a\in\mathbb{R}} L^a_{t+s_{\varepsilon,x}(\omega)}(X^x) - L^a_t(X^x)\right) \times 2L^*\left(X^{x+h}_t - X^x_t\right) \geq -\frac{\varepsilon}{1+\varepsilon}(X^{x+h}_t - X^x_t)$$

This inequality implies:

$$\inf_{r\in[t,t+s_{\varepsilon,x}(\omega)]}(X_r^{x+h}-X_r^x)\geq\frac{(X_t^{x+h}-X_t^x)}{1+\varepsilon}\tag{3}$$

In the same way we obtain:

$$\sup_{r\in[t,t+s_{\varepsilon,x}(\omega)]}(X_r^{x+h}-X_r^x)\leq(1+\varepsilon)(X_t^{x+h}-X_t^x)\leq\frac{(X_t^{x+h}-X_t^x)}{1-\varepsilon}\tag{4}$$

In particular a.s. $N_{\varepsilon,T,x}(\omega):=\lceil\frac{T}{s_{\varepsilon,x}(\omega)}\rceil<\infty$, and thus a.s. we have

$$X_T^{x+h}-X_T^x\geq h\left(\frac{1}{1+\varepsilon}\right)^{N_{\varepsilon,T,x}(\omega)}>0$$

□

Remark 2. Using corollary 3.3, with the same argument of the preceding proof, it is possible to prove that given an interval $[x,y]$, $\forall\varepsilon>0$ a.s. exists $s_{\varepsilon,x,y}(\omega)>0$ such that $s\in[0,s_{\varepsilon,x,y}(\omega)]$ implies $L^a_{t+s}(X^u)-L^a_t(X^u)<\frac{\varepsilon}{2L^*(1+\varepsilon)}$ $\forall a\in\mathbb{R}$, $t\in[0,T]$, and $u\in[x,y]$. This fact will be used in the next theorem, which is crucial to prove the existence of a flow of class $C^{1,\alpha}$.

Theorem 4.3. *Let $x,y,t\in\mathbb{R}$ such that $x<y$, and $t\geq0$. Then, a.s.*

$$\inf_{u\in[x,y]}\exp\left(\int_{\mathbb{R}}L^a_t(X^u)Db(da)\right)\leq\left(\frac{X_t^y-X_t^x}{y-x}\right)\leq\sup_{u\in[x,y]}\exp\left(\int_{\mathbb{R}}L^a_t(X^u)Db(da)\right)$$

Proof. **Step 1.** Fix $x<y$ and $t\geq0$. Fix $\varepsilon>0$, and let $s_{\varepsilon,x,y}(\omega)$ be defined as in remark 2. Moreover define $N_{\varepsilon,t,x,y}(\omega):=\lceil\frac{t}{s_{\varepsilon,x,y}(\omega)}\rceil$. Define , $\forall\,i\in\mathbb{N}$, $t_i(\omega)=(i\times s_{\varepsilon,x,y}(\omega))\wedge t$. Obviously we have, a.s., that $i\geq N_{\varepsilon,t,x,y}(\omega)$ implies $t_i(\omega)=t$. Define $g:\Omega\times\mathbb{R}^+\to R^+$ as:

$$g(h)=\sup_{r\in[0,h]}\sup_{z\in[x,y]}\sup_{s\in[0,t]}\sup_{a\in\mathbb{R}}|L_s^{r+a}(X^z)-L^a_s(X^z)|$$

Note that, with the same reasoning used in corollary 3.3 and remark 2, it is possible to show that g is a.s. continuous, increasing and vanishing in 0. Let $z,w\in[x,y]$ such that $z<w$. Then it holds:

$$\ln\left(\frac{X_t^w-X_t^z}{w-z}\right)=\int_0^t\frac{b(X_s^w)-b(X_s^z)}{X_s^w-X_s^z}ds=\sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}}\frac{b(X_s^w)-b(X_s^z)}{X_s^w-X_s^z}ds:=I_0^{\varepsilon,z,w}$$

Observe that in the last summation a.s. only a finite number of terms are different from 0. Define:

$$I_1^{\varepsilon,z,w}:=\sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}}\frac{b(X_s^w)-b(X_s^z)}{X_{t_i}^w-X_{t_i}^z}ds$$

$$I_2^{\varepsilon,z,w}:=\sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}}\frac{1}{X_{t_i}^w-X_{t_i}^z}\left[b\left(X_s^z+\left(\frac{X_{t_i}^w-X_{t_i}^z}{1+\varepsilon}\right)\right)-b(X_s^z)\right]ds$$

$$I_3^{\varepsilon,z,w} := \sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}} \frac{1}{X_{t_i}^w - X_{t_i}^z}\left[b^*\left(X_s^z + \left(\frac{X_{t_i}^w - X_{t_i}^z}{1-\varepsilon}\right)\right) - b^*\left(X_s^z + \left(\frac{X_{t_i}^w - X_{t_i}^z}{1+\varepsilon}\right)\right)\right] ds$$

Note that the following estimate holds:

$$I_2^{\varepsilon,z,w} - I_3^{\varepsilon,z,w} \le I_1^{\varepsilon,z,w} \le I_2^{\varepsilon,z,w} + I_3^{\varepsilon,z,w} \tag{5}$$

Finally define the random set:

$$A_{i,\varepsilon,z,w} := \left[-\frac{1}{1-\varepsilon}\left(X_{t_i}^w - X_{t_i}^z\right), -\frac{1}{1+\varepsilon}\left(X_{t_i}^w - X_{t_i}^z\right)\right]$$

and

$$\rho_{i,\varepsilon,z,w}^A(a) := \left(\frac{1-\varepsilon^2}{2\varepsilon\left(X_{t_i}^w - X_{t_i}^z\right)} \times \mathbb{1}_{A_{i,\varepsilon,z,w}}\right)(a)$$

The following properties are immediate: $\rho_{i,\varepsilon,z,w}^A \ge 0$, $\int_{\mathbb{R}} \rho_{i,\varepsilon,z,w}^A(a)da = 1$, and has support contained in $[-\frac{1}{1-\varepsilon}(w-z)e^{Kt}, 0]$. We will use the following notation: $\check{\rho}_{i,\varepsilon,z,w}^A(a) = \rho_{i,\varepsilon,z,w}^A(-a)$.
Similarly we define:

$$B_{i,\varepsilon,z,w} := \left[-\frac{1}{1+\varepsilon}\left(X_{t_i}^w - X_{t_i}^z\right), 0\right]$$

and

$$\rho_{i,\varepsilon,z,w}^B(a) := \left(\frac{1+\varepsilon}{\left(X_{t_i}^w - X_{t_1}^z\right)} \times \mathbb{1}_{B_{i,\varepsilon,z,w}}\right)(a)$$

$\rho_{i,\varepsilon,z,w}^B$ satisfies properties similar to those of $\rho_{i,\varepsilon,z,w}^A$. In particular its support is contained in $[-\frac{1}{1+\varepsilon}(w-z)e^{Kt}, 0]$.
Step 2. Thanks to estimates (3) and (4) we have:

$$|I_0^{\varepsilon,z,w} - I_1^{\varepsilon,z,w}| = \left|\sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}} \frac{b(X_s^w) - b(X_s^z)}{X_s^w - X_s^z} - \frac{b(X_s^w) - b(X_s^z)}{X_{t_i}^w - X_{t_i}^z} ds\right|$$

$$\le \varepsilon \sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}} \left|\frac{b(X_s^w) - b(X_s^z)}{X_{t_i}^w - X_{t_i}^z}\right| ds$$

Using the decomposition $b^* = 2b^1 - b$, and the relation, which holds for $\alpha \le \beta$, $|b(\beta) - b(\alpha)| \le b^*(\beta) - b^*(\alpha)$, we obtain:

$$|I_0^{\varepsilon,z,w} - I_1^{\varepsilon,z,w}| \le 2\varepsilon \sum_{i=0}^{\infty}\left(\int_{t_i}^{t_{i+1}} \frac{b^1(X_s^w) - b^1(X_s^z)}{X_{t_i}^w - X_{t_i}^z} ds\right) - \varepsilon \sum_{i=0}^{\infty}\left(\int_{t_i}^{t_{i+1}} \frac{b(X_s^w) - b(X_s^z)}{X_{t_i}^w - X_{t_i}^z} ds\right)$$

$$\le \frac{2\varepsilon}{1-\varepsilon} Kt - \varepsilon I_1^{\varepsilon,z,w} \tag{6}$$

Step 3. From the occupation time formula we have:

$$I_3^{\varepsilon,z,w} = \left| \sum_{i=0}^{\infty} \int_{\mathbb{R}} \frac{1}{(X_{t_i}^w - X_{t_i}^z)} b^* \left(a + \left(\frac{X_{t_i}^w - X_{t_i}^z}{1+\varepsilon} \right) \right) - b^* \left(a + \left(\frac{X_{t_i}^w - X_{t_i}^z}{1-\varepsilon} \right) \right) \right.$$
$$\left. \times (L_{t_{i+1}}^a(X^z) - L_{t_i}^a(X^z)) da \right|$$
$$= \left| \sum_{i=0}^{\infty} \int_{\mathbb{R}} \left(\frac{2\varepsilon}{1-\varepsilon^2} \right) D \left(b^* * \rho_{i,\varepsilon,z,w}^A \right)(a) \times (L_{t_{i+1}}^a(X^z) - L_{t_i}^a(X^z)) da \right|$$
$$= \left(\frac{2\varepsilon}{1-\varepsilon^2} \right) \left| \sum_{i=0}^{\infty} \int_{\mathbb{R}} \left(\check{\rho}_{i,\varepsilon,z,w}^A * (L_{t_{i+1}}^{\cdot}(X^z) - L_{t_i}^{\cdot}(X^z))(a) \right) Db^*(da) \right|$$
$$\leq \left(\frac{2\varepsilon}{1-\varepsilon^2} \right) \left| \sum_{i=0}^{\infty} \int_{\mathbb{R}} \left[\check{\rho}_{i,\varepsilon,z,w}^A * \left(L_{t_{i+1}}^{\cdot}(X^z) - L_{t_i}^{\cdot}(X^z) \right)(a) - \left(L_{t_{i+1}}^a(X^z) - L_{t_i}^a(X^z) \right) \right] \right.$$
$$\left. \times Db^*(da) \right| + \left(\frac{2\varepsilon}{1-\varepsilon^2} \right) \int_{\mathbb{R}} L_t^a(X^z) Db^*(da)$$

$$\leq \left(\frac{2\varepsilon}{1-\varepsilon^2} \right) \left| \int_{\mathbb{R}} L_t^a(X^z) Db^*(da) \right| + 4L^* \left(\frac{2\varepsilon}{1-\varepsilon^2} \right) g \left(\frac{1}{1-\varepsilon}(w-z)e^{Kt} \right) N_{\varepsilon,t,x,y} \tag{7}$$

Step 4. It holds:

$$I_2^{\varepsilon,z,w} = \sum_{i=0}^{\infty} \int_{\mathbb{R}} \frac{1}{(X_{t_i}^w - X_{t_i}^z)} \left[b \left(a + \left(\frac{X_{t_i}^w - X_{t_i}^z}{1+\varepsilon} \right) \right) - b(a) \right] \times \left[L_{t_{i+1}}^a(X^z) - L_{t_i}^a(X^z) \right] da$$
$$= \sum_{i=0}^{\infty} \int_{\mathbb{R}} D \left(b * \rho_{i,\varepsilon,z,w}^B \right)(a) \times \left[L_{t_{i+1}}^a(X^z) - L_{t_i}^a(X^z) \right] da$$
$$= \sum_{i=0}^{\infty} \int_{\mathbb{R}} \left(\left[L_{t_{i+1}}^{\cdot}(X^z) - L_{t_i}^{\cdot}(X^z) \right] * \check{\rho}_{i,\varepsilon,z,w}^B \right)(a) Db(da)$$

From this equality it follows:

$$|I_2^{\varepsilon,z,w} - \int_{\mathbb{R}} L_t^a(X^z) Db(da)|$$
$$= \left| \sum_{i=0}^{\infty} \int_{\mathbb{R}} \left\{ \left[L_{t_{i+1}}^{\cdot}(X^z) - L_{t_i}^{\cdot}(X^z) \right] * \check{\rho}_{i,\varepsilon,z,w}^B(a) - \left[L_{t_{i+1}}^a(X^z) - L_{t_i}^a(X^z) \right] \right\} Db(da) \right|$$

$$\leq 4L^* g \left(\frac{1}{1+\varepsilon}(w-z)e^{Kt} \right) N_{\varepsilon,t,x,y} \tag{8}$$

Step 5. From (5), and from (6), (7) and (8), follows that, assuming $\varepsilon \in (0, \frac{1}{2})$, there exists a constant M dependent only on x, y, t, and on the constants K, L and L^*, such that

$$|I_0^{\varepsilon,z,w} - \int_{\mathbb{R}} L_t^a(X^z)Db(da)| \leq M\varepsilon(1 + \sup_{u\in[x,y]} \sup_{a\in\mathbb{R}} L_t^a(X^u)) + Mg(M(w-z))N_{\varepsilon,t,x,y} \tag{9}$$

We call Δ a finite partition of $[x, y]$ if, for some $n \in \mathbb{N}$, $\Delta = \{x = z_0 < z_1 < \cdots < z_i < z_{i+1} < \cdots < z_n = y\}$. Moreover we define $|\Delta| := \max_{z_i\in\Delta\setminus y}(z_{i+1} - z_i)$. We denote by Λ the set of finite partition of $[x, y]$. Obviously it holds:

$$\sup_{\Delta\in\Lambda} \min_{z_i\in\Delta\setminus y} \left(\frac{X_t^{z_{i+1}} - X_t^{z_i}}{z_{i+1} - z_i} \right) \leq \left(\frac{X_t^y - X_t^x}{y - x} \right) \leq \inf_{\Delta\in\Lambda} \max_{z_i\in\Delta\setminus y} \left(\frac{X_t^{z_{i+1}} - X_t^{z_i}}{z_{i+1} - z_i} \right) \tag{10}$$

Using (9), we have, $\forall \varepsilon \in (0, \frac{1}{2})$:

$$\begin{aligned}
\inf_{\Delta\in\Lambda} \max_{z_i\in\Delta\setminus y} \left(\frac{X_t^{z_{i+1}} - X_t^{z_i}}{z_{i+1} - z_i} \right) &\leq \inf_{\Delta\in\Lambda} \max_{z_i\in\Delta\setminus y} \exp\left(\int_{\mathbb{R}} L_t^a(X^{z_i})Db(da) \right) \\
&\quad \times \exp\left(M\varepsilon(1 + \sup_{u\in[x,y]} \sup_{a\in\mathbb{R}} L_t^a(X^u)) + Mg(M|\Delta|)N_{\varepsilon,t,x,y} \right) \\
&\leq \sup_{u\in[x,y]} \exp\left(\int_{\mathbb{R}} L_t^a(X^u)Db(da) \right) \times \exp\left(M\varepsilon(1 + \sup_{u\in[x,y]} \sup_{a\in\mathbb{R}} L_t^a(X^u)) \right)
\end{aligned}$$

With the same reasoning we obtain:

$$\begin{aligned}
\sup_{\Delta\in\Lambda} \inf_{z_i\in\Delta\setminus y} \left(\frac{X_t^{z_{i+1}} - X_t^{z_i}}{z_{i+1} - z_i} \right) &\geq \inf_{z\in[x,y]} \exp\left(\int_{\mathbb{R}} L_t^a(X^z)Db(da) \right) \\
&\quad \times \exp\left(-M\varepsilon(1 + \sup_{u\in[x,y]} \sup_{a\in\mathbb{R}} L_t^a(X^u)) \right)
\end{aligned}$$

Thanks to the arbitrariness of ε, and thanks to relation (10) the proof is complete. □

Using the preceding theorem we can now prove the existence of the stochastic flow of class $C^{1,\alpha}$. All results we have proved refers to solutions starting from $t = 0$. This choice was made to simplify notations. In the next theorem we will treat the general case. So, we will consider solutions of the following equation:

$$\begin{cases} dX_t^{s,x} = b(X_t^{s,x})dt + dW_t \\ X^{s,x}(s) = x \end{cases} \tag{11}$$

Obviously for $X_t^{0,x}$ theorem 4.3 holds.

Theorem 4.4. *Assume conditions 1 and 2 of section 2, and let $T > 0$. Then there exists a map $(s, t, x, \omega) \to \phi_{s,t}(x)(\omega)$ defined for $0 \leq s \leq t \leq T$, $x \in \mathbb{R}$, $\omega \in \Omega$ with values in $\mathbb{R}$, such that*

1. *given any $0 \leq s \leq T$, $x \in \mathbb{R}$ the process $X^{s,x} = (X_t^{s,x}\ s \leq t \leq T)$ defined as $X_t^{s,x} = \phi_{s,t}(x)$ is a continuous $\mathscr{F}_{s,t}$−measurable solution of equation (11).*

2. *a.s. $\phi_{s,t}$ is a diffeomorphism $\forall 0 \le s \le t \le T$ and $\phi_{s,t}$, $\phi_{s,t}^{-1}$, $D\phi_{s,t}$, and $D\phi_{s,t}^{-1}$ are continuous in (s,t,x), and of class C^α in x, for $\alpha < \frac{1}{2}$.*
3. *a.s. $\phi_{s,t}(x) = \phi_{u,t}(\phi_{s,u}(x))$ $\forall 0 \le s \le u \le t \le T$ and $x \in \mathbb{R}$, and $\phi_{s,s}(x) = x$.*

Moreover an explicit expression for $D\phi_{s,t}(x)$ is given by:

$$D\phi_{s,t}(x) = \exp\left(\int_{\mathbb{R}} \left(L_t^a(X^{\phi^{-1}(0,s,x)}) - L_s^a(X^{\phi^{-1}(0,s,x)})\right) Db(da)\right)$$

Proof. **Step 1.** We first give the proof under the assumption $b^1 \in W^{1,\infty}$. It will take the first three steps. Let D be the set of dyadic numbers, and let $D^T := D \cap [0,T]$. Define $\forall x \in D$ and $\forall t \in D^T$, $\widetilde{X}_t^{0,x}(\omega) = X_t^x(\omega)$. Note the two following facts:

1. Being D a countable set, there exists a negligible set A_0 such that, $\forall \omega \notin A_0$ and $\forall x \in D$ $X_\cdot^x$ is a continuous solution of equation (1). In particular since it holds

$$X_t^x = x + \int_0^t b(X_s^x)ds + W_t$$

$\forall \omega \notin A_0$ and $x \in D$, the family of processes $\{X_\cdot^x\}_{x\in D}$ is uniformly equicontinuous.

2. Thanks to the countability of the set $D \times D \times D^T$, there exists a negligible set $A_1 \supset A_0$ such that $\forall \omega \notin A_1$ $\forall x, y \in D$ such that $x < y$, and $\forall t \in D^T$ it holds

$$\inf_{u\in[x,y]} \exp\left(\int_{\mathbb{R}} L_t^a(X^u)Db(da)\right) \le \left(\frac{\widetilde{X}_t^{0,y} - \widetilde{X}_t^{0,x}}{y-x}\right) \le \sup_{u\in[x,y]} \exp\left(\int_{\mathbb{R}} L_t^a(X^u)Db(da)\right)$$

These facts imply that, $\forall \omega \notin A_1$ is well-defined the continuous extension of $\widetilde{X}_\cdot^{0,\cdot}(\omega)$ on $\mathbb{R} \times [0,T]$. Denote by $\phi_{0,t}(x)(\omega)$ this extension. Note that, $\forall \omega \notin A_1$, the family $\{\phi_{0,\cdot}(x)(\omega)\}_{x\in\mathbb{R}}$ is uniformly equicontinuous; more precisely, $\forall \omega \notin A_1$, $\forall 0 \le s \le t \le T$

$$|\phi_{0,t}(x) - \phi_{0,s}(x)| \le (t-s)L + \sup_{r\in[0,T]} |W_{r+(t-s)} - W_r| \tag{12}$$

In particular we have that a.s. $|\phi_{0,t}(x) - x|$ is bounded in x. This fact, together with continuity in x, implies that, $\forall \omega \notin A_1$, $\phi_{0,t}(\cdot)$ is surjective. Moreover it's immediate to verify that, $\forall \omega \notin A_1$, and $\forall x, y \in \mathbb{R}$, such that $x < y$, it holds:

$$\inf_{u\in[x,y]} \exp\left(\int_{\mathbb{R}} L_t^a(X^u)Db(da)\right) \le \left(\frac{\phi_{0,t}(y) - \phi_{0,t}(x)}{y-x}\right) \le \sup_{u\in[x,y]} \exp\left(\int_{\mathbb{R}} L_t^a(X^u)Db(da)\right)$$

This fact, together with the continuity of $L_t^a(X^x)$ with respect to (a,t,x) implies that $\forall \omega \notin A_1$, $\forall 0 \le t \le T$, $\phi_{0,t}(x)(\omega)$ is differentiable in x, and its derivative is $\exp\left(\int_{\mathbb{R}} L_t^a(X^x)Db(da)\right)$. So $\forall t \in [0,T]$ a.s. $\phi_{0,t}$ is a surjective and differentiable function whose derivative is strictly positive everywhere, and therefore is a diffeomorphism. Moreover $\exp\left(\int_{\mathbb{R}} L_t^a(X^x)Db(da)\right)$ is of class C^α with respect to x for $\alpha < \frac{1}{2}$.

Step 2. We now prove that $\forall x \in \mathbb{R}$ a.s. $\phi_{0,t}(x) = X_t^x$ $\forall 0 \le t \le T$. Fix $x \in \mathbb{R}$. Because of the a.s. continuity of both $\phi_{0,t}(x)$ and X_t^x with respect to t, and because of the countability of D^T, it is sufficient to prove that $\forall t \in D^T$, a.s. $\phi_{0,t}(x) = X^x_t$. So fix $t \in D^T$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of dyadic numbers converging to x. By construction we have that $\forall \omega \notin A_1$ $\phi_{0,t}(x_n) = \widetilde{X}^{0,x_n}_t = X^{x_n}_t$ and $\phi_{0,t}(x) = \lim_{n\to\infty} \phi_{0,t}(x_n) = \lim_{n\to\infty} X^{x_n}_t$. Moreover thanks to theorem 4.3, there is a negligible set A^n_x such that $\forall \omega \notin A^n_x$ it holds

$$\inf_{u\in[x,x_n]} \exp\left(\int_{\mathbb{R}} L^a_t(X^u) Db(da)\right) \le \left(\frac{X^{x_n}_t - X^x_t}{x_n - x}\right) \le \sup_{u\in[x,x_n]} \exp\left(\int_{\mathbb{R}} L^a_t(X^u) Db(da)\right)$$

Denote by A_x the negligible set $\bigcup_{i=0}^{\infty} A^i_x$. Therefore $\forall \omega \notin A_x \cup A_1$ we have the equality

$$X^x_t = \lim_{n\to\infty} X^{x_n}_t = \phi_{0,t}(x)$$

This implies that $\phi_{0,t}(x)$ is a continuous $\mathscr{F}_{0,t}$−measurable solution of equation (1).
Step 3. We will denote by $\phi^{-1}_{0,t}(\omega)$ the inverse function of $\phi_{0,t}(\omega)$. Define $\forall 0 \le s \le t \le T$ $\phi_{s,t} = \phi_{0,t} \circ \phi^{-1}_{0,s}$. It's immediate to verify that property 3 if satisfied. It's also easy to show that $\phi_{s,t}$ is a diffeomorphism, and its inverse function is $\phi^{-1}_{s,t} = \phi_{0,s} \circ \phi^{-1}_{0,t}$. Moreover, let $\{(s_n, t_n, x_n)\}_{n\in\mathbb{N}}$ be a sequence such that $0 \le s_n \le t_n \le T$, $x_n \in \mathbb{R}$ and $(s_n, t_n, x_n) \to (s, t, x)$. Obviously we have

$$\phi_{s_n,t_n}(x_n) = (\phi_{0,t_n} \circ \phi^{-1}_{0,t}) \circ \phi_{s,t} \circ (\phi_{0,s} \circ \phi^{-1}_{0,s_n}(x_n))$$

Thanks to (12), $\phi_{0,t_n} \circ \phi^{-1}_{0,t}$ and $\phi_{0,s} \circ \phi^{-1}_{0,s_n}$ converge to the identity; this observation prove that $\phi_{s,t}(x)$ is jointly continuous in (s, t, x). This implies the continuity of $\phi^{-1}_{0,t}$ with respect to (t, x), and thus the continuity of $\phi^{-1}_{s,t}$ in (s, t, x). Having proved that the derivative of $\phi_{0,t}(x)$ is $\exp\left(\int_{\mathbb{R}} L^a_t(X^x) Db(da)\right)$, we have that

$$D\phi_{s,t}(x) = \exp\left(\int_{\mathbb{R}} \left(L^a_t(X^{\phi^{-1}(0,s,x)}) - L^a_s(X^{\phi^{-1}(0,s,x)})\right) Db(da)\right)$$

$$D\phi^{-1}_{s,t}(x) = \exp\left(-\int_{\mathbb{R}} \left(L^a_t(X^{\phi^{-1}(0,s,x)}) - L^a_s(X^{\phi^{-1}(0,s,x)})\right) Db(da)\right)$$

which are jointly continuous in (s, t, x), and of class C^α for $\alpha < \frac{1}{2}$. The property (2) is proved. We have already proved property (1) if $s = 0$. To prove property (1) in the general case, fix $x \in \mathbb{R}$ and $s \ge 0$. Thus applying the Itô formula, we obtain that $\phi_{s,t}(x)$ is a solution of equation (11). This fact, and the pathwise uniqueness imply property (1).
Step 4. We give now the proof under the condition $b^2 \in W^{1,\infty}$.
Let $\widetilde{W}_t := W_{T-t} - W_T$. Note that $\widetilde{W}$ is a Brownian motion and the completed σ-algebra generated by $\widetilde{W}_u - \widetilde{W}_r$ for $s \le r \le u \le t$ is $\widetilde{\mathscr{F}}_{s,t} := \mathscr{F}_{T-t,T-s}$. Note that $-b$ satisfies the conditions used in the first three steps. So there exists $\widetilde{\phi}_{s,t}(x)$, continuous $\widetilde{\mathscr{F}}_{s,t}$−measurable solution of the equation

$$\begin{cases} dX^{s,x}_t = -b(X^{s,x}_t)dt + d\widetilde{W}_t \\ X^{s,x}(s) = x \end{cases} \tag{13}$$

satisfying property (2) and (3).
Define, for $0 \le s \le t \le T$ $\psi_{s,t}(x)(\omega) := \widetilde{\phi}^{-1}_{T-t,T-s}(x)(\omega)$. It's immediate to verify that ψ satisfies property (2) and (3). Moreover it's easy to verify that, for any $x \in \mathbb{R}$ and $0 \le s \le T$, $\psi_{s,\cdot}(x)(\omega)$ is a continuous $\mathscr{F}_{s,t}$−measurable solution of the equation solution of problem (11). □

Remark 3. *A classic approach to the problem of the existence of a* $C^{1,\alpha}$ *stochastic flow is the use of Itô formula to remove the drift (see [9] and later works on this approach, or a variant in [3]). Nevertheless, under the assumptions of this paper, there are some difficulties to use this approach. Indeed, suppose it is possible to prove the existence of a solution of the following parabolic equation*

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\partial_{xx} u(t,x) + b(x)\partial_x u(t,x) = 0 \\ u(T,x) = x \end{cases} \tag{14}$$

and that the solution is sufficiently smooth, so that we can apply Itô formula:

$$du(t,X_t) = \partial_t u(t,X_t)dt + \partial_x u(t,X_t)b(X_t)dt + \partial_x u(t,X_t)dW_t + \frac{1}{2}\partial_{xx}u(t,X_t)dt = \partial_x u(t,X_t)dW_t$$

Suppose moreover that $u(t,\cdot)$ *is a diffeomorphism. Through the change of variable* $Y_t = u(t,X_t)$ *the original problem has been reduced to the problem of the existence of the flow of the following stochastic equation*

$$dY_t = \sigma(t,Y_t)dW_t$$

where $\sigma(t,y) = \partial_x u(t,u_t^{-1}(y))$*. To solve this problem using well-known theorems, we should prove* $\sigma \in C^{1,\alpha}$ *for some* $\alpha > 0$*. But* $\sigma \in C^{1,\alpha}$ *implies* $\partial_{xx}u(t,\cdot) \in C^\alpha$*. So the discontinuity of the term* $b(x)\partial_x u(t,x)$ *would be balanced by the term* $\partial_t u(t,x)$*. However, examples such that* $\partial_t u(t,x)$ *is continuous, and* $b(x)$ *is discontinuous can be shown. Consider, for example,* $b(x) = \mathrm{sgn}(x)$*. It is possible to prove that* $u(t,x) = E[X_T^{t,x}]$ *and* $\partial_t u(t,x) = P(X_T^{t,x} < 0) - P(X_T^{t,x} > 0)$*. This term, as shown in [4], is continuous in* x*. Thus we have obtained* $\partial_{xx}u(t,\cdot) \notin C^\alpha$*. So, even if it is possible to prove the existence and smoothness of the solution of equation* (14)*, it is not possible to prove that* $\sigma \in C^{1,\alpha}$*, and it is not possible to prove the existence of the stochastic flow through well-known theorems.*

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