

HSU-ROBBINS AND SPITZER'S THEOREMS FOR THE VARIATIONS OF FRACTIONAL BROWNIAN MOTION

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Submitted February 16, 2009, accepted in final form June 28, 2009

AMS 2000 Subject classification: 60G15, 60H05, 60F05, 60H07.

Keywords: multiple stochastic integrals, selfsimilar processes, fractional Brownian motion, Hermite processes, limit theorems, Stein's method.

Abstract

Using recent results on the behavior of multiple Wiener-Itô integrals based on Stein's method, we prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables related to the increments of the fractional Brownian motion.

1 Introduction

A famous result by Hsu and Robbins [7] says that if X_1, X_2, \dots is a sequence of independent identically distributed random variables with zero mean and finite variance and $S_n := X_1 + \dots + X_n$, then

$$\sum_{n \geq 1} P(|S_n| > \varepsilon n) < \infty$$

for every $\varepsilon > 0$. Later, Erdős ([3], [4]) showed that the converse implication also holds, namely if the above series is finite for every $\varepsilon > 0$ and X_1, X_2, \dots are independent and identically distributed, then $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. Since then, many authors extended this result in several directions. Spitzer's showed in [13] that

$$\sum_{n \geq 1} \frac{1}{n} P(|S_n| > \varepsilon n) < \infty$$

for every $\varepsilon > 0$ if and only if $\mathbf{E}X_1 = 0$ and $\mathbf{E}|X_1| < \infty$. Also, Spitzer's theorem has been the object of various generalizations and variants. One of the problems related to the Hsu-Robbins' and Spitzer's theorems is to find the precise asymptotic as $\varepsilon \rightarrow 0$ of the quantities $\sum_{n \geq 1} P(|S_n| > \varepsilon n)$ and $\sum_{n \geq 1} \frac{1}{n} P(|S_n| > \varepsilon n)$. Heyde [5] showed that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \sum_{n \geq 1} P(|S_n| > \varepsilon n) = \mathbf{E}X_1^2 \tag{1}$$

whenever $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. In the case when X is attracted to a stable distribution of exponent $\alpha > 1$, Spataru [12] proved that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{-\log \varepsilon} \sum_{n \geq 1} \frac{1}{n} P(|S_n| > \varepsilon n) = \frac{\alpha}{\alpha - 1}. \tag{2}$$

The purpose of this paper is to prove Hsu-Robbins and Spitzer’s theorems for sequences of correlated random variables, related to the increments of fractional Brownian motion, in the spirit of [5] or [12]. Recall that the fractional Brownian motion $(B_t^H)_{t \in [0,1]}$ is a centered Gaussian process with covariance function $R^H(t, s) = \mathbf{E}(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$. It can be also defined as the unique self-similar Gaussian process with stationary increments. Concretely, in this paper we will study the behavior of the tail probabilities of the sequence

$$V_n = \sum_{k=0}^{n-1} H_q \left(n^H \left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right) \right) \tag{3}$$

where B is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ (in the sequel we will omit the superscript H for B) and H_q is the Hermite polynomial of degree $q \geq 1$ given by $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} (e^{-\frac{x^2}{2}})$. The sequence V_n behaves as follows (see e.g. [9], Theorem 1; the result is also recalled in Section 3 of our paper): if $0 < H < 1 - \frac{1}{2q}$, a central limit theorem holds for the renormalized sequence $Z_n^{(1)} = \frac{V_n}{c_{1,q,H} \sqrt{n}}$ while if $1 - \frac{1}{2q} < H < 1$, the sequence $Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}}$ converges in $L^2(\Omega)$ to a Hermite random variable of order q (see Section 2 for the definition of the Hermite random variable and Section 3 for a rigorous statement concerning the convergence of V_n). Here $c_{1,q,H}, c_{2,q,H}$ are explicit positive constants depending on q and H .

We note that the techniques generally used in the literature to prove the Hsu-Robbins and Spitzer’s results are strongly related to the independence of the random variables X_1, X_2, \dots . In our case the variables are correlated. Indeed, for any $k, l \geq 1$ we have

$$\mathbf{E} \left(H_q(B_{k+1} - B_k) H_q(B_{l+1} - B_l) \right) = \frac{1}{(q!)^2} \rho_H(k - l)$$

where the correlation function is $\rho_H(k) = \frac{1}{2} \left((k + 1)^{2H} + (k - 1)^{2H} - 2k^{2H} \right)$ which is not equal to zero unless $H = \frac{1}{2}$ (which is the case of the standard Brownian motion). We use new techniques based on the estimates for the multiple Wiener-Itô integrals obtained in [2], [10] via Stein’s method and Malliavin calculus. Concretely, we study in this paper the behavior as $\varepsilon \rightarrow 0$ of the quantities

$$\sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n) = \sum_{n \geq 1} \frac{1}{n} P \left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n} \right), \tag{4}$$

and

$$\sum_{n \geq 1} P(V_n > \varepsilon n) = \sum_{n \geq 1} P \left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n} \right), \tag{5}$$

if $0 < H < 1 - \frac{1}{2q}$ and of

$$\sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n^{2-2q(1-H)}) = \sum_{n \geq 1} \frac{1}{n} P \left(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)} \right) \tag{6}$$

and

$$\sum_{n \geq 1} P(V_n > \varepsilon n^{2-2q(1-H)}) = \sum_{n \geq 1} P(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}) \tag{7}$$

if $1 - \frac{1}{2q} < H < 1$. The basic idea in our proofs is that, if we replace $Z_n^{(1)}$ and $Z_n^{(2)}$ by their limits (standard normal random variable or Hermite random variable) in the above expressions, the behavior as $\varepsilon \rightarrow 0$ can be obtained by standard calculations. Then we need to estimate the difference between the tail probabilities of $Z_n^{(1)}, Z_n^{(2)}$ and the tail probabilities of their limits. To this end, we will use the estimates obtained in [2], [10] via Malliavin calculus and we are able to prove that this difference converges to zero in all cases. We obtain that, as $\varepsilon \rightarrow 0$, the quantities (4) and (6) are of order of $\log \varepsilon$ while the functions (5) and (7) are of order of ε^2 and $\varepsilon^{1-q(1-H)}$ respectively.

The paper is organized as follows. Section 2 contains some preliminaries on the stochastic analysis on Wiener chaos. In Section 3 we prove the Spitzer’s theorem for the variations of the fractional Brownian motion while Section 4 is devoted to the Hsu-Robbins theorem for this sequence. Throughout the paper we will denote by c a generic strictly positive constant which may vary from line to line (and even on the same line).

2 Preliminaries

Let $(W_t)_{t \in [0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0, 1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W . The basic reference is [11].

Let $f \in \mathcal{S}_m$ be an elementary function with m variables that can be written as

$$f = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} \mathbf{1}_{A_{i_1} \times \dots \times A_{i_m}}$$

where the coefficients satisfy $c_{i_1, \dots, i_m} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}([0, 1])$ are disjoint. For such a step function f we define

$$I_m(f) = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} W(A_{i_1}) \dots W(A_{i_m})$$

where we put $W(A) = \int_0^1 \mathbf{1}_A(s) dW_s$ if $A \in \mathcal{B}([0, 1])$. It can be seen that the mapping I_n constructed above from \mathcal{S}_m to $L^2(\Omega)$ is an isometry on \mathcal{S}_m , i.e.

$$\mathbf{E} [I_n(f)I_m(g)] = n! \langle f, g \rangle_{L^2([0,1]^n)} \text{ if } m = n \tag{8}$$

and

$$\mathbf{E} [I_n(f)I_m(g)] = 0 \text{ if } m \neq n.$$

Since the set \mathcal{S}_n is dense in $L^2([0, 1]^n)$ for every $n \geq 1$ the mapping I_n can be extended to an isometry from $L^2([0, 1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension.

We will need the following bound for the tail probabilities of multiple Wiener-Itô integrals (see [8], Theorem 4.1)

$$P(|I_n(f)| > u) \leq c \exp\left(\left(\frac{-cu}{\sigma}\right)^{\frac{2}{n}}\right) \tag{9}$$

for all $u > 0, n \geq 1$, with $\sigma = \|f\|_{L^2([0,1]^n)}$.

The Hermite random variable of order $q \geq 1$ that appears as limit in Theorem 1, point ii. is defined as (see [9])

$$Z = d(q, H)I_q(L) \tag{10}$$

where the kernel $L \in L^2([0, 1]^q)$ is given by

$$L(y_1, \dots, y_q) = \int_{y_1 \vee \dots \vee y_q}^1 \partial_1 K^H(u, y_1) \dots \partial_1 K^H(u, y_q) du.$$

The constant $d(q, H)$ is a positive normalizing constant that guarantees that $\mathbf{E}Z^2 = 1$ and K^H is the standard kernel of the fractional Brownian motion (see [11], Section 5). We will not need the explicit expression of this kernel. Note that the case $q = 1$ corresponds to the fractional Brownian motion and the case $q = 2$ corresponds to the Rosenblatt process.

3 Spitzer’s theorem

Let us start by recalling the following result on the convergence of the sequence V_n (3) (see [9], Theorem 1).

Theorem 1. *Let $q \geq 2$ an integer and let $(B_t)_{t \geq 0}$ a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Then, with some explicit positive constants $c_{1,q,H}, c_{2,q,H}$ depending only on q and H we have*

i. *If $0 < H < 1 - \frac{1}{2q}$ then*

$$\frac{V_n}{c_{1,q,H} \sqrt{n}} \xrightarrow[n \rightarrow \infty]{\text{Law}} N(0, 1) \tag{11}$$

ii. *If $1 - \frac{1}{2q} < H < 1$ then*

$$\frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \xrightarrow[n \rightarrow \infty]{L^2} Z \tag{12}$$

where Z is a Hermite random variable given by (10).

In the case $H = 1 - \frac{1}{2q}$ the limit is still Gaussian but the normalization is different. However we will not treat this case in the present work.

We set

$$Z_n^{(1)} = \frac{V_n}{c_{1,q,H} \sqrt{n}}, \quad Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \tag{13}$$

with the constants $c_{1,q,H}, c_{2,q,H}$ from Theorem 1.

Let us denote, for every $\varepsilon > 0$,

$$f_1(\varepsilon) = \sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n) = \sum_{n \geq 1} \frac{1}{n} P(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}) \tag{14}$$

and

$$f_2(\varepsilon) = \sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n^{2-2q(1-H)}) = \sum_{n \geq 1} \frac{1}{n} P(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}) \tag{15}$$

Remark 1. It is natural to consider the tail probability of order $n^{2-2q(1-H)}$ in (15) because the L^2 norm of the sequence V_n is in this case of order $n^{1-q(1-H)}$.

We are interested to study the behavior of $f_i(\varepsilon)$ ($i = 1, 2$) as $\varepsilon \rightarrow 0$. For a given random variable X , we set $\Phi_X(z) = 1 - P(X < z) + P(X < -z)$.

The first lemma gives the asymptotics of the functions $f_i(\varepsilon)$ as $\varepsilon \rightarrow 0$ when $Z_n^{(i)}$ are replaced by their limits.

Lemma 1. Consider $c > 0$.

i. Let $Z^{(1)}$ be a standard normal random variable. Then as

$$\frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) \rightarrow_{\varepsilon \rightarrow 0} 2.$$

ii. Let $Z^{(2)}$ be a Hermite random variable of order q given by (10). Then, for any integer $q \geq 1$

$$\frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \rightarrow_{\varepsilon \rightarrow 0} \frac{1}{1-q(1-H)}.$$

Proof: The case when $Z^{(1)}$ follows the standard normal law is hidden in [12]. We will give the ideas of the proof. We can write (see [12])

$$\sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) = \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx - \frac{1}{2} \Phi_{Z^{(1)}}(c\varepsilon) - \int_1^\infty P_1(x) d \left[\frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) \right].$$

with $P_1(x) = [x] - x + \frac{1}{2}$. Clearly as $\varepsilon \rightarrow 0$, $\frac{1}{\log \varepsilon} \Phi_{Z^{(1)}}(c\varepsilon) \rightarrow 0$ because $\Phi_{Z^{(1)}}$ is a bounded function and concerning the last term it is also trivial to observe that

$$\begin{aligned} & \frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) d \left[\frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) \right] \\ &= \frac{1}{-\log c\varepsilon} \left(- \int_1^\infty P_1(x) \left(\frac{1}{x^2} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx + c\varepsilon \frac{1}{2} x^{-\frac{1}{2}} \frac{1}{x} \Phi'_{Z^{(1)}}(c\varepsilon\sqrt{x}) \right) dx \right) \rightarrow_{\varepsilon \rightarrow 0} 0 \end{aligned}$$

since $\Phi_{Z^{(1)}}$ and $\Phi'_{Z^{(1)}}$ are bounded. Therefore the asymptotics of the function $f_1(\varepsilon)$ as $\varepsilon \rightarrow 0$ will be given by $\int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx$. By making the change of variables $c\varepsilon\sqrt{x} = y$, we get

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx = \lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} 2 \int_{c\varepsilon}^\infty \frac{1}{y} \Phi_{Z^{(1)}}(y) dy = \lim_{\varepsilon \rightarrow 0} 2 \Phi_{Z^{(1)}}(c\varepsilon) = 2.$$

Let us consider now the case of the Hermite random variable. We will have as above

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{-\log c\varepsilon} \left(\int_1^\infty \frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx - \int_1^\infty P_1(x) d \left[\frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) \right] \right) \end{aligned}$$

By making the change of variables $c\epsilon x^{1-q(1-H)} = y$ we will obtain

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{1}{-\log c\epsilon} \int_1^\infty \frac{1}{x} \Phi_{Z^{(2)}}(c\epsilon x^{1-q(1-H)}) dx \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{-\log c\epsilon} \frac{1}{1-q(1-H)} \int_{c\epsilon}^\infty \frac{1}{y} \Phi_{Z^{(2)}}(y) dy = \lim_{\epsilon \rightarrow 0} \frac{1}{1-q(1-H)} \Phi_{Z^{(2)}}(c\epsilon) = \frac{1}{1-q(1-H)} \end{aligned}$$

where we used the fact that $\Phi_{Z^{(2)}}(y) \leq y^{-2} \mathbf{E}|Z^{(2)}|^2$ and so $\lim_{y \rightarrow \infty} \log y \Phi_{Z^{(2)}}(y) = 0$.

It remains to show that $\frac{1}{-\log c\epsilon} \int_1^\infty P_1(x) d \left[\frac{1}{x} \Phi_{Z^{(2)}}(c\epsilon x^{1-q(1-H)}) \right]$ converges to zero as ϵ tends to 0 (note that actually it follows from a result by [1] that a Hermite random variable has a density, but we don't need it explicitly, we only use the fact that $\Phi_{Z^{(2)}}$ is differentiable almost everywhere). This is equal to

$$\begin{aligned} & \lim_{\epsilon} \frac{1}{-\log c\epsilon} \int_1^\infty P_1(x) c\epsilon (1-q(1-H)) x^{-q(1-H)-1} \Phi'_{Z^{(2)}}(c\epsilon x^{1-q(1-H)}) dx \\ &= c \frac{\epsilon}{-\log \epsilon} (c\epsilon)^{\frac{q(1-H)}{1-q(1-H)}} \int_{c\epsilon}^\infty P_1 \left(\left(\frac{y}{c\epsilon} \right)^{\frac{1}{1-q(1-H)}} \right) \Phi'_{Z^{(2)}}(y) y^{-\frac{1}{1-q(1-H)}} dy \\ &\leq c \frac{1}{-\log \epsilon} \int_{c\epsilon}^\infty P_1 \left(\left(\frac{1}{c\epsilon} \right)^{\frac{1}{1-q(1-H)}} \right) \Phi'_{Z^{(2)}}(y) dy \end{aligned}$$

which clearly goes to zero since P_1 is bounded and $\int_0^\infty \Phi'_{Z^{(2)}}(y) dy = 1$. ■

The next result estimates the limit of the difference between the functions $f_i(\epsilon)$ given by (14), (15) and the sequence in Lemma 1.

Proposition 1. *Let $q \geq 2$ and $c > 0$.*

i. *If $H < 1 - \frac{1}{2q}$, let $Z_n^{(1)}$ be given by (13) and let $Z^{(1)}$ be standard normal random variable. Then it holds*

$$\frac{1}{-\log c\epsilon} \left[\sum_{n \geq 1} \frac{1}{n} P(|Z_n^{(1)}| > c\epsilon \sqrt{n}) - \sum_{n \geq 1} \frac{1}{n} P(|Z^{(1)}| > c\epsilon \sqrt{n}) \right] \rightarrow_{\epsilon \rightarrow 0} 0.$$

ii. *Let $Z^{(2)}$ be a Hermite random variable of order $q \geq 2$ and $H > 1 - \frac{1}{2q}$. Then*

$$\frac{1}{-\log c\epsilon} \left[\sum_{n \geq 1} \frac{1}{n} P(|Z_n^{(2)}| > c\epsilon n^{1-q(1-H)}) - \sum_{n \geq 1} \frac{1}{n} P(|Z^{(2)}| > c\epsilon n^{1-q(1-H)}) \right] \rightarrow_{\epsilon \rightarrow 0} 0.$$

Proof: Let us start with the point i. Assume $H < 1 - \frac{1}{2q}$. We can write

$$\begin{aligned} & \sum_{n \geq 1} \frac{1}{n} P(|Z_n^{(1)}| > c\epsilon \sqrt{n}) - \sum_{n \geq 1} \frac{1}{n} P(|Z^{(1)}| > c\epsilon \sqrt{n}) \\ &= \sum_{n \geq 1} \frac{1}{n} \left[P(Z_n^{(1)} > c\epsilon \sqrt{n}) - P(Z^{(1)} > c\epsilon \sqrt{n}) \right] + \sum_{n \geq 1} \left[\frac{1}{n} P(Z_n^{(1)} < -c\epsilon \sqrt{n}) - P(Z^{(1)} < -c\epsilon \sqrt{n}) \right] \\ &\leq 2 \sum_{n \geq 1} \frac{1}{n} \sup_{x \in \mathbb{R}} \left| P(Z_n^{(1)} > x) - P(Z^{(1)} > x) \right|. \end{aligned}$$

It follows from [10], Theorem 4.1 that

$$\sup_{x \in \mathbb{R}} \left| P(Z_n^{(1)} > x) - P(Z^{(1)} > x) \right| \leq c \begin{cases} \frac{1}{\sqrt{n}}, & H \in (0, \frac{1}{2}] \\ n^{H-1}, & H \in [\frac{1}{2}, \frac{2q-3}{2q-2}) \\ n^{qH-q+\frac{1}{2}}, & H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q}). \end{cases} \tag{16}$$

and this implies that

$$\sum_{n \geq 1} \frac{1}{n} \sup_{x \in \mathbb{R}} \left| P(Z_n^{(i)} > x) - P(Z^{(i)} > x) \right| \leq c \begin{cases} \sum_{n \geq 1} \frac{1}{n\sqrt{n}}, & H \in (0, \frac{1}{2}] \\ \sum_{n \geq 1} n^{H-2}, & H \in [\frac{1}{2}, \frac{2q-3}{2q-2}) \\ \sum_{n \geq 1} n^{qH-q-\frac{1}{2}}, & H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q}). \end{cases} \tag{17}$$

and the last sums are finite (for the last one we use $H < 1 - \frac{1}{2q}$). The conclusion follows.

Concerning the point ii. (the case $H > 1 - \frac{1}{2q}$), by using a result in Proposition 3.1 of [2] we have

$$\sup_{x \in \mathbb{R}} \left| P(Z_n^{(i)} > x) - P(Z^{(i)} > x) \right| \leq c \left(\mathbf{E} |Z_n^{(2)} - Z^{(2)}|^2 \right)^{\frac{1}{2q}} \leq cn^{1-\frac{1}{2q}-H} \tag{18}$$

and as a consequence

$$\sum_{n \geq 1} \frac{1}{n} P(|Z_n^{(2)}| > c\epsilon n^{1-q(1-H)}) - \sum_{n \geq 1} \frac{1}{n} P(|Z^{(2)}| > c\epsilon n^{1-q(1-H)}) \leq c \sum_{n \geq 1} n^{-\frac{1}{2q}-H}$$

and the above series is convergent because $H > 1 - \frac{1}{2q}$. ■

We state now the Spitzer’s theorem for the variations of the fractional Brownian motion.

Theorem 2. Let f_1, f_2 be given by (14), (15) and the constants $c_{1,q,H}, c_{2,q,H}$ be those from Theorem 1.

i. If $0 < H < 1 - \frac{1}{2q}$ then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\log(c_{1,H,q}^{-1} \epsilon)} f_1(\epsilon) = 2.$$

ii. If $1 > H > 1 - \frac{1}{2q}$ then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\log(c_{2,H,q}^{-1} \epsilon)} f_2(\epsilon) = \frac{1}{1 - q(1 - H)}.$$

Proof: It is a consequence of Lemma 1 and Proposition 1. ■

Remark 2. Concerning the case $H = 1 - \frac{1}{2q}$, note that the correct normalization of V_n (3) is $\frac{1}{(\log n)\sqrt{n}}$. Because of the appearance of the term $\log n$ our approach is not directly applicable to this case.

4 Hsu-Robbins theorem for the variations of fractional Brownian motion

In this section we prove a version of the Hsu-Robbins theorem for the variations of the fractional Brownian motion. Concretely, we denote here by, for every $\varepsilon > 0$

$$g_1(\varepsilon) = \sum_{n \geq 1} P(|V_n| > \varepsilon n) \tag{19}$$

if $H < 1 - \frac{1}{2q}$ and by

$$g_2(\varepsilon) = \sum_{n \geq 1} P(|V_n| > \varepsilon n^{2-2q(1-H)}) \tag{20}$$

if $H > 1 - \frac{1}{2q}$. and we estimate the behavior of the functions $g_i(\varepsilon)$ as $\varepsilon \rightarrow 0$. Note that we can write

$$g_1(\varepsilon) = \sum_{n \geq 1} P(|Z_n^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n}), \quad g_2(\varepsilon) = \sum_{n \geq 1} P(|Z_n^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)})$$

with $Z_n^{(1)}, Z_n^{(2)}$ given by (13).

We decompose it as: for $H < 1 - \frac{1}{2q}$

$$g_1(\varepsilon) = \sum_{n \geq 1} P(|Z^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n}) + \sum_{n \geq 1} \left[P(|Z_n^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n}) - P(|Z^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n}) \right].$$

and for $H > 1 - \frac{1}{2q}$

$$g_2(\varepsilon) = \sum_{n \geq 1} P(|Z^{(2)}| > \varepsilon c_{2,q,H}^{-1} n^{1-q(1-H)}) + \sum_{n \geq 1} \left[P(|Z_n^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}) - P(|Z^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}) \right].$$

We start again by consider the situation when $Z_n^{(i)}$ are replaced by their limits.

Lemma 2. *i. Let $Z^{(1)}$ be a standard normal random variable. Then*

$$\lim_{\varepsilon \rightarrow 0} (c\varepsilon)^2 \sum_{n \geq 1} P(|Z^{(1)}| > c\varepsilon \sqrt{n}) = 1.$$

ii. Let $Z^{(2)}$ be a Hermite random variable with $H > 1 - \frac{1}{2q}$. Then

$$\lim_{\varepsilon \rightarrow 0} (c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} P(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}) = \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}.$$

Proof: The part i. is a consequence of the result of Heyde [5]. Indeed take $X_i \sim N(0, 1)$ in (1). Concerning part ii. we can write

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} (c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \\ &= \lim_{\varepsilon \rightarrow 0} (c\varepsilon)^{\frac{1}{1-q(1-H)}} \left[\int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx - \int_1^\infty P_1(x) d[\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})] \right] \\ &:= \lim_{\varepsilon \rightarrow 0} (A(\varepsilon) + B(\varepsilon)) \end{aligned}$$

with $P_1(x) = [x] - x + \frac{1}{2}$. Moreover

$$\begin{aligned} A(\varepsilon) &= (c\varepsilon)^{\frac{1}{1-q(1-H)}} \int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx \\ &= \frac{1}{1-q(1-H)} \int_{c\varepsilon}^\infty \Phi_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}-1} dy. \end{aligned}$$

Since $\Phi_{Z^{(2)}}(y) \leq y^{-2}$ we have $\Phi_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}} \rightarrow_{y \rightarrow \infty} 0$ and therefore

$$A(\varepsilon) = -\Phi_{Z^{(2)}}(c\varepsilon)(c\varepsilon)^{\frac{1}{1-q(1-H)}} - \int_{c\varepsilon}^\infty \Phi'_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}} dy$$

where the first terms goes to zero and the second to $\mathbf{E} |Z^{(2)}|^{\frac{1}{1-q(1-H)}}$. The proof that the term $B(\varepsilon)$ converges to zero is similar to the proof of Lemma 2, point ii. ■

Remark 3. The Hermite random variable has moments of all orders (in particular the moment of order $\frac{1}{1-q(1-H)}$ exists) since it is the value at time 1 of a selfsimilar process with stationary increments.

Proposition 2. i. Let $H < 1 - \frac{1}{2q}$ and let $Z_n^{(1)}$ be given by (13). Let also $Z^{(1)}$ be a standard normal random variable. Then

$$(c\varepsilon)^2 \sum_{n \geq 1} [P(|Z_n^{(1)}| > c\varepsilon \sqrt{n}) - P(|Z^{(1)}| > c\varepsilon \sqrt{n})] \rightarrow_{\varepsilon \rightarrow 0} 0$$

ii. Let $H > 1 - \frac{1}{2q}$ and let $Z_n^{(2)}$ be given by (13). Let $Z^{(2)}$ be a Hermite random variable. Then

$$(c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} [P(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}) - P(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)})] \rightarrow_{\varepsilon \rightarrow 0} 0.$$

Remark 4. Note that the bounds (16), (18) does not help here because the series that appear after their use are not convergent.

Proof of Proposition 2: Case $H < 1 - \frac{1}{2q}$. We have, for some $\beta > 0$ to be chosen later,

$$\begin{aligned} & \varepsilon^2 \sum_{n \geq 1} [P(|Z_n^{(1)}| > c\varepsilon\sqrt{n}) - P(|Z^{(1)}| > c\varepsilon\sqrt{n})] \\ = & \varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} [P(|Z_n^{(1)}| > c\varepsilon\sqrt{n}) - P(|Z^{(1)}| > c\varepsilon\sqrt{n})] \\ & + \varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} [P(|Z_n^{(1)}| > c\varepsilon\sqrt{n}) - P(|Z^{(1)}| > c\varepsilon\sqrt{n})] \\ := & I_1(\varepsilon) + J_1(\varepsilon). \end{aligned}$$

Consider first the situation when $H \in (0, \frac{1}{2}]$. Let us choose a real number β such that $2 < \beta < 4$. By using (16),

$$I_1(\varepsilon) \leq c\varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} n^{-\frac{1}{2}} \leq c\varepsilon^2 \varepsilon^{-\frac{\beta}{2}} \rightarrow_{\varepsilon \rightarrow 0} 0$$

since $\beta < 4$. Next, by using the bound for the tail probabilities of multiple integrals and since $\mathbf{E}|Z_n^{(1)}|^2$ converges to 1 as $n \rightarrow \infty$

$$\begin{aligned} J_1(\varepsilon) &= \varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} P(Z_n^{(1)} > c\varepsilon\sqrt{n}) \leq c\varepsilon^{-2} \sum_{n > [\varepsilon^{-\beta}]} \exp\left(\frac{-c\varepsilon\sqrt{n}}{\left(\mathbf{E}|Z_n^{(1)}|^2\right)^{\frac{1}{2}}}\right)^{\frac{2}{q}} \\ &\leq \varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} \exp\left(\left(-cn^{-\frac{1}{\beta}}\sqrt{n}\right)^{\frac{2}{q}}\right) \end{aligned}$$

and since converges to zero for $\beta > 2$. The same argument shows that $\varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} P(Z^{(1)} > c\varepsilon\sqrt{n})$ converges to zero.

The case when $H \in (\frac{1}{2}, \frac{2q-3}{2q-2})$ can be obtained by taking $2 < \beta < \frac{2}{H}$ (it is possible since $H < 1$) while in the case $H \in (\frac{2q-3}{2q-2}, 1 - \frac{1}{2q})$ we have to choose $2 < \beta < \frac{2}{qH - q + \frac{3}{2}}$ (which is possible because $H < 1 - \frac{1}{2q}$!).

Case $H > 1 - \frac{1}{2q}$. We have, with some suitable $\beta > 0$

$$\begin{aligned} & \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} [P(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}) - P(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)})] \\ = & \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n=1}^{[\varepsilon^{-\beta}]} [P(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}) - P(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)})] \\ & + \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n \geq [\varepsilon^{-\beta}]} [P(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}) - P(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)})] \\ := & I_2(\varepsilon) + J_2(\varepsilon). \end{aligned}$$

Choose $\frac{1}{1-q(1-H)} < \beta < \frac{1}{(1-q(1-H))(2-H-\frac{1}{2q})}$ (again, this is always possible when $H > 1 - \frac{1}{2q}$!). Then

$$I_2(\varepsilon) \leq c\varepsilon^{\frac{1}{1-q(1-H)}} \varepsilon^{(-\beta)(2-H-\frac{1}{2q})} \rightarrow_{\varepsilon \rightarrow 0} 0$$

and by (9)

$$J_2(\varepsilon) \leq c \sum_{n > [\varepsilon^{-\beta}]} \exp \left(\left(\frac{-c\varepsilon n^{1-q(1-H)}}{\left(\mathbf{E} |Z_n^{(2)}|^2 \right)^{\frac{1}{2}}} \right)^{\frac{2}{q}} \right) \leq c \sum_{n > [\varepsilon^{-\beta}]} \exp \left(c n^{-\frac{1}{\beta}} n^{1-q(1-H)} \right)^{\frac{2}{q}} \xrightarrow{\varepsilon \rightarrow 0} 0$$

■

We state the main result of this section which is a consequence of Lemma 2 and Proposition 2.

Theorem 3. Let $q \geq 2$ and let $c_{1,q,H}, c_{2,q,H}$ be the constants from Theorem 1. Let $Z^{(1)}$ be a standard normal random variable, $Z^{(2)}$ a Hermite random variable of order $q \geq 2$ and let g_1, g_2 be given by (19) and (20). Then

i. If $0 < H < 1 - \frac{1}{2q}$, we have $(c_{1,q,H}^{-1} \varepsilon)^2 g_1(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 1 = \mathbf{E} Z^{(1)}$.

ii. If $1 - \frac{1}{2q} < H < 1$ we have $(c_{2,q,H}^{-1} \varepsilon)^{\frac{1}{1-q(1-H)}} g_2(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \mathbf{E} |Z^{(2)}|^{\frac{1}{1-q(1-H)}}$.

Remark 5. In the case $H = \frac{1}{2}$ we retrieve the result (1) of [5]. The case $q = 1$ is trivial, because in this case, since $V_n = B_n$ and $\mathbf{E} V_n^2 = n^{2H}$, we obtain the following (by applying Lemma 1 and 2 with $q = 1$)

$$\frac{1}{\log \varepsilon} \sum_{n \geq 1} \frac{1}{n} P(|V_n| > \varepsilon n^{2H}) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{H}$$

and

$$\varepsilon^2 \sum_{n \geq 1} P(|V_n| > \varepsilon n^{2H}) \xrightarrow{\varepsilon \rightarrow 0} \mathbf{E} |Z^{(1)}|^{\frac{1}{H}}.$$

Remark 6. Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of i.i.d. centered random variable with finite variance and let $(a_i)_{i \geq 1}$ a square summable real sequence. Define $X_n = \sum_{i \geq 1} a_i \varepsilon_{n-i}$. Then the sequence $S_N = \sum_{n=1}^N [K(X_n) - \mathbf{E} K(X_n)]$ satisfies a central limit theorem or a non-central limit theorem according to the properties of the measurable function K (see [6] or [14]). We think that our tools can be applied to investigate the tail probabilities of the sequence S_N in the spirit of [5] or [12] at least in particular cases (for example, when ε_i represents the increment $W_{i+1} - W_i$ of a Wiener process because in this case ε_i can be written as a multiple integral of order one and X_n can be decomposed into a sum of multiple integrals. We thank the referee for mentioning the references [6] and [14].

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