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DEVIATION INEQUALITIES AND MODERATE DEVIATIONS FOR ESTIMATORS OF PARAMETERS IN AN ORNSTEIN-UHLENBECK PROCESS WITH LINEAR DRIFT

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Abstract

Some deviation inequalities and moderate deviation principles for the maximum likelihood estimators of parameters in an Ornstein-Uhlenbeck process with linear drift are established by the logarithmic Sobolev inequality and the exponential martingale method.

1 Introduction and main results

1.1 Introduction

We consider the following Ornstein-Uhlenbeck process

$$dX_t = (-\theta X_t + \gamma)dt + dW_t, \qquad X_0 = x \tag{1.1}$$

where W is a standard Brownian motion and θ, γ are unknown parameters with $\theta \in (0, +\infty)$. We denote by $P_{\theta,\gamma,x}$ the distribution of the solution of (1.1).

It is known that the maximum likelihood estimators (MLE) of the parameters θ and γ are (cf.

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[15])

$$\hat{\theta}_T = \frac{-T\int_0^T X_t dX_t + (X_T - x)\int_0^T X_t dt}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} = \theta + \frac{W_T\hat{\mu}_T - \int_0^T X_t dW_t}{T\hat{\sigma}_T^2}, \tag{1.2}$$

$$\hat{\gamma}_T = \frac{-\int_0^T X_t dt \int_0^T X_t dX_t + (X_T - x)\int_0^T X_t^2 dt}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} = \gamma + \frac{W_T}{T} + \frac{\hat{\mu}_T(W_T\hat{\mu}_T - \int_0^T X_t dW_t)}{T\hat{\sigma}_T^2}, \tag{1.3}$$

where

$$\hat{\mu}_T = \frac{1}{T}\int_0^T X_t dt, \qquad \hat{\sigma}_T^2 = \frac{1}{T}\int_0^T X_t^2 dt - \hat{\mu}_T^2. \tag{1.4}$$

It is known that $\hat{\theta}_T$ and $\hat{\gamma}_T$ are consistent estimators of θ and γ and have asymptotic normality (cf. [15]).

For $\gamma \equiv 0$ case, Florens-Landais and Pham([9]) calculated the Laplace functional of $(\int_0^T X_t dX_t, \int_0^T X_t^2 dt)$ by Girsanov's formula and obtained large deviations for $\hat{\theta}_T$ by Gärtner-Ellis theorem. Bercu and Rouault ([1]) presented a sharp large deviation for $\hat{\theta}_T$. Lezaud ([14]) obtained the deviation inequality of quadratic functional of the classical OU processes. We refer to [8] and [11] for the moderate deviations of some non-linear functionals of moving average processes and diffusion processes. In this paper we use the logarithmic Sobolev inequality (LSI) to study the deviation inequalities and the moderate deviations of $\hat{\theta}_T$ and $\hat{\gamma}_T$ for $\gamma \neq 0$ case.

1.2 Main results

Throughout this paper, let $\lambda_T, T \geq 1$ be a positive sequence satisfying

$$\lambda_T \to \infty, \quad \frac{\lambda_T}{\sqrt{T}} \to 0. \tag{1.5}$$

Theorem 1.1. *There exist finite positive constants C_0, C_1, C_2 and C_3 such that for all $r > 0$ and all $T \geq 1$,*

$$P_{\theta,\gamma,x}\left(|\hat{\theta}_T - \theta| \geq r\right) \leq C_0 \exp\left\{-C_1 r T E_{\theta,\gamma,x}(\hat{\sigma}_T^2)\min\{1, C_2 r\}\right\} + C_0 \exp\left\{-C_3 T E_{\theta,\gamma,x}(\hat{\sigma}_T^2)\right\}$$

and

$$P_{\theta,\gamma,x}\left(|\hat{\gamma}_T - \gamma| \geq r\right) \leq C_0 \exp\left\{-C_1 r T E_{\theta,\gamma,x}(\hat{\sigma}_T^2)\min\{1, C_2 r\}\right\} + C_0 \exp\left\{-C_3 T E_{\theta,\gamma,x}(\hat{\sigma}_T^2)\right\}.$$

Remark 1.1. *In this theorem and the remainder of the paper, all the constants involved depend on θ, γ and the initial point x.*

Theorem 1.2. *(1).* $\left\{P_{\theta,\gamma,x}\left(\sqrt{\frac{T}{\lambda_T}}(\hat{\theta}_T-\theta)\in\cdot\right),T\geq 1\right\}$ *satisfies the large deviation principle with speed* λ_T *and rate function* $I_1(u)=\frac{u^2}{4\theta}$*, that is, for any closed set* F *in* $\mathbb{R}$*,*

$$\limsup_{n\to\infty}\frac{1}{\lambda_T}\log P_{\theta,\gamma,x}\left(\sqrt{\frac{T}{\lambda_T}}(\hat{\theta}_T-\theta)\in F\right)\leq -\inf_{u\in F}\frac{u^2}{4\theta}$$

and open set G *in* $\mathbb{R}$*,*

$$\liminf_{n\to\infty}\frac{1}{\lambda_T}\log P_{\theta,\gamma,x}\left(\sqrt{\frac{T}{\lambda_T}}(\hat{\theta}_T-\theta)\in G\right)\geq -\inf_{u\in G}\frac{u^2}{4\theta}.$$

(2). $\left\{P_{\theta,\gamma,x}\left(\sqrt{\frac{T}{\lambda_T}}(\hat{\gamma}_T-\gamma)\in\cdot\right),T\geq 1\right\}$ *satisfies the large deviation principle with speed* λ_T *and rate function* $I_2(u)=\frac{\theta u^2}{2(\theta+2\gamma^2)}$*, that is, for any closed set* F *in* $\mathbb{R}$*,*

$$\limsup_{n\to\infty}\frac{1}{\lambda_T}\log P_{\theta,\gamma,x}\left(\sqrt{\frac{T}{\lambda_T}}(\hat{\gamma}_T-\gamma)\in F\right)\leq -\inf_{u\in F}\frac{\theta u^2}{2(\theta+2\gamma^2)}$$

and open set G *in* $\mathbb{R}$*,*

$$\liminf_{n\to\infty}\frac{1}{\lambda_T}\log P_{\theta,\gamma,x}\left(\sqrt{\frac{T}{\lambda_T}}(\hat{\gamma}_T-\gamma)\in G\right)\geq -\inf_{u\in G}\frac{\theta u^2}{2(\theta+2\gamma^2)}.$$

In $\gamma=0$ case, the deviation inequalities of quadratic functionals of the classical OU process are obtained in [14]. For the large deviations and the moderate deviations of $\hat{\theta}_T$, we refer to [1], [9] and [11]. The proofs of Theorem 1.1 and Theorem 1.2 are based on the LSI with respect to L^2-norm in the Wiener space and Herbst's argument (cf. [10], [12]).

2 Deviation inequalities

In this section, we give some deviation inequalities for the estimators $\hat{\theta}_T$ and $\hat{\gamma}_T$ by the logarithmic Sobolev inequality and the exponential martingale method. For deviation bounds for additive functionals of Markov processes, we refer to [3] and [18].

2.1 Moments

It is known that the solution of equation (1.1) has the following expression:

$$X_t=\left(x-\frac{\gamma}{\theta}\right)e^{-\theta t}+\frac{\gamma}{\theta}+e^{-\theta t}\int_0^t e^{\theta s}dW_s. \tag{2.1}$$

From this expression, it is easily seen that for any $t\geq 0$,

$$\mu_t:=E_{\theta,\gamma,x}(X_t)=\left(x-\frac{\gamma}{\theta}\right)e^{-\theta t}+\frac{\gamma}{\theta}, \tag{2.2}$$

$$\sigma_t^2:=\mathrm{Var}_{\theta,\gamma,x}(X_t)=\frac{1}{2\theta}(1-e^{-2\theta t}) \tag{2.3}$$

and for any $0 \le s \le t$,

$$\mathrm{Cov}_{\theta,\gamma,x}(X_s, X_t) = \frac{1}{2\theta}(1 - e^{-2\theta s})e^{-\theta(t-s)}. \tag{2.4}$$

Therefore

$$E_{\theta,\gamma,x}(\hat{\mu}_T) = \frac{1}{T}E_{\theta,\gamma,x}\left(\int_0^T X_t dt\right) = \frac{1}{\theta T}\left(x - \frac{\gamma}{\theta}\right)(1 - e^{-\theta T}) + \frac{\gamma}{\theta}, \tag{2.5}$$

$$\begin{aligned}\mathrm{Var}_{\theta,\gamma,x}\left(\hat{\mu}_T\right) =& \frac{1}{T^2}E_{\theta,\gamma,x}\left(\left(\int_0^T e^{-\theta t}\int_0^t e^{\theta s} dW_s dt\right)^2\right) \qquad (2.6)\\ =& \frac{1}{\theta^2 T^2}\left(T - \frac{1}{2\theta}(e^{-2\theta T} - 1) + \frac{2}{\theta}(e^{-\theta T} - 1)\right)\end{aligned}$$

and so for all $T \ge 1$,

$$\mathrm{Var}_{\theta,\gamma,x}\left(\hat{\mu}_T\right) \le \frac{1}{2\theta^3 T}(2\theta + 1) \tag{2.7}$$

and

$$\begin{aligned}E_{\theta,\gamma,x}(\hat{\sigma}_T^2) =& \frac{1}{2\theta} + \frac{1}{4\theta^2 T}(1 - e^{-2\theta T})\left(-1 + 2\theta\left(x - \frac{\gamma}{\theta}\right)^2\right)\\ &- \frac{1}{\theta^2 T^2}(1 - e^{-\theta T})^2\left(x - \frac{\gamma}{\theta}\right)^2(1 - e^{-\theta T})\\ &- \frac{1}{\theta^2 T^2}\left(T - \frac{1}{2\theta}(e^{-2\theta T} - 1) + \frac{2}{\theta}(e^{-\theta T} - 1)\right)\end{aligned}$$

which implies

$$\left|E_{\theta,\gamma,x}(\hat{\sigma}_T^2) - \frac{1}{2\theta}\right| \le \frac{1}{\theta^2 T}\left(\theta\left(x - \frac{\gamma}{\theta}\right)^2 + \frac{2}{\theta}\right). \tag{2.8}$$

Lemma 2.1. *For any $0 \le \alpha \le \theta^2/4$, for all $T \ge 1$,*

$$E_{\theta,\gamma,x}\left(\exp\left(\alpha\int_0^T X_t^2 dt\right)\right) < \infty,$$

and there exist finite positive constants L_1 and L_2 such that for all $0 \le \alpha \le \theta^2/4$ and $T \ge 1$,

$$E_{\theta,\gamma,x}\left(\exp\left(\alpha\int_0^T X_t^2 dt\right)\right) \le L_1 e^{L_2 \alpha T}.$$

Proof. For any $0 \le \alpha \le \theta^2/4$, set $\kappa = \sqrt{\theta^2 - 2\alpha}$. Then by Girsanov theorem, we have

$$\frac{dP_{\theta,\gamma,x}}{dP_{\kappa,\gamma,x}} = \exp\left\{-\int_0^T (\theta - \kappa)X_t dX_t - \int_0^T (\alpha X_t^2 - \gamma(\theta - \kappa)X_t)dt\right\}$$

and so

$$\begin{aligned}
&E_{\theta,\gamma,x}\left(\exp\left(\alpha\int_0^T X_t^2 dt\right)\right)\\
=&E_{\kappa,\gamma,x}\left(\frac{dP_{\theta,\gamma,x}}{dP_{\kappa,\gamma,x}}\exp\left\{\alpha\int_0^T X_t^2 dt\right\}\right)\\
=&E_{\kappa,\gamma,x}\left(\exp\left\{(-\theta+\kappa)\int_0^T X_t dX_t+\gamma\int_0^T(\theta-\kappa)X_t dt\right\}\right)\\
=&E_{\kappa,\gamma,x}\left(\exp\left\{\frac{-(\theta-\kappa)}{2}(X_T^2-T)+\gamma\int_0^T(\theta-\kappa)X_t dt\right\}\right)\\
\leq&\exp\left\{\frac{(\theta-\kappa)T}{2}\right\}E_{\kappa,\gamma,x}\left(\exp\left\{\gamma\int_0^T(\theta-\kappa)X_t dt\right\}\right)
\end{aligned}$$

where the last inequality is due to $\theta\geq\kappa$. Now we have to estimate $E_{\kappa,\gamma,x}(\exp\{\gamma\int_0^T(\theta-\kappa)X_t dt\})$. Since under $P_{\kappa,\gamma,x}$,

$$\hat{\mu}_T\sim N\left(\frac{1}{\kappa T}(x-\frac{\gamma}{\kappa})(1-e^{-\kappa T})+\frac{\gamma}{\kappa},\frac{1}{\kappa^2T^2}\left(T-\frac{1}{2\kappa}(e^{-2\kappa T}-1)+\frac{2}{\kappa}(e^{-\kappa T}-1)\right)\right),$$

we have

$$\begin{aligned}
&E_{\kappa,\gamma,x}\left(\exp\left\{\gamma\int_0^T(\theta-\kappa)X_t dt\right\}\right)\\
=&\exp\left\{\frac{\gamma(\theta-\kappa)}{\kappa}\left(\left(x-\frac{\gamma}{\kappa}\right)(1-e^{-\kappa T})+\gamma T\right)\right\}\\
&\cdot\exp\left\{\frac{\gamma^2(\theta-\kappa)^2}{2\kappa^2}\left(T-\frac{1}{2\kappa}(e^{-2\kappa T}-1)+\frac{2}{\kappa}(e^{-\kappa T}-1)\right)\right\}.
\end{aligned}$$

Noting $\theta/\sqrt{2}\leq\kappa\leq\theta$, $0\leq\theta-\kappa=2\alpha/(\theta+\kappa)\leq 2\alpha/\theta$ and $(\theta-\kappa)^2\leq\alpha\theta$ for all $0\leq\alpha\leq\theta^2/4$, we complete the proof of the lemma.

□

2.2 Logarithmic Sobolev inequality

Since the LSI with respect to the Cameron-Martin metric does not produce the concentration inequality of correct order in large time T for the functionals

$$F(X):=\frac{1}{\sqrt{T}}\left(\int_0^T g(X_s)ds-\mathbb{E}\left(\int_0^T g(X_s)ds\right)\right),$$

in order to get the concentration inequality of correct order for the functionals $F(X)$, as pointed out by Djellout, Guillin and Wu ([7]) we should establish the LSI with respect to the L^2-metric. Let us introduce the logarithmic Sobolev inequality on W with respect to the gradient in $L^2([0,T],\mathbb{R})$ ([10]). Let μ be the Wiener measure on $W=C([0,T],\mathbb{R})$. A function $f:W\to\mathbb{R}$ is said to be differentiable with respect to the L^2-norm, if it can be extend to $L^2([0,T],\mathbb{R})$ and for any $w \in W$, there exists a bounded linear operator $Df(w): g \to D_g f(w)$ on $L^2([0,T],\mathbb{R})$ such that

$$\lim_{\|g\|_{L_2}\to 0} \frac{|f(w+g)-f(w)-D_g f(w)|}{\|g\|_{L_2}} = 0.$$

If $f : W \to \mathbb{R}$ is differentiable with respect to the L^2-norm, then there exists a unique element $\nabla f(w) = (\nabla_t f(w), t \in [0,T])$ in $L^2([0,T],\mathbb{R})$ such that

$$D_g f(w) = \langle \nabla f(w), g\rangle_{L^2}, \ \ for\ all\ g \in L^2([0,T],\mathbb{R}).$$

Denote by $C_b^1(W/L^2)$ the space of all bounded function f on W, differentiable with respect to the L^2-norm, such that ∇f is also continuous and bounded from W equipped with L^2-norm to $L^2([0,T],\mathbb{R})$. Applying Theorem 2.3 in [10] to the Ornstein-Uhlenbeck process with linear drift, we have

$$Ent_{P_{\theta,\gamma,x}}(f^2) \le \frac{2}{\theta^2} E_{\theta,\gamma,x}\left(\int_0^T |\nabla_t f|^2 dt\right), \quad f \in C_b^1(W/L^2) \tag{2.9}$$

where the entropy of f^2 is given by

$$Ent_{P_{\theta,\gamma,x}}(f^2) = E_{\theta,\gamma,x}(f^2 \log f^2) - E_{\theta,\gamma,x}(f^2)\log E_{\theta,\gamma,x}(f^2).$$

Lemma 2.2. *For any* $|\alpha| \le \theta^2/4$,

$$E_{\theta,\gamma,x}\left(\exp\left\{\alpha\left(\int_0^T X_t^2 dt - E_{\theta,\gamma,x}\left(\int_0^T X_t^2 dt\right)\right)\right\}\right) \le E_{\theta,\gamma,x}\left(\exp\left\{\frac{4\alpha^2}{\theta^2}\int_0^T X_t^2 dt\right\}\right)$$

and

$$E_{\theta,\gamma,x}\left(\exp\left\{\alpha T\left(\hat{\mu}_T^2 - E_{\theta,\gamma,x}(\hat{\mu}_T^2)\right)\right\}\right) \le E_{\theta,\gamma,x}\left(\exp\left\{\frac{4\alpha^2}{\theta^2}\int_0^T X_t^2 dt\right\}\right).$$

Proof. We apply Theorem 2.7 in [12] to prove the conclusions of the lemma. Take $\mathscr{A}_1 = \{\alpha f;\ |\alpha| \le \theta^2/4\}$ and $\mathscr{A}_2 = \{\alpha h;\ |\alpha| \le \theta^2/4\}$, where

$$f(w) = \int_0^T w_t^2 dt, \quad h(w) = \frac{1}{T}\left(\int_0^T w_t dt\right)^2.$$

Define

$$\Gamma_1(g_1) = \frac{4}{\theta^2}\frac{g_1^2}{f},\ g_1 \in \mathscr{A}_1; \qquad \Gamma_2(g_2) = \frac{4}{\theta^2}\frac{g_2^2}{h},\ g_2 \in \mathscr{A}_2.$$

Then for any $\lambda \in [-1,1]$, $g_1 \in \mathscr{A}_1$ and $g_2 \in \mathscr{A}_2$, $\lambda g_1 \in \mathscr{A}_1$, $\lambda g_2 \in \mathscr{A}_2$, $\Gamma_1(\lambda g_1) = \lambda^2\Gamma_1(g_1)$, $\Gamma_2(\lambda g_2) = \lambda^2\Gamma_2(g_2)$ and by Lemma 2.1

$$E_{\theta,\gamma,x}\left(\exp\{\lambda\Gamma_1(g_1)\}\right) < \infty, \quad E_{\theta,\gamma,x}\left(\exp\{\lambda\Gamma_2(g_2)\}\right) < \infty.$$

Choose a sequence of real C^∞-functions $\Phi_n, n \ge 1$ with compact support such that $\lim_{n\to\infty}\sup_{|x|\le M}|\Phi_n(x) - e^x| = 0$ for all $M \in (0,\infty)$. For any $g_1 = \alpha f \in \mathscr{A}_1$ and $g_2 = \alpha h \in \mathscr{A}_2$, set

$$F_n(w) = \Phi_n\left(g_1(w)/2\right), \quad H_n(w) = \Phi_n\left(g_2(w)/2\right).$$

Then for any $g \in L^2([0,T],\mathbb{R})$,

$$\lim_{\|g\|_{L_2}\to 0} \frac{|F_n(w+g) - F_n(w) - \alpha\Phi_n'\left(g_1(w)/2\right)\langle w, g\rangle_{L^2}|}{\|g\|_{L^2}} = 0$$

and

$$\lim_{\|g\|_{L_2}\to 0} \frac{|H_n(w+g) - H_n(w) - \alpha\Phi_n'\left(g_2(w)/2\right)\frac{1}{T}\int_0^T w_t dt \int_0^T g_t dt|}{\|g\|_{L^2}} = 0.$$

Therefore, $F_n, H_n \in C_b^1(W/L^2)$, $\nabla F_n = \alpha\Phi_n'\left(g_1(w)/2\right) w$, and

$$\nabla H_n = \frac{\alpha}{T}\int_0^T w_t dt \Phi_n'\left(g_2(w)/2\right)$$

and so by (2.9), we have

$$Ent_{P_{\theta,\gamma,x}}\left(F_n^2\right) \le \frac{2}{\theta^2} E_{\theta,\gamma,x}\left(\int_0^T |\alpha w_t|^2 dt \left(\Phi_n'\left(g_1(w)/2\right)\right)^2\right)$$

and

$$Ent_{P_{\theta,\gamma,x}}\left(H_n^2\right) \le \frac{2}{\theta^2} E_{\theta,\gamma,x}\left(\frac{1}{T}\left(\alpha\int_0^T w_t dt\right)^2 \left(\Phi_n'\left(g_2(w)/2\right)\right)^2\right).$$

Letting $n \to \infty$ and by Lemma 2.1, we get

$$Ent_{P_{\theta,\gamma,x}}(e^{g_1}) \le \frac{1}{2} E_{\theta,\gamma,x}\left(\Gamma_1(g_1)e^{g_1}\right), \quad Ent_{P_{\theta,\gamma,x}}(e^{g_2}) \le \frac{1}{2} E_{\theta,\gamma,x}\left(\Gamma_2(g_2)e^{g_2}\right), \tag{2.10}$$

and so the conclusions of the lemma hold by Theorem 2.7 in [12] and $T\hat{\mu}_T^2 \le \int_0^T X_t^2 dt$.

□

2.3 Deviation inequalities

Since $X_T \sim N\left(\mu_T, \sigma_T^2\right)$, and under $P_{\theta,\gamma,x}$

$$\hat{\mu}_T \sim N\left(\frac{1}{\theta T}(x - \frac{\gamma}{\theta})(1 - e^{-\theta T}) + \frac{\gamma}{\theta}, \frac{1}{\theta^2 T^2}\left(T - \frac{1}{2\theta}(e^{-2\theta T} - 1) + \frac{2}{\theta}(e^{-\theta T} - 1)\right)\right),$$

it is easily to get from Chebyshev inequality, for any $r > 0$,

$$P_{\theta,\gamma,x}\left(\left|X_T - E_{\theta,\gamma,x}(X_T)\right| \ge r\right) \le 2\exp\left\{-\theta r^2\right\}, \tag{2.11}$$

$$P_{\theta,\gamma,x}\left(\left|\hat{\mu}_T - E_{\theta,\gamma,x}(\hat{\mu}_T)\right| \ge r\right) \le 2\exp\left\{-\frac{\theta^3 T r^2}{2\theta + 1}\right\} \tag{2.12}$$

where we used (2.7).

Lemma 2.3. *There exist finite positive constants C_0, C_1, C_2 such that for all $r > 0$ and all $T \geq 1$,*

$$P_{\theta,\gamma,x}\left(\left|\int_0^T X_t^2 dt - E_{\theta,\gamma,x}\left(\int_0^T X_t^2 dt\right)\right| \geq rT\right) \leq C_0 \exp\left\{-C_1 rT \min\left\{1, C_2 r\right\}\right\}$$

and

$$P_{\theta,\gamma,x}\left(\left|\hat{\mu}_T^2 - E_{\theta,\gamma,x}(\hat{\mu}_T^2)\right| \geq r\right) \leq C_0 \exp\left\{-C_1 rT \min\left\{1, C_2 r\right\}\right\}.$$

In particular, there exist finite positive constants C_0, C_1, C_2 such that for all $r > 0$ and all $T \geq 1$,

$$P_{\theta,\gamma,x}\left(|\hat{\sigma}_T^2 - E_{\theta,\gamma,x}(\hat{\sigma}_T^2)| \geq r\right) \leq C_0 \exp\left\{-C_1 rT \min\left\{1, C_2 r\right\}\right\}.$$

Proof. We only prove the first inequality. By Lemma 2.2 and Lemma 2.1, there exist finite positive constants L_1 and L_2 such that for all $T \geq 1$, for any $|\alpha| \leq \theta^2/4$,

$$E_{\theta,\gamma,x}\left(\exp\left\{\alpha\left(\int_0^T X_t^2 dt - E_{\theta,\gamma,x}\left(\int_0^T X_t^2 dt\right)\right)\right\}\right) \leq L_1 e^{L_2\alpha^2 T}.$$

Therefore, by Chebyshev inequality, for any $r > 0$, $T \geq 1$ and $|\alpha| \leq \theta^2/4$,

$$P_{\theta,\gamma,x}\left(\int_0^T X_t^2 dt - E_{\theta,\gamma,x}(\int_0^T X_t^2 dt) \geq rT\right) \leq L_1 e^{-(\alpha r - L_2\alpha^2)T}$$

and

$$P_{\theta,\gamma,x}\left(\int_0^T X_t^2 dt - E_{\theta,\gamma,x}(\int_0^T X_t^2 dt) \leq -rT\right) \leq L_1 e^{-(\alpha r - L_2\alpha^2)T}.$$

Now, by

$$\sup_{|\alpha|\leq\theta^2/4}\{\alpha r - L_2\alpha^2\} \geq \frac{\theta^2 r}{8}\min\left\{1, \frac{2r}{L_2\theta^2}\right\},$$

we obtain the first inequality of the lemma from the above estimates.

□

Lemma 2.4. *There exist finite positive constants C_0, C_1 and C_2 such that for all $r > 0$ and all $T \geq 1$,*

$$P_{\theta,\gamma,x}\left(\left|W_T\left(\hat{\mu}_T - \frac{\gamma}{\theta}\right)\right| \geq rT\right) \leq C_0 \exp\left\{-C_1 rT \min\left\{1, C_2 r\right\}\right\}.$$

Proof. Since for any $r > 0$ and $T \geq 1$,

$$\begin{aligned}
&\left\{\left|W_T\left(\hat{\mu}_T - \frac{\gamma}{\theta}\right)\right| \geq rT\right\} \\
&\subset \left\{\left|W_T(\hat{\mu}_T - E_{\theta,\gamma,x}(\hat{\mu}_T))\right| \geq rT/2\right\} \cup \left\{\left|W_T\left(E_{\theta,\gamma,x}(\hat{\mu}_T) - \frac{\gamma}{\theta}\right)\right| \geq rT/2\right\} \\
&\subset \left\{|W_T| \geq \sqrt{r}T/2\right\} \cup \left\{\left|(\hat{\mu}_T - E_{\theta,\gamma,x}(\hat{\mu}_T))\right| \geq \sqrt{r}\right\} \cup \left\{\left|W_T\right| \geq \frac{\theta rT}{2|\left(x - \frac{\gamma}{\theta}\right)|}\right\},
\end{aligned}$$

by (2.12) and $W_T \sim N(0,T)$, we get

$$\begin{aligned}&P_{\theta,\gamma,x}\left(\left|W_T(\hat{\mu}_T-\frac{\gamma}{\theta})\right|\geq rT\right)\\ \leq& 2\exp\left\{-\frac{Tr}{8}\right\}+2\exp\left\{-\frac{\theta^3 Tr}{2\theta+1}\right\}+2\exp\left\{-\frac{\theta^2r^2T}{8\left(x-\frac{\gamma}{\theta}\right)^2}\right\}.\end{aligned}$$

□

Lemma 2.5. *For each $\beta\in\mathbb{R}$ fixed, there exist finite positive constants C_0, C_1, C_2 such that for all $r>0$ and all $T\geq 1$,*

$$P_{\theta,\gamma,x}\left(\left|\int_0^T (X_t-\beta)\,dW_t\right|\geq rT\right)\leq C_0\exp\left\{-C_1 rT\min\left\{1,C_2 r\right\}\right\}.$$

Proof. It is known that for $\alpha\in\mathbb{R}$,

$$M_T^{(\alpha)}=\exp\left\{\alpha\int_0^T (X_t-\beta)\,dW_t-\frac{\alpha^2}{2}\int_0^T (X_t-\beta)^2\,dt\right\},\quad T\geq 0$$

is $\mathscr{F}_T$-martingale, where $\mathscr{F}_T:=\sigma(W_t,t\leq T)$. Therefore, by Hölder inequality, we can get that for any $\epsilon\in(0,1]$,

$$\begin{aligned}&E_{\theta,\gamma,x}\left(\exp\left\{\alpha\int_0^T (X_t-\beta)\,dW_t\right\}\right)\\ \leq&\left(E_{\theta,\gamma,x}\left(\exp\left\{\frac{(1+\epsilon)^2\alpha^2}{2\epsilon}\int_0^T (X_t-\beta)^2\,dt\right\}\right)\right)^{\frac{\epsilon}{1+\epsilon}}\left(E_{\theta,\gamma,x}\left(M_T^{((1+\epsilon)\alpha)}\right)\right)^{\frac{1}{1+\epsilon}}\\ =&\left(E_{\theta,\gamma,x}\left(\exp\left\{\frac{(1+\epsilon)^2\alpha^2}{2\epsilon}\int_0^T (X_t-\beta)^2\,dt\right\}\right)\right)^{\frac{\epsilon}{1+\epsilon}}.\end{aligned}$$

In particular, take $\epsilon=1$, then by Lemma 2.1, there exists finite positive constants $L_1=L_1(\theta,\beta,\gamma,x)$ and $L_2=L_2(\theta,\beta,\gamma,x)$ such that for all $T\geq 1$, for any $\alpha^2\leq\theta^2/16$, by Cauchy-Schwartz inequality,

$$\begin{aligned}&E_{\theta,\gamma,x}\left(\exp\left\{\alpha\int_0^T (X_t-\beta)\,dW_t\right\}\right)\\ \leq&\left(E_{\theta,\gamma,x}\left(\exp\left\{2\alpha^2\int_0^T (X_t-\beta)^2dt\right\}\right)\right)^{\frac{1}{2}}\\ \leq&\left(E_{\theta,\gamma,x}\left(\exp\left\{4\alpha^2\int_0^T X_t^2dt\right\}\right)\right)^{\frac{1}{4}}\left(E_{\theta,\gamma,x}\left(\exp\left\{4\alpha^2\int_0^T (-2\beta X_t+\beta^2)dt\right\}\right)\right)^{\frac{1}{4}}\\ \leq& L_1e^{L_2\alpha^2T}.\end{aligned}$$

Therefore, by Chebyshev inequality, the conclusion of the lemma holds.

□

Proof of Theorem 1.1

We only show the first inequality. The second one is similar. By

$$\hat{\theta}_T - \theta = \frac{W_T\left(\hat{\mu}_T - \frac{\gamma}{\theta}\right) - \int_0^T \left(X_t - \frac{\gamma}{\theta}\right) dW_t}{T\hat{\sigma}_T^2}$$

for any $r > 0$ and $T \geq 1$,

$$\begin{aligned} &P_{\theta,\gamma,x}\left(|\hat{\theta}_T - \theta| \geq r\right) \\ \leq &P_{\theta,\gamma,x}\left(\left|\hat{\sigma}_T^2 - E_{\theta,\gamma,x}(\hat{\sigma}_T^2)\right| \geq E_{\theta,\gamma,x}(\hat{\sigma}_T^2)/2\right) \\ &+ P_{\theta,\gamma,x}\left(\left|W_T\left(\hat{\mu}_T - \frac{\gamma}{\theta}\right) - \int_0^T \left(X_t - \frac{\gamma}{\theta}\right) dW_t\right| \geq E_{\theta,\gamma,x}(\hat{\sigma}_T^2) rT/2\right) \end{aligned}$$

Therefore, by Lemmas 2.3, 2.4 and 2.5, we obtain the first inequality of the theorem.

□

3 Moderate deviations

In this section, we show Theorem 1.2. By (1.2) and (1.3), we have the following estimates

$$\begin{aligned} &\left|(\hat{\theta}_T - \theta) + \frac{2\theta}{T}\int_0^T \left(X_t - \frac{\gamma}{\theta}\right) dW_t\right| \\ \leq &\frac{|W_T\left(\hat{\mu}_T - \frac{\gamma}{\theta}\right)|}{T\hat{\sigma}_T^2} + \frac{|2\theta\hat{\sigma}_T^2 - 1|\left|\int_0^T \left(X_t - \frac{\gamma}{\theta}\right) dW_t\right|}{T\hat{\sigma}_T^2} \end{aligned} \tag{3.1}$$

and for

$$\begin{aligned} &\left|(\hat{\gamma}_T - \gamma) - \frac{W_T}{T} + \frac{2\gamma}{T}\int_0^T \left(X_t - \frac{\gamma}{\theta}\right) dW_t\right| \\ \leq &\frac{|\hat{\mu}_T||W_T\left(\hat{\mu}_T - \frac{\gamma}{\theta}\right)|}{T\hat{\sigma}_T^2} + \frac{|2\gamma\hat{\sigma}_T^2 - \hat{\mu}_T|\left|\int_0^T \left(X_t - \frac{\gamma}{\theta}\right) dW_t\right|}{T\hat{\sigma}_T^2}. \end{aligned} \tag{3.2}$$

Lemma 3.1. *(1). For any $r > 0$,*

$$\limsup_{T\to\infty} \frac{1}{\lambda_T} \log P_{\theta,\gamma,x}\left(\left|\hat{\mu}_T - \frac{\gamma}{\theta}\right|\left|W_T\right| \geq \sqrt{T\lambda_T}r\right) = -\infty,$$

$$\limsup_{T\to\infty} \frac{1}{\lambda_T} \log P_{\theta,\gamma,x}\left(\left|\hat{\mu}_T - \frac{\gamma}{\theta}\right|\left|\int_0^T \left(X_t - \frac{\gamma}{\theta}\right) dW_t\right| \geq \sqrt{T\lambda_T}r\right) = -\infty$$

and

$$\limsup_{T\to\infty} \frac{1}{\lambda_T} \log P_{\theta,\gamma,x}\left(\left|\hat{\sigma}_T^2 - \frac{1}{2\theta}\right|\left|\int_0^T \left(X_t - \frac{\gamma}{\theta}\right) dW_t\right| \geq \sqrt{T\lambda_T}r\right) = -\infty.$$

(2). For any $\delta > 0$,

$$\limsup_{T\to\infty} \frac{1}{\lambda_T} \log P_{\theta,\gamma,x}\left(\left| (\hat{\theta}_T - \theta) - \frac{2\theta}{T} \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta \sqrt{\frac{\lambda_T}{T}} \right) = -\infty$$

and

$$\limsup_{T\to\infty} \frac{1}{\lambda_T} \log P_{\theta,\gamma,x}\left(\left| (\hat{\gamma}_T - \gamma) - \frac{W_T}{T} - \frac{2\gamma}{T} \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta \sqrt{\frac{\lambda_T}{T}} \right) = -\infty.$$

Proof. (1). We only give the proof of the third assertion in (1). The rest is similar. For any $L > 0$,

$$\left\{ \left| \hat{\sigma}_T^2 - \frac{1}{2\theta} \right| \left| \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \sqrt{T\lambda_T} r \right\}$$

$$\subset \left\{ \left| \hat{\sigma}_T^2 - \frac{1}{2\theta} \right| \geq \frac{r}{L} \right\} \cup \left\{ \frac{1}{\sqrt{T\lambda_T}} \left| \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq L \right\}.$$

By Lemma 2.3, and Lemma 2.5, we have

$$\limsup_{T\to\infty} \frac{1}{\lambda_T} \log P_{\theta,\gamma,x}\left(\left| \hat{\sigma}_T^2 - \frac{1}{2\theta} \right| \geq \frac{r}{L} \right) = -\infty$$

and

$$\limsup_{T\to\infty} \frac{1}{\lambda_T} \log P_{\theta,\gamma,x}\left(\frac{1}{\sqrt{T\lambda_T}} \left| \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq L \right) \leq -L^2 C_1 C_2.$$

Hence,

$$\limsup_{T\to\infty} \frac{1}{\lambda_T} \log P_{\theta,\gamma,x}\left(\left| \hat{\sigma}_T^2 - \frac{1}{2\theta} \right| \left| \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \sqrt{T\lambda_T} r \right) \leq -L^2 C_1 C_2.$$

Letting $L \to \infty$, we obtain the third conclusion.
(2). It follows from (3.1) and (3.2) that

$$\left(\left| (\hat{\theta}_T - \theta) - \frac{2\theta}{T} \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta \sqrt{\frac{\lambda_T}{T}} \right)$$

$$\subset \left\{ |W_T \left(\hat{\mu}_T - \frac{\gamma}{\theta} \right)| \geq \delta \hat{\sigma}_T^2 \frac{\sqrt{T\lambda_T}}{2} \right\} \cup \left\{ |2\theta \hat{\sigma}_T^2 - 1| \left| \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta \hat{\sigma}_T^2 \frac{\sqrt{T\lambda_T}}{2} \right\}$$

$$\subset \left\{ |W_T \left(\hat{\mu}_T - \frac{\gamma}{\theta} \right)| \geq \delta E_{\theta,\gamma,x}(\hat{\sigma}_T^2) \frac{\sqrt{T\lambda_T}}{4} \right\} \cup \left\{ |\hat{\sigma}_T^2 - E_{\theta,\gamma,x}(\hat{\sigma}_T^2)| \geq E_{\theta,\gamma,x}(\hat{\sigma}_T^2)/2 \right\}$$

$$\cup \left\{ |2\theta \hat{\sigma}_T^2 - 1| \left| \int_0^T \left(X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta E_{\theta,\gamma,x}(\hat{\sigma}_T^2) \frac{\sqrt{T\lambda_T}}{4} \right\}$$

and

$$
\begin{aligned}
&\left(\left|(\hat{\gamma}_T-\gamma)-\frac{2\gamma}{T}\int_0^T\left(X_t-\frac{\gamma}{\theta}\right)dW_t\right|\geq\delta\sqrt{\frac{\lambda_T}{T}}\right)\\
&\subset\left\{|\hat{\mu}_T||W_T\left(\hat{\mu}_T-\frac{\gamma}{\theta}\right)|\geq\delta\hat{\sigma}_T^2\frac{\sqrt{T\lambda_T}}{2}\right\}\cup\left\{|2\gamma\hat{\sigma}_T^2-\hat{\mu}_T|\left|\int_0^T\left(X_t-\frac{\gamma}{\theta}\right)dW_t\right|\geq\delta\hat{\sigma}_T^2\frac{\sqrt{T\lambda_T}}{2}\right\}\\
&\subset\left\{\left|\hat{\mu}_T-\frac{\gamma}{\theta}\right|\geq\frac{\gamma}{2\theta}\right\}\cup\left\{|\hat{\sigma}_T^2-E_{\theta,\gamma,x}(\hat{\sigma}_T^2)|\geq E_{\theta,\gamma,x}(\hat{\sigma}_T^2)/2\right\}\\
&\quad\cup\left\{\frac{3\gamma}{2\theta}|W_T\left(\hat{\mu}_T-\frac{\gamma}{\theta}\right)|\geq\delta E_{\theta,\gamma,x}(\hat{\sigma}_T^2)\frac{\sqrt{T\lambda_T}}{4}\right\}\\
&\quad\cup\left\{\left(\left|2\gamma\hat{\sigma}_T^2-\frac{\gamma}{\theta}\right|+\left|\hat{\mu}_T-\frac{\gamma}{\theta}\right|\right)\left|\int_0^T\left(X_t-\frac{\gamma}{\theta}\right)dW_t\right|\geq\delta E_{\theta,\gamma,x}(\hat{\sigma}_T^2)\frac{\sqrt{T\lambda_T}}{4}\right\}.
\end{aligned}
$$

Therefore, by Lemmas 2.3 and (1), we get the conclusions.

□

Lemma 3.2. *For each $\beta,\kappa\in\mathbb{R}$ fixed, $\left\{P_{\theta,\gamma,x}\left(\frac{\kappa}{\sqrt{T\lambda_T}}\int_0^T(X_t-\beta)\,dW_t\in\cdot\right),T\geq1\right\}$ satisfies the LDP with speed λ_T and rate function $J(u)=\frac{\theta^2u^2}{\kappa^2(\theta+2(\gamma-\theta\beta)^2)}$.*

Proof. By (2.12) and Lemma 2.3, we can get for any $\delta>0$,

$$
\lim_{T\to\infty}\frac{1}{T}\log P_{\theta,\gamma,x}\left(\left|\frac{1}{T}\int_0^T(X_t-\beta)^2\,dt-\left(\frac{1}{2\theta}+\frac{1}{\theta^2}(\gamma-\theta\beta)^2\right)\right|\geq\delta\right)<0. \tag{3.3}
$$

Therefore, Proposition 1 in [4] yields the conclusion of the lemma.

□

Proof of Theorem 1.2

By Lemma 3.1, $\{P_{\theta,\gamma,x}(\sqrt{\frac{T}{\lambda_T}}(\hat{\theta}_T-\theta)\in\cdot),T\geq1\}$ and $\{P_{\theta,\gamma,x}(\sqrt{\frac{T}{\lambda_T}}(\hat{\gamma}_T-\gamma)\in\cdot),T\geq1\}$ are exponential equivalent to

$$
\left\{P_{\theta,\gamma,x}\left(\sqrt{\frac{T}{\lambda_T}}\frac{2\theta}{T}\int_0^T\left(X_t-\frac{\gamma}{\theta}\right)dW_t\in\cdot\right),T\geq1\right\}
$$

and

$$
\left\{P_{\theta,\gamma,x}\left(\sqrt{\frac{T}{\lambda_T}}\left(\frac{W_T}{T}+\frac{2\gamma}{T}\int_0^T\left(X_t-\frac{\gamma}{\theta}\right)dW_t\right)\in\cdot\right),T\geq1\right\},
$$

respectively. Noting for $\gamma\neq0$, $\frac{W_T}{T}+\frac{2\gamma}{T}\int_0^T\left(X_t-\frac{\gamma}{\theta}\right)dW_t=\frac{2\gamma}{T}\int_0^T\left(X_t-\frac{\gamma}{\theta}+\frac{1}{2\gamma}\right)dW_t$, Theorem 1.2 follows from Lemma 3.2.

□

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