

A NOTE ON ERGODIC TRANSFORMATIONS OF SELF-SIMILAR VOLTERRA GAUSSIAN PROCESSES

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Abstract

We derive a class of ergodic transformations of self-similar Gaussian processes that are Volterra, i.e. of type $X_t = \int_0^t z_X(t, s) dW_s$, $t \in [0, \infty)$, where z_X is a deterministic kernel and W is a standard Brownian motion.

1 Introduction

Let $(X_t)_{t \in [0, \infty)}$ be a continuous *Volterra* Gaussian process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This means that

$$X_t = \int_0^t z_X(t, s) dW_s, \text{ a.s., } t \in [0, \infty), \quad (1.1)$$

where the kernel $z_X \in L^2_{\text{loc}}([0, \infty)^2)$ is *Volterra*, i.e. $z_X(t, s) = 0$, $s \geq t$, and $(W_t)_{t \in [0, \infty)}$ is a standard Brownian motion. Clearly, X is centered and

$$R^X(s, t) := \text{Cov}_{\mathbb{P}}(X_s, X_t) = \int_0^s z_X(s, u) z_X(t, u) du, \quad 0 \leq s \leq t < \infty.$$

We assume that X is β -self-similar for some $\beta > 0$, i.e.

$$(X_{at})_{t \in [0, \infty)} \stackrel{d}{=} (a^\beta X_t)_{t \in [0, \infty)}, \quad a > 0, \quad (1.2)$$

where $\stackrel{d}{=}$ denotes equality of finite-dimensional distributions. Furthermore, we assume that z_X is *non-degenerate* in the sense that the family $\{z_X(t, \cdot) \mid t \in (0, \infty)\}$ is linearly independent and generates a dense subspace of $L^2([0, \infty))$. Then

$$\Gamma_t(X) := \overline{\text{span}\{X_s \mid s \in [0, t]\}} = \Gamma_t(W), \quad t \in (0, \infty), \quad (1.3)$$

where the closure is in $L^2(\mathbb{P})$, or equivalently,

$$\mathbb{F}^X = \mathbb{F}^W,$$

where $\mathbb{F}^X := (\mathcal{F}_t^X)_{t \in [0, \infty)}$ denotes the completed natural filtration of X .

We assume implicitly that $(\Omega, \mathcal{F}, \mathbb{P})$ is the *coordinate space* of X , which means that $\Omega = \{\omega : [0, \infty) \rightarrow \mathbb{R} \mid \omega \text{ is continuous}\}$, $\mathcal{F} = \mathcal{F}_\infty^X := \sigma(X_t \mid t \in [0, \infty))$ and $\mathbb{P}$ is the probability measure with respect to which the *coordinate process* $X_t(\omega) = \omega(t)$, $\omega \in \Omega$, $t \in [0, \infty)$, is a centered Gaussian process with covariance function R^X . Recall that a measurable map

$$\begin{aligned} \mathcal{Z} : (\Omega, \mathcal{F}, \mathbb{P}) &\rightarrow (\Omega, \mathcal{F}, \mathbb{P}) \\ X(\omega) &\mapsto \mathcal{Z}(X(\omega)) \end{aligned}$$

is a *measure-preserving transformation*, or *endomorphism*, on $(\Omega, \mathcal{F}, \mathbb{P})$ if $\mathbb{P}^\mathcal{Z} = \mathbb{P}$, or equivalently, if $\mathcal{Z}(X) \stackrel{d}{=} X$. If $\mathcal{Z}$ is also bijective and $\mathcal{Z}^{-1}$ is measurable, then it is an *automorphism*.

Processes of the above type are a natural generalization of the nowadays in connection with finance and telecommunications extensively studied *fractional Brownian motion with Hurst index* $H \in (0, 1)$, or *H-fBm*. The *H-fBm*, denoted by $(B_t^H)_{t \in [0, \infty)}$, is the continuous, centered Gaussian process with covariance function

$$R^{B^H}(s, t) = \frac{1}{2} (s^{2H} + t^{2H} - |s - t|^{2H}), \quad s, t \in [0, \infty).$$

For $H = \frac{1}{2}$, fBm is standard Brownian motion. *H-fBm* is *H*-self-similar and has stationary increments. The non-degenerate Volterra kernel is given by

$$z_{B^H}(t, s) = c(H)(t - s)^{H - \frac{1}{2}} \cdot {}_2F_1\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, 1 - \frac{t}{s}\right), \quad 0 < s < t < \infty,$$

where $c(H) := \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}\right)^{\frac{1}{2}}$ with Γ denoting the Gamma function, and ${}_2F_1$ is the Gauss hypergeometric function. In 2003, Molchan (see [8]) showed that the transformation

$$\mathcal{Z}_t(B^H) := B_t^H - 2H \int_0^t \frac{B_s^H}{s} ds, \quad t \in [0, \infty), \quad (1.4)$$

is measure-preserving and satisfies

$$\Gamma_T(\mathcal{Z}(B^H)) = \Gamma_T(Y^H), \quad T > 0, \quad (1.5)$$

where

$$Y_t^H := M_t^H - \frac{t}{T} \xi_T^H, \quad t \in [0, T]. \quad (1.6)$$

Here, $M_t^H := \sqrt{2 - 2H} \int_0^t s^{\frac{1}{2} - H} dW_s$, $t \in [0, \infty)$, is the *fundamental martingale* of B^H and $\xi_T^H := 2H \int_0^T \left(\frac{s}{T}\right)^{2H - 1} dM_s^H$.

In this work, we present a class of measure-preserving transformations (on the coordinate space) of X , which generalizes this result. Moreover, we show that these measure-preserving transformations are ergodic.

2 Ergodic transformations

First, we introduce the class of measure-preserving transformations:

Theorem 2.1. *Let $\alpha > \frac{-1}{2}$. Then the transformation*

$$\mathcal{Z}_t^\alpha(X) := X_t - (2\alpha + 1)t^{\beta-\alpha-\frac{1}{2}} \int_0^t s^{\alpha-\beta-\frac{1}{2}} X_s ds, \quad t \in [0, \infty), \quad (2.1)$$

is an automorphism on the coordinate space of X . The inverse is given by

$$\mathcal{Z}_t^{\alpha,-1}(X) = X_t - (2\alpha + 1)t^{\alpha+\beta+\frac{1}{2}} \int_t^\infty X_s s^{-\beta-\alpha-\frac{3}{2}} ds, \quad a.s., \quad t \in [0, \infty). \quad (2.2)$$

The integrals on the right-hand sides are $L^2(\mathbb{P})$ -limits of Riemann sums.

Proof. First, note that R^X is continuous. Furthermore, by combining Hölder's inequality and (1.2), we have that

$$|R^X(s, t)| \leq \mathbb{E}_{\mathbb{P}}(X_1)^2 s^\beta t^\beta, \quad s, t \in (0, \infty).$$

Hence, the double Riemann integrals

$$\int_0^t \int_0^t (us)^{\alpha-\beta-\frac{1}{2}} R^X(u, s) du ds$$

and

$$\int_t^\infty \int_t^\infty (us)^{-\beta-\alpha-\frac{3}{2}} R^X(u, s) du ds$$

are finite. Thus, the integrals in (2.1) and (2.2) are well-defined (see [5], section 1).

Second, we show that $\mathcal{Z}^\alpha$ is a measure-preserving transformation. Let $Y_t := \exp(-\beta t) X_{\exp(t)}$ and $Y_t^\alpha := \exp(-\beta t) \mathcal{Z}_{\exp(t)}^\alpha(X)$, $t \in \mathbb{R}$, denote the Lamperti transforms of X and $\mathcal{Z}^\alpha(X)$, respectively. Hence, the process $(Y_t)_{t \in \mathbb{R}}$ is stationary. By substituting $v := \ln(s)$, we obtain that

$$\begin{aligned} Y_t^\alpha &= \exp(-\beta t) \left(X_{\exp(t)} - (2\alpha + 1) \exp \left(\left(\beta - \alpha - \frac{1}{2} \right) t \right) \int_0^{\exp(t)} s^{\alpha-\beta-\frac{1}{2}} X_s ds \right) \\ &= Y_t - (2\alpha + 1) \exp \left(\left(-\alpha - \frac{1}{2} \right) t \right) \int_{-\infty}^t \exp \left(v \left(\alpha - \beta + \frac{1}{2} \right) \right) X_{\exp(v)} dv \\ &= Y_t - (2\alpha + 1) \exp \left(\left(-\alpha - \frac{1}{2} \right) t \right) \int_{-\infty}^t \exp \left(v \left(\alpha + \frac{1}{2} \right) \right) Y_v dv \\ &= \int_{-\infty}^\infty h^\alpha(t - v) Y_v dv, \quad a.s., \quad t \in \mathbb{R}, \end{aligned}$$

where

$$h^\alpha(x) := \delta_0(x) - (2\alpha + 1) 1_{(0, \infty)}(x) \exp \left(- \left(\alpha + \frac{1}{2} \right) x \right), \quad x \in \mathbb{R}.$$

Thus, Y^α is a linear, non-anticipative, time-invariant transformation of Y . The spectral distribution function of Y^α is given by (see [13], p. 151)

$$dF^\alpha(\lambda) = |H^\alpha(\lambda)|^2 dF(\lambda), \quad \lambda \in \mathbb{R},$$

where $H^\alpha(\lambda) := \int_{\mathbb{R}} \exp(-i\lambda x) h^\alpha(x) dx$ denotes the Fourier transform of h^α and F is the spectral distribution function of Y . We have that (see [3], p. 14 and p. 72)

$$|H^\alpha(\lambda)| = \left| 1 - (2\alpha + 1) \left(\frac{\alpha + \frac{1}{2} - i\lambda}{(\frac{1}{2} + \alpha)^2 + \lambda^2} \right) \right| = \left| \frac{-\frac{1}{2} - \alpha + i\lambda}{\frac{1}{2} + \alpha + i\lambda} \right| = 1, \lambda \in \mathbb{R},$$

i.e. $F^\alpha \equiv F$. It follows from this that $(Y_t^\alpha)_{t \in \mathbb{R}} \stackrel{d}{=} (Y_t)_{t \in \mathbb{R}}$, or equivalently, $(\mathcal{Z}_t^\alpha(X))_{t \in [0, \infty)} \stackrel{d}{=} (X_t)_{t \in [0, \infty)}$.

Third, by splitting integrals and using Fubini's theorem, we obtain that

$$\mathcal{Z}_t^{\alpha, -1}(\mathcal{Z}^\alpha(X)) = X_t = \mathcal{Z}_t^\alpha(\mathcal{Z}^{\alpha, -1}(X)), \text{ a.s., } t \in [0, \infty). \quad \square$$

Remark 2.2. Theorem 2.1 generalizes (1.4). In fact, $\mathcal{Z}^{H-\frac{1}{2}}(B^H) = \mathcal{Z}(B^H)$, $H \in (0, 1)$.

Remark 2.3. Theorem 2.1 holds true for general continuous centered β -self-similar Gaussian processes.

Next, we present two auxiliary lemmas concerning the structure of z_X :

Lemma 2.4. Let $(X_t)_{t \in [0, \infty)}$ be a Volterra Gaussian process with a non-degenerate Volterra kernel z_X . Then the following are equivalent:

1. X is β -self-similar, i.e.

$$\int_0^s z_X(at, au) z_X(as, au) du = a^{2\beta-1} \int_0^s z_X(t, u) z_X(s, u) du, \quad 0 < s \leq t < \infty, \quad a > 0.$$

2. It holds that

$$z_X(at, as) = a^{\beta-\frac{1}{2}} z_X(t, s), \quad 0 < s < t < \infty, \quad a > 0.$$

3. There exists $F_X \in L^2((0, 1), (1-x)^{2\beta-1} dx)$ such that

$$z_X(t, s) = (t-s)^{\beta-\frac{1}{2}} F_X\left(\frac{s}{t}\right), \quad 0 < s < t < \infty.$$

Proof. $1 \Rightarrow 2$: For $a > 0$, let $z_{Y(a)}(t, s) := a^{\frac{1}{2}-\beta} z_X(at, as)$, $0 < s < t < \infty$, and let $Y_t(a) := \int_0^t z_{Y(a)}(t, s) dW_s$, $t \in [0, \infty)$. Clearly, $z_{Y(a)}$ is non-degenerate. From (1.3), we obtain that $\Gamma_t(Y(a)) = \Gamma_t(W)$, $t \in (0, \infty)$. From part 1, it follows that $X \stackrel{d}{=} Y(a)$. Thus, the process $W'_t := \int_0^t z_X^*(t, s) dY_s(a)$, $t \in [0, \infty)$, where z_X^* is the reciprocal of z_X and the integral is an abstract Wiener integral, is a standard Brownian motion with $\Gamma_t(W') = \Gamma_t(Y(a))$, $t \in (0, \infty)$. Hence, $\Gamma_t(W) = \Gamma_t(W')$, i.e. W and W' are indistinguishable. Therefore, $Y_t(a) = \int_0^t z_X(t, s) dW_s$, a.s., $t \in (0, \infty)$, i.e. $Y_t(a) = X_t$, a.s., $t \in [0, \infty)$. In particular, $0 = \mathbb{E}_{\mathbb{P}}(Y_t(a) - X_t)^2 = \int_0^t (z_{Y(a)}(t, s) - z_X(t, s))^2 ds$, $t \in (0, \infty)$. Thus, $z_X(t, \cdot) \equiv z_{Y(a)}(t, \cdot)$, $t \in (0, \infty)$.

$2 \Rightarrow 3$: Let $G_X(t, s) := (t-s)^{\frac{1}{2}-\beta} z_X(t, s)$, $0 < s < t < \infty$. From part 2, it follows that $G_X(at, as) = G_X(t, s)$, $0 < s < t < \infty$, $a > 0$. Hence, for every (t, s) , $s < t$, the function G_X is constant on the line $\{(at, as) | a \in (0, \infty)\}$, which depends only on the slope $\frac{s}{t}$. Thus, $G_X(t, s) = F_X(\frac{s}{t})$, $0 < s < t < \infty$, for some $F_X \in L^2((0, 1), (1-x)^{2\beta-1} dx)$.

$3 \Rightarrow 1$: This is trivial. $\square$

Lemma 2.5. Let $\alpha > \frac{-1}{2}$. Then we have that

$$t^{\beta-\alpha-\frac{1}{2}} \int_s^t u^{\alpha-\beta-\frac{1}{2}} z_X(u, s) du = s^\alpha \int_s^t z_X(t, u) u^{-\alpha-1} du, \quad 0 < s < t < \infty.$$

Proof. From Lemma 2.4, it follows that the Volterra kernel of X can be written as

$$z_X(t, s) = (t - s)^{\beta - \frac{1}{2}} F_X\left(\frac{s}{t}\right), \quad 0 < s < t < \infty,$$

for some function F_X . By substituting first $x := \frac{s}{u}$ and then $v := tx$, we obtain that

$$\begin{aligned} t^{\beta - \alpha - \frac{1}{2}} \int_s^t u^{\alpha - \beta - \frac{1}{2}} z_X(u, s) du &= t^{\beta - \alpha - \frac{1}{2}} \int_s^t u^{\alpha - \beta - \frac{1}{2}} (u - s)^{\beta - \frac{1}{2}} F_X\left(\frac{s}{u}\right) du \\ &= t^{\beta - \alpha - \frac{1}{2}} s^\alpha \int_{\frac{s}{t}}^1 x^{-\alpha - 1} (1 - x)^{\beta - \frac{1}{2}} F_X(x) dx \\ &= s^\alpha \int_s^t (t - v)^{\beta - \frac{1}{2}} F_X\left(\frac{v}{t}\right) v^{-\alpha - 1} dv \\ &= s^\alpha \int_s^t z_X(t, v) v^{-\alpha - 1} dv. \quad \square \end{aligned}$$

The next lemma is the key result for deriving the ergodicity of the measure-preserving transformations:

Lemma 2.6. *Let $\alpha > \frac{-1}{2}$. Then*

$$\mathcal{Z}_t^\alpha(X) = \int_0^t z_X(t, s) d\mathcal{Z}_s^\alpha(W), \quad a.s., \quad t \in [0, \infty).$$

Proof. By combining (1.1) and the stochastic Fubini theorem, using Lemma 2.5, again the stochastic Fubini theorem, and finally using partial integration, we obtain that

$$\begin{aligned} \mathcal{Z}_t^\alpha(X) &= X_t - (2\alpha + 1) \int_0^t \left(t^{\beta - \alpha - \frac{1}{2}} \int_u^t s^{\alpha - \beta - \frac{1}{2}} z_X(s, u) ds \right) dW_u \\ &= X_t - (2\alpha + 1) \int_0^t \left(u^\alpha \int_u^t z_X(t, s) s^{-\alpha - 1} ds \right) dW_u \\ &= X_t - (2\alpha + 1) \int_0^t z_X(t, s) s^{-\alpha - 1} \int_0^s u^\alpha dW_u ds \\ &= X_t - (2\alpha + 1) \int_0^t z_X(t, s) \left((-\alpha) s^{-\alpha - 1} \int_0^s u^{\alpha - 1} W_u du ds + s^{-1} W_s ds \right) \\ &= \int_0^t z_X(t, s) d\mathcal{Z}_s^\alpha(W), \quad a.s., \quad t \in (0, \infty). \quad \square \end{aligned}$$

In the following, let $\mathcal{Z}^{\alpha, n} := (\mathcal{Z}^\alpha)^n$ denote the n -th iterate of $\mathcal{Z}^\alpha$, $n \in \mathbb{Z}$. Also, let

$$\Gamma_\infty(X) := \overline{\text{span}\{X_t \mid t \in [0, \infty)\}}.$$

For $\alpha > -\frac{1}{2}$, let

$$N_t^\alpha := \int_0^t s^\alpha dW_s, \quad t \in [0, \infty).$$

Clearly, N^α is an $(\alpha + \frac{1}{2})$ -self-similar $\mathbb{F}^X$ -martingale. From Lemma 2.6, it follows that

$$\mathcal{Z}_t^\alpha(X) = \int_0^t z_X(t, s) s^{-\alpha} d\mathcal{Z}_s^\alpha(N^\alpha), \quad a.s., \quad t \in (0, \infty). \quad (2.3)$$

The next lemma is an auxiliary result, which was obtained in [7], section 3.2 and Theorem 5.2. (The automorphism $\mathcal{Z}^\alpha$ on the coordinate space of N^α here corresponds to the (ergodic) automorphism $\mathcal{T}^{(1)}$ on the coordinate space of the martingale M with $M := N^\alpha$ in [7].)

Lemma 2.7. *Let $\alpha > \frac{-1}{2}$ and $T > 0$.*

1. *It holds that*

$$\Gamma_T(\mathcal{Z}^\alpha(N^\alpha)) = \Gamma_T(N^{\alpha,T}),$$

where $N_t^{\alpha,T} := N_t^\alpha - \left(\frac{t}{T}\right)^{2\alpha+1} N_T^\alpha$, $t \in [0, T]$, is a bridge of N^α , i.e. a process satisfying $\text{Law}_{\mathbb{P}}(N^{\alpha,T}) = \text{Law}_{\mathbb{P}}(N^\alpha \mid N_T^\alpha = 0)$, and $\Gamma_T(N^{\alpha,T}) := \overline{\text{span}\{N_t^{\alpha,T} \mid t \in [0, T]\}}$.

2. *We have that*

$$\Gamma_T(N^\alpha) = \overline{\perp_{n \in \mathbb{N}_0} \text{span}\{\mathcal{Z}_T^{\alpha,n}(N^\alpha)\}} \quad (2.4)$$

and

$$\Gamma_\infty(N^\alpha) = \overline{\perp_{n \in \mathbb{Z}} \text{span}\{\mathcal{Z}_T^{\alpha,n}(N^\alpha)\}}. \quad (2.5)$$

Here, $\perp$ denotes the orthogonal direct sum.

By combining (2.3) and part 1 of Lemma 2.7, we obtain the following:

Lemma 2.8. *Let $\alpha > \frac{-1}{2}$ and $T > 0$. Then*

$$\Gamma_T(\mathcal{Z}^\alpha(X)) = \Gamma_T(N^{\alpha,T}).$$

Remark 2.9. Lemma 2.8 is a generalization of identity (1.5). Indeed, we have that $Y_t^H = \sqrt{2-2H} \int_0^t s^{1-2H} dN_s^{H-\frac{1}{2},T}$, a.s., $t \in [0, T]$, where Y^H is the process defined in (1.6). Y^H is a bridge (of some process) if and only if $H = \frac{1}{2}$.

The following generalizes part 2 of Lemma 2.7:

Lemma 2.10. *Let $\alpha > \frac{-1}{2}$ and $T > 0$. Then we have that*

$$\Gamma_T(X) = \overline{\oplus_{n \in \mathbb{N}_0} \text{span}\{\mathcal{Z}_T^{\alpha,n}(X)\}}$$

and

$$\Gamma_\infty(X) = \overline{\oplus_{n \in \mathbb{Z}} \text{span}\{\mathcal{Z}_T^{\alpha,n}(X)\}}.$$

Proof. We assume that $X \neq N^\alpha$. By iterating (2.3) and (2.4), we obtain that

$$\mathcal{Z}_T^{\alpha,n}(X) \in \Gamma_T(\mathcal{Z}^{\alpha,n}(X)) = \Gamma_T(\mathcal{Z}^{\alpha,n}(N^\alpha)) = \overline{\perp_{i \geq n} \text{span}\{\mathcal{Z}_T^{\alpha,i}(N^\alpha)\}}, \quad n \in \mathbb{Z}.$$

Moreover, $X_T \not\perp N_T^\alpha$, hence $\mathcal{Z}_T^{\alpha,n}(X) \not\perp \mathcal{Z}_T^{\alpha,n}(N^\alpha)$, $n \in \mathbb{Z}$, and therefore,

$$\mathcal{Z}_T^{\alpha,n}(X) \notin \overline{\perp_{i \geq n+1} \text{span}\{\mathcal{Z}_T^{\alpha,i}(N^\alpha)\}}, \quad n \in \mathbb{Z}.$$

From (2.4) and (2.5), it follows that the systems $\{\mathcal{Z}_T^{\alpha,n}(X)\}_{n \in \mathbb{N}_0}$ and $\{\mathcal{Z}_T^{\alpha,n}(X)\}_{n \in \mathbb{Z}}$ are free and complete in $\Gamma_T(X)$ and $\Gamma_\infty(X)$, respectively. $\square$

Remark 2.11. The process X is an $\mathbb{F}^X$ -Markov process if and only if there exists $\alpha > \frac{-1}{2}$ and a constant $c(X)$, such that

$$X_t = c(X) \cdot t^{\beta-\frac{1}{2}-\alpha} \int_0^t s^\alpha dW_s, \quad a.s., \quad t \in (0, \infty). \quad (2.6)$$

The free complete system $\{\mathcal{Z}_T^{\alpha,n}(X)\}_{n \in \mathbb{Z}}$ is orthogonal if and only if (2.6) is satisfied.

From Lemma 2.10, we obtain the following:

Corollary 2.12. *Let $\alpha > \frac{-1}{2}$ and $T > 0$. Then*

$$\mathcal{F}_T^X = \bigvee_{n \in \mathbb{N}_0} \sigma(\mathcal{Z}_T^{\alpha, n}(X)).$$

Furthermore,

$$\mathcal{F} = \mathcal{F}_\infty^X = \bigvee_{n \in \mathbb{Z}} \sigma(\mathcal{Z}_T^{\alpha, n}(X)).$$

Recall that an automorphism $\mathcal{Z}$ is a *Kolmogorov automorphism*, if there exists a σ -algebra $\mathcal{A} \subseteq \mathcal{F}$, such that $\mathcal{Z}^{-1}\mathcal{A} \subseteq \mathcal{A}$, $\bigvee_{m \in \mathbb{Z}} \mathcal{Z}^m \mathcal{A} = \mathcal{F}$ and $\bigcap_{m \in \mathbb{N}_0} \mathcal{Z}^{-m} \mathcal{A} = \{\Omega, \emptyset\}$. A Kolmogorov automorphism is strongly mixing and hence ergodic (see [11], Propositions 5.11 and 5.9 on p. 63 and p. 62). The ergodicity of $\mathcal{Z}^\alpha$ is hence a consequence of the following:

Theorem 2.13. *Let $\alpha > \frac{-1}{2}$ and $T > 0$. The automorphisms $\mathcal{Z}^\alpha$ and $\mathcal{Z}^{\alpha, -1}$ are Kolmogorov automorphisms with $\mathcal{A} = \bigvee_{n \in -\mathbb{N}} \sigma(\mathcal{Z}_T^{\alpha, n}(X))$ and $\mathcal{A} = \mathcal{F}_T^X$, respectively.*

Proof. $\mathcal{Z}^\alpha$ is a Kolmogorov automorphism with $\mathcal{A} = \bigvee_{n \in -\mathbb{N}} \sigma(\mathcal{Z}_T^{\alpha, n}(X))$:

First, $\mathcal{Z}^{\alpha, -1} \mathcal{A} = \bigvee_{n \in -\mathbb{N}} \sigma(\mathcal{Z}_T^{\alpha, n-1}(X)) \subseteq \mathcal{A}$.

Second, $\bigvee_{m \in \mathbb{Z}} \mathcal{Z}^{\alpha, m} \mathcal{A} = \bigvee_{m \in \mathbb{Z}} \bigvee_{n \in -\mathbb{N}} \sigma(\mathcal{Z}_T^{\alpha, m+n}(X)) = \mathcal{F}$.

Third, let $\{Y_n\}_{n \in -\mathbb{N}}$ denote the Hilbert basis of $\overline{\bigoplus_{n \in -\mathbb{N}} \text{span}\{\mathcal{Z}_T^{\alpha, n}(X)\}}$ which is obtained from $\{\mathcal{Z}_T^{\alpha, n}(X)\}_{n \in -\mathbb{N}}$ via Gram-Schmidt orthonormalization. By using Kolmogorov's zero-one law (see [12], p. 381), we obtain that $\bigcap_{m \in \mathbb{N}_0} \mathcal{Z}^{\alpha, -m} \mathcal{A} = \bigcap_{m \in \mathbb{N}_0} (\bigvee_{n \leq -m-1} \sigma(\mathcal{Z}_T^{\alpha, n}(X))) = \bigcap_{m \in \mathbb{N}_0} (\bigvee_{n \leq -m-1} \sigma(Y_n)) = \{\Omega, \emptyset\}$.

Similarly, one shows that $\mathcal{Z}^{\alpha, -1}$ is a Kolmogorov automorphism with $\mathcal{A} = \mathcal{F}_T^X$. $\square$

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