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APPROXIMATE MODEL EQUATIONS FOR WATER WAVES*

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Abstract. We present two new model equations for the unidirectional propagation of long waves in dispersive media for the specific purpose of modeling water waves. The derivation of the new equations uses a Padé (2,2) approximation of the phase velocity that arises in the linear water wave theory. Unlike the Korteweg-deVries (KdV) equation and similarly to the Benjamin-Bona-Mahony (BBM) equation, our models have a bounded dispersion relation. At the same time, the equations we propose provide the best approximation of the phase velocity for small wave numbers that can be obtained with third-order equations. We note that the new model equations can be transformed into previously studied models, such as the BBM and the Burgers-Poisson equations. It is therefore straightforward to establish the existence and uniqueness of solutions to the new equations. We also show that the distance between the solutions of one of the new equations, the KdV equation, and the BBM equation, is of the small order that is formally neglected by all models.

Key words. KdV equation, BBM equation, Padé approximation, water waves

AMS subject classifications. 35Q53

1. Introduction

The first published observation of a solitary wave (a single and localized wave) was made by John Scott Russel on the Edinburgh-Glasgow canal in 1834. An account of his observation is given in his 1844 report to the British Association [11]. In this report, Russel describes how he followed a solitary water wave for more than one mile on horseback, observing that the wave preserves its original shape. He also noted that higher waves travel faster, that an initial profile evolves into several waves which then move apart and approach solitary waves as time $t \to \infty$, and that solitary waves that move with different speeds, undergo a nonlinear interaction from which they emerge in their original shape.

In 1872, Joseph Boussinesq proposed a variety of possible models for describing the propagation of water waves in shallow channels [3], including what is now referred to as the Korteweg-deVries (KdV) equation

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.1}$$

The form of equation (1.1) is non-dimensional: the physical parameters are scaled into the definition of space x, time t, and the water velocity $u(x,t)$. In his work, Boussinesq found the first members of what is now known to be an infinite hierarchy of conservation laws for (1.1). He also gave some evidence that the solutions to his model resemble Russel's solitary wave. In 1895 Korteweg and de Vries re-derived Eq. (1.1) [10]. Their main contribution was in paying specific attention to the solitary wave solution of (1.1).

While the KdV equation has remarkable properties [6], some other aspects of this equation are less favorable. This includes, e.g., an unbounded dispersion relation, that is obviously non-physical. Several noticeable attempts to improve the KdV model were

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taken over the years. In particular, we would like to mention two such models: the Benjamin-Bona-Mahony (BBM) equation [1] and the Camassa-Holm (CH) equation [4, 8]. The BBM equation,

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.2}$$

replaces the third-order derivative in (1.1) by a mixed derivative, $-u_{xxt}$, which, in turn, results in a bounded dispersion relation. This boundedness was utilized to prove existence, uniqueness, and regularity results for solutions of the BBM equation [1]. Most of these results are by now also known for the KdV equation (see [5] and the references therein). A second model, the so-called Camassa-Holm (CH) equation,

$$u_t + ku_x + 3uu_x - u_{xxt} - (2u_x u_{xx} + uu_{xxx}) = 0, \quad k \geq 0, \tag{1.3}$$

was derived from an asymptotic expansion of the Hamiltonian for the incompressible Euler equations in the shallow water regime [4]. The CH equation is bi-Hamiltonian, completely integrable and has an infinite number of conservation laws. BBM-like and CH-like equations play a major role in the present study.

In this work we are interested in developing approximate models for water waves. We present a framework for deriving such models, all of which are based on approximating the linear theory dispersion relation. Some of the models that result from our approach are well-known, such as the KdV, the BBM, and the Burgers-Poisson equations. A couple of other (also third-order) models are derived in this paper for the first time. These models are based on second-order Padé approximations. They provide a better approximation of the dispersion relation for small wave numbers.

The structure of the paper is as follows: we start in Section 2 with a short review of the linear water wave theory. The main goal here is to recall the dispersion relation that is associated with this problem. A unified approach to weakly nonlinear waves is then discussed in Section 3. Here, we use two different strategies to approximate the dispersion relation derived in Section 2. First, we invoke the procedure that was proposed by Whitham [13] for deriving a nonlinear equation from a given dispersion relation. Using this strategy we obtain the KdV equation from a second-order Taylor approximation, the Burgers-Poisson equation from a (0,2) Padé approximation (similarly to what was done in [7]), and a new (third-order) model equation from a (2,2) Padé approximation. This equation is referred to as the Padé-I equation. We then follow Benjamin *et al.* [1], and use asymptotic expansions arguments to simplify the model equation. In practice, this amounts to removing the same nonlinear terms that one has to add to the BBM equation in order to turn it into the Burgers-Poisson equation [7]. This procedure yields a simplified version of the new model equation, which we refer to as the Padé-II equation.

In Section 4 we formulate the existence and uniqueness theory for the new Padé-I and Padé-II equations. These results follow from a change of variables that transforms the two equations into the BBM and the Burgers-Poisson equations. We also provide a comparison between solutions of the KdV, BBM, and the Padé-II equations. This analysis is based on the ideas of [2].

2. Linear water wave theory

The water wave equations. We consider a two-dimensional inviscid incompressible fluid in a constant gravitational field. The space coordinates are (x,y) and the gravitational acceleration g is in the negative y direction. Let h_0 be the undisturbed depth of the fluid and let $y=\eta(x,t)$ represent the free surface of the fluid (see Figure 2.1). We also assume that the motion is irrotational and let $\varphi(x,y,t)$ denote the velocity potential ($u=\nabla\varphi$).

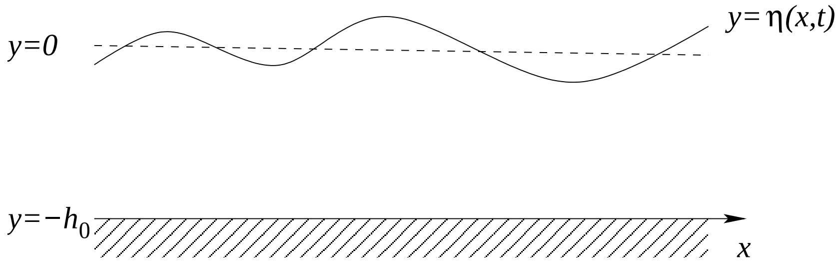

FIG. 2.1. *Geometrical configuration for water waves*

The divergence-free condition on the velocity field implies that the velocity potential φ satisfies the Laplace's equation [9]:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0, \qquad \text{for } -h_0<y<\eta(x,t). \tag{2.1}$$

On a solid fixed boundary, the normal velocity of the fluid must vanish. For a horizontal flat bottom, h_0 is constant and we have

$$\frac{\partial\varphi}{\partial y}=0, \quad \text{on } y=-h_0. \tag{2.2}$$

The boundary conditions at the free surface $y=\eta(x,t)$ are given by

$$\frac{\partial\varphi}{\partial y}-\frac{\partial\eta}{\partial t}-\frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial x}=0, \tag{2.3a}$$

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right]+g\eta=0. \tag{2.3b}$$

Equation (2.3a) is a kinematic boundary condition, while (2.3b) represents the continuity of pressure at the free surface, as derived from Bernoulli's equation.

The linearized theory. We assume small perturbations of the water surface that is initially at rest. In this case, η and φ are small, and the equations can be linearized. The Laplace equation (2.1) and the boundary conditions (2.2) on the bottom are already linear and are independent of η. Moreover, η can be eliminated from the linear versions of (2.3a) and (2.3b). The linear problem for φ alone is:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0, \qquad \text{for } -h_0<y<0, \tag{2.4a}$$

$$\frac{\partial\varphi}{\partial y}=0, \qquad \text{on } y=-h_0, \tag{2.4b}$$

$$\frac{\partial^2\varphi}{\partial t^2}+g\frac{\partial\varphi}{\partial y}=0, \qquad \text{on } y=0. \tag{2.4c}$$

We assume a solution in the form of a sinusoidal wave

$$\varphi(x,y,t)=\hat{\varphi}(y)e^{i(kx-\omega t)}. \tag{2.5}$$

Then, (2.4a) yields the solution

$$\hat{\varphi}(y) = A\cosh k(y+h_0) + B\sinh k(y+h_0), \tag{2.6}$$

where A and B are arbitrary constants. The boundary condition (2.4b) implies $B=0$, while the remaining condition (2.4c) gives the dispersion relation:

$$\omega^2 = gk\tanh kh_0. \tag{2.7}$$

This dispersion relation, (2.7), will play a critical role in the following sections.

3. A unified approach to weakly nonlinear waves

In this section we consider various approximations of the dispersion relation (2.7). These approximations are then used to derive nonlinear PDEs for which the approximation of (2.7) is the exact dispersion relation of the corresponding linear PDE. Clearly, without any additional constraints on the nonlinear terms (or on the equation as a whole), there is no unique correspondence between a PDE and a given approximate dispersion relation. Our starting point is the systematic way of inverting the dispersion relation (and hence, picking a particular form of the PDE) given by Whitham [13].

Following Whitham's strategy, for a given arbitrary dispersion relation

$$\frac{\omega}{k} = c(k), \tag{3.1}$$

a corresponding linear equation is given in [13, Section 13.14] as the integro-differential equation

$$u_t + \int_{-\infty}^{\infty} K(x-\xi)u_\xi(\xi,t)d\xi = 0, \tag{3.2}$$

with a kernel, $K(x)$, that is given by

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dx. \tag{3.3}$$

The dispersive effects can be then combined with nonlinear effects (see [13, Section 13.14]) to give:

$$u_t + \frac{3}{2}\frac{c_0}{h_0}uu_x + \int_{-\infty}^{\infty} K(x-\xi)u_\xi(\xi,t)d\xi = 0. \tag{3.4}$$

Here, h_0 is the depth of the fluid and $c_0 = \sqrt{gh_0}$.

We are now ready to consider several small k approximations of the phase velocity, $c(k)$ (given by the linear water wave theory (2.7)),

$$c(k) = c_0\sqrt{\frac{\tanh(kh_0)}{kh_0}}. \tag{3.5}$$

We will discuss three such approximations: one truncated Taylor series and two Padé approximations.

Taylor expansion. We consider the Taylor expansion of (3.5) truncated to second order:

$$c(k)=c_0\left(1-\frac{1}{6}(kh_0)^2\right)=c_0-\frac{1}{6}c_0h_0^2k^2. \tag{3.6}$$

Using (3.6) in (3.3) yields

$$K(x)=c_0\delta(x)+\frac{1}{6}c_0h_0^2\delta''(x). \tag{3.7}$$

For $K(x)$ given by (3.7), the PDE (3.4) reduces to the KdV equation:

$$u_t+c_0u_x+\frac{3}{2}\frac{c_0}{h_0}uu_x+\frac{1}{6}c_0h_0^2u_{xxx}=0. \tag{3.8}$$

Padé (0,2). The Padé (0,2) approximation of (3.5) is given by

$$c(k)=c_0\frac{1}{1+\frac{1}{6}h_0^2k^2}. \tag{3.9}$$

The kernel $K(x)$ that corresponds to (3.9) can be easily computed by recalling that $G(x)=\frac{1}{2\alpha}e^{-\frac{|x|}{\alpha}}$ is the Green's function for the operator $L=\mathrm{Id}-\alpha^2\partial_x^2$. Its Fourier transform $\hat{G}$ is given by $\hat{G}(k)=\frac{1}{1+\alpha^2k^2}$. Hence,

$$K(x)=\frac{c_0\sqrt{6}}{2h_0}e^{-\frac{-\sqrt{6}|x|}{h_0}}. \tag{3.10}$$

With this kernel, the nonlinear dispersive wave equation (3.4) is the Burgers-Poisson (BP) equation

$$u_t+c_0u_x+\frac{3}{2}\frac{c_0}{h_0}uu_x-\frac{1}{6}h_0^2u_{xxt}-\frac{1}{4}c_0h_0(3u_xu_{xx}+uu_{xxx})=0. \tag{3.11}$$

REMARK 3.1. *Local and global existence for a scaled version of the BP equation (3.11) were recently studied in [7]. This work contains also a detailed study of traveling wave solutions of (3.11). Interestingly, this model was already considered by Whitham in [13, Section 13.14], where he proposes to use the approximate kernel (3.10) in the nonlinear wave equation (3.4). Nevertheless, we note that the equation (3.11) is not explicitly written in [13]. It is also demonstrated in [13] that kernels like (3.10) translate into PDE's that feature wave breaking in finite time. The reason is that the dispersive term in (3.11) is milder, making a sufficiently asymmetric hump break in the typical hyperbolic manner.*

Padé (2,2). The dispersion relation (3.5) is an even function, and hence the Padé (1,2) is identical to the (0,2) case. The next case of interest is therefore the Padé (2,2) approximation of (3.5), which is

$$c(k)=c_0\frac{1+\frac{3}{20}h_0^2k^2}{1+\frac{19}{60}h_0^2k^2}. \tag{3.12}$$

The kernel $K(x)$ that corresponds to (3.12) is

$$K(x)=c_0\left[\frac{9}{19}\delta(x)+\frac{10}{19}\frac{1}{2\beta}e^{-\frac{|x|}{\beta}}\right], \tag{3.13}$$

where $\beta^2 = \frac{19}{60}h_0^2$. Substituting (3.13) into (3.4), we obtain

$$u_t + c_0 u_x + \frac{3}{2}\frac{c_0}{h_0} u u_x - \frac{19}{60} h_0^2 u_{xxt} - \frac{3}{20} h_0^2 c_0 u_{xxx} - \frac{19}{40} c_0 h_0 (3 u_x u_{xx} + u u_{xxx}) = 0. \quad (3.14)$$

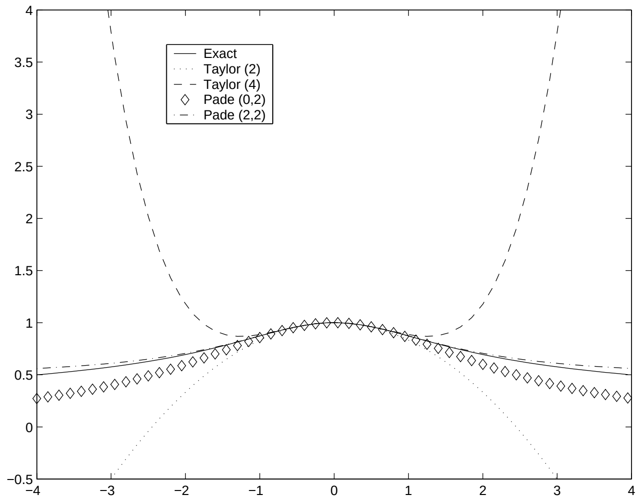

FIG. 3.1. *Approximations of the scaled phase velocity* $c(k) = \sqrt{\frac{\tanh k}{k}}$ *in the range of small* k.

REMARKS 3.2.

1. *Figure 3.1 compares the different approximations of the function* $c(k) = \sqrt{\frac{\tanh k}{k}}$. *It is important to note that the Padé (2,2) approximation (3.12) provides a better small* k *approximation for the phase velocity (3.5), while keeping the phase velocity positive and the group velocity bounded.*
2. *The special interest we have in the Padé* (2,2) *approximation is due to the fact that we can seemingly improve the quality of the approximation of the dispersion relation for small* k *while still considering a third-order equation. Higher-order Padé and Taylor approximations correspond to equations of order higher than three. Taylor approximations of higher order have an associated non-bounded phase velocity which is the main source of the theoretical difficulties that are associated with the KdV equation.*
3. *The Padé* (2,2) *model, (3.14), does not seem to have been previously considered.*

In order to simplify the nonlinear dispersive equations (3.8), (3.11) and (3.14), we rescale u, x and t by $\frac{2}{3}$, $\frac{1}{\sqrt{6}}$ and $\frac{1}{\sqrt{6}}$, respectively, and set $h_0 = 1$ and $c_0 = 1$. The equations then become, respectively:

$$u_t + u_x + u u_x + u_{xxx} = 0, \quad (3.15)$$

$$u_t + u_x + u u_x - u_{xxt} - (3 u_x u_{xx} + u u_{xxx}) = 0, \quad (3.16)$$

$$u_t + u_x + u u_x - \frac{9}{10} u_{xxx} - \frac{19}{10} u_{xxt} - \frac{19}{10} (3 u_x u_{xx} + u u_{xxx}) = 0. \quad (3.17)$$

For future reference we label equation (3.17) the Padé-I equation.

Other water wave models. We briefly overview a couple of other water wave models that are related to this work.

BBM equation. The Benjamin-Bona-Mahony model, given by (1.2), was introduced in [1] as an alternative to the KdV equation.

As stated in [1], the BBM equation represents, in important respects, a more satisfactory model than the KdV equation. The main argument is that the phase velocity ω/k and the group velocity $d\omega/dk$ in the KdV model (see (3.6)) are not bounded from below (as functions of k), which means that there is no control over the rate at which the fine scale features are transmitted in the negative $(-x)$ direction. In contrast, the BBM equation has a phase velocity (see (3.9)) and a group velocity both of which are bounded for all k. They also approach zero as $k \to \infty$.

Camassa-Holm equation. The Camassa-Holm equation (see (1.3)) was first obtained by Fokas and Fuchssteiner [8] as a formally integrable bi-Hamiltonian nonlinear PDE. Its physical derivation as a model for shallow water waves is due to Camassa and Holm [4]. The derivation uses an asymptotic expansion in the Hamiltonian for incompressible Euler equations in the shallow water regime.

While the terms in (1.3) correspond to those in (3.16), equation (1.3) has some additional appealing features: it is completely integrable, it is bi-Hamiltonian and it possesses an infinite number of conservation laws. Similarly to (3.16), for $k=0$, the Camassa-Holm equation (1.3) admits peaked solitary wave solutions (peakons) given by $u(x,t)=\sigma\exp(-|x-\sigma t|)$. In addition to permanent wave forms, it also features breaking phenomena, which makes it an attractive model for shallow water waves.

BBM, KdV or Padé?

The derivation of the KdV equation in [1] uses a scaling of the variables u, x and t, and a perturbation expansion argument in such a way that the dispersive and the nonlinear effects become small. Such a scaled KdV equation reads:

$$u_t + u_x + \epsilon(uu_x + u_{xxx}) = 0, \tag{3.18}$$

where ϵ is a small perturbation parameter.

The main argument that was used in [1] to derive the BBM equation is that, to the first order in ϵ, the scaled KdV equation (3.18) is equivalent to

$$u_t + u_x + \epsilon(uu_x - u_{xxt}) = 0, \tag{3.19}$$

as both equations reduce to $u_t + u_x = 0$ at zero order.

While the derivation presented in [1] is formally valid, it is important to note that the particular choice of the mixed derivative $-u_{xxt}$ as a replacement of u_{xxx} may seem arbitrary from the point of view of asymptotic expansions. Indeed, any admissible combination[1] of these two terms could be valid based on the zero-order correspondence between the derivatives in space and time $(u_t = -u_x)$.

An approximation theory point of view sheds a somewhat different light on this argument. Based on the Padé approximations (3.9) and (3.12) of the linear water wave phase velocity $c(k)$ given by (3.5), it is clear that not every combination of the two third-order terms is valid. In fact, the only valid choices (in addition to KdV), are precisely (3.9) and (3.12).

[1]An admissible combination is a sum $\alpha u_{xxx} - \beta u_{xxt}$, with $\alpha+\beta=1$ (where α and β are not necessarily positive).

The Padé $(0,2)$ approximation used in this BBM-type derivation yields the BBM equation (3.19). By similar arguments, the same procedure can be repeated for the Padé $(2,2)$ approximation, to yield (compare with (3.17)):

$$u_t + u_x + \epsilon\left(uu_x - \frac{9}{10}u_{xxx} - \frac{19}{10}u_{xxt}\right) = 0. \tag{3.20}$$

The parameter ϵ can be eliminated from equations (3.19) and (3.20) by rescaling u, x and t with $\frac{1}{\epsilon}$, $\sqrt{\epsilon}$ and $\sqrt{\epsilon}$, respectively. With such a scaling, one obtains the form (1.2) and

$$u_t + u_x + uu_x - \frac{9}{10}u_{xxx} - \frac{19}{10}u_{xxt} = 0. \tag{3.21}$$

We would like to emphasize that (3.21) is the third-order equation that provides the best approximation of the dispersion relation (2.7) for small k. Together with the Padé-I equation, (3.17), it will serve as the main model equation that will be investigated in the next section. For future reference we label equation (3.21) as Padé-II.

We conclude this section by noting that the linear part of the BBM equation (1.2) can be obtained from (3.2) where $K(x)$ is the Green's function of the operator $\mathrm{Id} - \partial_x^2$. The nonlinear term uu_x, however, has to be added separately, unlike the case of the Burgers-Poisson equation (3.16), which was derived using the unified Whitham approach provided by (3.4). This fact emphasizes again the non-uniqueness of the nonlinear terms in the various dispersive water wave models.

4. Analysis

We now discuss the analytical properties of the two PDE's ((3.21) and (3.17)) that were derived from a Padé $(2,2)$ approximation of the linear phase velocity.

Change of variables. Using the notation

$$a^2 = \frac{9}{10}, \quad b^2 = \frac{19}{10}, \quad p = \frac{a^2}{b^2}, \tag{4.1}$$

equations (3.21) and (3.17) read, respectively:

$$u_t + u_x + uu_x - a^2 u_{xxx} - b^2 u_{xxt} = 0, \tag{4.2}$$

$$u_t + u_x + uu_x - a^2 u_{xxx} - b^2 u_{xxt} - b^2(3u_x u_{xx} + uu_{xxx}) = 0. \tag{4.3}$$

We now consider the following change of variables:

$$v(x,t) = u(bx + bpt, bt). \tag{4.4}$$

Then (4.2) becomes

$$v_t + (1-p)v_x + vv_x - v_{xxt} = 0, \tag{4.5}$$

and (4.3) becomes

$$v_t + (1-p)v_x + vv_x - v_{xxt} - (3u_x u_{xx} + uu_{xxx}) = 0. \tag{4.6}$$

Moreover, the coefficient $1-p$ can also be eliminated from (4.5) and (4.6) by substituting

$$v(x,t) = (1-p)U, \quad t = \frac{T}{1-p}. \tag{4.7}$$

This change of variables transforms (4.5) into the BBM equation (1.2) and (4.6) into the Burgers-Poisson equation (3.16).

Existence and Uniqueness. As a result of this transformation, the existence and uniqueness results for (3.21) and (3.17) (or (4.2) and (4.3) with the notations (4.1)) follow from the known results for the BBM and Burgers-Poisson equations [1, 7]. We summarize these results below.

We start with an existence and uniqueness result for the Cauchy problem associated with (4.2):

$$\begin{cases} u_t+u_x+uu_x-a^2u_{xxx}-b^2u_{xxt}=0, \\ u(x,t=0)=u_0(x). \end{cases} \tag{4.8}$$

We denote by $\mathcal{C}_T$ the space of functions, $v(x,t)$, that are continuous and uniformly bounded on $\mathbb{R}\times[0,T]$. The space $\mathcal{C}_T^{l,m}$ is the set of function $v(x,t)$ such that $\partial_x^i\partial_t^j v\in\mathcal{C}_T$ for $0\le i\le l$, $0\le j\le m$. Using these notations, Theorems 1 and 4 in [1] imply

PROPOSITION 4.1. *Let $u_0\in C^2(\mathbb{R})$ and assume that $\int_{-\infty}^{\infty}(u_0^2+u_0'^2)dx<\infty$. Then (4.8) has a unique solution $u\in\mathcal{C}_\infty^{2,\infty}$.*

REMARK 4.2. *Note that Proposition 4.1 holds for any $b\neq 0$.*

We now consider the Cauchy problem associated with the Padé-I equation (4.3):

$$\begin{cases} u_t+u_x+uu_x-a^2u_{xxx}-b^2u_{xxt}-b^2(3u_xu_{xx}+uu_{xxx})=0, \\ u(x,t=0)=u_0(x). \end{cases} \tag{4.9}$$

Using the change of variables (4.4), (4.7), the problem (4.9) reduces to the Cauchy problem for the Burgers-Poisson equation (3.16). Hence, the results from [7, Theorem 4.1] imply the following

PROPOSITION 4.3. *Assume $u_0\in H^k(\mathbb{R})$ with $k>\frac{3}{2}$. Then, there exists a time $T>0$ such that (4.9) has a unique solution $u\in L^\infty((0,T);H^k(\mathbb{R}))\cap C([0,T];H^{k-1}(\mathbb{R}))$.*

Whitham (see [13, Section 13.14]) presents an argument due to Selinger [12] according to which a solution with sufficiently asymmetric initial data for (3.16) would break in finite time. Clearly, a similar result holds for the Padé-I equation (3.17) as well.

Global weak solutions of the Cauchy problem for (3.16) were studied in [7]. According to [7, Theorem 4.2], for initial data in $BV(\mathbb{R})$, the equation (3.16) has a global weak solution $u\in L^\infty_{loc}([0,\infty);BV(\mathbb{R}))$. Moreover, the weak solution satisfies an appropriate entropy condition. The analog of this theorem for the Cauchy problem (4.9) is

PROPOSITION 4.4. *If $u_0\in BV(\mathbb{R})$, there exists a global weak solution $u\in L^\infty_{loc}([0,\infty);BV(\mathbb{R}))$ of (4.9), satisfying the entropy condition*

$$\left(u^2\right)_t+\left[\frac{2}{3}u^3+\frac{a^2}{b^2}u^2+\left(1-\frac{a^2}{b^2}\right)\left(\psi^2-b^2\psi_x^2\right)\right]_x\le 0, \tag{4.10}$$

where

$$\psi-b^2\psi_{xx}=u.$$

Proof. The regularity result follows immediately from [7, Theorem 4.2]. As for the entropy condition, we rewrite (4.3) as

$$u_t + uu_x + \varphi[u]_x = 0, \tag{4.11}$$

where

$$\varphi[u] = K * u.$$

The kernel $K(x)$ is

$$K(x) = \frac{a^2}{b^2}\delta(x) + \left(1 - \frac{a^2}{b^2}\right)\frac{1}{2b}e^{-\frac{|x|}{b}}. \tag{4.12}$$

Note that the kernel (4.12) is a scaled version of (3.13). The entropy condition (4.10) can be then obtained from (4.11) by standard arguments. ∎

BBM, KdV or Padé? (revisited). We are interested in comparing solutions of the Cauchy problems for (3.18), (3.19) and (3.20), with the same initial data $g(x)$. The technique follows [2].

We first note that, rescaling u, x and t in (3.18), (3.19) and (3.20) by $\frac{1}{\epsilon}$, $\sqrt{\epsilon}$ and $\sqrt{\epsilon}$, respectively, reduces the problem to comparing the solutions of (3.15), (1.2) and (3.21) with identical initial data

$$g_0(x) = \epsilon g(\epsilon^{\frac{1}{2}} x). \tag{4.13}$$

Following [2], we consider the initial-value problem for the KdV equation

$$\begin{cases} \eta_t^\epsilon + \eta_x^\epsilon + \eta^\epsilon \eta_x^\epsilon + \eta_{xxx}^\epsilon = 0, & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \eta^\epsilon(x,0) = \epsilon g(\epsilon^{1/2} x), \end{cases} \tag{4.14}$$

and the corresponding initial-value problem for the BBM equation

$$\begin{cases} \xi_t^\epsilon + \xi_x^\epsilon + \xi^\epsilon \xi_x^\epsilon - \xi_{xxt}^\epsilon = 0, & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \xi^\epsilon(x,0) = \epsilon g(\epsilon^{1/2} x). \end{cases} \tag{4.15}$$

In addition, we consider the Cauchy problem associated with the Padé-II equation (3.21):

$$\begin{cases} \mu_t^\epsilon + \mu_x^\epsilon + \mu^\epsilon \mu_x^\epsilon - \frac{9}{10}\mu_{xxx}^\epsilon - \frac{19}{10}\mu_{xxt}^\epsilon = 0, & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \mu^\epsilon(x,0) = \epsilon g(\epsilon^{1/2} x). \end{cases} \tag{4.16}$$

The proximity of the solutions of (4.16) and (4.14) follows from the results of [2] concerning the estimates on the distance between the solutions of (4.14) and (4.15). The following change of variables

$$x \mapsto \frac{x}{\sqrt{\epsilon}} + \frac{t}{\epsilon\sqrt{\epsilon}}, \quad t \mapsto \frac{t}{\epsilon\sqrt{\epsilon}}, \quad u = \frac{\eta^\epsilon}{\epsilon}, \quad v = \frac{\xi^\epsilon}{\epsilon}, \quad w = \frac{\mu^\epsilon}{\epsilon},$$

transforms the Cauchy problems (4.14), (4.15), and (4.16) into

$$u_t + uu_x + u_{xxx} = 0, \tag{4.17}$$

$$v_t + vv_x + v_{xxx} - \epsilon v_{xxt} = 0, \tag{4.18}$$

$$w_t + ww_x + w_{xxx} - \epsilon \frac{19}{10} w_{xxt} = 0, \tag{4.19}$$

respectively. The initial data becomes $u(x,0) = v(x,0) = w(x,0) = g(x)$. The only difference between (4.18) and (4.19) is the coefficient of the mixed derivative term. In view of [2, Theorem 1] we have

PROPOSITION 4.5. *Let $g \in H^{k+5}$ with $k \geq 0$. Let $\epsilon > 0$ and η^ϵ, μ^ϵ be the unique solutions of the Cauchy problems (4.14), (4.16). Then, there is an $\epsilon_0 > 0$ and constants M_j such that if $0 < \epsilon \leq \epsilon_0$, then*

$$||\partial_x^j \eta^\epsilon(\cdot,t) - \partial_x^j \mu^\epsilon(\cdot,t)||_2 \leq M_j \epsilon^{\frac{j}{2}+\frac{13}{4}} t, \tag{4.20}$$

at least for $0 \leq t \leq \epsilon^{-\frac{3}{2}}$, where $0 \leq j \leq k$.

Proof. We note that the existence and uniqueness for (4.14) is standard (see [2] for instance), and the existence and uniqueness for (4.16) is guaranteed by Proposition 4.1. The proof is identical to the proof of [2, Theorem 1] taking into account the different constants. □

REMARKS 4.6.

1. *A simpler comparison analysis can be done for the linear versions of (4.14), (4.15) and (4.16). The technique is to estimate the distance between the solutions of the linear equations in the Fourier space and then transform these estimates back to the physical space (see [2] for how this technique is used to compare solutions of the linearized (4.14) and (4.15)).*
2. *Clearly, a result identical to (4.20) holds with respect to the distance between solutions of (4.15) and (4.16).*

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