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Toda Lattice Hierarchy and Generalized String Equations

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Abstract: String equations of the p^{th} generalized Kontsevich model and the compactified $c = 1$ string theory are re-examined in the language of the Toda lattice hierarchy. As opposed to a hypothesis postulated in the literature, the generalized Kontsevich model at $p = -1$ does not coincide with the $c = 1$ string theory at self-dual radius. A broader family of solutions of the Toda lattice hierarchy including these models is constructed, and shown to satisfy generalized string equations. The status of a variety of $c \leqq 1$ string models is discussed in this new framework.

1. Introduction

The so-called "string equations" play a key role in various applications of integrable hierarchies to low dimensional string theories. The most fundamental integrable hierarchy in this context is the KP hierarchy [31] that provides a universal framework for dealing with many KdV-type hierarchies. String equations for "(p, q) models" of two-dimensional quantum gravity can be treated in a unified manner in this language. In contrast, the status of the Toda lattice hierarchy [37], which is another universal integrable hierarchy, had remained relatively obscure until rather recent years. The Toda lattice hierarchy was pointed out to be an integrable structure of the one- and multi-matrix models [13], but these matrix models (matrix integrals) were only considered as an intermediate step towards the continuous (double scaling) limit to two-dimensional quantum gravity.

In the last few years, the Toda lattice hierarchy has come to be studied from renewed points of view, such as $c = 1$ strings [7, 23, 27, 28], two-dimensional topological strings [14, 16, 10, 34, 5], the topological CP^1 sigma model and its variations related to affine Coxeter groups [11, 18, 9]. As opposed to the (p, q) models in the KP hierarchy, these are related to string theories with a true continuous target space. Our goal in this paper is to elucidate the structure of those string equations, in particular, those of $c = 1$ strings in a more general framework.

It will be instructive to recall the relationship between the (p, q) models and the KP hierarchy. String equations of these models were first discovered in the form of the Douglas equation [8]

$$[P,Q]=1\,, \tag{1.1}$$

where P and Q are ordinary differential operators of the form

$$\begin{aligned} P &= \partial_x^p + a_2\partial_x^{p-2} + \cdots + a_p\,, \\ Q &= \partial_x^q + b_2\partial_x^{q-2} + \cdots + b_q\,. \end{aligned} \tag{1.2}$$

These two operators were later found to be related to an extended Lax formalism of the KP hierarchy. The extended Lax formalism, developed by Orlov and his coworkers [29], is based on the ordinary Lax operator L and a secondary Lax operator (Orlov–Shulman operator) M. They are pseudo-differential operators of the form

$$\begin{aligned} L &= \partial_x + \sum_{n=1}^{\infty} u_{n+1}\partial_x^{-n}\,, \\ M &= \sum_{n=2}^{\infty} nt_nL^{n-1} + x + \sum_{n=1}^{\infty} v_nL^{-n-1}\,, \end{aligned} \tag{1.3}$$

and obey the Lax equations

$$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad \frac{\partial M}{\partial t_n} = [B_n, M] \tag{1.4}$$

and the canonical commutation relation

$$[L,M]=1\,. \tag{1.5}$$

The operators B_n are given by

$$B_n = (L^n)_{\geqq 0}\,, \tag{1.6}$$

where "$(\quad)_{\geqq 0}$" denotes the projection onto the space spanned by nonnegative powers of ∂_x; similarly, we shall use "$(\quad)_{\leqq -1}$" for the projection onto the space spanned by negative powers of ∂_x. To reproduce the Douglas equation, we define P and Q as

$$\begin{aligned} P &= L^p\,, \\ Q &= -\frac{1}{p}ML^{1-p} + \frac{p-1}{2p}L^{-p} + L^q\,, \end{aligned} \tag{1.7}$$

require the constraints

$$(P)_{\leqq -1} = 0, \qquad (Q)_{\leqq -1} = 0, \tag{1.8}$$

and restrict the range of the time variables as

$$t_{p+q} = t_{p+q+1} = \cdots = 0\,. \tag{1.9}$$

The Douglas equation follows automatically from the construction of P and Q – (1.8) and (1.9) ensure that P and Q are differential operators of the required form. Note, in particular, that (1.8) give constraints on the Lax and Orlov–Shulman operators, the first one being a reduction condition to the p^{th} generalized KdV hierarchy. Thus the (p,q) model can be understood as a "constrained KP hierarchy." From the standpoint of the KP hierarchy, therefore, it is these constraints on L and M rather than the Douglas equation itself that plays a more essential role. In our terminology, "string equations" mean such constraints as (1.8).

This correspondence with the KP hierarchy allows one to use many powerful tools developed for the study of the KP hierarchy, such as the Sato Grassmannian, the Hirota equations, $W_{1+\infty}$ algebras, etc. [32, 15, 40, 12]. Since this raises many interesting mathematical issues, generalizations to multi-component KP hierarchies have been also attempted [38].

The case of $(p,1)$ (or $(1,q)$) models has been studied with particular interest in recent studies. This is the case where the models becomes "topological," i.e., describes topological strings of A_k $(k = p-2)$ type [6]. Unlike the other (p,q) models, the string equation for these models can be solved more explicitly in terms of a "matrix Airy function" (or "Kontsevich integral") [1], and by virtue of this matrix integral construction, it is rigorously proven (at least for the case of $p = 2$) that the corresponding τ function is a generating function of intersection numbers on a moduli space of Riemann surfaces [21]. The (generalized) Kontsevich integral is an integral over the space of $N \times N$ Hermitian matrices of the form

$$Z(\Lambda) = C(\Lambda)\int dM \exp \mathrm{Tr}\left(\Lambda^p M - \frac{1}{p+1} M^{p+1}\right), \tag{1.10}$$

where Λ is another $N \times N$ Hermitian matrix, and $C(\Lambda)$ is a normalization factor that also plays an important role. Kontsevich's observation is that this integral has two-fold interpretations. The first interpretation, revealed by "fat graph" expansion, is that this is a generating function of intersection numbers. The second is that this is a τ function of the KP hierarchy,

$$Z(\Lambda) = \tau(t), \tag{1.11}$$

where Λ and t are connected by the so called "Miwa transformation:"

$$t_n = \frac{1}{n} \mathrm{Tr}\, \Lambda^{-n} = \sum_{i=1}^{N} \lambda_i^{-n}. \tag{1.12}$$

Here $\lambda_1, \ldots, \lambda_N$ are eigenvalues of Λ.

As for the Toda lattice hierarchy, our knowledge on string equations is far more fragmental, but simultaneously suggests richer possibilities. The most fundamental and well understood cases are the one- and two-matrix models [19, 4, 25]. String equations of these two models are quite different. Let us specify this in more detail. As we shall review in the next section, the Lax formalism of the Toda lattice hierarchy uses two Lax operators $L, \bar{L}$ and two Orlov–Shulman operators $M, \bar{M}$ [35, 2]. These operators are "difference" operators obeying a set of Lax equations and a twisted canonical commutation relations of the form

$$[L, M] = L, \qquad [\bar{L}, \bar{M}] = \bar{L}. \tag{1.13}$$

(M should not be confused with the matrix variable M in the Kontsevich integral.) String equations are formulated as algebraic relations between the two pairs (L, M) and $(\bar{L}, \bar{M})$. Roughly speaking, string equations of the one-matrix model are written

$$L = \bar{L}^{-1}, \qquad ML^{-1} = -\bar{M}\bar{L}, \tag{1.14}$$

and those of the two-matrix model are given by

$$L = -\bar{M}\bar{L}, \qquad ML^{-1} = -\bar{L}^{-1}. \tag{1.15}$$

The point is that both (L, ML^{-1}) and $(\bar{L}^{-1}, \bar{M}\bar{L})$ are a canonical conjugate pair, and that string equations are canonical transformations between them. The difference between the one- and two-matrix models is that whereas string equations (1.14) of the one-matrix model are coordinate-to-coordinate and momentum-to-momentum relations, string equations (1.15) of the two-matrix model mix coordinate and momentum variables. Note that the latter is also a characteristic of Fourier transformations. This is actually related to the fact that the Kontsevich integral is essentially a matrix version of Fourier transformations [25].

In fact, most examples of string equations in the Toda lattice hierarchy are variants of the above two. For instance, string equations of $c = 1$ strings [7, 23, 27] and their topological versions [14, 16, 10, 34, 5] are of the two-matrix model type. The generalized Kontsevich models, too, may be considered as a solution of the Toda lattice hierarchy obeying string equations of the two-matrix model type [19]. Meanwhile, string equations of the topological CP^1 model and its variants [11, 18, 9] are of the one-matrix type. Furthermore, the deformed $c = 1$ theory in the presence of black hole backgrounds [28] are known to obey more involved string equations, though this case, too, is essentially of the two-matrix model type.

In this paper, we are mostly concerned with string equations of the two-matrix model type in the above sense. We re-examine the generalized Kontsevich models and the $c = 1$ string theory in detail, and present a broader family of solutions that includes these two examples as special cases. This will clarify the status of a variety of $c \leqq 1$ string models. For instance, we shall show that the generalized Kontsevich model at $p = -1$ does not reproduce the $c = 1$ string theory, but is rather related to the Penner model [30]. This poses a question on recent attempts in the literature [22] that treat two-dimensional topological strings as "A_k strings at $k = -3$."

This paper is organized as follows. Section 2 is a brief review on necessary tools and results from the theory of the Toda lattice hierarchy. Sections 3 and 4 deal with the generalized Kontsevich models and the $c = 1$ string theory. In Sect. 5, our new family of solutions and string equations are presented. Section 6 is devoted to conclusion and discussion.

2. Preliminaries on Toda Lattice Hierarchy

We first present necessary tools and results [36] in an $\hbar$-independent form (i.e., letting $\hbar = 1$), and show an $\hbar$-dependent formulation in the end of this section. Throughout this section, s denotes a discrete variable ("lattice coordinate") with values in $\mathbf{Z}$, and $t = (t_1, t_2, \ldots)$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, \ldots)$ two sets of continuous variables that play the role of "time variables" in the Toda lattice hierarchy.

2.1. Difference operators. The Lax and Orlov–Shulman operators of the Toda lattice hierarchy are difference operators of the form

$$L = e^{\partial_s} + \sum_{n=0}^{\infty} u_{n+1} e^{-n\partial_s},$$

$$M = \sum_{n=1}^{\infty} nt_n L^n + s + \sum_{n=1}^{\infty} v_n L^{-n} ,$$

$$\bar{L} = \tilde{u}_0 e^{\partial_s} + \sum_{n=0}^{\infty} \tilde{u}_{n+1} e^{(n+2)\partial_s} ,$$

$$\bar{M} = -\sum_{n=1}^{\infty} n\bar{t}_n \bar{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n , \tag{2.1}$$

where $e^{n\partial_s}$ are shift operators that act on a function of s as $e^{n\partial_s} f(s) = f(s+n)$. The coefficients u_n, v_n, $\tilde{u}_n$ and $\bar{v}_n$ are functions of $(t, \bar{t}, q)$, $u_n = u_n(t, \bar{t}, s)$, etc. These operators obey the twisted canonical commutation relations

$$[L, M] = L, \qquad [\bar{L}, \bar{M}] = \bar{L} , \tag{2.2}$$

and the Lax equations

$$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad \frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L] ,$$

$$\frac{\partial M}{\partial t_n} = [B_n, M], \qquad \frac{\partial M}{\partial \bar{t}_n} = [\bar{B}_n, M] ,$$

$$\frac{\partial \bar{L}}{\partial t_n} = [B_n, \bar{L}], \qquad \frac{\partial \bar{L}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{L}] ,$$

$$\frac{\partial \bar{M}}{\partial t_n} = [B_n, \bar{M}], \qquad \frac{\partial \bar{M}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{M}] , \tag{2.3}$$

where the Zakharov–Shabat operators B_n and $\bar{B}_n$ are given by

$$B_n = (L^n)_{\geqq 0}, \qquad \bar{B}_n = (\bar{L}^{-n})_{<0} , \tag{2.4}$$

and $(\quad)_{\geqq 0, <0}$ denotes the projection

$$\left(\sum_n a_n e^{n\partial_s}\right)_{\geqq 0} = \sum_{n \geqq 0} a_n e^{n\partial_s}, \qquad \left(\sum_n a_n e^{n\partial_s}\right)_{<0} = \sum_{n<0} a_n e^{n\partial_s} . \tag{2.5}$$

We call (2.2) "twisted" because it is rather the "untwisted" operators ML^{-1} and $\bar{M}\bar{L}^{-1}$ that give canonical conjugate variable of L and $\bar{L}$:

$$[L, ML^{-1}] = 1, \qquad [\bar{L}, \bar{M}\bar{L}^{-1}] = 1 . \tag{2.6}$$

Another important set of difference operators are the so-called "dressing operators" of the form

$$W = 1 + \sum_{n=1}^{\infty} w_n e^{-n\partial_s}, \qquad w_n = w_n(t, \bar{t}, s) ,$$

$$\bar{W} = \bar{w}_0 + \sum_{n=1}^{\infty} \bar{w}_n e^{n\partial_s}, \qquad \bar{w}_n = \bar{w}_n(t, \bar{t}, s) \tag{2.7}$$

that "undress" the Lax and Orlov–Shulman operators as

$$L = We^{\partial_s}W^{-1}, \qquad M = W\left(s + \sum_{n=1}^{\infty} nt_n e^{n\partial_s}\right)W^{-1},$$

$$\bar{L} = \bar{W}e^{\partial_s}\bar{W}^{-1}, \qquad \bar{M} = \bar{W}\left(s - \sum_{n=1}^{\infty} n\bar{t}_n e^{-n\partial_s}\right)\bar{W}^{-1}. \tag{2.8}$$

This does not determine W and $\bar{W}$ uniquely, and one can select a suitable pair of W and $\bar{W}$ such that the following equations are satisfied:

$$\frac{\partial W}{\partial t_n} = -(We^{n\partial_s}W^{-1})_{<0}W, \qquad \frac{\partial W}{\partial \bar{t}_n} = (\bar{W}e^{-n\partial_s}\bar{W}^{-1})_{<0}W,$$

$$\frac{\partial \bar{W}}{\partial t_n} = (We^{n\partial_s}W^{-1})_{\geqq 0}\bar{W}, \qquad \frac{\partial \bar{W}}{\partial \bar{t}_n} = -(\bar{W}e^{-n\partial_s}\bar{W}^{-1})_{\geqq 0}\bar{W}. \tag{2.9}$$

These equations of flows in the space of dressing operators can be "linearized" as follows. A clue is the "operator ratio"

$$U(t,\bar{t}) = W(t,\bar{t})^{-1}\bar{W}(t,\bar{t}) \tag{2.10}$$

of the dressing operators $W = W(t,\bar{t})$ and $\bar{W} = \bar{W}(t,\bar{t})$. One can indeed easily see that $U = U(t,\bar{t})$ satisfies the "linear equations"

$$\frac{\partial U}{\partial t_n} = e^{n\partial_s}U, \qquad \frac{\partial U}{\partial \bar{t}_n} = -Ue^{-n\partial_s}, \tag{2.11}$$

so that the flows in the space of U operators are given by simple exponential operators:

$$U(t,\bar{t}) = \exp\left(\sum_{n=1}^{\infty} t_n e^{n\partial_s}\right)U(0,0)\exp\left(-\sum_{n=1}^{\infty} \bar{t}_n e^{-n\partial_s}\right). \tag{2.12}$$

Furthermore, the passage from $(W, \bar{W})$ to U is reversible. Namely, given such an operator $U(t,\bar{t})$, one can solve the above "factorization problem" (2.10) to obtain two operators $W = W(t,\bar{t})$ and $\bar{W} = \bar{W}(t,\bar{t})$, which then automatically satisfy (2.9). This is a Toda lattice version of Mulase's factorization problem for the KP hierarchy [26]. Actually, this factorization problem is solved by reformulating the problem in the language of infinite matrices.

2.2. Infinite matrices. Note the following one-to-one correspondence between the sets of difference operators and of infinite ($\mathbf{Z} \times \mathbf{Z}$) matrices:

$$\sum_n a_n(s)e^{n\partial_s} \leftrightarrow \sum_n \mathrm{diag}[a_n(i)]\Lambda^n, \tag{2.13}$$

where

$$\mathrm{diag}[a_n(i)] = (a_n(i)\delta_{ij}), \qquad \Lambda^n = (\delta_{i,j-n}), \tag{2.14}$$

and the indices i (row) and j (column) run over $\mathbf{Z}$. (This Λ should not be confused with the finite matrix Λ in the partition function of generalized Kontsevich models!)

The projectors $(\quad)_{\geqq 0,<0}$ are replaced by the projectors onto upper and (strictly) lower triangular matrices:

$$(A)_{\geqq 0} = (\theta(j-i)a_{ij}), \qquad (A)_{<0} = ((1-\theta(j-i))a_{ij})\,. \tag{2.15}$$

All the Lax, Orlov–Shulman, Zakharov–Shabat and dressing operators have their counterparts in infinite matrices,

$$L \leftrightarrow \mathbf{L}, \quad M \leftrightarrow \mathbf{M}, \quad \bar{L} \leftrightarrow \bar{\mathbf{L}}, \quad \bar{M} \leftrightarrow \bar{\mathbf{M}}, \quad W \leftrightarrow \mathbf{W}, \quad \bar{W} \leftrightarrow \bar{\mathbf{W}}\,, \tag{2.16}$$

and they obey the same equations as their counterparts. For instance, dressing relation (2.8) turns into a matrix relation of the form

$$\mathbf{L} = \mathbf{W}\Lambda\mathbf{W}^{-1}, \qquad \mathbf{M} = \mathbf{W}\left(\Delta + \sum_{n=1}^{\infty} nt_n\Lambda^n\right)\mathbf{W}^{-1},$$

$$\bar{\mathbf{L}} = \bar{\mathbf{W}}\Lambda\bar{\mathbf{W}}^{-1}, \qquad \bar{\mathbf{M}} = \bar{\mathbf{W}}\left(\Delta - \sum_{n=1}^{\infty} n\bar{t}_n\Lambda^{-n}\right)\bar{\mathbf{W}}^{-1}. \tag{2.17}$$

Here Δ is the infinite matrix

$$\Delta = (i\delta_{ij}) \tag{2.18}$$

that represents the multiplication operator s. Similarly, the difference operator $U(t,\bar{t})$ has an infinite matrix counterpart $\mathbf{U}(t,\bar{t})$, and its time evolutions are generated by exponential matrices:

$$\mathbf{U}(t,\bar{t}) = \exp\left(\sum_{n=1}^{\infty} t_n\Lambda^n\right)\mathbf{U}(0,0)\exp\left(-\sum_{n=1}^{\infty} \bar{t}_n\Lambda^{-n}\right). \tag{2.19}$$

Factorization relation (2.10) is now converted into a factorization problem of infinite matrices of the form

$$\mathbf{U}(t,\bar{t}) = \mathbf{W}(t,\bar{t})^{-1}\,\bar{\mathbf{W}}(t,\bar{t})\,. \tag{2.20}$$

This is an infinite matrix version of the Gauss decomposition, because $\mathbf{W} = \mathbf{W}(t,\bar{t})$ and $\bar{\mathbf{W}} = \bar{\mathbf{W}}(t,\bar{t})$ are lower and upper triangular matrices. As in the ordinary finite dimensional cases, this Gauss decomposition can be solved explicitly by (an infinite matrix version of) the Cramer formula [33]. In particular, the matrix elements $w_n(t,\bar{t},s)$ and $\bar{w}_n(t,\bar{t},s)$ can be written as a quotient of two semi-infinite determinants. The denominators eventually turn out to give τ functions of the Toda lattice hierarchy:

$$\tau(t,\bar{t},s) = \det(u_{ij}(t,\bar{t})\ (-\infty < i,j < s))\,. \tag{2.21}$$

This gives a Toda lattice version of a similar formula in the KP hierarchy [31]. The matrix elements $u_{ij}(t,\bar{t})$ of $\mathbf{U}(t,\bar{t})$ are connected with their "initial values" $u_{ij}(0,0)$ by

$$u_{ij}(t,\bar{t}) = \sum_{m,n=0}^{\infty} S_m(t)u_{i+m,\,j+n}(0,0)S_n(-\bar{t})\,, \tag{2.22}$$

where S_n are the fundamental Schur functions

$$\sum_{n=0}^{\infty} S_n(t)\lambda^n = \exp\left(\sum_{n=1}^{\infty} t_n\lambda^n\right). \tag{2.23}$$

Eventually, the infinite matrix (a $GL(\infty)$ element) $\mathbf{U}(0,0)$ persists as arbitrary constants in a general solution of the Toda lattice hierarchy. It is thus the $GL(\infty)$ group that plays the role of the Sato Grassmannian in the Toda lattice hierarchy.

2.3. Constraints. "String equations" in our concern are derived from linear relations of the form

$$A(\Delta,\Lambda)\mathbf{U}(0,0)=\mathbf{U}(0,0)\bar{A}(\Delta,\Lambda)\,, \tag{2.24}$$

among the matrix elements of $\mathbf{U}(0,0)$. Here $A(\Delta,\Lambda)$ and $\bar{A}(\Delta,\Lambda)$ are linear combinations of monomials of Δ,Λ and Λ^{-1}:

$$A(\Delta,\Lambda)=\sum_{m,n}a_{mn}\Delta^m\Lambda^n\,,$$

$$\bar{A}(\Delta,\Lambda)=\sum_{m,n}\bar{a}_{mn}\Delta^m\Lambda^n\,, \tag{2.25}$$

m runing over nonnegative integers and n over all integers. A fundamental result is the following [35, 2, 28]:

Theorem 1. *The above linear relations of* $\mathbf{U}(0,0)$ *is equivalent to the constraint*

$$A(\mathbf{M},\mathbf{L})=\bar{A}(\bar{\mathbf{M}},\bar{\mathbf{L}}) \tag{2.26}$$

of the Lax and Orlov–Shulman operators, where

$$A(\mathbf{M},\mathbf{L})=\sum_{m,n}a_{mn}\mathbf{M}^m\mathbf{L}^n,\qquad \bar{A}(\bar{\mathbf{M}},\bar{\mathbf{L}})=\sum_{m,n}\bar{a}_{mn}\bar{\mathbf{M}}^m\bar{\mathbf{L}}^n\,. \tag{2.27}$$

These constraints may be interpreted as a fixed point condition under $W_{1+\infty}$ symmetries of the Toda lattice hierarchy [35, 2, 28]. These matrices are in one-to-one correspondence with difference operators,

$$A(\Delta,\Lambda)\leftrightarrow A(s,e^{\partial_s}),\qquad \bar{A}(\Delta,\Lambda)\leftrightarrow\bar{A}(s,e^{\partial_s})\,, \tag{2.28}$$

and give a closed Lie algebra with the fundamental commutation relation

$$[\Delta,\Lambda]=\Delta\ \leftrightarrow\ [e^{\partial_s},s]=e^{\partial_s}\,. \tag{2.29}$$

This is essentially a (centerless) $W_{1+\infty}$ algebra with generators

$$W_n^{(k)}=(\Delta\Lambda^{-1})^k\Lambda^{n+k}\ \leftrightarrow\ (se^{-\partial_s})^ke^{(n+k)\partial_s}\,. \tag{2.30}$$

It is this $W_{1+\infty}$ algebra that underlies the above constraints.

These constraints can be further converted into linear constraints for the τ functions of the form [35, 2, 28]

$$X_A\tau(t,\bar{t},s)=\bar{X}_{\bar{A}}\tau(t,\bar{t},s)+\text{const.}\ \tau(t,\bar{t},s)\,, \tag{2.31}$$

where $X_A=X_A(t,s,\partial_t)$ and $\bar{X}_{\bar{A}}(\bar{t},s,\partial_{\bar{t}})$ are linear differential operators in t and $\bar{t}$ that represent $W_{1+\infty}$ symmetries acting on the τ functions, and "const" is a constant that also depends on A and $\bar{A}$. This constant may emerge due to a nonvanishing central charge in the $W_{1+\infty}$ algebra of symmetries acting on the τ functions. Fixing this constant requires a subtle trick [12, 38]. We shall not deal with this issue in this paper.

2.4. τ functions in Miwa variables. A clue connecting generalized Kontsevich models and the KP hierarchy [21, 1] is the Miwa variable representation of the KP τ function. If the matrix $\mathbf{U}(0,0)$ is upper or lower triangular, a similar representation of the Toda lattice τ functions [19] is also available as we show below. (In fact, such a representation persists without this condition, though the relation to $\mathbf{U}$ then becomes more complicated.)

Theorem 2. (i) *Suppose that* $\mathbf{U}(0,0)$ *is lower triangular. Then by the Miwa transformation*

$$t_n = \frac{1}{n}\sum_{i=1}^{N} \lambda_i^{-n} \tag{2.32}$$

of t *with arbitrary parameters* λ_i, *the* τ *functions* $\tau(t,\bar{t},s)$ *can be written*

$$\tau(t,\bar{t},s) = \frac{\det(u_{s-j}(0,\bar{t},\lambda_i)\ (1 \leqq i,j \leqq N))}{\det(\lambda_i^{-(s-j)-1}\ (1 \leqq i,j \leqq N))} \prod_{i=-\infty}^{s-N-1} u_{ii}\,, \tag{2.33}$$

where $u_j(0,\bar{t},\lambda)$ *are the following generating functions of matrix elements* $u_{ij}(0,\bar{t})$ *of* $\mathbf{U}(0,\bar{t})$:

$$u_j(0,\bar{t},\lambda) = \sum_{i=j}^{\infty} \lambda^{-i-1} u_{ij}(0,\bar{t})\,. \tag{2.34}$$

(ii) *Suppose that* $\mathbf{U}(0,0)$ *is upper triangular. Then by the Miwa transformation*

$$\bar{t}_n = -\frac{1}{n}\sum_{i=1}^{N} \mu_i^{n} \tag{2.35}$$

of $\bar{t}$ *with arbitrary parameters* μ, *the* τ *function can be written*

$$\tau(t,\bar{t},s) = \frac{\det(\bar{u}_{s-i}(t,0,\mu_j)\ (1 \leqq i,j \leqq N))}{\det(\mu_j^{s-i}\ (1 \leqq i,j \leqq N))} \prod_{i=-\infty}^{s-N-1} u_{ii}\,, \tag{2.36}$$

where $\bar{u}_i(t,0,\mu)$ *are the following generating functions of the matrix elements* $u_{ij}(t,0)$ *of* $\mathbf{U}(t,0)$:

$$\bar{u}_i(t,0,\mu) = \sum_{j=i}^{\infty} u_{ij}(t,0)\mu^j\,. \tag{2.37}$$

The infinite products of u_{ii} and $\bar{u}_{ii}$ in the above formulae are interpreted as follows. Note that the matrices $\mathbf{U}(0,\bar{t})$ in (i) and $\mathbf{U}(t,0)$ in (ii) are still triangular. The Laurent series $u_j(0,\bar{t},\lambda)$ and $\bar{u}(t,0,u)$ thereby take such a form as

$$u_j(0,\bar{t},\lambda) = \sum_{i=j}^{\infty} \lambda^{-i-1} u_{ij}(0,\bar{t}), \qquad \bar{u}_i(t,0,\mu) = \sum_{j=i}^{\infty} u_{ij}(t,0)\mu^j\,. \tag{2.38}$$

As opposed to an ordinary setting [19], we do not assume that the leading coefficients are normalized to be 1. The infinite product then arises. This factor is essentially the same as encountered in the treatment of semi-infinite determinant (2.21) [33], and can be interpreted as:

$$\prod_{-\infty}^{n} u_{ii} = \begin{cases} \text{const.}\ u_{00}\dots u_{nn} & (n \geqq 0) \\ \text{const.} & (n = -1) \\ \text{const.}\ /(u_{n+1,n+1}\dots u_{-1,-1}) & (n \leqq -2) \end{cases} \tag{2.39}$$

with an overall renormalization constant "const."

2.5. $\hbar$-dependent formulation. An $\hbar$-dependent formulation of Toda lattice hierarchy can be achieved by inserting $\hbar$ in front of all derivatives in the previous equations as:

$$\frac{\partial}{\partial t_n} \to \frac{\partial}{\partial t_n}, \qquad \frac{\partial}{\partial \bar{t}_n} \to \frac{\partial}{\partial \bar{t}_n}, \qquad e^{\partial_s} \to e^{\hbar\partial_s}. \tag{2.40}$$

The discrete variable s now takes values in $\hbar\mathbf{Z}$. Accordingly, Lax and Orlov–Shulman operators are difference operators of the form

$$\begin{aligned}
L &= e^{\hbar\partial_s} + \sum_{n=0}^{\infty} u_{n+1} e^{-n\hbar\partial_s}, \\
M &= \sum_{n=1}^{\infty} n t_n L^n + s + \sum_{n=1}^{\infty} v_n L^{-n}, \\
\bar{L} &= \tilde{u}_0 e^{\hbar\partial_s} + \sum_{n=0}^{\infty} \tilde{u}_{n+1} e^{(n+2)\hbar\partial_s}, \\
\bar{M} &= -\sum_{n=1}^{\infty} n \bar{t}_n \bar{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n,
\end{aligned} \tag{2.41}$$

where the coefficients are functions of $(\hbar, t, \bar{t}, s)$, and obey the twisted canonical commutation relations

$$[L, M] = \hbar L, \qquad [\bar{L}, \bar{M}] = \hbar \bar{L}, \tag{2.42}$$

and Lax equations

$$\begin{aligned}
\hbar\frac{\partial L}{\partial t_n} &= [B_n, L], & \hbar\frac{\partial L}{\partial \bar{t}_n} &= [\bar{B}_n, L], \\
\hbar\frac{\partial M}{\partial t_n} &= [B_n, M], & \hbar\frac{\partial M}{\partial \bar{t}_n} &= [\bar{B}_n, M], \\
\hbar\frac{\partial \bar{L}}{\partial t_n} &= [B_n, \bar{L}], & \hbar\frac{\partial \bar{L}}{\partial \bar{t}_n} &= [\bar{B}_n, \bar{L}], \\
\hbar\frac{\partial \bar{M}}{\partial t_n} &= [B_n, \bar{M}], & \hbar\frac{\partial \bar{M}}{\partial \bar{t}_n} &= [\bar{B}_n, \bar{M}].
\end{aligned} \tag{2.43}$$

B_n and $\bar{B}_n$ are defined in the the same way as in the case of $\hbar = 1$.

Note that such an $\hbar$-dependent Toda lattice hierarchy emerges if one starts from an $\hbar$-independent formulation and rescales variables as

$$t_n \to \hbar^{-1} t_n, \qquad \bar{t}_n \to \hbar^{-1} \bar{t}_n, \qquad s \to \hbar^{-1} s \tag{2.44}$$

and

$$M \to \hbar^{-1} M, \qquad \bar{M} \to \hbar^{-1} \bar{M}. \tag{2.45}$$

This is indeed the case for solutions that we shall construct in subsequent sections. The $\hbar$-dependent formulation, however, also admits solutions that cannot be obtained by the above rescaling of $(t, \bar{t}, s)$.

3. Generalized Kontsevich Models

Our first example is the generalized Kontsevich models that have been studied in the context of the KP hierarchy. We introduce "negative times" $\bar{t}$ and reconsider these models in the language of the Toda lattice hierarchy. The idea is more or less parallel to the ITEP-Lebedev group [19], who however treated these models as solutions of a "forced" hierarchy on a semi-infinite lattice $\mathbf{Z}_{\geqq 0}$. We rather attempt to interpret these models as solutions of the hierarchy on the bi-infinite lattice $\mathbf{Z}$.

3.1. Partition function as KP τ function. In an $\hbar$-dependent formulation, the partition function of the p^{th} generalized Kontsevich model is given by

$$Z(\Lambda) = C(\Lambda)\int dM \exp \hbar^{-1} \operatorname{Tr}\left(\Lambda^p M - \frac{1}{p+1} M^{p+1}\right), \tag{3.1}$$

where the normalization constant $C(\Lambda)$ is given by

$$\begin{aligned} C(\Lambda) = \text{const.} \prod_{i>j} \frac{\lambda_i^p - \lambda_j^p}{\lambda_i - \lambda_j} \\ \times (\det \Lambda)^{(p-1)/2} \exp \hbar^{-1} \operatorname{Tr}\left(\frac{p}{p+1}\Lambda^{p+1}\right). \end{aligned} \tag{3.2}$$

By the standard method using the Harish–Chandra–Itzykson–Zuber formula [25], the partition function $Z(\Lambda)$ can be rewritten as a quotient of two determinants,

$$Z(\Lambda) = \frac{\det(u_{-j}(\lambda_i)\ (1 \leqq i,j \leqq N))}{\det(\lambda_i^{j-1}\ (1 \leqq i,j \leqq N))}, \tag{3.3}$$

where

$$\begin{aligned} u_j(\lambda) &= c(\lambda)\int d\mu \mu^{-j-1} \exp \hbar^{-1}\left(\lambda^p \mu - \frac{1}{p+1}\mu^{p+1}\right), \\ c(\lambda) &= \text{const.}\ \mu^{(p-1)/2} \exp \hbar^{-1}\left(\frac{p}{p+1}\lambda^{p+1}\right). \end{aligned} \tag{3.4}$$

The right-hand side of the definition of $u_j(\lambda)$ is an integral over the whole real axis. By the standard saddle point method, one can show that $u_j(\lambda)$ has asymptotic expansion of the form

$$u_j(\lambda) \sim \sum_{i=j}^{\infty} \lambda^{-i-1} u_{ij} \tag{3.5}$$

as $\lambda \to +\infty$. It is these asymptotic series rather than the functions $u_j(\lambda)$ themselves that are eventually relevant for the interpretation of $Z(\Lambda)$ as a KP τ function. Namely, we insert this asymptotic expansion into the right-hand side of (3.3) and consider it as a function (or, rather, formal power series) of t by the Miwa transformation

$$t_n = \frac{\hbar}{n}\sum_{i=1}^{N} \lambda_i^{-n} = \frac{\hbar}{n}\operatorname{Tr}\Lambda^{-n}. \tag{3.6}$$

3.2. Extension to Toda lattice τ function. We now extend the above construction to u_j's with negative index j. Apparently, this appears problematical, because the integral (over the real axis) becomes singular at $\mu = 0$; actually, what we need is asymptotic series rather than the functions $u_j(\lambda)$ themselves. As we shall discuss later in a more general setting, the asymptotic expansion as $\lambda \to +\infty$ is determined only by a small neighborhood of the saddle point at $\mu = \lambda$. The other part of the path of integration may be slightly deformed to avoid the singularity at $\mu = 0$, or even cut off! With this deformation or cut-off, the integral itself will be varied, but the difference is subdominant and does not affect the asymptotic series. Thus upon suitably modifying the definition of $u_j(\lambda)$, we can show that $u_j(\lambda)$ for $j < 0$, too, has asymptotic expansion of the form

$$u_j(\lambda) \sim \sum_{i=j}^{\infty} \lambda^{-i-1} u_{ij} \,. \tag{3.7}$$

We can thus define u_{ij} for all integers, and hence a $\mathbf{Z} \times \mathbf{Z}$ matrix $\mathbf{U} = \mathbf{U}(0,0)$. According to the general construction in the last section, this determines a Toda lattice τ function $\tau(t, \bar{t}, s)$.

Since $\mathbf{U}(0,0)$ is lower triangular, the τ function should have a finite determinant representation by the Miwa transformation

$$t_n = \frac{\hbar}{n} \sum_{i=1}^{N} \lambda_i^{-n} = \frac{\hbar}{n} \operatorname{Tr} \Lambda^{-n} \,. \tag{3.8}$$

One can easily see that the generating functions $u_j(0, \bar{t}, \lambda)$ is given by

$$u_j(0, \bar{t}, \lambda) = c(\lambda) \int d\mu \mu^{-j-1} \exp \hbar^{-1} \left(\lambda^p \mu - \frac{1}{p+1} \mu^{p+1} - \sum_{n=1}^{\infty} \bar{t}_n \mu^{-n} \right) \tag{3.9}$$

with the same normalization factor $c(\lambda)$ and the same path of integration (deformed or cut off in the aforementioned sense) as in the definition of $u_j(\lambda) = u_j(0, 0, \lambda)$. The $\tau(t, \bar{t}, s)$ can eventually be written

$$\tau(t, \bar{t}, s) = \frac{\det(u_{\hbar^{-1}s-j}(0, \bar{t}, \lambda_i) \ (1 \leqq i, j \leqq N))}{\det(\lambda_i^{\hbar^{-1}s-j-1} \ (1 \leqq i, j \leqq N))} \,. \tag{3.10}$$

(Note that, along with t and $\bar{t}$, the lattice coordinate s, too, has to be rescaled as $s \to \hbar^{-1} s$ in an $\hbar$-dependent formulation of the Toda lattice hierarchy. s now runs over the $\hbar$-spaced lattice $\hbar \mathbf{Z}$.)

Formally, this τ function can be derived from the following matrix integral that extends $Z(\Lambda)$:

$$Z(\Lambda, \bar{t}, s) = C(\Lambda) \int dM \exp \hbar^{-1} \operatorname{Tr} \left(\Lambda^p M - \frac{1}{p+1} M^{p+1} - \sum_{n=1}^{\infty} \bar{t}_n M^{-n} + s \log M \right) \,. \tag{3.11}$$

3.3. String equations. It is now rather straightforward to derive string equations – we first derive linear relations among $u_{ij} = u_{ij}(0,0)$, then convert them into relations among $(L, M, \bar{L}, \bar{M})$.

Linear relations can be derived from linear functional or differential relations among u_j's. This is just to apply well known calculations [19, 25] to u_j's with negative indices as follows.

First, by integrating by part,

$$\begin{aligned}\lambda^p u_j(\lambda) &= c(\lambda) \int d\mu \hbar \frac{\partial}{\partial \lambda}(\exp(\hbar^{-1}\lambda^p \mu)) \times \mu^{-j-1} \exp \hbar^{-1}\left(-\frac{1}{p+1}\mu^{p+1}\right) \\ &= c(\lambda) \int d\mu (\hbar(j+1)\mu^{-j-2} + \mu^{-j+p-1}) \times \exp \hbar^{-1}\left(\lambda^p \mu - \frac{1}{p+1}\mu^{p+1}\right) \\ &= \hbar(j+1)u_{j+1}(\lambda) + u_{j-p}(\lambda) \,. \end{aligned} \tag{3.12}$$

Here we have implicitly assumed that the path of integration connects two points at infinity, so that no boundary term emerges. If the path has a finite endpoint, such as the semi-infinite line $[\varepsilon, \infty)$, $\varepsilon > 0$, the boundary term is subdominant compared to the main asymptotic series arising from the saddle point at $\mu = \lambda$. Since we shall eventually derive linear relations among u_{ij}'s, such subdominant terms are negligible.

Similarly (but without integration by part),

$$\begin{aligned} u_{j-1}(\lambda) &= c(\lambda) \int d\mu \hbar \lambda^{1-p} \frac{1}{p}\frac{\partial}{\partial \lambda}\left(\mu^{-j-1} \exp \hbar^{-1}\left(\lambda^p \mu - \frac{1}{p+1}\mu^{p+1}\right)\right) \\ &= c(\lambda)\hbar\lambda^{1-p}\frac{1}{p}\frac{\partial}{\partial\lambda}(c(\lambda)^{-1}u_j(\lambda)) \\ &= \left(\hbar\lambda^{1-p}\frac{1}{p}\frac{\partial}{\partial\lambda} - \hbar\frac{p-1}{2p}\lambda^{-p} + \lambda\right)u_j(\lambda). \end{aligned} \tag{3.13}$$

These relations imply the linear relations

$$\begin{aligned} u_{i+p,j} &= \hbar(j+1)u_{i,j+p} + u_{i,j-1} \,, \\ u_{i,j-1} &= \hbar\left(-\frac{i}{p} + \frac{p-1}{2p}\right)u_{i-p,j} + u_{i+1,j} \end{aligned} \tag{3.14}$$

among the coefficients of asymptotic expansion. In terms of $\mathbf{U} = \mathbf{U}(0,0)$, they can be rewritten

$$\begin{aligned} \Lambda^p \mathbf{U} &= \mathbf{U}(\hbar\Delta\Lambda^{-1} + \Lambda^p) \,, \\ \mathbf{U}\Lambda &= \left(-\frac{\hbar}{p}\Delta\Lambda^{-p} + \hbar\frac{p-1}{2p}\Lambda^{-p} + \Lambda\right)\mathbf{U} \,. \end{aligned} \tag{3.15}$$

Finally, we replace

$$\Lambda \to L, \bar{L}, \qquad \hbar\Delta \to M, \bar{M} \tag{3.16}$$

(taking into account the rescaling due to the presence of $\hbar$) and obtain the following result on string equations:

Theorem 3. *The Lax and Orlov–Shulman operators of this solution obey the string equations*

$$L^p = \bar{M}\bar{L}^{-1} + \bar{L}^p,$$
$$\bar{L} = -\frac{1}{p}ML^{-p} + \hbar\frac{p-1}{2p}L^{-p} + L\,. \tag{3.17}$$

3.4. Generalized Kontsevich model at $p = -1$. Let us consider the case of $p = -1$. The string equations then become

$$L^{-1} = \bar{M}\bar{L}^{-1} + \bar{L}^{-1}\,,$$
$$\bar{L} = ML + \hbar L + L\,. \tag{3.18}$$

As we shall argue in the next section, these string equations do not agree with string equations of two-dimensional (or $c = 1$) strings (compactified at self-dual radius).

This is a puzzling consequence of our analysis. Some of recent studies on the two-dimensional topological string theory [22] are based on the hypothesis that the topological A_k model at $k = -3$ (which corresponds to $p = -1$) gives the two-dimensional string theory. Our result implies that such an identification is somewhat problematical.

The model at $p = -1$, as Dijkgraaf et al. [7] remarked, is rather related to the Penner model. The partition function for $p = -1$ is given by

$$Z(\Lambda, \bar{t}, 0) = C(\Lambda) \int dM \exp \hbar^{-1} \operatorname{Tr}\left(\Lambda^{-1}M - \log M - \sum_{n=1}^{\infty} \bar{t}_n M^{-n}\right)\,. \tag{3.19}$$

If $\bar{t} = 0$, this function factorizes into a power of $\det \Lambda$ and $Z(1,0,0)$, and that the second factor $Z(1,0,0)$ coincides with the partition function of the Penner model [30]. Curiously, however, Dijkgraaf et al. identified this matrix integral with the $c = 1$ string partition function. Actually, as we have mentioned above, this matrix integral does not correspond to the $c = 1$ string partition function.

What physical meaning do the "negative times" $\bar{t}$ possess? It seems likely that $\bar{t}$ are nothing else but the coupling constants of "anti-states" that Montano and Rivlis [24] postulate in their topological interpretation of Ward identities for $(1, q)$ models. This is in fact a main idea that lies at the heart of the work of the aforementioned studies [22] on the two-dimensional topological string theory. Although we consider their interpretation of the $(1, q = -1)$ model rather problematical, their work contains many intriguing ideas. We shall return to this point in the final section.

4. Compactified $c = 1$ String Theory

Our second example is the compactified $c = 1$ string theory formulated in the language of the Toda lattice hierarchy [7, 23, 27]. This string theory possesses a discrete parameter $\beta = 1, 2, \ldots$, and $\beta = 1$ corresponds to the case at self-dual radius.

4.1. Partition function of $c = 1$ *strings.* The partition function of compactified $c = 1$ strings becomes the Toda lattice τ function with a diagonal $\mathbf{U} = \mathbf{U}(0,0)$ matrix of the form [7, 23, 27]

$$u_{ij} = (-\hbar)^{(i+\frac{1}{2})/\beta} \frac{\Gamma\left(\frac{1}{2} - \frac{1}{\hbar} + \frac{i+\frac{1}{2}}{\beta}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{\hbar}\right)} \delta_{ij} . \tag{4.1}$$

In fact, this is a somewhat simplified version of a true $c = 1$ string partition function [7]. The above definition is substantially the same as used by Nakatsu [27] and leads to the same string equations (except that $\hbar$ in his definition is replaced by $-\hbar$ here). String equations of the original $c = 1$ string partition function are presented in Ref. [28] in a more general setting including black hole backgrounds.

Since $\mathbf{U} = \mathbf{U}(0,0)$ is a diagonal matrix, the τ functions have two Miwa variable representations with respect to t and $\bar{t}$. In both representations, the τ functions are written in terms of a finite determinant including the generating functions $u_j(0, \bar{t}, \lambda)$ and $\bar{u}_i(t, 0, \mu)$. By the familiar integral representation of the Gamma function, one can easily obtain the following integral representation of these generating functions:

$$\begin{aligned} u_j(0, \bar{t}, \lambda) = {} & \frac{\beta(-\hbar)^{\frac{1}{\hbar}-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \frac{1}{\hbar}\right)} \lambda^{(\beta-1)/2} \exp(-\hbar^{-1}\beta \log \lambda) \\ & \times \int_0^\infty d\mu \mu^{(-\beta-1)/2-j-1} \exp \hbar^{-1} \left((\lambda/\mu)^\beta + \beta \log \mu - \sum_{n=1}^\infty \bar{t}_n \mu^{-n} \right) , \\ \bar{u}_i(t, 0, \mu) = {} & \frac{\beta(-\hbar)^{\frac{1}{\hbar}-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \frac{1}{\hbar}\right)} \mu^{(-\beta-1)/2} \exp(\hbar^{-1}\beta \log \mu) \\ & \times \int_0^\infty d\lambda \lambda^{(\beta-1)/2+i} \exp \hbar^{-1} \left((\lambda/\mu)^\beta - \beta \log \lambda + \sum_{n=1}^\infty t_n \lambda^n \right) . \end{aligned} \tag{4.2}$$

4.2. Integral representation of Kontsevich type. If $\beta = 1$ (i.e., in the case at self-dual radius), the τ function has a matrix integral representation of Kontsevich type. For simplicity, let us consider the case of $s = \hbar(N - 1)$ (though a similar matrix integral representation persists in a general case). By the Miwa transformation

$$\bar{t}_n = -\frac{\hbar}{n} \sum_{i=1}^N \mu_i^n = -\frac{\hbar}{n} \mathrm{Tr} M^n , \tag{4.3}$$

the τ function $\tau(t, \bar{t}, \hbar(N - 1))$ can be written

$$\tau(t, \bar{t}, \hbar(N - 1)) = \text{const.} \frac{\det(\bar{u}_{N-i-1}(t, 0, \mu_j) \ (1 \leqq i, j \leqq N))}{\det(\mu_j^{N-i-1} \ (1 \leqq i, j \leqq N))} . \tag{4.4}$$

As in the case of generalized Kontsevich models, this can be converted into a matrix integral of the form

$$\tau(t, \bar{t}, \hbar(N - 1)) = \bar{C}(M) \int d\Lambda \exp \hbar^{-1} \mathrm{Tr} \left(\Lambda M^{-1} - \log \Lambda + \sum_{n=1}^\infty t_n \Lambda^n \right) \tag{4.5}$$

with a suitable normalization factor $\bar{C}(M)$.

Although very similar, this matrix integral is substantially different from the generalized Kontsevich model at $p = -1$. The latter rather corresponds to $\beta = -1$. (Mathematically, the present construction is also valid for $\beta = -1, -2, \ldots$.) If $\beta = -1$, one can indeed derive a matrix integral representation of the form

$$\tau(t, \bar{t}, 0) = C(\Lambda) \int dM \exp \hbar^{-1} \operatorname{Tr} \left(\Lambda^{-1} M - \log M - \sum_{n=1}^{\infty} \bar{t}_n M^{-n} \right) \tag{4.6}$$

in the Miwa variables

$$t_n = \frac{\hbar}{n} \sum_{i=1}^{N} \lambda_i^{-n} = \frac{\hbar}{n} \operatorname{Tr} \Lambda^{-n} . \tag{4.7}$$

(We have put $s = 0$ just for simplicity. A similar expression persists for a general value of s.) This is nothing else but the generalized Kontsevich model at $p = -1$ given by (3.19).

Thus, as opposed to the claim of Dijkgraaf et al. [7], we conclude that the generalized Kontsevich model at $p = -1$ does not correspond to the $c = 1$ string theory of $\beta = 1$, but falls into the case of the strange value $\beta = -1$. We shall reconfirm this puzzling conclusion in the language of string equations below.

4.3. String equations. By the recursion relation $\Gamma(x + 1) = x\Gamma(x)$ of the Gamma function, one can easily derive the following algebraic relations for $\mathbf{U} = \mathbf{U}(0, 0)$:

$$\begin{aligned} \Lambda^{\beta} \mathbf{U} &= \mathbf{U} \left(-\frac{\hbar}{\beta} \Delta - \hbar \frac{\beta + 1}{2\beta} + 1 \right) \Lambda^{\beta} , \\ \mathbf{U} \Lambda^{-\beta} &= \left(-\frac{\hbar}{\beta} \Delta + \hbar \frac{\beta - 1}{2\beta} + 1 \right) \Lambda^{-\beta} \mathbf{U} . \end{aligned} \tag{4.8}$$

Here Λ and Δ are the infinite matrices introduced in Sect. 2. To obtain the associated string equations, we have only to resort to the substitution rule

$$\Lambda \to L, \bar{L}, \qquad \hbar \Delta \to M, \bar{M} . \tag{4.9}$$

We can thus reproduce the following result of Nakatsu [27] (who derives this result by a somewhat different method).

Theorem 4. *The Lax and Orlov–Shulman operators of this solution obey the string equations*

$$\begin{aligned} L^{\beta} &= \left(-\frac{1}{\beta} \bar{M} - \hbar \frac{\beta + 1}{2\beta} + 1 \right) \bar{L}^{\beta} , \\ \bar{L}^{-\beta} &= \left(-\frac{1}{\beta} M + \hbar \frac{\beta - 1}{2\beta} + 1 \right) L^{-\beta} . \end{aligned} \tag{4.10}$$

As promised, this result clearly shows that string equations (3.18) of the generalized Kontsevich model at $p = -1$ agree with none of the $c = 1$ string theory with $\beta \geqq 1$. The only possible value is $\beta = -1$.

One can rewrite these string equations into linear constraints ($W_{1+\infty}$ constraints) on the τ functions. To do this, just take the n^{th} power ($n = 1, 2, \ldots$) of both sides of the above string equations as

$$L^{n\beta} = (\cdots)^n, \qquad \bar{L}^{n\beta} = (\cdots)^n , \tag{4.11}$$

and apply the relation between nonlinear constraints (2.26) on $(L, M, \bar{L}, \bar{M})$ and linear constraints (2.31) on the τ function. If $\beta = 1, 2, \ldots$, one will thus obtain a set of $W_{1+\infty}$ constraints of the form

$$\begin{aligned} \frac{\partial \tau(t, \bar{t}, s)}{\partial t_{n\beta}} &= \bar{X}_{\beta,n}(\bar{t}, s, \partial_{\bar{t}})\tau(t, \bar{t}, s) , \\ \frac{\partial \tau(t, \bar{t}, s)}{\partial \bar{t}_{n\beta}} &= X_{\beta,n}(t, s, \partial_t)\tau(t, \bar{t}, s) , \end{aligned} \tag{4.12}$$

where $X_{n,\beta}$ and $\bar{X}_{\beta,n}$ are linear differential operators as in (2.31) that give a $W_{1+\infty}$ symmetry of the Toda lattice hierarchy. If $\beta = 1$, this is exactly the $W_{1+\infty}$ constraints that Dijkgraaf et al. [7] present in the former half of their paper. (Thus, as far as we understand, they deal with two distinct cases in the same paper – the $\beta = 1$ case in the former half and the $\beta = -1$ case in the latter half!) Meanwhile, if $\beta = -1, -2, \ldots$, the $W_{1+\infty}$ constraints are of the form

$$\begin{aligned} n|\beta| t_{n|\beta|}\tau(t, \bar{t}, s) &= \bar{X}_{\beta,n}(\bar{t}, s, \partial_{\bar{t}})\tau(t, \bar{t}, s) , \\ n|\beta| \bar{t}_{n|\beta|}\tau(t, \bar{t}, s) &= X_{\beta,n}(t, s, \partial_t)\tau(t, \bar{t}, s) . \end{aligned} \tag{4.13}$$

As a final remark, we would like to note that if $\hbar$ is replaced by $-\hbar$ in the definition of u_{ij}, the string equations become

$$\begin{aligned} L^{\beta} &= \left(\frac{1}{\beta}\bar{M} + \hbar\frac{\beta+1}{2\beta} + 1\right)\bar{L}^{\beta} , \\ \bar{L}^{-\beta} &= \left(\frac{1}{\beta}M - \hbar\frac{\beta-1}{2\beta} + 1\right)L^{-\beta} . \end{aligned} \tag{4.14}$$

In several papers, string equations of $c = 1$ or two-dimensional topological strings are given in this form.

5. Synthesis – Generalized String Equations

We now present our new family of special solutions and associated string equations. These solutions have two discrete parameters $(p, \bar{p})$. The generalized Kontsevich models and the compactified $c = 1$ string theory can be reproduced by letting these parameters to special values. In fact, there are two apparently different constructions starting from the generating functions $u_j(\lambda)$ and $\bar{u}_i(\mu)$, respectively, which eventually lead to the same string equations. Unfortunately, we have been unable to see if these two constructions give the same solution. We mostly present the first construction based on $u_j(\lambda)$, and just briefly mention the second one.

5.1. Generating functions and string equations. The generating functions $u_j(\lambda)$ of the new special solution are given by

$$u_j(\lambda) = c(\lambda) \int d\mu\, \mu^{(\bar{p}-1)/2-j-1} \exp \hbar^{-1} \left(\lambda^p \mu^{\bar{p}} - \frac{\bar{p}}{p+\bar{p}} \mu^{p+\bar{p}} \right),$$

$$c(\lambda) = \text{const.}\ \lambda^{(p-1)/2} \exp \hbar^{-1} \left(\frac{-p}{p+\bar{p}} \lambda^{p+\bar{p}} \right). \tag{5.1}$$

The case of $p + \bar{p} = 0$ is also included here by replacing

$$\frac{p}{p+\bar{p}} \lambda^{p+\bar{p}} \to p \log \lambda \tag{5.2}$$

in the limit as $\bar{p} \to -p$. In fact, $u_j(\lambda)$ then becomes monomials of λ, and reduce to the generating functions in the $c = 1$ string theory. The case of $(p, \bar{p}) = (p, 1)$, meanwhile, is nothing but the p^{th} generalized Kontsevich model. The integrands of the above integrals may have singularities at the origin, but this apparent difficulty can be remedied by the same trick as discussed in the previous cases. Furthermore, in much the same way, one can derive the following relations for general $(p, \bar{p})$:

$$\lambda^p u_j(\lambda) = \hbar \left(\frac{j+1}{\bar{p}} + \frac{\bar{p}-1}{2\bar{p}} \right) u_{j+\bar{p}}(\lambda) + u_{j-p}(\lambda),$$

$$u_{j-\bar{p}}(\lambda) = \left(\frac{\hbar}{p} \lambda^{1-p} \frac{\partial}{\partial \lambda} - \hbar \frac{p-1}{2p} \lambda^{-p} + \lambda^{\bar{p}} \right) u_j(\lambda). \tag{5.3}$$

The solution of the Toda lattice hierarchy in question is determined by asymptotic expansion of $u_j(\lambda)$. In fact, whereas the case of $p + \bar{p} > 0$ is more or less parallel to generalized Kontsevich models, the case of $p + \bar{p} < 0$ is quite distinct – we have to consider asymptotic expansion as $\lambda \to +0$ rather than $\lambda \to +\infty$. To see this, let us change the variable of integration from μ to a new variable z as:

$$\mu = \lambda z. \tag{5.4}$$

Then

$$u_j(\lambda) = \text{const.}\ \lambda^{(p+\bar{p})/2-j-1} \exp \hbar^{-1} \left(\frac{-p}{p+\bar{p}} \lambda^{p+\bar{p}} \right)$$

$$\times \int dz z^{(\bar{p}-1)/2-j-1} \exp(\hbar^{-1} \lambda^{p+\bar{p}}) \left(z^{\bar{p}} - \frac{\bar{p}}{p+\bar{p}} z^{p+\bar{p}} \right). \tag{5.5}$$

Thus the saddle point analysis has to be done in a region where $\hbar^{-1} \lambda^{p+\bar{p}} \to +\infty$. The most dominant contribution comes from the saddle point $z = 1$, which corresponds to $\mu = \lambda$ in the original integral. Contributions from other saddle points (and from the endpoint of the path of integration, if the path is cut off in a neighborhood of the origin) are subdominant and negligible in asymptotic expansion of $u_j(\lambda)$. The prefactor on the right-hand side of (5.5) cancels the leading quasi-classical contribution from the integral, so that $u_j(\lambda)$ has asymptotic expansion in negative integral powers of $\lambda^{p+\bar{p}}$. In summary, we obtain the following result:

Theorem 5. (i) *If* $p + \bar{p} > 0$, $u_j(\lambda)$ *has asymptotic expansion of the form*

$$u_j(\lambda) \sim \sum_{i=j}^{\infty} \lambda^{-i-1} u_{ij} \quad (\lambda \to +\infty) . \tag{5.6}$$

In particular, $\mathbf{U}$ *is lower triangular.*

(ii) *If* $p + \bar{p} < 0$, $u_j(\lambda)$ *has asymptotic expansion of the form*

$$u_j(\lambda) \sim \sum_{i=-\infty}^{j} \lambda^{-i-1} u_{ij} \quad (\lambda \to +0) . \tag{5.7}$$

In particular, $\mathbf{U}$ *is upper triangular.*

(iii) *In both cases,* $u_{ij} = 0$ *if* $i - j \not\equiv 0 \bmod p + \bar{p}$.

We shall examine this asymptotic expansion in more detail in the next subsection. Anyway, the infinite matrix $\mathbf{U} = \mathbf{U}(0,0)$ determines a solution of the Toda lattice hierarchy. By the triangular form of $\mathbf{U}$, the τ function is ensured to have a Miwa variable representation. One should however note that one cannot use $u_j(\lambda)$ in the case of $p + \bar{p} < 0$. If $p + \bar{p} < 0$ (so that $\mathbf{U}$ is upper triangular), it is $\bar{u}_i(\mu)$ that emerge in the Miwa variable representation. Unfortunately, we have been unable to give a closed expression to $\bar{u}_i(\mu)$ in the present setting. [This should not be confused with the "dual construction" discussed later, which starts from an explicit construction of $\bar{u}_i(\mu)$. In that case, the corresponding $u_j(\lambda)$ remains unknown in turn.] This is an incomplete aspect of the present construction.

The aforementioned relations among the generating functions imply the following linear relations among the coefficients u_{ij} of the asymptotic expansion:

$$\begin{aligned} u_{i+p,j} &= \hbar \left(\frac{j+1}{\bar{p}} + \frac{\bar{p}-1}{2\bar{p}} \right) u_{i,j+\bar{p}} + u_{i,j-p} , \\ u_{i,j-\bar{p}} &= \hbar \left(-\frac{i}{p} + \frac{p-1}{2p} \right) u_{i-p,j} + u_{i+\bar{p},j} . \end{aligned} \tag{5.8}$$

In terms of the matrix $\mathbf{U} = (u_{ij})$,

$$\begin{aligned} \Lambda^p \mathbf{U} &= \mathbf{U} \left(\frac{\hbar}{\bar{p}} \Delta \Lambda^{-\bar{p}} - \hbar \frac{\bar{p}-1}{2\bar{p}} \Lambda^{-\bar{p}} + \Lambda^p \right) , \\ \mathbf{U} \Lambda^{\bar{p}} &= \left(-\frac{\hbar}{p} \Delta \Lambda^{-p} + \hbar \frac{p-1}{2p} \Lambda^{-p} + \Lambda^{\bar{p}} \right) \mathbf{U} . \end{aligned} \tag{5.9}$$

String equations can be readily derived from these relations:

Theorem 6. *The Lax and Orlov–Shulman operators of this solution obey the string equations*

$$\begin{aligned} L^p &= \frac{1}{\bar{p}} \bar{M} \bar{L}^{-\bar{p}} - \hbar \frac{\bar{p}-1}{2\bar{p}} \bar{L}^{-\bar{p}} + \bar{L}^p , \\ \bar{L}^{\bar{p}} &= -\frac{1}{p} M L^{-p} + \hbar \frac{p-1}{2p} L^{-p} + L^{\bar{p}} . \end{aligned} \tag{5.10}$$

The string equations of generalized Kontsevich models and $c = 1$ strings can be reproduced by letting $(p, \bar{p}) = (p, 1)$ and $(p, \bar{p}) = (\beta, -\beta)$. The case of $(p, \bar{p}) = (1, -1)$ is also essentially the same as the string equations of the two-matrix model; the extra term L and $\bar{L}$ on the right-hand side can be absorbed into a shift of s. (More precisely, it is a "forced Toda lattice hierarchy" confined to a semi-infinite lattice [19] that arises here.) The string equations for general values of $(p, \bar{p})$, too, relates "coordinates" L and $\bar{L}$ to "momenta" $\bar{M}\bar{L}^{-\bar{p}}/\bar{p}$ and ML^{-p}/p. In this respect, these string equations may be referred to as "two-matrix model type," as mentioned in the Introduction.

5.2. Systematic method of asymptotic expansion. Let us examine the coefficients u_{ij} of the asymptotic expansion in more detail. We would eventually like to evaluate them in an explicit form. Such an explicit formula will be useful in considering the relation between the present construction and its "dual" form discussed later.

Since this issue can be treated in a more general context, we now consider an integral of the general form

$$I(k) = \int dz g(z) e^{-kf(z)} , \tag{5.11}$$

where k is a large positive number, $f(z)$ and $g(z)$ are holomorphic functions, and $f(z)$ is supposed to have no saddle point other than a simple nondegenerate saddle point at $z = z_0$ along the path of integration. By the assumption, $f(z)$ can be written

$$f(z) = f_0 + f_1(z - z_0)^2 + O((z - z_0)^3), \quad f_1 \neq 0 , \tag{5.12}$$

in a neighborhood of z_0. Then $I(k)$ should have asymptotic expansion of the following form as $k \to +\infty$:

$$I(k) \sim e^{-kf_0} \sum_{n=0}^{\infty} a_n k^{-n-(1/2)} . \tag{5.13}$$

The problem is how to evaluate these coefficients. From the field theoretical standpoint, the most familiar method would be the expansion into "Feynman graphs." Namely, one separates $f(z)$ into the quadratic part and the rest, as shown above, treating the latter as a "perturbation," and evaluate the final series of integrals as Gaussian integrals. This is actually a very inefficient way, because a_n is then given by a sum of various Feynman graphs, and evaluating the sum is a hard task. A more efficient method is to resort to the Laplace method. A modernized version of this method is presented by Berry and Howls [3] (whose main concern rather consists in global issues like Stokes phenomena). This method yields a closed formula for a_n, as we now briefly present below.

A clue of the latter method is to rewrite the integral $I(k)$ by the new variable

$$y = (f(z) - f_0)^{1/2} = f_1^{1/2}(z - z_0) + O((z - z_0)^2) . \tag{5.14}$$

Note that it is just a neighborhood of z_0 that actually contributes to the above asymptotic expansion; "modulo subdominant terms," one can replace the full path of integration by a small piece in a neighborhood of z_0. y is a local coordinate in such a neighborhood of z_0. The integral $I(k)$ can now be written into a Gaussian integral:

$$I(k) = e^{-kf_0} \int dy \frac{dz}{dy} g(z) e^{-ky^2} + \text{subdom} . \tag{5.15}$$

The subdominant terms "subdom." are of course invisible in asymptotic expansion, and negligible in the following calculations. The Jacobian dz/dy and the function $g(z)$ are both holomorphic functions of y in a neighborhood of $y = 0$. Let us write the Taylor expansion of their product as:

$$\frac{dz}{dy} g(z) = \sum_{n=0}^{\infty} h_n y^n . \tag{5.16}$$

The asymptotic expansion of the above integral can be obtained by inserting this Taylor expansion, formally interchanging the order of summation and integration, and finally extending the range of integration to the full real axis. Final integrals are to be evaluated by the familiar formula

$$\int_{-\infty}^{\infty} dy y^{2n} e^{-ky} = \Gamma\left(n + \frac{1}{2}\right) k^{-n-(1/2)}, \qquad \int_{-\infty}^{\infty} dy y^{2n-1} e^{-ky} = 0 . \tag{5.17}$$

Thus a_n are given by

$$a_n = \Gamma\left(n + \frac{1}{2}\right) h_{2n} . \tag{5.18}$$

Furthermore, the Taylor coefficients h_{2n} are given by an integral of the form

$$h_{2n} = \frac{1}{2\pi i} \oint_{y=0} dy \frac{dz}{dy} \frac{g(z)}{y^{2n+1}} \tag{5.19}$$

along a small contour around $y = 0$. This contour is mapped onto a contour around $z = z_0$ on the z plane, because $y \to z$ is a coordinate transformation. Thus the last integral can be rewritten into a contour integral on the z plane:

$$h_{2n} = \frac{1}{2\pi i} \oint_{z=z_0} dz \frac{g(z)}{(f(z) - f_0)^{n+(1/2)}} . \tag{5.20}$$

In summary, we obtain the following formula for the coefficients a_n:

$$a_n = \frac{\Gamma\left(n + \frac{1}{2}\right)}{2\pi i} \oint_{z=z_0} dz \frac{g(z)}{(f(z) - f_0)^{n+(1/2)}} . \tag{5.21}$$

In the case of our $u_j(\lambda)$, $f(z)$ and $g(z)$ are given by

$$f(z) = \frac{\bar{p}}{p + \bar{p}} z^{p+\bar{p}} - z^{\bar{p}}, \qquad g(z) = z^{(\bar{p}-1)/2-j-1} . \tag{5.22}$$

The relevant saddle point is at $z = 1$; the others are p^{th} roots of unity and the origin, and all subdominant if we consider the asymptotics as $\hbar^{-1}\lambda^{p+\bar{p}} \to +\infty$. The above contour integrals are still hard to evaluate explicitly, but anyway, we thus obtain a closed formula for the coefficients u_{ij}.

5.3. Dual construction. A parallel construction is possible by starting from the generating functions

$$\bar{u}_i(\mu) = \bar{c}(\mu) \int d\lambda \lambda^{(p-1)/2+i} \exp \hbar^{-1} \left(\lambda^p \mu^{\bar{p}} - \frac{p}{p+\bar{p}} \lambda^{p+\bar{p}} \right) ,$$

$$\bar{c}(\mu) = \text{const. } \mu^{(\bar{p}-1)/2} \exp \hbar^{-1} \left(\frac{-\bar{p}}{p+\bar{p}} \mu^{p+\bar{p}} \right) . \tag{5.23}$$

Previous calculations on $u_j(\lambda)$ can be repeated in a fully parallel manner. The generating functions $\bar{u}_i(\mu)$ turn out to have asymptotic expansion of the form

$$\bar{u}_i(\mu) \sim \sum_{j=-\infty}^{i} \bar{u}_{ij} \mu^j \quad (\mu \to +\infty) \tag{5.24}$$

if $p + \bar{p} > 0$, and

$$\bar{u}_i(\mu) \sim \sum_{j=i}^{\infty} \bar{u}_{ij} \mu^j \quad (\mu \to +0) \tag{5.25}$$

if $p + \bar{p} < 0$. (The case of $p + \bar{p} = 0$ again reduces to the $c = 1$ string theory.) The coefficients $\bar{u}_{ij}$ satisfy exactly the same linear relations as the coefficients u_{ij} in the previous construction. In other words, this "dual" construction leads to a solution of the same string equations.

We have been, however, unable to prove (or disprove) that these two constructions in fact give the same solution. An obstruction is the fact that the linear relations among u_{ij} appear to allow more ambiguities than the overall rescaling $u_{ij} \to \text{const. } u_{ij}$. Nevertheless, it seems likely that the above solutions are rather special and coincide. The contour integral representation in the previous subsection will be useful in pursuing this issue.

5.4. Concluding remark of this section. The contents of this section is inspired by the work of Kharchev and Marshakov [20] on the (p,q) duality in $c < 1$ strings. Our $\mathbf{U}$ is a matrix representation of their Fourier–Laplace integral operator that gives the (p,q) duality. Namely, if V_{pq} denotes the point representing the (p,q) model in the Sato Grassmannian, $\mathbf{U}$ acts on the Sato Grassmannian and interchanges the (p,q) and (q,p) models:

$$\mathbf{U} V_{qp} = V_{pq} . \tag{5.26}$$

Note that everything is formulated in the language of the KP hierarchy.

Although closely related with their work, our usage of $\mathbf{U}$ is substantially different. We rather interpret $\mathbf{U}$ as a $GL(\infty)$ element that determines a solution of the Toda lattice hierarchy. Only in the case of $q = 1$, their work and ours are directly related via generalized Kontsevich models. Recall, however, that even in that case, our interpretation of generalized Kontsevich models is slightly different from the ITEP-Lebedev group [19]. We have pointed out that the generalized Kontsevich models with "negative times" give solutions on the full (bi-infinite) lattice rather than a half (semi-infinite) lattice.

6. Conclusion and Discussion

We have constructed a new family of special solutions of the Toda lattice hierarchy, and derived string equations of these solutions. These solutions have two discrete parameters $(p, \bar{p})$, and include already known solutions as follows:

- $(p, \bar{p}) = (p, 1)$, $p > 0$ – the p^{th} generalized model with "negative times" $\bar{t}$ and a "discrete time" s.
- $(p, \bar{p}) = (\beta, -\beta)$, $\beta > 0$ – the compactified $c = 1$ string theory. $\beta = 1$ corresponds to the self-dual radius case.
- $(p, \bar{p}) = (-1, 1)$ – related to the Penner model. This case may be interpreted as the generalized Kontsevich model at $p = -1$, and also as the $c = 1$ string theory "at $\beta = -1$."

In the course of reviewing known examples, we have also posed several questions on the relation [7, 22] between the generalized Kontsevich models and the $c = 1$ string theory. In particular, as opposed to the hypothesis postulated in the literature [22], the naive extrapolation of the p^{th} generalized Kontsevich model to $p = -1$ does not give the $c = 1$ string theory at self-dual radius. The $p = -1$ limit is rather related to the Penner model [30]. Actually, these three models are mutually distinct and obey different string equations. Thus, although there is a lot of evidence [39] that the A_{k+1} model at $k = -3$ is anyhow related to $c = 1$ (or two-dimensional) strings, the approach to $c = 1$ by the generalized Kontsevich models seems to be problematical.

A natural extrapolation of the $c = 1$ string theory at self-dual radius will be given by the case of $p = 1$ and $\bar{p} \leqq -1$. This model is ensured to reduced to the $c = 1$ model by letting $\bar{p} \to -1$. Furthermore, remarkably, this model has a matrix integral representation of Kontsevich type, as follows. The matrix $\mathbf{U} = \mathbf{U}(0, 0)$ of this model is upper triangular; the τ function has a Miwa variable representation by the Miwa transformation

$$\bar{t}_n = -\frac{\hbar}{n} \sum_{i=1}^{N} \mu_i^{-n} = -\frac{\hbar}{n} \operatorname{Tr} M^{-n} . \tag{6.1}$$

This Miwa variable representation can be rewritten into a matrix integral of the form

$$\tau(t, \bar{t}, \hbar(N-1)) = \bar{C}(M) \int d\Lambda \exp \hbar^{-1} \operatorname{Tr} \left(\Lambda M^{\bar{p}} - \frac{1}{1+\bar{p}} \Lambda^{1+\bar{p}} + \sum_{n=1}^{\infty} t_n \Lambda^n \right), \tag{6.2}$$

where we have considered the case of $s = \hbar(N-1)$ for simplicity. (This expression can be readily extended to general s.)

Unlike the ordinary generalized Kontsevich models, the Kontsevich potential $\Lambda^{1+\bar{p}}/(1 + \bar{p})$ in this matrix integral is a negative power of Λ. Therefore the integral should be suitably interpreted (e.g., by suitable regularization to avoid essential singularities as $\det \Lambda \to 0$). The τ function itself, as a function (or formal power series) of $(t, \bar{t})$, is however independent of such a detail of regularization, and only determined by the contribution from the saddle point $\Lambda = M$. The above matrix integral representation might lead to a topological interpretation of this model by means of "fat graph" expansion.

Another important difference from the ordinary generalized Kontsevich models is that the string equations of this model never reduces to string equations in the KP hierarchy. In the case of generalized Kontsevich models, the "negative times" are coupling constants of unphysical states ("anti-states" in the language of Montano and Rivlis [24]), which decouple from the theory as $\bar{t} \to 0$, and the string equations turn into string equations in the KP hierarchy in this limit. This does not occur in the case of $p = 1$ and $\bar{p} \leqq -1$. In other words, the model persists to be "$c = 1$" in the whole space of the coupling constants.

We, however, do not know what physical interpretation the string equations actually have for general values of $(p, \bar{p})$ including the above case of $p = 1$ and $\bar{p} \leqq -1$. Do they all correspond to a physical model (hopefully, of strings)? A first step towards such deeper understanding will be to show a connection with moduli spaces of Riemann surfaces. The works of Montano and Rivlis [24] and Lavi et al. [22] will provide useful ideas in that direction.

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Note added After this paper was completed, the author learned that the same issue as we have considered on $c = 1$ strings was studied by Imbimbo and Mukhi in a recent preprint [17]. They noted the inconsistency between the Kontsevich-type representation of Dijkgraaf et al [7] and the $W_{1+\infty}$ constraints, and proposed an alternative matrix model Our matrix integral representation of the $c = 1$ partition function in Sect. 4 seems to be essentially the same as this model The author would like to thank Sunil Mukhi for pointing out the existence of their paper

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