Commun. Math. Phys. 179, 599–622 (1996)

Communications in
Mathematical
Physics

A Rohlin Property for One-Parameter Automorphism Groups

A. Kishimoto
Department of Mathematics, Hokkaido University, Sapporo 060, Japan

Received: 12 June 1995 / Accepted: 5 January 1996

Abstract: We define a Rohlin property for one-parameter automorphism groups of unital simple C^*-algebras and show that for such an automorphism group any cocycle is almost a coboundary. We apply the same method to the single automorphism case and show that if an automorphism of a unital simple C^*-algebra with a certain condition has a central sequence of approximate eigen-unitaries for any complex number of modulus one, then any cocycle is almost a coboundary, or the automorphism has the stability. We also show that if a one-parameter automorphism group of a unital separable purely infinite simple C^*-algebra has the Rohlin property then the crossed product is simple and purely infinite.

1. Introduction

A C^*-dynamical system is a C^*-algebra with an action of a locally compact group by automorphisms. To analyse such a system the notion of Rohlin property was introduced and exploited at least when the group is the integer group $\mathbf{Z}$ or perhaps an amenable discrete group [8, 12, 13, 5, 3, 4, 22, 26, 19, 20]. We here introduce a Rohlin property for one-parameter automorphism groups; if α is a strongly continuous one-parameter automorphism group of a unital simple C^*-algebra A, α is said to have the Rohlin property if for any real number $p \in \mathbf{R}$ there is a central sequence $\{v_n\}$ of unitaries in A such that $\alpha_t(v_n) - e^{ipt}v_n$ converges to zero uniformly in t on every bounded subset of $\mathbf{R}$. In this case the spectral projections of v_n would be periodically transformed, in a sense, under α_t with period $2\pi/p$; so this is an analogue of the Rohlin property for single automorphisms. The main result (Theorem 2.1) will show that then for any α-cocycle u, i.e., a continuous family $u(t)$ of unitaries with $u(s)\alpha_s(u(t)) = u(s+t)$, $s, t \in \mathbf{R}$ is almost a coboundary, i.e., has a sequence $\{w_n\}$ of unitaries with $w_n\alpha_t(w_n^*) \to u(t)$ uniformly in t on every bounded subset of $\mathbf{R}$. (Here a small condition on u should be imposed; see 2.1 for details.) The only natural examples we can give of one-parameter automorphism groups with the Rohlin property are on simple non-commutative tori (Proposition 2.5). (Others may

Dedicated to: Prof. H. Hasegawa

be obtained by considering infinite tensor products.) Note that any one-parameter automorphism group of an AF algebra does not have the Rohlin property since K_1 is trivial (the unitaries $\{v_n\}$ above must give non-trivial elements in K_1). This is very much different from the situation for single automorphisms [20]. We also note that the crossed product by a one-parameter automorphism group with the Rohlin property must be simple (Proposition 2.4).

We employ a similar method to show that if an automorphism α of a unital simple C^*-algebra with a certain condition (which is satisfied by AF algebras and by purely infinite C^*-algebras if $\alpha_* = id$ on K_0), has the property that for any complex number λ of modulus one there is a central sequence $\{v_n\}$ of unitaries such that $\alpha(v_n) - \lambda v_n \to 0$, then for any unitary u in the connected component of 1 of the unitary group of A there is a sequence $\{w_n\}$ of unitaries such that $w_n\alpha(w_n^*) \to u$ (Theorem 3.2). If the C^*-algebra is AF and the automorphism induces the trivial action on K_0, the above property is equivalent to the Rohlin property. (This is perhaps not surprising because the Rohlin property may be obtained by the property that any non-zero power of the automorphism is not weakly inner in any tracial representation [20], which follows easily from the above property.)

If a one-parameter automorphism group α has the Rohlin property then there are α-covariant irreducible representations [17]. If the C^*-algebra has real rank zero and satisfies the condition referred to above, we shall show that there is a decreasing sequence of almost α-invariant projections whose limit is a minimal projection in the second dual (Theorem 4.1). (A similar result for single automorphisms with trivial action on K_0 can be obtained by using part of the arguments for 4.1 if the automorphism satisfies the property that all non-zero powers are outer; a substantially weaker property than the Rohlin property in general.) We have an example of one-parameter automorphism groups where the conclusion of Theorem 4.1 does not hold. (In this example α_1 is inner.) Then we shall show that the crossed product of a unital separable purely infinite simple C^*-algebra by a one-parameter automorphism group with the Rohlin property is simple and purely infinite (Theorem 4.8).

2. One-Parameter Automorphism Groups

Let A be a unital C^*-algebra and let

$$A^\infty = l^\infty(\mathbf{N},A)/c_0(\mathbf{N},A)\,,$$

which is a unital C^*-algebra; $x = (x_n) + c_0(\mathbf{N},A)$ has norm $\limsup \|x_n\|$. Embedding A into $l^\infty(\mathbf{N},A)$ by $x \to (x,x,\ldots)$ and also into A^∞, we denote $A^\infty \cap A'$ by A_∞. Let α be a strongly continuous one-parameter group of automorphisms of A; then α acts on $l^\infty(\mathbf{N},A)$ in the natural way and leaves $c_0(\mathbf{N},A)$ and A invariant. Let

$$l^\infty_\alpha(\mathbf{N},A) = \{x \in l^\infty(\mathbf{N},A); t \to \alpha_t(x) \text{ is continuous}\}$$

which is a C^*-subalgebra of $l^\infty(\mathbf{N},A)$ containing $c_0(\mathbf{N},A)$ and A, and let

$$A^\infty_\alpha = l^\infty_\alpha(\mathbf{N},A)/c_0(\mathbf{N},A), \qquad A_{\infty,\alpha} = A^\infty_\alpha \cap A'\,.$$

By an α-cocycle u in $A_{\infty,\alpha}$ we mean a continuous family $u(t)$, $t \in \mathbf{R}$, of unitaries in $A_{\infty,\alpha}$ such that for $s,t \in \mathbf{R}$,

$$u(s)\alpha_s(u(t)) = u(s+t)\,.$$

For a unitary $u \in A$, if u is in the connected component of 1 of the unitary group of A, let $l(u)$ be the infimum of the lengths of rectifiable paths from u to 1 and otherwise let $l(u) = \infty$. If A has real rank zero, then either $l(u) \leqq \pi$ or $l(u) = \infty$ [24].

Theorem 2.1. *Let A be a unital separable simple C^*-algebra and let α be a strongly continuous one-parameter automorphism group of A. Then the following conditions are equivalent*:

1. *For each $p \in \mathbf{R}$ there exists a unitary $v \in A_{\infty,\alpha}$ such that $\alpha_t(v) = e^{itp}v$.*
2. *For each α-cocycle u in $A_{\infty,\alpha}$ such that $t^{-1}l(u(t)) \to 0$ as $t \to 0$, there exists a unitary $w \in A_{\infty,\alpha}$ such that $u(t) = w\alpha_t(w^*)$.*

In this case for each α-cocycle u in A such that $t^{-1}l(u(t)) \to 0$ as $t \to 0$, there is a sequence $\{w_n\}$ of unitaries in A such that $\|u(t) - w_n\alpha(w_n^*)\| \to 0$ uniformly in t on each compact subset of $\mathbf{R}$.

A one-parameter automorphism group α of a unital simple C^*-algebra A is said to have the *Rohlin property* if α satisfies the condition (1) in the above theorem.

Let T be a subset of A. We say that T is equicontinuous with respect to α if the family of continuous functions $t \to \alpha_t(x)$ with $x \in T$ is equicontinuous, i.e., for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|t| < \delta$ and $x \in T$, then $\|\alpha_t(x) - x\| < \varepsilon$.

Lemma 2.2. *Let $u(t)$, $t \in \mathbf{R}$ be a continuous family of unitaries in $A_{\infty,\alpha}$. Then there exists a sequence $\{u_n(t)\}$ of continuous families of unitaries in A such that $(u_1(t), u_2(t), \ldots)$ represents $u(t)$, $\{u_n(\cdot)\}$ is equicontinuous on every compact subset of $\mathbf{R}$ and $\{u_n(t)\}$ is equicontinuous with respect to α for each $t \in \mathbf{R}$. Moreover if u satisfies that $\|u(s) - u(t)\| < C|s - t|$ for distinct $s, t \in [-r, r]$ with some $C, r > 0$, then $\{u_n\}$ can be chosen so that $\|u_n(s) - u_n(t)\| < C|s - t|$ for distinct $s, t \in [-r, r]$, for all sufficiently large n.*

Proof. For each $t \in \mathbf{R}$ there is a sequence $\{x_n\}$ in A representing $u(t)$ such that x_n's are equicontinuous with respect to α. Since $x_n^*x_n \to 1$ and $x_nx_n^* \to 1$. $u_n = x_n|x_n|^{-1}$ is well-defined for large n and $\{u_n\}$ satisfies the same properties as $\{x_n\}$. Thus we can assume that x_n's are unitaries.

For each $t \in \mathbf{R}$ let $\{\tilde{u}_n(t)\}$ be a sequence of unitaries in A representing $u(t)$ such that $\tilde{u}_n(t)$'s are equicontinuous with respect to α. Let $n_0 = 0$. For each $k \in \mathbf{N}$ we choose an $n_k \in \mathbf{N}$ such that $n_k > n_{k-1}$ and if $s, t \in [-k, k]$ satisfies $|s - t| < 2^{-n_k}$ then $\|u(s) - u(t)\| < 1/3k$. Let $m_0 = 0$. Then we choose an $m_k \in \mathbf{N}$ such that $m_k > m_{k-1}$ and for any $l \geqq m_k$ and $s, t \in P_k \equiv \{j \in [-k, k] \mid 2^{n_k}j \in \mathbf{Z}\}$,

$$\|\tilde{u}_l(s) - \tilde{u}_l(t)\| < \|u(s) - u(t)\| + 1/3k \ .$$

For l with $m_k \leqq l < m_{k+1}$ and $s \in P_k$ define $h_{l,s} = h_{l,s}^* \in A$ of small norm by

$$\tilde{u}_l(s + 2^{-n_k}) = e^{ih_{l,s}}\tilde{u}_l(s) \ .$$

Let

$$u_l(t) = e^{i(t-s)2^{n_k}h_{l,s}}\tilde{u}_l(s), \quad t \in [s, s + 2^{-n_k}] \ ,$$

$u_l(t) = \tilde{u}_l(-k)$, $t \leqq -k$, and $u_l(t) = \tilde{u}_l(k)$, $t \geqq k$. Thus we obtain the continuous functions u_n on $\mathbf{R}$ for $n \geqq m_1$. We assert that $\{u_n\}$ satisfies the required properties.

For $t \in 2^{-m}\mathbf{Z}$ with $m \in \mathbf{N}$, $\{u_n(t)\}$ represents $u(t)$. Let $s, t \in \mathbf{R}$. For a sufficiently large k with $s, t \in [-k, k]$ we have that for $n \geqq m_k$,

$$\begin{aligned}\|u_n(s) - u_n(t)\| &\leqq \|u_n(s) - u_n(s')\| + \|u_n(s') - u_n(t')\| + \|u_n(t') - u_n(t)\| \\ &< \|u(s) - u(t)\| + 1/k ,\end{aligned}$$

where $s', t' \in P_k$ with $|s - s'| < 2^{-n_k}$, $|t - t'| < 2^{-n_k}$. Hence $\{u_n(\cdot)\}$ is equicontinuous on each bounded interval of $\mathbf{R}$. In particular $\{u_n(t)\}$ represents $u(t)$ for each $t \in \mathbf{R}$.

If $s \in 2^{-m}\mathbf{Z}$ with $m \in \mathbf{N}$, then $\{u_n(s)\}$ is equicontinuous with respect to α. Since $\{u_n(\cdot)\}$ is equicontinuous, this is the case for any $t \in \mathbf{R}$.

To show the last assertion choose an increasing sequence $\{n_k\}$ such that for $n \in \{n_k, n_k + 1, \ldots, n_{k+1} - 1\}$ and $l = -2^k + 1, -2^k + 2, \ldots, 2^k$,

$$\varepsilon_n \equiv \|u_n(r2^{-k}(l-1)) - u_n(r2^{-k}l)\| < Cr2^{-k} .$$

Suppose that $Cr2^{-k} < 2$. We define u_n' by: For $t \in [r2^{-k}(l-1), r2^{-k}l]$,

$$u_n'(t) = e^{i(2^k t - r(l-1))/r \cdot h} u_n(r2^{-k}(l-1)) ,$$

where h is defined as before by $\log(u_n(r2^{-k}l)u_n(r2^{-k}(l-1))^*)$ with branch along the negative real axis. Then since $\|h\| \leqq 2 \arcsin \varepsilon_n/2$, the new $\{u_n'\}$ satisfies the required properties.

Lemma 2.3. *Let A be a unital simple C^*-algebra and let $\{v_n\}$ be a central sequence of unitaries in A such that for any $\varepsilon > 0$, $\mathrm{Sp}(v_n) + (0, \varepsilon) = \mathbf{T}$ (regarded as $\mathbf{R}/\mathbf{Z}$) for all sufficiently large n. Define a linear map of the algebraic tensor product $A \odot C(\mathbf{T})$ into A by*

$$\phi_n(a \otimes f) = af(v_n) .$$

Then $\{\phi_n\}$ is an approximate homomorphism, i.e.,

$$\|\phi_n(x)^* - \phi_n(x^*)\|, \quad \|\phi_n(xy) - \phi_n(x)\phi_n(y)\|$$

converge to zero for any $x, y \in A \odot C(\mathbf{T})$, and for any $x \in A \odot C(\mathbf{T})$,

$$\lim \|\phi_n(x)\| = \|x\| ,$$

where $\|x\|$ is the C^*-norm of $x \in A \otimes C(\mathbf{T})$.

Proof. It follows by easy computations that $\{\phi_n\}$ is an approximate homomorphism. Then the map $x \mapsto (\phi_1(x), \phi_2(x), \ldots)$ defines a homomorphism of $A \odot C(\mathbf{T})$ into A^∞ and so

$$\gamma(x) = \limsup \|\phi_n(x)\|$$

defines a C^*-seminorm on $A \odot C(\mathbf{T})$. (See [23, 9] for similar arguments.)

Let I be an open interval in $\mathbf{T}$ and let for $a \in A$,

$$\delta(a) = \sup\{\gamma(a \otimes f) | 0 \leqq f \leqq \chi_I\} ,$$

where χ_I is the characteristic function of I. Then it follows that δ is a C^*-seminorm on A. Since $\delta(1) = 1$ by the assumption on $\mathrm{Sp}(v_n)$ and since A is simple, it follows that $\delta(a) = \|a\|$, $a \in A$.

Note that any non-zero closed two-sided ideal of the tensor product $A \otimes C(\mathbf{T})$ contains a non-zero element $a \otimes f$. If γ is not a norm, there is a non-zero element $a \otimes f$ such that

$$\gamma(a^*a \otimes f^*f) = 0 .$$

Since f^*f dominates $c\chi_I$ for some $c > 0$ and $I \neq \emptyset$, this contradicts that δ is a norm. Since $A \odot C(\mathbf{T})$ has a unique C^*-norm, we obtain that

$$\gamma(x) = \|x\|, \quad x \in A \odot C(\mathbf{T}) ,$$

which suffices to conclude the proof.

Proof of Theorem 2.1. For each $p \in \mathbf{R}$, $t \mapsto e^{ipt}$ can be regarded as an α-cocycle in $A_{\infty,\alpha}$. Thus (1) is a special case of (2). We shall prove that (1) implies (2).

Let u be an α-cocycle in $A_{\infty,\alpha}$ and $\varepsilon > 0$. First we choose an $N \in \mathbf{N}$ so that $l(N)/N < \varepsilon$. Let $\{u_n(\cdot)\}$ be a sequence of continuous families of unitaries which represents u as in Lemma 2.2. Then it follows that

$$u_n(s)\alpha_s(u_n(t)) - u_n(s+t)$$

converges to zero uniformly in s,t on each compact subset of $\mathbf{R}$. Here we may suppose that $u_n(0) = 1$ and $l(u_n(N)) < \varepsilon N$. We choose a sufficiently large $n \in \mathbf{N}$ so that for $s,t \in [0,2N]$,

$$\|u_n(s)\alpha_s(u_n(t)) - u_n(s+t)\| < \varepsilon ,$$

and we let $U = u_n$. By Lemma 2.2 we may further assume that there is a continuous family $x(t), t \in [0,N]$ of unitaries such that $x(0) = 1$, $x(N) = U(N)$ and for distinct $s,t \in [0,N]$,

$$\|x(s) - x(t)\| < \varepsilon|s-t| .$$

Define a unitary W in $A \otimes C(\mathbf{T})$ by

$$W(s) = U(Ns)\alpha_{N(s-1)}(x(Ns)^*)$$

for $s \in [0,1]$. Since $W(0) = 1 = W(1)$, W is in fact in $A \otimes C(\mathbf{T})$.

Define a one-parameter automorphism group γ on $C(\mathbf{T})$ by $(\gamma_t f)(s) = f(s-t)$. Suppose that $0 \leqq s < 1$ and $0 < t \leqq N$. If $Ns > t$, then

$$(W\alpha_t \otimes \gamma_{t/N}(W^*))(s) = U(Ns)\alpha_{N(s-1)}(x(Ns)^*x(Ns-t))\alpha_t(U(Ns-t))^* .$$

Since $\|x(Ns)^*x(Ns-t) - 1\| < \varepsilon t$ and $\|u(Ns) - U(t)\alpha_t(U(Ns-t)\| < \varepsilon$, it follows that

$$\|(W\alpha_t \otimes \gamma_{t/N}(W^*))(s) - U(t)\| < \varepsilon(t+1) .$$

If $Ns < t$, then

$$\begin{aligned}&(W\alpha_t \otimes \gamma_{t/N}(W^*))(s)\\&\quad = U(Ns)\alpha_{N(s-1)}(x(Ns)^*)\alpha_{Ns}(x(N+Ns-t))\alpha_t(U(N+Ns-t)^*) .\end{aligned}$$

Since $\|x(Ns) - 1\| < \varepsilon Ns$ and $\|x(N+Ns-t) - U(N)\| < \varepsilon(t-Ns)$, it follows that

$$\begin{aligned}\|W\alpha_t \otimes \gamma_{t/N}(W^*)(s) - U(t)\| &< \varepsilon t + \|U(Ns)\alpha_{Ns}(U(N))\alpha_t(U(N+Ns-t))^* - U(t)\|\\&< \varepsilon(t+2) .\end{aligned}$$

Hence we obtain for $t \in [0,N]$,

$$\|W\alpha_t \otimes \gamma_{t/N}(W^*) - U(t) \otimes 1\| < \varepsilon(t+2).$$

By (1) there is a unitary $v \in A_{\infty,\alpha}$ such that $\alpha_t(v) = e^{-2\pi it/N}v$. Let $\{v_n\}$ be a sequence of unitaries in A representing v such that $\{v_n\}$ is equicontinuous with respect to α. We define a linear map ϕ_n of $A \odot C(\mathbf{T})$ into A by

$$\phi_n(a \otimes f) = af(v_n)$$

as in Lemma 2.3.

Note that for $a \in A$ and $f \in C(\mathbf{T})$,

$$\phi_n \circ \alpha_t \otimes \gamma_{t/N}(a \otimes f) = \alpha_t(a)f(e^{-2\pi it/N}v_n) \approx \alpha_t(af(v_n)),$$

i.e., for $x \in A \odot C(\mathbf{T})$,

$$\lim \|\phi_n \circ \alpha_t \otimes \gamma_{t/N}(x) - \alpha_t \circ \phi_n(x)\| = 0.$$

We find a $W_1 \in A \odot C(\mathbf{T}) \subset A \otimes C(\mathbf{T})$ such that $\|W_1 - W\| < \varepsilon$ and find an $n \in \mathbf{N}$ such that for $t \in [0,N)$,

$$\|\phi_n(W_1)\phi_n(W_1^*) - 1\| < \varepsilon + \|W_1W_1^* - 1\|,$$

$$\|\phi_n(W_1^*)\phi_n(W_1) - 1\| < \varepsilon + \|W_1^*W_1 - 1\|,$$

$$\|\phi_n \circ \alpha_t \otimes \gamma_{t/N}(W_1) - \alpha_t \circ \phi_n(W_1)\| < \varepsilon,$$

$$\|\phi_n(W_1)\phi_n \circ \alpha_t \otimes \gamma_{t/N}(W_1^*) - \phi_n(W_1\alpha \otimes \gamma_{t/N}(W_1^*)\| < \varepsilon,$$

$$\|\phi_n(W_1\alpha_t \otimes \gamma_{t/N}(W_1^*)) - \phi_n(U(t) \otimes 1)\| < \varepsilon + \|W_1\alpha_t \otimes \gamma_{t/N}(W_1^*) - U(t) \otimes 1\|.$$

Then for $\tilde{W} = \phi_n(W_1)$ we obtain that

$$\begin{aligned}
\|\tilde{W}\alpha_t(\tilde{W}^*) - U(t)\| &< \varepsilon + \|\phi_n(W_1)\phi_n \circ \alpha_t \otimes \gamma_{t/N}(W_1)^* - U(t)\| \\
&< 2\varepsilon + \|\phi_n(W_1\alpha_t \otimes \gamma_{t/N}(W_1^*)) - \phi_n(U(t) \otimes 1)\| \\
&< 3\varepsilon + \|W_1\alpha_t \otimes \gamma_{t/N}(W_1^*) - U(t) \otimes 1\| \\
&< 5\varepsilon + \|W\alpha_t \otimes \gamma_{t/N}(W^*) - U(t) \otimes 1\| \\
&< 7\varepsilon + \varepsilon t.
\end{aligned}$$

Since $\tilde{W}$ is close to a unitary, the unitary w obtained by the polar decomposition of $\tilde{W}$ has the desired properties.

For each sufficiently large n we specify N, $x(t)$ and v_m and then construct $w_n = w$ in the above way. If A is separable, it is easy to make $\{w_n\}$ central. (We have assumed the separability only for this reason.) This concludes the proof of $(1) \Rightarrow (2)$.

The last statement follows from the same proof as above. We do not need the separability for this statement.

Proposition 2.4. *Let A be a unital simple C^*-algebra and let α be a strongly continuous one-parameter automorphism group with the Rohlin property. Then the crossed product $A \times \mathbf{R}$ is simple.*

Proof. The dual action $\hat{\alpha}$ of $\mathbf{R}$ on $A \times \mathbf{R}$ is defined by $\hat{\alpha}_p(a) = a$, $\hat{\alpha}_p(\lambda_t) = e^{ipt}\lambda_t$, where $a \in A$ and the canonical unitary group λ implementing α on A are elements in the multiplier algebra. Then it soon follows that each $\hat{\alpha}_p$ is approximately inner, i.e., $\hat{\alpha}_p = \lim \mathrm{Ad}\, u_n \,|\, A \times \mathbf{R}$, where $\{u_n\}$ is a central sequence of unitaries in A with $\lim \|\alpha_t(u_n) - e^{-ipt}u_n\| = 0$ uniformly in t on every compact subset of $\mathbf{R}$. Hence any closed two-sided ideal of $A \times \mathbf{R}$ is left invariant under $\hat{\alpha}$. Since A is simple, this implies that $A \times \mathbf{R}$ is simple.

Proposition 2.5. *Let A be a simple non-commutative n-torus, i.e., the universal C^*-algebra generated by n unitaries $u_1, \ldots, u_n$ with $u_i u_j u_i^* u_j^* = e^{2\pi i \theta_{ij}} 1 \in \mathbf{C}1$ such that the anti-symmetric matrix $\Theta = (\theta_{ij})$ satisfies that $\Theta m \notin \mathbf{Z}^n$ for any non-zero $m \in \mathbf{Z}^n$. Let α be a one-parameter automorphism group of A such that*

$$\alpha_t(u_j) = e^{2\pi i p_j t} u_j$$

and any α_t with $t \neq 0$ is not inner, i.e., $(\mathbf{Z}^n + \Theta\mathbf{Z}^n) \cap \mathbf{R}p = \{0\}$ with $p = (p_1, \ldots, p_n)^t$. Then α has the Rohlin property.

Proof. This is noted in [18] in a different context. Let $q \in \mathbf{R}$. We shall find a sequence $\{m_k\}$ in $\mathbf{Z}^n$ such that

$$\mathrm{dist}\,(\Theta m_k, \mathbf{Z}^n) \to 0, \qquad p^t m_k \to q\,.$$

Then the sequence of unitaries

$$u^{m_k} = u_1^{m_{k1}} u_2^{m_{k2}} \cdots u_n^{m_{kn}}$$

is central and satisfies that $\alpha_t(u^{m_k}) - e^{2\pi i q t} u^{m_k} \to 0$.

This follows since

$$G = \{(\Theta m + k, p^t m) \,|\, k, m \in \mathbf{Z}^n\}$$

is dense in $\mathbf{R}^{n+1}$. To prove the density of G suppose that $\bar{G} \neq \mathbf{R}^{n+1}$. Then since G is a subgroup of $\mathbf{R}^{n+1}$, there must be a non-zero $\xi = (\xi_0, \xi_1) \in \mathbf{R}^n \times \mathbf{R}$ such that $\langle \xi, g \rangle \in \mathbf{Z}$ for any $g \in \bar{G}$, i.e.,

$$\langle -\Theta\xi_0, m \rangle + \langle \xi_0, k \rangle + \langle \xi_1 p, m \rangle \in \mathbf{Z}\,,$$

which implies that $\xi_0 \in \mathbf{Z}^n$ and

$$\langle -\Theta\xi_0 + \xi_1 p, m \rangle \in \mathbf{Z}\,.$$

Then it follows that $\xi_1 \neq 0$ and $\xi_1 p \in \mathbf{Z}^n + \Theta\mathbf{Z}^n$, a contradiction.

3. Single Automorphisms

Let A be a unital C^*-algebra and let α be an automorphism of A. Let

$$A_\alpha = \{f \in C[0,1] \otimes A \,|\, f(1) = \alpha(f(0))\}\,.$$

Condition 3.1. There is an increasing function $f : \mathbf{R}_+ \to \mathbf{R}_+$ such that for any u in the connected component of 1 of the unitary group of A_α there is a continuous path $\tilde{u}_s, s \in [0,1]$ of unitaries in A_α such that $\tilde{u}_0 = u$, $\tilde{u}_1 = 1$, and the length $l(\tilde{u})$ of $\tilde{u}$ is bounded by $f(L(u))$, where $L(u)$ is the length of $u(s)$, $s \in [0,1]$.

We shall consider this condition later in Propositions 3.4 and 3.5.

Theorem 3.2. *Let A be a unital simple C^*-algebra and let α be an automorphism of A. Suppose that Condition* 3.1 *is satisfied and that for any $\mu \in \mathbf{T}$ there is a unitary $v \in A_\infty$ such that $\alpha(v) = \mu v$. Then for any u in the connected component of* 1 *of the unitary group of A there is a sequence $\{v_n\}$ of unitaries in A such that $u = \lim v_n \alpha(v_n^*)$.*

Proof. Let $u \in A$ be a unitary in the connected component of 1 and let $u(1) = u$ and $u(k) = u\alpha(u(k-1))$ for $k = 2, 3, \dots$, i.e., u is an α-cocycle in the sense that $u(k)\alpha^k(u(m)) = u(k+m)$ for $k, m \in \mathbf{Z}$. Let v_0 be a rectifiable path of unitaries in A such that

$$v_0(0) = 1, \qquad v_0(1) = u\,.$$

Let, for $k = 1, 2, \dots, n-1$,

$$v_k = u(k)\alpha^k(v_0) \in C[0,1] \otimes A\,.$$

Note that $s \in [0,1] \mapsto v_0(s\,\cdot\,)^* v_0(s)(n)\alpha^n(v_0(s\,\cdot\,))$ is a path from 1 to $v_0^* u(n)\alpha^n(v_0)$ in the unitary group of A_α, where $v_0(s)(n)$ is defined in the same way as $u(n)$, based on $v_0(s)$ instead of u, and that the length $L(v_0^* u(n)\alpha^n(v_0))$ as a function on $[0,1]$ is at most $2l(v_0)$, independent of n. Using Condition 3.1 let $w_0 = 1, w_1, \dots, w_{n-2}$, $w_{n-1} = \alpha^{-n}(v_0^* u(n)\alpha^n(v_0))$ be a sequence of unitaries in A_α such that

$$\|w_k - w_{k-1}\| < C/n, \quad k = 1, 2, \dots, n-1\,,$$

where $C = f(2l(v_0))$. For $k = 0, 1, \dots, n-2$ let $\tilde{w}_k \in C[0,1] \otimes A$ be a unitary such that

$$\tilde{w}_k(t) = w_k(t), \quad t \in [0, 1/2]\,,$$

$$\tilde{w}_k(1) = w_{k+1}(1)\,,$$

$$\|\tilde{w}_k - w_k\| < C/n$$

and let $\tilde{w}_{n-1} = w_{n-1}$.

Let $\tilde{v}_k = v_k \alpha^k(\tilde{w}_k^*)$. We define a unitary $v \in C(\mathbf{T}) \otimes A$ as follows: for $t \in [k/n, (k+1)/n)$,

$$v(t) = \tilde{v}_k(nt - k)\,.$$

Then v is indeed continuous in $t \in \mathbf{T}$ because for $k = 0, 1, \dots, n-1$,

$$\tilde{v}_k(1) = u(k)\alpha^k(u)\alpha^k(w_{k+1}(1)^*) = u(k+1)\alpha^k(w_{k+1}(1)^*)\,,$$

$$\tilde{v}_{k+1}(0) = u(k+1)\alpha^{k+1}(w_{k+1}(0)^*) = u(k+1)\alpha^k(w_{k+1}(1)^*)\,,$$

and

$$\tilde{v}_0(0) = 1\,,$$

$$\tilde{v}_{n-1}(1) = u(n-1)\alpha^{n-1}(u)\alpha^{-1}(u^* u(n)\alpha^n(u)) = u(n)\alpha^{-1}(u^* u(n+1)) = 1\,.$$

Note also that

$$\begin{aligned}\tilde{v}_{k+1}\alpha(\tilde{v}_k^*) &= v_{k+1}\alpha^{k+1}(\tilde{w}_{k+1}^*)\alpha(v_k\alpha^k(\tilde{w}_k)^*)^* \\ &= u(k+1)\alpha^{k+1}(v_0)\alpha^{k+1}(\tilde{w}_{k+1}^*\tilde{w}_k)\alpha^{k+1}(v_0^*)\alpha(u(k)^*)\,, \\ \tilde{v}_0\alpha(\tilde{v}_{n-1}^*) &= v_0\tilde{w}_0^*\alpha(v_{n-1}\alpha^{n-1}(\tilde{w}_{n-1}^*))^* \\ &= v_0\tilde{w}_0^*\alpha^n(\tilde{w}_{n-1})\alpha^n(v_0^*)\alpha(u(n-1)^*) \\ &= v_0\tilde{w}_0^*v_0^*u\,.\end{aligned}$$

Thus it follows that

$$\|v(t)\alpha(v(t-1/n)) - u\| < 3C/n\,.$$

Let $\{u_m\}$ be a central sequence of unitaries in A such that $\|\alpha(u_m) - e^{-2\pi i/n}u_m\| \to 0$. Define a linear map ϕ_m of $A \odot C(\mathbf{T})$ into A by

$$\phi_m(a \otimes f) = af(u_m)\,.$$

Since $\|v\alpha \otimes \gamma_{1/n}(v^*) - u \otimes 1\| < 3C/n$ and $\phi_m \circ \alpha \otimes \gamma_{1/n} \approx \alpha \circ \phi_m$, we have, as in the proof of Theorem 2.1, that for a sufficiently large m, a unitary w which is close to "$\phi_m(v)$" satisfies that

$$\|w\alpha(w^*) - u\| < 3C/n\,.$$

This concludes the proof.

We have not proved an obvious adaptation of Theorem 2.1 to this case, i.e., the equivalence of the following two conditions:

1. For any $\mu \in \mathbf{T}$ there is a unitary $v \in A_\infty$ such that $\alpha(v) = \mu v$.
2. For any u in the connected component of the unitary group of A_∞ there is a unitary $v \in A_\infty$ such that $u = v\alpha(v^*)$.

To prove this we would need an obvious condition involving central sequences which is stronger than Condition 3.1, and which we could not prove unless A is an AF algebra. We shall now consider Condition 3.1.

Lemma 3.3. *Let A be a unital C^*-algebra of real rank zero and let $u(s)$, $s \in [0,1]$ be a continuous path of unitaries in A with $[u(0)] = 0$ in $K_1(A)$. Then for any $\varepsilon > 0$ there is a continuous function h of $[0,1]$ into the self-adjoint part of A such that for all $s \in [0,1]$*

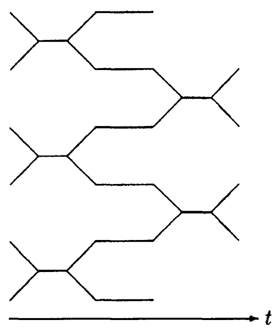
$$\|u(s) - e^{ih(s)}\| < \varepsilon\,.$$

Furthermore if $u(0) = 1 = u(1)$, then h can be chosen such that $h(0) = 0$, $e^{ih(1)} = 1$, and $\|h(t)\| < C_1 l(u) + C_2$, where C_1 and C_2 are constants independent of u (depending on ε).

Proof. For any $\varepsilon > 0$ there exist a $\delta \in (0, \varepsilon/2)$ and $N \in \mathbf{N}$ such that if $\|u_0 - u_1\| < \delta$ with u_0, u_1 unitaries of finite spectra, the unitaries

$$\tilde{u}_i = \sum_{k=0}^{2N-1} e^{\pi i k/N} P_i([k/2N - 1/4N, k/2N + 1/4N))\,,$$

where $P_i(\cdot)$ is the spectral measure of u_i on $\mathbf{T}$, can be connected by a continuous path $\tilde{u}_t$ of unitaries such that $\|\tilde{u}_t - u_i\| < \varepsilon/2$, $\mathrm{Sp}(\tilde{u}_t)$ is finite, and $G = \{(t, \lambda) | \lambda \in \mathrm{Sp}(\tilde{u}_t)\}$ looks like:



Some of the end points may not exist (all $e^{\pi ik/N}$'s may not be eigenvalues of $\mathrm{Sp}(\tilde{u}_i)$); so some line segments should be removed from the above picture (cf. [2]; here we have used freedom to change the pattern of eigenvalues slightly). In particular, $(0, e^{\pi ik/N})$ may be connected to at most four points of $(1, e^{\pi ik/N + \pi il/N})$, $l = -2, -1, 0, 1, 2$ in G as t increases.

Let L be the length $l(u)$ of u. If $M \in \mathbf{N}$ satisfies that $L/M < \delta \leqq L/(M-1)$ one finds a sequence $t_0 = 0 < t_1 < t_2 < \cdots < t_M = 1$ such that for $s, t \in [t_{i-1}, t_i]$,

$$\|u(s) - u(t)\| < \delta .$$

Then composing the $\tilde{u}$ constructed for the pair $u(t_{i-1})$, $u(t_i)$, we obtain a path $\tilde{u}(s)$, $s \in [0,1]$ such that $\|u(s) - \tilde{u}(s)\| < \varepsilon$ and $G_i = \{(t, \lambda) \,|\, t \in [t_{i-1}, t_i],\ \lambda \in \mathrm{Sp}(\tilde{u}(t))\}$ satisfies the condition as above. Then starting from $h(0) = -i \log \tilde{u}(0)$ with $\|h(0)\| \leqq \pi$ we can continuously define $h(t) = -i \log \tilde{u}(t)$; the definition of $h(t)$ (or the eigenvalues of $h(t)$) is by no means unique and in general the spectral projections of $h(t)$ are finer than those of $\tilde{u}(t)$.

To be more precise, if $\tilde{u}(t) = \sum_i \mu_i(t) p_i(t)$, $p_i(t)$'s are projections with $\sum_i p_i(t) = 1$, and $\mu_i(t)$'s are mutually distinct at a neighbourhood of $t = t_0$, $h(t)$ may be defined as

$$\sum_i \sum_k (-i \log \mu_i(t) + 2\pi k) p_{ik}(t) \qquad (*)$$

in that neighbourhood, where $\sum_k$ is the sum over a finite set of integers, $\log \mu_i(t)$ is defined as a continuous function, and $p_{ik}(t)$'s are projections with $\sum_k p_{ik}(t) = p_i(t)$.

If $\mu_i(t)$'s are mutually distinct for $t \leqq t_0$ except for $\mu_{2i-1}(t_0) = \mu_{2i}(t_0)$, and $\tilde{u}(t) = \sum_i \mu_i'(t) p_i'(t)$ for $t \geqq t_0$, where $\mu_i'(t_0) = \mu_{2i-1}(t_0) = \mu_{2i}(t_0)$ and $p_i'(t_0) = p_{2i-1}(t_0) + p_{2i}(t_0)$, then $h(t)$ is defined up to $t = t_0$ as in $(*)$ and by assuming $\log \mu_{2i-1}(t_0) = \log \mu_{2i}(t_0)$, $h(t)$ may be defined as

$$\sum_i \sum_k (-i \log \mu_i'(t) + 2\pi k) p_{ik}'(t) \qquad (**)\,,$$

where $p_{ik}'(t)$'s are projections with $\sum_k p_{ik}'(t) = p_i'(t)$.

If $\tilde{u}(t) = \sum_i \mu_i''(t) p_i''(t)$ for $t \geqq t_0$, $p_i''(t)$'s are projections with $\sum_i p_i''(t) = 1$, $\mu_i''(t)$'s are mutually distinct for $t \geqq t_0$ except for $\mu_{2i+1}''(t_0) = \mu_{2i}''(t_0) = \mu_i(t_0)$, then

$h(t)$ is defined up to $t = t_0$ and $\sum_k p_{ik}(t_0) = p''_{2i+1}(t_0) + p''_{2i}(t_0)$. Let $t_1 < t_0$ be close to t_0. By the Riesz decomposition property, we find subprojections q_{ik} of $p''_i(t_0)$ such that

$$[p_{ik}(t_1)] = [q_{2i+1,k}] + [q_{2i,k}] ,$$

$$\sum_k q_{ik} = p''_i(t_0) .$$

Then we find a path $\tilde{p}_{ik}(t)$, $t \in [t_1, t_0]$ of projections from $p_{ik}(t_1)$ to $q_{2i+1,k} + q_{2i,k}$ at $t = t_0$ for each k such that $\sum_k \tilde{p}_{ik}(t) = p_i(t)$. By using these paths we change $h(t)$ for $t \in [t_1, t_0]$ keeping the relation $\tilde{u}(t) = e^{ih(t)}$. Then we can continue to define $h(t)$ for $t > t_0$ by finding paths $p''_{ik}(t)$ such that $p''_{ik}(t_0) = q_{ik}$ and $\sum_k p''_{ik}(t) = p''_i(t)$.

If $u(0) = 1 = u(1)$, we assume that $\tilde{u}(0) = 1 = \tilde{u}(1)$. Starting with $h(0) = 0$, we can estimate the norm of $h(t)$: If $t_{k-1} \leqq t < t_k$,

$$\|h(t)\| \leqq 2\pi k/N ,$$

and hence for any $t \in [0, 1]$,

$$\|h(t)\| \leqq 2\pi M/N \leqq 2\pi(L/\delta + 1)/N = \frac{2\pi}{\delta N} L + \frac{2\pi}{N} .$$

We note that if $K_0(A)$ is totally ordered, then the above h can be defined in a unique way by requiring that the *ramification* can occur at most on one eigenvalue of $\tilde{u}(t)$. (This requirement makes the choice of subprojections q_{ik} unique when we use the Riesz decomposition property in the above proof.) In this case if $u(0) = 1 = u(1)$ we have that $h(1) \in 2\pi\mathbf{Z}1$.

We also note that if A is purely infinite and simple, we can also impose the above requirement, though in this case this does not remove the freedom we have when applying the Riesz decomposition property. (If $\mathrm{Sp}(\tilde{u})$ is full, we can use this freedom to control the norm of $h(t)$ to just over π.)

Proposition 3.4. *Let A be a unital simple C^*-algebra of real rank zero such that A has a weakly unperforated ordered group with the Riesz decomposition property as $K_0(A)$ and has the cancellation property. Let α be an automorphism of A such that $\alpha_* = id$ on $K_0(A)$. Then A_α satisfies Condition* 3.1.

Proof. Let $u \in A_\alpha$ be a unitary with $[u] = 0$. Since $u(0)$ can be approximated by a unitary of finite spectrum, we may assume that $\mathrm{Sp}(u(0))$ is finite. Let $h = h^* \in A$ be such that $\|h\| \leqq \pi$ and $u(0) = e^{ih}$. Let

$$u_s(t) = u(t)e^{-\imath(1-t)sh - its\alpha(h)} .$$

Then $u_0 = u$, $u_1(0) = 1 = u_1(1)$, the length of u_s, $s \in [0, 1]$ is $\|h\|$, and the length $L(u_1)$ of u_1 as a function on $[0, 1]$ is at most $2\|h\|$ greater than $L(u)$. Hence we may assume that $u(0) = 1 = u(1)$ and we can regard u as an element of $(SA)^+ = (C(0, 1) \otimes A)^+$.

Suppose that u_s, $s \in [0, 1]$ is a path from u to 1 in the unitary group of A_α. By Lemma 3.3 $u_s(1)$ can be approximated by $e^{ih(s)}$ with $h(0) = 0$ and $e^{ih(1)} = 1$. Hence there is another $h'(s)$ with $\|h'(s)\| \approx 0$ such that

$$u_s(0) = e^{ih(s)}e^{ih'(s)}, \qquad h'(0) = 0 = h'(1) .$$

By replacing u_s by

$$t \mapsto u_s(t)e^{-i(1-t)h'(s)-it\alpha(h'(s))}e^{-i(1-t)h(s)-it\alpha(h(s))}$$

we can assume that $u_s(0) = 1$. Thus u is connected to

$$u' : t \mapsto e^{-i(1-t)h(1)-it\alpha(h(1))}$$

in $(SA)^+$. Let $h(1) = \sum_k 2\pi k p_k$, where $\{p_k\}$ is an orthogonal family of projections in A. Then the class of u' is equal to

$$\sum_k k[p_k] - \sum_k k[\alpha(p_k)] = 0$$

in $K_1(SA)$ which is identified with $K_0(A)$, where we have used that $[p_k] = [\alpha(p_k)]$. Hence u (with $u(0) = 1 = u(1)$) is connected to 1 in the unitary group of $(SA)^+$.

By applying Lemma 3.3 to u with $\varepsilon = 2$, there are continuous functions h, h' of $[0, 1]$ into the self-adjoint part of A such that $h(0) = 0 = h'(0)$, $h'(1) = 0$, $\|h(t)\| \leqq C_1L + C_2$, $\|h'(t)\| < \pi$, and

$$u(t) = e^{ih(t)}e^{ih'(t)} .$$

Note that u and the unitary v defined by $v(t) = e^{ih(t)}$ can be connected by a path of unitaries in $(SA)^+$ of length at most π and that v and the unitary w defined by $w(t) = e^{ith(1)}$ can be connected by the path of unitaries $e^{i(1-s)h(t)+isth(1)}$ whose length is at most $2(C_1L + C_2)$. We shall now show that w can be connected to 1 by a path of unitaries in $(SA)^+$ of length at most $4\pi(C_1L + C_2)$.

Let

$$h(1) = \sum_{k=-K}^{K} 2\pi k p_k ,$$

where $\{p_k\}$ are mutually orthogonal projections in A and $|K| \leqq C_1L + C_2$. Since $[w] = 0$ in $K_1(SA)$, we must have that

$$\sum k[p_k] = 0 .$$

Let K_0 be the maximum of $|k|$ with $p_k \neq 0$. If $K_0 = 0$ then $h(1) = 0$ and there is nothing to prove. If $K_0 > 0$, suppose that $p_{K_0} \neq 0$. Since

$$K_0[p_{K_0}] + \sum_{k=-1}^{-K_0} k[p_k] \leqq 0 ,$$

it follows that

$$K_0[p_{K_0}] \leqq K_0 \sum_{k=-1}^{-K_0} [p_k] .$$

Suppose that $[p_{K_0}] \leqq \sum_{k=-1}^{-K_0}[p_k]$ which follows from the strict inequality in the above formula by $K_0(A)$ being weakly unperforated; then there are subprojections q_k of p_k for $k = -1, \ldots, -K_0$ such that

$$[p_{K_0}] = \sum_{k=-1}^{K_0} [q_k] .$$

By using the cancellation property we then find a partial isometry W such that

$$WW^* = p_{K_0}, \qquad W^*W = \sum_{k=-1}^{-K_0} q_k .$$

Let

$$U_\theta = \cos\theta \left(p_{K_0} + \sum_{k=-1}^{-K_0} q_k \right) + \sin\theta(W - W^*) + 1 - p_{K_0} - \sum_{k=-1}^{-K_0} q_k \,.$$

Then

$$U_\theta e^{2\pi i t p_{K_0}} U_\theta^* e^{ith(1) - 2\pi i t p_{K_0}}$$

connects w to $e^{ith'}$, where

$$h' = 2\pi \sum_{k=0}^{K_0-1} k p_k + 2\pi \sum_{k=-K_0}^{-1} k(p_k - q_k) + 2\pi \sum_{-K_0}^{-1} (k+1) q_k$$

$$+ 2\pi(K_0 - 1)(p_{K_0} + p_{K_{0-1}}) \,.$$

If $[p_{K_0}] \not\leqq \sum_{k=-1}^{-K_0} [p_k]$, i.e., $K_0 \geqq 2$ and $K_0[p_{K_0}] = K_0 \sum_{k=-1}^{-K_0} [p_k]$, which implies that $p_k = 0$ for $k = -1, \dots, -K_0 + 1$, let p'_{K_0} be a non-zero subprojection of p_{K_0}. Then $[p'_{K_0}] \leqq [p_{-K_0}]$, and we apply the above procedure with p'_{K_0} in place of p_{K_0}. Then we again apply the above procedure to the resulting h' with $p_{K_0} - p'_{K_0}$ in place of p_{K_0}, to transform $e^{ith'}$ to $e^{ith''}$ with

$$h'' = 2\pi \sum_{k=-K_0}^{K_0-1} k p''_k$$

with no K_0^{th} term.

If $p_{-K_0} \neq 0$, we can use a similar argument to remove the $-K_0^{\text{to}}$ term. By repeating this argument we find a path of unitaries which connects w to 1. Since each argument requires a path of length π, the resulting path has length at most

$$4K\pi \leqq 4\pi(C_1 L + C_2) \,.$$

Proposition 3.5. *Let A be a unital purely infinite simple C^*-algebra and let α be an automorphism of A such that $\alpha_* = id$ on $K_0(A)$. Then A_α satisfies Condition* 3.1.

Proof. A purely infinite simple C^*-algebra has real rank zero [28] and satisfies that any non-zero projections p and q are equivalent if $[p] = [q]$ in $K_0(A)$ [11]. Hence this can be proved in the same way as Proposition 3.4.

Actually this case is simpler. When we apply Lemma 3.4 in the proof of the above proposition, we impose the extra condition indicated just after the lemma, and we obtain the self-adjoint h which may end up with

$$h(1) = 2\pi k p + 2\pi(k+1) q \,,$$

where $p + q = 1$ and

$$k[p] + (k+1)[q] = 0 \,.$$

We have to estimate the length of a path which connects $w : w(t) = e^{ith(1)}$ to 1. Suppose $k > 0$. Then since $[p] = (k+1)[1]$, we find $k+1$ non-zero subprojections $e_1, \dots, e_{k+1}$ of p such that $e_1 + \cdots e_{k+1} = p$ and $[e_i] = 1$. Then w and the unitary y defined by $y(t) = e^{2\pi(k+1)t(1-e_1)}$ can be connected by a path of length $k\pi$. Since $[1 - e_1] = 0$, y can be connected to 1 by a path of length $(k+2)\pi$. Thus the estimate is $(2k+2)\pi = 2\|h(1)\|$. The other case can be treated in a similar way.

4. Real Rank Zero C^*-Algebras

Theorem 4.1. *Let A be a unital separable simple C^*-algebra of real rank zero and let α be a one-parameter automorphism group of A. Suppose that α has the Rohlin property and that A has a weakly unperforated ordered group with the Riesz decomposition property as K_0 and satisfies the cancellation property, or A is purely infinite. Then for any $\varepsilon > 0$, there is a decreasing sequence $\{e_n\}$ of projections in A and an α-cocycle u in A such that*

$$\mathrm{Ad}u(t) \circ \alpha_t(e_n) = e_n ,$$

$$\|u(t) - 1\| < \varepsilon, \quad t \in [0, 1] ,$$

and the limit of e_n in the second dual A^{**} is a minimal projection.

Proof. Let $\{x_n\}$ be a dense sequence in the unit ball of the self-adjoint part of A. Let $\varepsilon > 0$, $e_0 = 1$, $\alpha^{(0)} = \alpha$, and $u_0(t) = 1$ for all $t \in \mathbf{R}$. We shall construct a decreasing sequence $\{e_n\}$ of non-zero projections in A and a sequence $\{u_n\}$ with u_n an $\alpha^{(n-1)}$-cocycle such that

$$u_n(1 - e_{n-1}) = 1 - e_{n-1} ,$$

$$\|u_n(t) - 1\| < 2^{-n}\varepsilon, \quad t \in [0, 1] ,$$

$$\alpha^{(n)} = \mathrm{Ad}u_n(t) \circ \alpha_t^{(n-1)} ,$$

$$\alpha_t^{(n-1)}(e_{n-1}) = e_{n-1} ,$$

$$D_{e_n}(e_n x_m e_n) < 1/n, \quad m = 1, 2, \ldots, n ,$$

where if e is a projection in A and h is a self-adjoint element of eAe, $D_e(h)$ denotes $\max \sigma_e(h) - \min \sigma_e(h)$ with $\sigma_e(h)$ the spectrum of h in eAe. If this is done let

$$u(t) = \lim_{n \to \infty} u_n(t) \cdots u_1(t) ,$$

which exists for all $t \in \mathbf{R}$, is an α-cocycle, and satisfies

$$\|u(t) - 1\| < \varepsilon, \quad t \in [0, 1] .$$

Note also that $\mathrm{Ad}u(t) \circ \alpha_t(e_n) = e_n$ for all n. Let ϕ be a state of A such that $\phi(e_n) = 1$ for all n. Then by the condition that $D_{e_n}(e_n x e_n) \to 0$ as $n \to 0$, ϕ is uniquely determined. Thus ϕ is a pure state, which proves that $\lim e_n$ is a minimal projection in A^{**}.

Suppose that we have constructed e_k's and u_k's up to $k = n - 1$. Let $\beta = \alpha^{(n-1)}$ and let $t_0 > 0$ be such that for $t \in [0, t_0]$,

$$\|\beta_t(x_m) - x_m\| < 1/2n$$

for $m = 1, 2, \ldots, n$. Let $N \in \mathbf{N}$ and $\delta > 0$. Then by Lemma 4.2 below we find a non-zero projection p in $e_{n-1}Ae_{n-1}$ such that $D_p(p\beta_{-lt_0}(x_m)p) < \delta$ for $m = 1, \ldots, n$ and $l = 0, 1, \ldots, N$. Noting that all non-zero powers of β_{t_0} are outer and applying Lemma 4.3, we may further assume, by replacing p by a smaller projection, that if $k, l = 0, 1, \ldots, N$, and $k \neq l$,

$$\|\beta_{kt_0}(p) x_m \beta_{lt_0}(p)\| < \delta$$

for $m = 0, \dots, n$ where $x_0 = 1$. By Lemma 4.4 below we find an orthogonal family $\{q_0, q_1, \dots, q_N\}$ of projections and a unitary v such that

$$\begin{gathered} q_0 = p , \\ v(1 - e_{n-1}) = 1 - e_{n-1} , \\ \mathrm{Ad}v \circ \beta_{t_0}(q_{k-1}) = q_k, \quad k = 1, \dots, N , \\ \|\beta_{kt_0}(p) - q_k\| < \varepsilon(\delta, N), \quad k = 0, \dots, N , \\ \|v - 1\| < \varepsilon(\delta, N) , \end{gathered}$$

where $\varepsilon(\delta, N) \to 0$ as $\delta \to 0$. Let $\{q_{ij}\}$ be matrix units such that

$$\begin{aligned} q_{ii} &= q_i , \\ q_{ij} &= \mathrm{Ad}v \circ \beta_{t_0}(q_{i-1,j-1}), \quad i,j = 1, \dots, N . \end{aligned}$$

Let

$$E = \frac{1}{N+1} \sum_{i,j=0}^{N} q_{ij} .$$

Then E is a projection in $e_{n-1}Ae_{n-1}$ and satisfies that

$$\|E - \beta_{t_0}(E)\| < 2/\sqrt{N} + 2\varepsilon(\delta, N) \equiv \varepsilon_1 ,$$

and that for $\lambda_k \in \sigma_{q_k}(q_k x_m q_k)$,

$$\begin{aligned} &\left\| E x_m E - \left(\frac{1}{N+1} \sum_{l=0}^{N} \lambda_l \right) E \right\| \\ &\leqq \left\| \frac{1}{(N+1)^2} \sum_{l \neq i} q_{kl} x_m q_{ij} \right\| + \left\| \frac{1}{(N+1)^2} \sum q_{kl} x_m q_{lj} - \left(\frac{1}{N+1} \sum_{l=0}^{N} \lambda_l \right) E \right\| \\ &< (N+1)^2 \delta' + \left\| \frac{1}{(N+1)^2} \sum q_{kl}(x_m - \lambda_l) q_{lj} \right\| \\ &\leqq (N+1)^2 \delta' + \sum_{l=0}^{N} D_{q_l}(q_l x_m q_l) , \end{aligned}$$

where $\delta' = \delta + 2\varepsilon(\delta, N)$ and

$$\begin{aligned} D_{q_l}(q_l x_m q_l) &\leqq D_p(p \beta_{-lt_0}(x_m) p) + 3\varepsilon(\delta, N) \\ &\leqq \delta + 3\varepsilon(\delta, N) . \end{aligned}$$

Thus we have that

$$D_E(E x_m E) \leqq (N+1)(N+2)(\delta + 3\varepsilon(\delta, N)) \equiv \varepsilon_2 .$$

Both ε_1 and ε_2 defined above can be made arbitrarily small by making N sufficiently large and then $\delta > 0$ sufficiently small.

By Lemma 4.5 there is a β-cocycle u such that

$$\mathrm{Ad}u_t \circ \beta_t(p) = p .$$

Let $v(0) = 1$ and let for $k = 1, 2, \ldots,$

$$v(k) = v\beta_{t_0}(v) \cdots \beta_{t_0}^{k-1}(v)$$

and define

$$w_t = \sum_{j=0}^{N} \beta_t(v(j))\beta_{jt_0}(u_t^* p)v(j)^* .$$

Since $\beta_t(v(j))\beta_{jt_0}(u_t^* p)v(j)^*$ is a partial isometry with initial projection q_j and final projection $\beta_t(q_j)$, w_t is also a partial isometry. Note that

$$w_t E w_t^* = \frac{1}{N+1} \sum \beta_t(v(i))\beta_{it_0}(u_t^* p)v(i)^* q_{ij} v(j)\beta_{jt_0}(u_t^* p)\beta_t(v(j)^*) .$$

We assume that

$$q_{10} = v u_{t_0}^* p .$$

Then

$$\begin{aligned} q_{i0} &= \mathrm{Ad}v(i-1) \circ \beta_{(i-1)t_0}(q_{10}) \mathrm{Ad}v(i-2) \circ \beta_{(i-2)t_0}(q_{10}) \cdots q_{10} \\ &= v(i-1)\beta_{(i-1)t_0}(q_{10})\beta_{(i-2)t_0}(v^* q_{10}) \cdots \beta_{t_0}(v^* q_{10}) v^* q_{10} \\ &= v(i) u_{it_0}^* p , \end{aligned}$$

and so

$$q_{ij} = v(i) u_{it_0}^* p u_{jt_0} v(j)^* .$$

By using this we obtain that

$$\begin{aligned} w_t E w_t^* &= \frac{1}{N+1} \sum \beta_t(v(i))\beta_{it_0}(u_t^* p) u_{it_0}^* p u_{jt_0} \beta_{jt_0}(u_t^* p)^* \beta_t(v(j)^*) \\ &= \frac{1}{N+1} \sum \beta_t(v(i)) u_{it_0+t}^* p u_{jt_0+t} \beta_t(v(j)^*) \\ &= \frac{1}{N+1} \sum \beta_t(v(i))\beta_t(u_{it_0}^*)\beta_t(p)\beta_t(u_{jt_0})\beta_t(v(j)^*) \\ &= \beta_t(E) . \end{aligned}$$

Note that

$$\begin{aligned} w_{t_0} E &= \frac{1}{N+1} \sum_{i,j=0}^{N} \beta_{t_0}(v(i))\beta_{it_0}(u_{t_0}^* p) u_{it_0}^* p u_{jt_0} v(j)^* \\ &= \frac{1}{N+1} \sum v^* v(i+1) u_{(i+1)t_0}^* p u_{jt_0} v(j)^* , \end{aligned}$$

and so

$$v w_{t_0} E - E = \frac{1}{N+1} \sum_{j=0}^{N} v(N+1) u_{(N+1)t_0}^* p u_{jt_0} v(j)^* - \frac{1}{N+1} \sum_{j=0}^{A} p u_{jt_0} v(j)^* .$$

Hence

$$\| v w_{t_0} E - E \| \leqq \frac{2}{\sqrt{N+1}} ,$$

and thus $Ew_{t_0}E$ is invertible in EAE and close to E. By Lemma 4.6 below there is a β-cocycle u in A such that

$$u(t)(1-e_{n-1}) = 1-e_{n-1}\,,$$

$$\mathrm{Ad}u(t)\circ\beta_t(E) = E\,,$$

$$\|u(t_0)-1\| < \gamma(\varepsilon_1)\,,$$

where $\gamma(\varepsilon)\to 0$ as $\varepsilon\to 0$ as in the lemma.

Let $x(t)$, $t\in[0,t_0]$ be a path of unitaries in A such that

$$x(0)=1,\ x(t_0)=u_{t_0}\,,$$

$$x(t)(1-e_{n-1}) = 1-e_{n-1}\,,$$

$$\|x(s)-x(t)\| < \frac{\gamma(\varepsilon_1)}{t_0}|s-t|,\ \ s,t\in[0,t_0]\,.$$

Define a unitary $W\in A\otimes C(\mathbf{T})$ by

$$W(s) = u(t_0s)\beta_{t_0(s-1)}(x(t_0s)^*)\,.$$

Then

$$\begin{aligned}(W^*(E\otimes 1)W)(s) &= W^*(s)EW(s)\\ &= \beta_{t_0(s-1)}(x(t_0s))u(t_0s)^*Eu(t_0s)\beta_{t_0(s-1)}(x(t_0s)^*)\\ &= \beta_{t_0(s-1)}(x(t_0s))\beta_{t_0s}(E)\beta_{t_0(s-1)}(x(t_0s)^*)\,,\end{aligned}$$

and for $\lambda\in\sigma_E(Ex_mE)$

$$\begin{aligned}&\|W^*(E\otimes 1)W(x_m\otimes 1)W^*(E\otimes 1)W-\lambda W^*(E\otimes 1)W\|\\ &= \max_{s\in[0,1]}\|\beta_{t_0s}(E)\beta_{t_0(s-1)}(x(t_0s)^*)x_m\beta_{t_0(s-1)}(x(t_0s))\beta_{t_0s}(E)-\lambda\beta_{t_0s}(E)\|\\ &\leqq 2\gamma(\varepsilon_1)+1/2n+\varepsilon_2\,.\end{aligned}$$

As in the proof of Theorem 2.1,

$$W\beta_{t_0t}\otimes\gamma_t(W^*)(s)\approx u(t_0t)$$

with error up to $\gamma(\varepsilon_1)t$. This is true for $t\in[0,1]$; then it follows that for all $t\in\mathbf{R}\backslash\{0\}$,

$$\|W\beta_t\otimes\gamma_{t/t_0}(W^*)-u(t)\otimes 1\| < \frac{\gamma(\varepsilon_1)}{t_0}|t|\,.$$

Now we use an approximate homomorphism $\{\phi_m\}$ of $A\odot C(\mathbf{T})$ into A as defined in Lemma 2.3 such that $\phi_m\circ\beta_{t_0t}\otimes\gamma_t\approx\beta_{t_0t}\circ\phi_m$ and we let w be a unitary obtained from a suitable image of W in A so that the following conditions are satisfied: for $e_n = wEw^*$,

$$\|\beta_t(e_n)-e_n\| < \frac{\gamma(\varepsilon_1)}{t_0}(|t|+1)\,,$$

$$D_{e_n}(e_nx_me_n) < 1/n,\quad m=1,\dots,n\,.$$

By Lemma 4.7 we then find a β-cocycle u_n such that

$$u_n(t)(1-e_{n-1}) = 1-e_{n-1}\ ,$$

$$\mathrm{Ad}u_n(t)\circ\beta_t(e_n) = e_n\ ,$$

$$u_n(t)\approx 1\ .$$

This concludes the proof.

Lemma 4.2. *Let $\{x_1,\ldots,x_n\}$ be a finite sequence of self-adjoint elements in A. For any $\varepsilon>0$ there is a non-zero projection p in A such that $D_p(px_mp)<\varepsilon$, $m=1,\ldots,n$.*

Proof. Here we use the assumption that A has real rank zero (cf. [7]). If $D_1(x_1)=\max\sigma_1(x_1)-\min\sigma_1(x_1)\geqq\varepsilon$, let $\lambda\in\sigma_1(x_1)$, f a non-zero non-negative continuous function on $\mathbf{R}$ such that $\mathrm{supp}\, f\subset(\lambda-\varepsilon/2,\lambda+\varepsilon/2)$, and let p_1 be a non-zero projection in the hereditary C^*-subalgebra generated by $f(x_1)$. Then $D_{p_1}(p_1x_1p_1)<e$. If $\sigma_{p_1}(p_1x_2p_1)\geqq\varepsilon$, we repeat this procedure for $p_1x_2p_1\in p_1Ap_1$ in place of $x_1\in A$ to obtain a non-zero subprojection p_2 of p_1 such that $D_{p_2}(p_2x_2p_2)<\varepsilon$. Note that $D_{p_2}(p_2x_1p_2)\leqq D_{p_1}(p_1x_1p_1)<\varepsilon$. We repeat this $n-2$ more times to obtain a projection $p=p_n$.

Lemma 4.3. *Let β be an automorphism of A such that β^k is outer for any $k=1,\ldots,N$, and let $\{x_1,\ldots,x_n\}$ be a finite sequence in A. For any non-zero projection e in A and $\varepsilon>0$ there exists a non-zero subprojection p of e such that for $k\neq l$ in $\{0,1,\ldots,N\}$ and $m=1,\ldots,n$,*

$$\|\beta^k(p)x_m\beta^l(p)\|<\varepsilon\ .$$

Proof. This follows from [15]. By using the assumption that A has real rank zero, we can take for p a projection instead of a positive element of norm one.

Lemma 4.4. *For each $N\in\mathbf{N}$ and sufficiently small $\delta>0$ there exists an $\varepsilon(\delta,N)>0$ such that $\lim_{\delta\to 0}\varepsilon(\delta,N)=0$ and the following conditions are satisfied: For any non-zero projection p in A with*

$$\|p\beta^k(p)\|<\delta,\quad k=1,2,\ldots,N\ ,$$

where β is an automorphism of A, there is an orthogonal family $\{q_0,q_1,\ldots,q_N\}$ of projections in A and a unitary $v\in A$ such that

$$q_0=p\ ,$$

$$\mathrm{Ad}v\circ\beta(q_{k-1})=q_k\ ,$$

$$\|\beta^k(p)-q_k\|<\varepsilon(\delta,N)\ ,$$

$$\|v-1\|<\varepsilon(\delta,N)$$

for $k=1,\ldots,N$.

Proof. This is standard. After construction $q_0,q_1,\ldots,q_{k-1}$, we construct q_k, by functional calculus, from

$$(1-q_0-\cdots-q_{k-1})\beta^k(p)(1-q_0-\cdots-q_{k-1})\ ,$$

which is close to $\beta^k(p)$. If $q_0,\dots,q_N$ are obtained, the unitary v is obtained, by polar decomposition, from

$$\sum_{k=0}^{N} q_k\beta^k(p) + \left(1 - \sum_{k=0}^{N} q_k\right)\left(1 - \sum_{k=0}^{N}\beta^k(p)\right) .$$

Lemma 4.5. *For any projection E in A there is an α-cocycle u in A such that*

$$\mathrm{Ad}u(t)\circ\alpha_t(E) = E .$$

Proof. Let E' be a projection in A such that $\|E - E'\| < 1/2$ and E' is in the domain of the generator δ of α. Let W be the unitary obtained by the polar decomposition of

$$EE' + (1-E)(1-E') .$$

Then $WE'W = E$. Let $h = [\delta(E'),E'] = \delta(E')E' - E'\delta(E')$. Then $[h,E'] = \delta(E')$ and so $(\delta - \delta_h)(E') = 0$, where $\delta_h(x) = [h,x]$. Define an α-cocycle v by

$$\frac{d}{dt}v(t) = -v(t)\alpha_t(h), \quad v(0) = 1 .$$

Then it follows that $\mathrm{Ad}v(t)\circ\alpha_t(E') = E'$. Let $u(t) = Wv(t)\alpha_t(W^*)$. Then u is the desired α-cocycle.

Lemma 4.6. *For each small $\varepsilon > 0$ there exists a $\gamma(\varepsilon) > 0$ such that* $\lim_{\varepsilon\to 0}\gamma(\varepsilon) = 0$ *and the following conditions are satisfied: For each projection E in A and an α-cocycle u in A with*

$$\|\alpha_{t_0}(E) - E\| < \varepsilon \quad \textit{for some } t_0 > 0 ,$$

$$\mathrm{Ad}u(t)\circ\alpha_t(E) = E ,$$

$$Eu(t_0)E \textit{ is connected to } 1 \textit{ in the invertible elements of } EAE ,$$

there is an α-cocycle v in A such that

$$\mathrm{Ad}v(t)\circ\alpha_t(E) = E ,$$

$$\|v(t_0) - 1\| < \gamma(\varepsilon) .$$

Proof. There exists a unitary W such that $W \approx 1$ and $\mathrm{Ad}W\circ\alpha_{t_0}(E) = E$. Since $[u(t_0)W^*E] = 0$ in $K_1(EAE)$ and $\mathrm{Ad}W\circ\alpha_{t_0}|EAE$ has the stability by 3.2, there exists, for any $\varepsilon' > 0$, a unitary V_1 in EAE such that

$$\|u(t_0)W^*E - V_1^*\mathrm{Ad}W\circ\alpha_{t_0}(V_1)\| < \varepsilon' .$$

In the same way there is a unitary V_2 in $(1-E)A(1-E)$ such that

$$\|u(t_0)W^*(1-E) - V_2^*\mathrm{Ad}W\circ\alpha_{t_0}(V_2)\| < \varepsilon' .$$

Let

$$v(t) = (V_1 + V_2)u(t)\alpha_t(V_1 + V_2)^* .$$

Then v is an α-cocycle and

$$\mathrm{Ad}v(t)\circ\alpha_t(E) = E ,$$

$$\|v(t_0) - 1\| < \varepsilon' + 3\|W - 1\| .$$

Lemma 4.7. *Let $\varepsilon > 0$ be sufficiently small and e a non-zero projection in A such that $\|\alpha_t(e) - e\| < \varepsilon(|t| + 1)$. Then there exists an α-cocycle u such that*

$$\mathrm{Ad}u(t) \circ \alpha_t(e) = e\,,$$
$$\|u(t) - 1\| < \delta(\varepsilon), \quad t \in [0, 1]\,,$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. Let f be a non-negative C^∞ function on $\mathbf{R}$ with compact support such that $\int f(t)dt = 1$. Let

$$C_1 = \int |t| f(t)dt, \qquad C_2 = \int |f'(t)|dt\,.$$

Then for $f_\gamma(t) = f(\gamma t)\gamma$ with $\gamma > 0$,

$$\int f_\gamma(t)dt = 1, \qquad \int |t| f_\gamma(t)dt = C_1/\gamma, \qquad \int |f_\gamma'(t)|dt = \gamma C_2\,.$$

Let h be a C^∞-function on $\mathbf{R}$ with compact support such that $h(t) = 0$ on $(-\infty, 1/2 - 1/2^{3/2}]$ and $h(t) = 1$ on $[1/2 + 1/2^{3/2}, 1]$. Let $\gamma > 0$ be such that

$$\varepsilon(C_1/\gamma + 1) = \sqrt{\varepsilon} \quad \text{or} \quad \gamma = \frac{C_1\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}}\,.$$

Let

$$x = \int f_\gamma(t)\alpha_t(e)dt\,.$$

Then $\|x - e\| < \varepsilon(C_1/\gamma + 1) = \sqrt{\varepsilon}$, $\|x\| \leqq 1$, and so $\|x^2 - x\| \leqq 2\|x - e\| < 2\sqrt{\varepsilon}$. If $2\sqrt{\varepsilon} \leqq 1/8$, then $\mathrm{Sp}(x) \subset [0, 1/2 - 1/2^{3/2}] \cup [1/2 + 1/2^{3/2}, 1]$ and $p = h(x)$ is a projection such that

$$\|p - x\| \leqq (1 - \sqrt{1 - 8\sqrt{\varepsilon}})/2 \leqq 4\sqrt{\varepsilon}\,.$$

Let δ be the generator of α, and let

$$\hat{h}(p) = \frac{1}{2\pi}\int h(t)e^{-ipt}dt\,.$$

Since

$$h(x) = \int \hat{h}(p)e^{ipx}dp\,,$$

it follows that

$$\|\delta(p)\| \leqq C_3\|\delta(x)\| \leqq C_2C_3\gamma = \frac{C_1C_2C_3\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}}\,,$$

where

$$C_3 = \int |\hat{h}(p)p|dp\,.$$

(See [6, 27] for details.) Since $\|p - e\| < 5\sqrt{\varepsilon}$, there is a unitary w such that $wpw^* = e$ and $\|w - 1\| < 30\sqrt{\varepsilon}$ (if $10\sqrt{\varepsilon} < 1/2$). By the proof of 4.5 there is an α-cocycle v such that

$$\|v(t) - 1\| \leqq 2\|\delta(p)\|\,|t|\,,$$
$$\mathrm{Ad}v(t) \circ \alpha_t(p) = p\,.$$

Then $u(t) = wv(t)\alpha_t(w^*)$ is the desired α-cocycle.

We present examples concerning the conclusion of Theorem 4.1; there is a non-unital purely infinite simple C^*-algebra B and a periodic one-parameter automorphism group β of B with period 1 such that if e is a non-zero projection in B then $\|\Delta(e)\| \geqq 1$, where Δ is the generator of β, and that there is a unital purely infinite simple C^*-algebra and a one-parameter automorphism group α with α_1 inner such that the conclusion of Theorem 4.1 does not hold. Note that for this periodic β there are many invariant pure states [16].

Let O_n be the Cuntz algebra generated by n isometries $S_1,\dots,S_n$ [10] and let α be a one-parameter automorphism group of O_n such that

$$\alpha_t(S_i) = e^{ip_it}S_i\ .$$

If $\{p_1,\dots,p_n,-p_i\}$ generates $\mathbf{R}$ as a closed subsemigroup for any $i=1,\dots,n$, then the crossed product $O_n\times_\alpha\mathbf{R}$ is simple [14] and furthermore if p_i's are all positive (or all negative), $O_n\times_\alpha\mathbf{R}$ has no non-zero projections [21]. (Since O_n has a KMS state for α in this case, α does not have the Rohlin property. We could not decide whether α has the Rohlin property in the other cases.) Let A be the crossed product of O_n by $\alpha_n, n\in\mathbf{Z}$. Then A is a unital purely infinite simple C^*-algebra [22]. We extend the action α to an action $\bar\alpha$ on A in a natural way; then $\bar\alpha_1$ is inner, i.e., $\bar\alpha_1=\mathrm{Ad}U$, where U is the canonical unitary in A which implements α_1 on O_n. Let f be a continuous function on $\mathbf{T}$ with $\operatorname{supp} f$ contained in a small neighbourhood of 0, and let B be the hereditary C^*-subalgebra of A generated by $f(U)$. Then $\bar\alpha_1|B$ is close to the identity with $\delta=\log(\bar\alpha_1|B)$ well-defined as a *-derivation ([25], 8.7.7). Let Δ be the generator of $\bar\alpha|B$ and let β be the one-parameter automorphism group of B generated by $\Delta_1=\Delta-\delta$. Then, since δ commutes with α it follows that $\beta_1=id$. Since $A\times_{\bar\alpha}\mathbf{R}\cong O_n\times_\alpha\mathbf{R}\otimes C(\mathbf{T})$ and $B\times_{\bar\alpha}\mathbf{R}\cong B\times_\beta\mathbf{R}$ is a continuous field over $\mathbf{T}$ to the C^*-algebra $B\times_\beta\mathbf{T}$, it follows that $B\times_\beta\mathbf{T}$ is isomorphic to a hereditary C^*-subalgebra of $O_n\times\mathbf{R}$. Hence it also follows that the fixed point algebra B^β is isomorphic to a hereditary C^*-subalgebra of $O_n\times\mathbf{R}$. This implies that for any non-zero projection e in the domain of the generator $\Delta_1=\Delta-\delta$,

$$\|\Delta_1(e)\| \geqq 1\ .$$

Because if $\|\Delta_1(e)\|<1$, then let $h=[\Delta_1(e),e]$ and let $\Delta_2=\Delta_1-\delta_h$ with $\delta_h(x)=[h,x]$. Then for $t>0$,

$$\|e^{t\Delta_2}-e^{t\Delta_1}\| = \left\|\int_0^1 \frac{d}{ds}e^{st\Delta_2}e^{(1-s)t\Delta_1}ds\right\| \leqq 2\|h\|t\ .$$

Since $\|h\|<1$ it follows that $\|e^{\Delta_2}-id\|<2$. Then $\delta_1=\log(e^{\Delta_2})$ can be defined as a *-derivation. Then for $\Delta_3=\Delta_2-\delta_1=\Delta_1-\delta_h-\delta_1$, we obtain that $\Delta_3(e)=e$ and that

$$e^{\Delta_3}=e^{\Delta_2}e^{-\delta_1}=id\ .$$

Since $B\times_{\beta_3}\mathbf{T}\cong B\times_\beta\mathbf{T}$ with $\beta_{3t}=e^{t\Delta_3}, B^{\beta_3}$ is isomorphic to a hereditary C^*-subalgebra of $O_n\times\mathbf{R}$. Since B^{β_3} has the non-zero projection e, this is a contradiction.

We assert that the conclusion of Theorem 4.1 does not hold for $(A,\bar\alpha)$ above. Suppose that there exists a decreasing sequence $\{e_n\}$ of projections and an $\bar\alpha$-cocycle u such that

$$\mathrm{Ad}u(t)\circ\bar\alpha_t(e_n)=e_n\ ,$$

and the limit of $\{e_n\}$ is a minimal projection in A^{**}. Let ϕ be a pure state of A such that $\phi(e_n) = 1$ for all n. There is a one-parameter unitary group U implementing $\bar{\alpha}$ in the GNS representation π_ϕ associated with ϕ such that $u(t)U_t\xi = \xi$, where π_ϕ is omitted and ξ is the cyclic vector associated with ϕ. For any $\varepsilon > 0$ we can find an α-cocycle v in $B + \mathbf{C}1$ and a unit vector η in the range of B such that

$$\|v(t) - 1\| < \varepsilon, \quad t \in [0,1],$$
$$v(t)U_t\eta = e^{ipt}\eta$$

for some $p \in \mathbf{R}$ (cf. [18]). By Kadison's transitivity theorem there is an $x \in BA$ such that $x\xi = \eta$ and $\|x\| = 1$. Let f be a continuous non-negative function on $\mathbf{R}$ with integral 1 and let

$$x_f = \int e^{-ipt} f(t)v(t)\bar{\alpha}_t(x)u(t)^* dt .$$

Then $x_f\xi = \eta$ and $\|x_f\| = 1$. Thus we may suppose that

$$\|e^{ipt}v(t)\bar{\alpha}_t(x)u(t)^* - x\| < \varepsilon, \quad t \in [0,1].$$

We can now see that $\{xe_nx^*\}$ are approximately projections in B (i.e., there is a sequence $\{p_n\}$ of projections in B such that $\|xe_nx^* - p_n\| \to 0$) (cf. [22]) and that

$$\|\mathrm{Ad}v(t) \circ \bar{\alpha}_t(xe_nx^*) - xe_nx^*\| < 2\varepsilon, \quad t \in [0,1].$$

Hence $\|\bar{\alpha}_t(xe_nx^*) - xe_nx^*\| < 4\varepsilon$, $t \in [0,1]$. This implies that B contains an almost $\bar{\alpha}$-invariant projection (cf. 4.6), which is a contradiction if $\|\delta\|$ is sufficiently small.

Theorem 4.8. *Let A be a unital separable purely infinite simple C^*-algebra and let α be a one-parameter automorphism group of A. If α has the Rohlin property then the crossed product $A \times_\alpha \mathbf{R}$ is a purely infinite simple C^*-algebra.*

Proof. The simplicity is shown in Proposition 2.4.

Let e_1, e_2 be non-zero projections in A such that $e_1e_2 = 0$ and $[e_i] = [1]$. By 4.5 we obtain an α-cocycle u in A such that

$$\mathrm{Ad}u(t) \circ \alpha_t(e_i) = e_i .$$

By replacing α by $\mathrm{Ad}u(t) \circ \alpha_t$, we assume that $\alpha_t(e_i) = e_i$. Let U be an isometry in A such that $UU^* = e_1$. Let $p > 0$ and define an α-cocycle u by

$$u(t) = e^{-ipt}U^*\alpha_t(U) .$$

Then for any $\varepsilon > 0$ there exists a unitary V in A such that

$$\|u(t) - V\alpha_t(V^*)\| < \varepsilon, \quad t \in [0,1].$$

Thus for $U_1 = UV$,

$$\|\alpha_t(U_1) - e^{ipt}U_1\| < \varepsilon, \quad t \in [0,1].$$

In the same way we have an isometry U_2 such that $U_2U_2^* = e_2$ and

$$\|\alpha_t(U_2) - e^{-ipt}U_2\| < \varepsilon, \quad t \in [0,1].$$

Let h_1, h_2 be C^∞-functions on $\mathbf{R}$ such that

$$0 \leqq h_1 \leqq 1, \quad 0 \leqq h_2 \leqq 1,$$
$$\mathrm{supp}\, h_1 \subset [-5p/2, p/2], \quad \mathrm{supp}\, h_2 \subset [-p/2, 5p/2],$$
$$h_1^2 + h_2^2 = 1 \quad \text{on } [-2p, 2p].$$

Let, as in the proof of Lemma 6 in [22],

$$x = U_1 h_1(H) + U_2 h_2(H)\,,$$

where H is the generator of the canonical unitary group λ_t in the multiplier algebra of $A \times \mathbf{R}$, and

$$h(H) = \int \hat{h}(t)\lambda_t dt\,,$$

$$\hat{h}(t) = \frac{1}{2\pi}\int e^{-itq}h(q)dq\,.$$

Then

$$xx^* = h_1^2(H) + h_2^2(H)\,,$$

$$xx^* \approx h_1^2(H-p)e_1 + h_2^2(H+p)e_2 + h_1h_2(H-p)V_1V_2^* + h_1h_2(H+p)V_2V_1^*\,.$$

If f is a continuous function on $\mathbf{R}$ such that $f(t) = 1$ on $[-3p/2, 3p/2]$ and $\operatorname{supp}(f) \subset [-2p, 2p]$, then for $y = f(x)$, it follows that $x^*xy = y$ and $\|yxx^* - xx^*\|$ is small depending on ε. If $\varepsilon > 0$ is sufficiently small then $\|yf_{1/8}(|x^*|) - f_{1/8}(|x^*|)\| < 1/4$, where f_s is a continuous function on $\mathbf{R}$ such that $f_s(t) = 1$ for $t \geqq s$, $f_s(t) = 0$ for $t \leqq s/2$, and linear on $[s/2, s]$. Then it follows that $\|f_{1/2}(|x|)f_{1/8}(x^*) - f_{1/8}(|x^*|)\| < 1/4$, i.e., x is an approximate scaling element. By Lemma 4.2 of [1] $A \times \mathbf{R}$ contains an infinite projection. The rest of the proof goes exactly as the proof of Theorem 2 of [22] by using Theorems 4.1 and 2.1.

References

1. Blackadar, B., Cuntz, J.: The structure of stable algebraically simple C^*-algebras. Amer. J. Math. **104**, 813–822 (1982)
2. Bratteli, O., Elliott, G.A., Evans, D.E., Kishimoto, A.: Homotopy of a pair of approximately commuting unitaries in a simple unital C^*-algebra. Preprint
3. Bratteli, O., Evans, D.E., Kishimoto, A.: The Rohlin property for quasi-free automorphisms of the Fermion algebra. Proc. London Math. Soc. **71**, 675–694 (1995)
4. Bratteli, O., Evans, D.E., Kishimoto, A.: Almost shift invariant projections in infinite tensor products. In: Quantum and Non-Commutative Analysis. H. Araki et al. eds. Amsterdam: Kluwer, 1993, pp. 427–434
5. Bratteli, O., Kishimoto, A., Rørdam, M., Størmer, E.: The crossed product of a UHF algebra by a shift. Ergod. Th. & Dynam. Sys. **13**, 615–626 (1993)
6. Bratteli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics I. Berlin, Heidelberg, New York: Springer 1979
7. Brown, L., Pedersen, G.K.: C^*-algebras of real rank zero. J. Funct. Anal. **99**, 131–149 (1991)
8. Connes, A.: Outer conjugacy class of automorphisms of factors. Ann. Scient. Ec. Norm. Sup., 4^e serie, **8**, 383–420 (1975)
9. Connes, A., Higson, N.: Deformations, morphismes asymptotiques et K-theorie bivariante. C.R. Acad. Sci. Par. Ser, I Math. **310**, 101–106 (1990)
10. Cuntz, J.: Simple C^*-algebras generated by isometries. Commun. Math. Phys. **57**, 173–185 (1977)
11. Cuntz, J.: K-theory for certain C^*-algebras. Ann. Math. **113**, 181–197 (1981)
12. Herman, R.H., Ocneanu, A.: Stability for integer actions on UHF C^*-algebras. J. Funct. Anal. **59**, 132–144 (1984)
13. Herman, R.H., Ocneanu, A.: Spectral analysis for automorphisms of UHF C^*-algebras. J. Funct. Anal. **66**, 1–10 (1986)
14. Kishimoto, A.: Simple crossed products by locally compact abelian groups. Yokohama Math. J. **28**, 69–85 (1980)
15. Kishimoto, A.: Outer automorphisms and reduced crossed products of simple C^*-algebras. Commun. Math. Phys. **81**, 429–435 (1981)

16. Kishimoto, A.: Automorphisms and covariant irreducible representations. Yokohama Math. J. **31**, 159–168 (1983)
17. Kishimoto, A.: Type I orbits in the pure states of a C^*-dynamical system, I, II. Publ. RIMS, Kyoto Univ. **23**, 321–336, 517–526 (1987)
18. Kishimoto, A.: Outer automorphism subgroups of a compact abelian ergodic action. J. Operator Theory **20**, 59–67 (1988)
19. Kishimoto, A.: The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras. J. Funct. Anal. (to appear)
20. Kishimoto, A.: The Rohlin property for automorphisms of UHF algebras. J. reine angew. Math. **465**, 183–196 (1995)
21. Kishimoto, A., Kumjian, A.: Simple stably projectionless C^*-algebras arising as crossed products. Canadian J. Math. (to appear)
22. Kishimoto, A., Kumjian, A.: Crossed products of Cuntz algebras by quasi-free automorphisms. Proceedings of the Fields Institute (to appear)
23. Kishimoto, A., Robinson, D.W.: Dissipations, derivations, dynamical systems, and asymptotic abelianess. J. Operator Theory **13**, 155–187 (1985)
24. Lin, H.: Exponential rank of C^*-algebras with real rank zero and the Brown–Pedersen conjecture. J. Funct. Anal. **114**, 1–11 (1993)
25. Pedersen, G.K.: C^*-algebras and their automorphism groups. London, New York, San Fransisco: Academic Press, 1979
26. Rørdam, M.: Classification of certian infinite simple C^*-algebras. J. Funct. Anal. (to appear)
27. Sakai, S.: Operator algebras in dynamical systems. Cambridge: Cambridge Univ. Press, 1991
28. Zhang, S.: A property of purely infinite simple C^*-algebras. Proc. Am. Math. Soc. **109**, 717–720 (1990)

Communicated by H. Araki