

The Boltzmann–Grad Limit for a One-Dimensional Boltzmann Equation in a Stationary State

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Abstract: In this paper we consider a one-dimensional model of interacting particles in a bounded interval with (possibly not homogeneous) diffusive boundary conditions. We prove that, when the number of particles N goes to infinity and the interaction is suitably rescaled (the Boltzmann–Grad limit), the one-particle distribution function of the unique invariant measure for the particle system, converges to the unique solution of the Boltzmann equation of the model, provided that the mean free path is sufficiently large.

1. Introduction

One of the most important problems in nonequilibrium statistical mechanics is the analysis of stationary nonequilibrium states. For instance one can couple the system under consideration to thermal reservoirs which are maintained at different constant temperatures. After an initial transient time, the system is expected to approach a stationary state in which there is a steady flux of energy from the hot to the cold reservoir.

Such a state can be described at various levels: macroscopically by means of the hydrodynamical equations, mesoscopically, through the kinetic (Boltzmann) equations, microscopically according to the basic Liouville equation. The three levels of descriptions are related by various scaling limits some of which (see for instance [1] as regards the deduction of the stationary hydrodynamical solutions from the Boltzmann equation) have been successfully investigated. However very little is known about stationary nonequilibrium states in the microscopic description. Namely what we know about stationary solutions of the Liouville equation for realistic systems with a non-constant profile of temperature on the boundary is, at most, existence and uniqueness (see e.g. [2]), without any additional property which could allow to go further in analyzing the scaling limits. Even worse is our knowledge as regards the Boltzmann equation: we have only partial existence results (see [3–5]) for stationary nonequilibrium solutions. In this situation a rigorous analysis of the Boltzmann–Grad limit seems hopeless, that is a derivation of the stationary

Boltzmann equation from the stationary solutions of the Liouville equation under a suitable scaling. Therefore a careful analysis of simplified models is useful, in which some of the features of more realistic systems are maintained and for which the problem can more easily be settled.

In this paper we want to study a one-dimensional model of interacting particles proposed by Ianiro and Lebowitz in [6] for the study of the stationary solutions of the corresponding Boltzmann equation. We prove that, under suitable assumptions and for large values of the mean free path, the stationary solutions of the Liouville (Master) equation, converge in the Boltzmann–Grad limit, to the (unique) solution of the Boltzmann equation.

The basic ingredient of our proof is a cluster expansion technique used by the same authors of the present paper for the time dependent problem (see [7]) on the whole line, which allows us the control of the interaction, combined with a mixing property of the free (Knudsen) flow. The full use of this last property makes necessary a lower bound for the allowed velocities which is, probably, only a technical restriction.

The paper is organized as follows. In Sect. 2 we introduce the model. In Sect. 3 we establish some preliminary results and the technical setup necessary for the proof of the main result. Here we introduce the v -functions, a sort of truncated correlation functions (see [8, 9] and references quoted therein), which play an important role in the convergence proof presented in Sect. 4. Two technical Appendices conclude the paper.

Some final comments are in order. After Lanford’s result [10], it is well known that the Boltzmann equation can be rigorously derived, for short times, in terms of the classical particle dynamics (see also [11] for further comments and results). Even proving the validity of the Boltzmann equation for arbitrary times (but this seems much beyond our present knowledge) it is strongly doubtful whether such a validation could work uniformly in time, for general situations. As a consequence, even knowing the trend to a nonequilibrium stationary state for the Boltzmann dynamics (which is, incidentally, not known) we could not conclude anything about the approximation of the nonequilibrium stationary state in terms of the corresponding invariant state for the microscopic dynamics in the Boltzmann–Grad limit since we are not authorized to commute this limit with $t \rightarrow \infty$.

In this paper we approach the problem differently for the simplified model under consideration. We evaluate the distance between the invariant measure for the particle system and the stationary solution to the Boltzmann equation (which both are known to exist), in terms of the v -functions and find that the time invariance condition on a scale of time sufficiently large to take advantage of the mixing property of the free flow, and sufficiently small to control the interaction (for a sufficiently large mean free path), gives rise to a contraction property which allows us to achieve the proof.

To our knowledge this is the first result in this direction for a non-trivial model. Our assumptions allow us to work without knowing much about the microscopic invariant state and we believe that extensions including more realistic models are indeed very difficult. Our analysis is essentially perturbative, therefore it works for a large mean free path. One can ask whether the present result can be extended for arbitrary mean free paths. A preliminary problem to solve is then to prove (or disprove) the uniqueness of the stationary solution to the Boltzmann equation without any smallness assumption. This problem has not yet been solved even in our simplified model (see [6]).

2. The Model

Let us consider a system of N identical particles in the interval $[0, 1]$. Denote by $Z_N = (X_N, V_N) = \{x_1 v_1, \dots, x_N v_N\}$ a state of the system, where x_i and v_i are the position and the velocity of the i^{th} particle. $Z_N \in \Lambda_N = [0, 1]^N \times \mathcal{V}^N$, the phase space of the system, where $\mathcal{V} = [-2, -1] \cup [1, 2]$ is the set of all possible velocities. A collision between two particles is defined in the following way. Denoting by v_1, v_2 and by v'_1, v'_2 the ingoing and the outgoing velocities respectively, then

$$v'_1 = -v_2, \quad v'_2 = -v_1 \quad \text{if } \text{sgn}(v_1) \neq \text{sgn}(v_2), \quad (2.1.1)$$

$$v'_1 = v_1, \quad v'_2 = v_2 \quad \text{if } \text{sgn}(v_1) = \text{sgn}(v_2). \quad (2.1.2)$$

Notice that, due to the fact that the particles are identical, the above collision rule is equivalent to a specular reflection of each colliding particle. However we prefer a labelling for which the direction of the velocity does not change in the collision because this will simplify some technicalities later on. Moreover, according to (2.1), collisions between particles travelling in the same direction are not considered.

Notice also that the relative velocity is preserved during the collision:

$$|v_i - v_j| = |v'_i - v'_j|. \quad (2.2)$$

The dynamics of the system is stochastic. The particles move freely up to the first time in which two of them arrive at the same point. Then they collide (independently) with probability ε and go ahead with probability $1 - \varepsilon$. Moreover each particle is stochastically reflected by the boundary $\{0\} \cup \{1\}$ according to two given probability distributions, \tilde{H}_0 and \tilde{H}_1 , for the outgoing velocities from 0 and 1 respectively. Namely, if a particle hits 0 (obviously with negative velocity) it is reflected with positive velocity v distributed according to the distribution $\tilde{H}_0 : [1, 2] \rightarrow \mathbb{R}^+$ with $\int_1^2 dv \tilde{H}_0 = 1$. The negative velocity outgoing from a collision in 1 is distributed according to the distribution $\tilde{H}_1 : [-2, -1] \rightarrow \mathbb{R}^+$ with $\int_{-2}^{-1} dv \tilde{H}_1 = 1$.

The distributions \tilde{H}_0 and \tilde{H}_1 play the role of the Maxwellians of the walls and mimic two reservoirs at possibly different temperatures.

The stochastic process introduced above defines a semigroup P'_N in the following way. Let $\nu_0 = \nu_0(Z_N)$ be a symmetric probability density. Then $\nu(Z_N, t) = P'_N \nu_0(Z_N)$ is the time evolved measure defined by

$$\int \nu(Z_N, t) \varphi(Z_N) dZ_N = \int \nu_0(Z_N) \mathbb{E} \varphi(T^t Z_N) dZ_N, \quad (2.3)$$

where φ is any smooth test function, $T^t Z_N$ is the N -particle stochastic process starting almost surely from Z_N and \mathbb{E} denotes the expectation. It is easy to see that P'_N acts isometrically from $L_1(dZ_N)$ into itself. Formally ν satisfies

$$\left(\partial_t + \sum_{i=1}^N v_i \partial_i \right) \nu(Z_N, t) = \frac{1}{2} \varepsilon \sum_{i \neq j} \delta(x_i - x_j) \chi(i, j) |v_i - v_j| \{ \nu(Z'_N(i, j), t) - \nu(Z_N, t) \}, \quad (2.4)$$

where ∂_i is the derivative with respect to x_i , $\delta(\cdot)$ is the δ -function centered at zero. $\chi(i, j) = 1$ if $\text{sgn}(v_i) \neq \text{sgn}(v_j)$ and 0 otherwise,

$$Z'_N(i, j) = (x_1 v_1, \dots, x_i v'_i, \dots, x_j v'_j, \dots, x_N v_N)$$

if

$$Z_N = (x_1 v_1, \dots, x_i v_i, \dots, x_j v_j, \dots, x_N v_N).$$

Equation (2.4) must be complemented by the initial condition

$$v(Z_N, 0) = v_0(Z_N),$$

and the boundary conditions which we are going to make explicit. To this purpose it is convenient to introduce

$$\begin{aligned} \mathcal{F}_0^i &= \{Z_N \in A_N | x_i = 0\}, \\ \mathcal{F}_1^i &= \{Z_N \in A_N | x_i = 1\}. \end{aligned}$$

Then for $(X_N, V_N) \in \mathcal{F}_0^i$ with $v_i > 0$, $V_N = \{v_1, \dots, v_N\}$ we have

$$v(X_N, V_N, t) = -H_0(v_i) \int_{w_i < 0} w_i v(X_N, v_1 \cdots w_i \cdots v_N, t) dw_i. \quad (2.5)$$

For $(X_N, V_N) \in \mathcal{F}_1^i$ with $v_i < 0$ we have

$$v(X_N, V_N, t) = H_1(v_i) \int_{w_i > 0} w_i v(X_N, v_1 \cdots w_i \cdots v_N, t) dw_i. \quad (2.6)$$

The above boundary conditions are written in terms of H_0 and H_1 . They are related to the Maxwellians \tilde{H}_0 and \tilde{H}_1 by a constant, in such a way that

$$\int H_i(v) v dv = 1, \quad i = 1, 2,$$

that is $H_{0,1} = \tilde{H}_{0,1} / \int_{v \geq 0} v \tilde{H}_{0,1} dv$. In this way the conservation of the probability is guaranteed.

It is a consequence of general arguments from the theory of Markov processes (see [2]) that P_N^t has a unique invariant measure, denoted by μ^N , which is also the (formal) solution of the equation:

$$\sum_{i=1}^N v_i \partial_i \mu^N(Z_N) = \frac{1}{2} \varepsilon \sum_{i \neq j} \delta(x_i - x_j) \chi(i, j) |v_i - v_j| \{ \mu^N(Z_N'(i, j)) - \mu(Z_N) \}. \quad (2.7)$$

In this paper we are interested in the behavior of the stationary solution μ^N in the limit $N \rightarrow \infty, \varepsilon \rightarrow 0$ in such a way that the combination $\varepsilon N = \lambda$ is constant. We call this limit the Boltzmann–Grad limit. The expected result is that the measure μ^N is going to factorize in this limit (propagation of chaos) and that the one-particle distribution function approximates a solution of the stationary Boltzmann equation relative to the model under consideration. This is given by

$$v \partial_x g(x, v) = \lambda \int dv_1 |v - v_1| \{ g(x, v') g(x, v'_1) - g(x, v) g(x, v_1) \}, \quad (2.8)$$

where the integral is on the set $\text{sgn}(v_1) \neq \text{sgn}(v)$, with the boundary conditions

$$g(0, v) = -H_0(v) \int_{w < 0} w g(0, w) dw, \quad v > 0,$$

and

$$g(1, v) = H_1(v) \int_{w > 0} w g(1, w) dw, \quad v < 0.$$

We also fix the total mass equal to one:

$$\int g(x, v) dx dv = 1. \quad (2.9)$$

The solutions to the above Boltzmann equation have been investigated by Ianiro and Lebowitz in [6] by rather explicit computations. In particular they obtained existence of bounded solutions and uniqueness for λ small.

To give a formal idea of how to get the Boltzmann–Grad limit for this system, it is convenient to introduce the j -particle distribution functions for the measure μ^N and introduce a hierarchy of equations (the BBKGY hierarchy of the model) as for the time dependent case (see [7]). Then *assuming* the propagation of chaos (indeed this property must be proven), one can check that the formal limit holds. This heuristic derivation is standard and absolutely identical to the time dependent case so that we omit it here.

3. Definitions and Preliminaries

For a fixed N , we denote by $I_k = \{1, 2, \dots, k\}$, $k \leq N$, the set of the first k integers while I, J , etc. will stand for any subset of indices of I_N . We will use also the notation $Z_I = \{x_i, v_i\}_{i \in I}$ and $|I|$ for the cardinality of I .

We denote by $K^t Z_I$ the Knudsen process, that is free dynamics of the group of particles with indices in I , starting almost surely from Z_I , with the diffusive boundary conditions described in the previous sections. In other words $K^t Z_I$ is the same as $T^t Z_I$ for $\varepsilon = 0$.

Associated to K^t we also introduce \bar{P}_I^t the (Knudsen) semigroup defined by

$$\langle \bar{P}_I^t f, \varphi \rangle = \int f(Z_I) \mathbb{E}(\varphi(K^t Z_I)) dZ_I \tag{3.1}$$

for $f \in L_1$ and $\varphi \in L_\infty$. It is easy to see that \bar{P}_I^t acts isometrically from $L_1(dZ_I)$ into itself.

We now establish an important asymptotic property of the free flow K^t which will be crucial in what follows.

Proposition 3.1. *For all $\eta > 0$ there exists $t_0 > 0$, such that for all $t > t_0$ and all $u \in L_1(dZ_I)$ such that*

$$\int u(Z_I) dz_i = 0, \quad i \in I, \tag{3.2}$$

then

$$\|\bar{P}_I^t u\|_1 \leq \eta^j \|u\|_1, \tag{3.3}$$

where $j = |I|$ and $\|\cdot\|_1$ denotes the L_1 norm.

Proposition 3.1 follows by a careful analysis of the Knudsen flow carried out by Arkeryd et al. in [12]. Notice that, in proving (3.3) we actually need the condition $|v| \geq c > 0$. For further comments, see Appendix A where the proof of Proposition 3.1 is sketched.

We now discuss the second preliminary result which we need later on. Let g_j denote the j -fold product of solutions of the stationary Boltzmann equation (2.8), i.e.

$$g_j(x_1 v_1 \cdots x_j v_j) = \prod_{i=1}^j g(x_i, v_i). \tag{3.4}$$

Denote by $g_j^N(t)$ the j -particle distribution function of the symmetric measure $P_N^t g_N$, that is

$$g_j^N(x_1 v_1 \cdots x_j v_j; t) = \int dx_{j+1} dv_{j+1} \cdots dx_N dv_N P_N^t g_N(x_1 v_1 \cdots x_N v_N). \tag{3.5}$$

Setting

$$r_j^N(t) = g_j^N(t) - g_j, \quad (3.6)$$

we have:

Proposition 3.2. *There exist positive constants A, B and D such that if*

$$\lambda \|g\|_\infty \tau \leq A e^{-B\tau}, \quad (3.7)$$

then

$$\sup_{0 \leq t \leq \tau} \|r_j^N(t)\|_1 \leq \frac{e^{Dj}}{N}. \quad (3.8)$$

The proof is based on a Lanford type argument (see [10, 11]) and will be proven in Appendix B.

Notice that the stationary solution g found in [6] obviously depends on the parameter λ . However from [6] we also argue that $\|g\|_\infty$ is uniformly bounded for λ in a neighbourhood of the origin so that (3.7) is certainly satisfied for λ small.

We now introduce the important notion of cluster which will play an important role in the sequel. With this scope, as in [7], we find it convenient to give a partially explicit representation of the stochastic process T^t in a suitable sample space. Define

$$\Omega = \{0, 1\}^{\frac{rN(N-1)}{2}}, \quad (3.9)$$

where r is a positive integer. Then $\omega \in \Omega$, $\omega = \omega(i, j; \alpha) = \omega(j, i; \alpha)$, $i \neq j$, where $\alpha = 1 \dots r$, is a function defined on the set of all pairs of particles and all integers less than or equal to r , taking values 0 and 1. Define the stochastic process

$$T_\omega^t : \Lambda_N \rightarrow \Lambda_N; \quad t \leq \frac{r-1}{4} \quad (3.10)$$

in the following way. $T_\omega^t Z_N$ is the free motion with diffusive boundary conditions (that is the N -particle Knudsen flow) unless at time t two particles, say i and j , are at the same point with opposite velocities. Suppose that in the time interval $[0, t)$ they have met, with opposite velocities, $\alpha - 1$ times. Then if $\omega(i, j; \alpha) = 1$, the two particles collide according to the law (2.1), otherwise they keep their free motion, up to the next collision. In other words T_ω^t is deterministic as regards the collisions among the particles and stochastic as regards the collisions with the boundary. Notice that, due to the boundedness of the velocities, each pair of particles can meet at most r times up to the time $t \leq \frac{r-1}{4}$.

T_ω^t is almost everywhere defined with respect to the Lebesgue measure on Λ_N . Indeed it is not defined on the sets of all Z_N 's which deliver triple collisions and on the set of configurations for which, for some $i, j \in N$, it is $x_i = x_j$, $\text{sgn}(v_i) \neq \text{sgn}(v_j)$ and $\omega(i, j; 1) = 1$, being, in this last case, not possible to distinguish between pre- and post-collisional configurations. However these sets have measure zero so that we disregard them.

Notice also that T_ω^t is not a Markov process because to know whether a collision between two given particles takes place, we must know how many times the two particles have met in the past.

We define the probability of a single event $\omega \in \Omega$ by

$$p(\omega) = \prod_{i,j} \prod_{\alpha=1}^r \varepsilon^{\omega(i,j;\alpha)} (1 - \varepsilon)^{1 - \omega(i,j;\alpha)}, \quad (3.11)$$

where $\prod_{i,j}$ denotes the product on all pairs. $\{\Omega, p\}$ is a probability space and we can write:

$$\mathbb{E}(\varphi(T^t Z_N)) = \mathbb{E}_\Omega \mathbb{E}_B(\varphi(T'_\omega Z_N)), \quad (3.12)$$

where φ is a test function, \mathbb{E}_B denotes the expectation with respect to the distributions of the outgoing velocities after the collisions with the boundary and \mathbb{E}_Ω is the expectation with respect to $\{\Omega, p\}$.

Given $\omega \in \Omega$, we denote by $\text{ch}(i)$ (“chain” starting from i) any set $I \subset I_N/i$ of indices such that

$$\omega(i, i_1; \alpha_1) \omega(i_1, i_2; \alpha_2) \cdots \omega(i_{k-1}, i_k; \alpha_k) = 1$$

for some ordering i_1, \dots, i_k of the set I and some sequence $\alpha_1 \cdots \alpha_k$, $\alpha_i = 1 \cdots r$. The union of all chains starting from the particle i will be called “cluster of i ” and denoted by

$$\text{cl}(i) = \bigcup \text{ch}(i). \quad (3.13)$$

Notice that the index i is not included in $\text{cl}(i)$. Moreover we denote by

$$\text{cl}(I) = \bigcup_{i \in I} \text{cl}(i)/I, \quad (3.14)$$

the cluster of the set I . Notice that $\text{cl}(I)$ is the set of all particles which can influence the dynamics of the set I (excluded the set I itself).

Finally we define

$$n(I) = \sum_{i,j \in I} \sup_{1 \leq \alpha \leq r} \omega(i, j; \alpha), \quad (3.15)$$

that is the number of the active (interacting) pairs among the particles in I .

We conclude the notational part of this section by introducing the so-called v -functions.

We define (for $|I| = j$):

$$f^N(Z_I) = \int \mu^N(Z_I, Z_{I_N/I}) dZ_{I_N/I} \quad (3.16)$$

the j -particle distribution function of the symmetric measure μ^N , the unique invariant measure of the process T^t . We shall also use the equivalent notation $f^N(Z_I) \equiv f_j^N(Z_I) \equiv f_j^N(Z_I)$. Moreover, let

$$g(Z_I) = \prod_{i \in I} g(z_i), \quad (3.17)$$

be the product of stationary solutions to the Boltzmann equation (see (3.4)). We set

$$v^N(Z_I) = \sum_{J \subseteq I} (-1)^{|J|} g(Z_J) f^N(Z_{I/J}), \quad (3.18)$$

where $\sum_{J \subseteq I}$ means the sum over all subsets of I (including I), assuming also that

$$v^N(\emptyset) = g(\emptyset) = f^N(\emptyset) = 1. \quad (3.19)$$

Notice that $\sum_{J \subseteq I} (-1)^{|J|} = 0$ if I is not the empty set and 1 otherwise. Thus we have

$$f^N(Z_I) = \sum_{J \subseteq I} g(Z_J) v^N(Z_{I/J}). \quad (3.20)$$

If f^N would factorize

$$f^N(Z_I) = \prod_{i \in I} f(z_i), \quad (3.21)$$

then:

$$v^N(Z_I) = \prod_{i \in I} (f(z_i) - g(z_i)). \quad (3.22)$$

The v -functions defined in (3.18) give a measure of the deviation of f_j^N from g_j . Indeed if $v_j^N \rightarrow 0$ as $N \rightarrow \infty$, for all $j \geq 1$ (as we shall prove for $\lambda = \varepsilon N$ small enough), then also $f_j^N \rightarrow g_j$ for all $j = 1, 2, \dots$, as follows by (3.20) and (3.19). As we shall see later on, it seems more convenient to estimate the v -functions in place of the differences $f_j^N - g_j$ (roughly speaking the product of the differences rather than the differences of the products). Indeed by the property

$$\int v^N(Z_I) dz_i = 0 \quad i \in I. \quad (3.23)$$

(which follows by the definition) and by Proposition 3.1, we have that

$$\|\bar{P}_I^t v^N\|_1 \leq \eta^j \|v^N\|_1, \quad |I| = j, \quad (3.24)$$

while the same arguments of Proposition 3.1 give nothing better than

$$\|\bar{P}_I^t (f_j^N - g_j)\|_1 \leq \eta^j \|f_j^N - g_j\|_1. \quad (3.25)$$

Therefore $\|\bar{P}_I^t v^N\|_1$ decays, for fixed t and j large, much better than $\|\bar{P}_I^t (f_j^N - g_j)\|_1$, and this will be essential in the proof of the main theorem in the next section.

4. The Main Result

In this section we establish the main result of this paper, namely the convergence in the Boltzmann–Grad limit of the j -particle distribution functions associated to the stationary measure μ^N to the j -fold products of the stationary solution of the corresponding Boltzmann equation.

Theorem 4.1. *For $\lambda = \varepsilon N$ sufficiently small:*

$$\lim_{N \rightarrow \infty} \|v_j^N\|_1 = 0 \quad (4.1)$$

and, consequently:

$$\lim_{N \rightarrow \infty} \|f_j^N - g_j\|_1 = 0. \quad (4.2)$$

Proof. We use the shorthand notation:

$$(g * v^N)(Z_I) \quad (4.3)$$

for the right-hand side of (3.20). Accordingly

$$v^N(Z_I) = (g^\perp * f^N)(Z_I), \quad (4.4)$$

where

$$g^\perp(Z_I) = (-1)^{|I|} g(Z_I). \quad (4.5)$$

By the time invariance condition we have

$$\langle f_I^N, \varphi_j \rangle = \int dZ_N \mu^N(Z_N) \mathbb{E}(\varphi(T^t Z_N)), \quad (4.6)$$

where $\varphi_j = \varphi_j(Z_I)$ is a smooth test function, $j = |I|$ and we use the notation $\varphi(Z_K) = \varphi_j(Z_I)$ for $I \subseteq K$.

We now expand the expression in the right-hand side of (4.6) in clusters:

$$\langle f_I^N, \varphi_j \rangle = \sum_{\substack{S \subseteq I_N \\ S \cap I = \emptyset}} \int dZ_{I \cup S} f^N(Z_{I \cup S}) \mathbb{E}(\varphi(T^t Z_{I \cup S}) \chi(\text{cl}(I) = S)). \quad (4.7)$$

Here $\chi(\text{something})$ denotes the characteristic function of the event *something*. The right-hand side of (4.7) follows from the fact that, if $\chi(\text{cl}(I) = S) = 1$ then

$$\varphi(T^t Z_N) = \varphi(T^t Z_{I \cup S}) \quad (4.8)$$

since, by definition of cluster, the dynamics of the particles with labels in $(I \cup S)^c$ cannot influence those with labels in I .

Defining $\tilde{f}_I^N(t)$ by

$$\begin{aligned} \langle \tilde{f}_I^N, \varphi_j \rangle &= \sum_{\substack{S \subseteq I_N \\ S \cap I = \emptyset}} \sum_{\substack{S_1 \dots S_j: \\ \cup_i S_i = S \\ S_r \cap S_k = \emptyset, r \neq k}} \int dZ_{I \cup S} f^N(Z_{I \cup S}) \\ &\mathbb{E}(\varphi(T^t Z_{I \cup S}) \prod_{i \in I} \chi(\text{cl}(i) = S_i) \chi(n(\bar{S}_i) = |S_i|)), \end{aligned} \quad (4.9)$$

where $\bar{S}_i = \{i\} \cup S_i$ and n is given by (3.15), we have

$$f_I^N = \tilde{f}_I^N(t) + R_I^1(t) \quad (4.10)$$

with

$$\|R_I^1(t)\|_1 \leq \frac{C_1^j}{N} \quad (4.11.1)$$

with some $C_1 > 0$ for λr sufficiently small.

The decomposition (4.10) and estimate (4.11.1) is analogous to those given in [7] (see (5.3) and (5.24)) which we shall comment later on, after estimate (4.31) of the present paper. For the moment we proceed algebraically. We operate on $g^N(t)$ the same decomposition as for f^N :

$$g_I^N(t) = \tilde{g}_I^N(t) + R_I^2(t), \quad (4.12)$$

where \tilde{g}_I^N is given by the analogous expansion (4.9) (here g^N replaces f^N) and where

$$\langle g_I^N(t), \varphi_j \rangle = \int dZ_N g^N(Z_N) \mathbb{E}(\varphi(T^t Z_N)), \quad |I| = j. \quad (4.13)$$

Then

$$\begin{aligned} v^N(Z_I) &= (g^\perp * \tilde{f}^N(t))(Z_I) + (g^\perp * R^1(t))(Z_I) \\ &= (\tilde{g}^N(t)^\perp * \tilde{f}^N(t))(Z_I) + (g^\perp * R^1(t))(Z_I) \\ &\quad + ((g - g^N(t))^\perp * \tilde{f}^N(t))(Z_I) + (R^2(t) * \tilde{f}^N(t))(Z_I). \end{aligned} \quad (4.14)$$

Notice that the expansion (4.9) is done in terms of disjoint clusters with a minimal number of interacting pairs and that the remainder R^1 is negligible. Again from [7] we can prove

$$\|R_I^2(t)\|_1 \leq \frac{C_1^j}{N}. \quad (4.11.2)$$

Thanks to estimates (4.11) and to the identity $R_\emptyset^i(t) = 0$, $i = 1, 2$, the second and the fourth terms in the right-hand side of (4.14) vanish in the limit $N \rightarrow \infty$ in the L_1 sense and the same does the third term by Proposition 3.2. Thus we concentrate ourselves on the first term. Define $\bar{v}^N(t)$ by the following expression:

$$\begin{aligned} \langle \bar{v}_I^N(t), \varphi_j \rangle &= \sum_{\substack{S \subseteq I_N \\ S \cap I = \emptyset}} \sum_{\substack{S_1 \cdots S_j \\ \cup_i S_i = S \\ S_r \cap S_k = \emptyset, r \neq k}} \sum_{J \subseteq I} (-1)^{|J|} \\ &\int dZ_{I \cup S} \mathbb{E}(\varphi(T^t Z_{I \cup S}) \prod_{i \in I} \chi(\text{cl}(i) = S_i) \chi(n(\bar{S}_i) = |S_i|)) \\ &f^N \left(Z_{\underline{I \cup S(J)}} \right) g(Z_{J \cup S(J)}), \end{aligned} \quad (4.15)$$

where

$$S(K) = \bigcup_{i \in K} S_i. \quad (4.16)$$

As we shall see at the end of this section, the difference

$$R_I^3(t) \equiv \bar{g}_I^N(t)^\perp * \bar{f}_I^N(t) - \bar{v}_I^N(t) \quad (4.17)$$

is vanishing as $N \rightarrow \infty$, so that the essential part in the right-hand side of (4.14) is just $\bar{v}^N(t)$ which we are going to estimate. To this purpose we introduce the norm

$$\|v^N\|_\alpha = \sup_I e^{-\alpha|I|} \|v_I^N\|_1 \quad (4.18)$$

with α to be determined later on.

In (4.15) we write

$$f^N \left(Z_{\underline{I \cup S(J)}} \right) g(Z_{J \cup S(J)}) = \sum_{M \subseteq \underline{I \cup S(J)}} v^N(Z_M) g \left(Z_{\underline{I \cup S(M)}} \right) \quad (4.19)$$

and interchange the sums:

$$\sum_{J \subseteq I} \sum_{M \subseteq \underline{I \cup S(J)}} (-1)^{|J|} = \sum_{M \subseteq I \cup S} \sum_{J \subseteq \underline{M \cup I(M)}} (-1)^{|J|} = \sum_{\substack{M \subseteq I \cup S \\ M \cup I(M) \supseteq I}}, \quad (4.20)$$

where

$$I(M) = \{i \in I \mid i \notin M, S_i \cap M \neq \emptyset\}. \quad (4.21)$$

We now make explicit, in the expansion (4.15), those clusters S_i which are empty:

$$\langle \bar{v}_I^N(t), \varphi_j \rangle = \sum_{k=0}^j \sum_{\substack{K \subset I \\ |K|=k}} \sum_{s \geq j-k} \sum_{\substack{s_1 \cdots s_j: \\ \sum s_i = s}} \sum_{S \subseteq I_N/I} \sum_{\substack{S_1 \cdots S_j: \\ \cup_i S_i = S \\ S_r \cap S_k = \emptyset, r \neq k \\ |S_i| = s_i}} \sum_{m \geq j} \sum_{\substack{M \subseteq I \cup S \\ M \cup M(I) \supseteq I \\ |M|=m}} \quad (4.22)$$

$$\int dZ_{I \cup S} \mathbb{E}(\varphi_j(T^t Z_{I \cup S}) \prod_{i \in I} \chi(\text{cl}(i) = S_i) \chi(n(\bar{S}_i) = s_i))$$

$$\prod_{i \in K^c} \chi(|\text{cl}(i)| > 0) \prod_{i \in K} \chi(\text{cl}(i) = \emptyset)) v^N(Z_M) g\left(\frac{Z_{I \cup S}}{M}\right).$$

Notice that, since $M \cup M(I) \supseteq I$ then $m = |M| \geq j$ and $K \subseteq M$. Indeed if $i \in I$, either $i \in M$ or $S_i \cap M \neq \emptyset$. Moreover

$$(\varphi_j(T^t Z_{I \cup S})) = \left(\varphi_j \left(K^t Z_K, T^t Z_{I \cup S} \right) \right) \quad (4.23)$$

and, by Proposition 3.1 we can choose an arbitrary small η , to be chosen later, and a sufficiently large t (and hence r), such that

$$\|\bar{P}_K^t v_M^N\|_1 \leq \eta^k \|v_M^N\|_1, \quad |K| = k. \quad (4.24)$$

Therefore, from (4.22) we have the following estimate:

$$\begin{aligned} |\langle \bar{v}_I^N(t), \varphi_j \rangle| &\leq \|\varphi_j\|_\infty \sum_{k=0}^j \eta^k \binom{j}{k} \sum_{s \geq j-k} \sum_{\substack{s_1 \cdots s_j: \\ \sum s_i = s}} \sum_{m=j}^{j+s} \binom{N}{s} \binom{j+s}{m} \\ &\times \frac{s!}{s_1! \cdots s_j!} \|v_m^N\|_1 \left(\frac{\lambda r}{N} \right)^s \prod_{i=1}^j s_i! e^{3s}. \end{aligned} \quad (4.25)$$

Indeed the sums over the subsets give rise to the above combinatorial coefficients. To control the expectation

$$\mathbb{E} \left(\prod_{i=1}^j \chi(\text{cl}(i) = S_i) \chi(n(\bar{S}_i) = s_i) \right) \quad (4.26)$$

we use the same argument as in [7]. The probability of having $\omega(i, j; \alpha) = 1$ for some $\alpha = 1 \dots r$ is bounded by $r\varepsilon$. So that (4.25) is bounded by $(r\lambda/N)^s$ times the number of ways we have to construct j clusters with $s_1 \cdots s_j$ given particles. Such a number has been estimated in [7] by $\prod_{i=1}^j s_i! e^{3s}$ (see (5.12) and (5.13) of [7]). From (4.25):

$$\|\bar{v}_I^N(t)\|_1 \leq \|v^N\|_\alpha \sum_{k=0}^j \eta^k \binom{j}{k} e^{\alpha j} e^{2j} \sum_{s \geq j-k} s e^{\alpha s} e^{4s} 2^s (\lambda r)^s. \quad (4.27)$$

The above estimate follows by the elementary inequalities:

$$\binom{N}{s} \frac{s!}{N^s} \leq 1, \quad \sum_{\substack{s_1 \cdots s_j: \\ \sum s_i = s}} 1 \leq e^{s+j}, \quad (4.28)$$

$$\binom{j+s}{m} \leq 2^{j+s}, \quad \sum_{m=j}^{j+s} e^{\alpha m} \leq s e^{\alpha(j+s)}. \quad (4.29)$$

Notice that the condition $\prod_{i \in K^c} \chi(|\text{cl}(i)| > 0) = 1$ implies that $s \geq j - k$.

Finally, for $\lambda r < e^{-(6+\alpha)}$:

$$\begin{aligned} \|\bar{v}_J^N(t)\|_1 &\leq \|v^N\|_\alpha \frac{e^{\alpha j} e^{2j}}{1 - \lambda r e^{6+\alpha}} \sum_{k=0}^j \eta^k \binom{j}{k} (\lambda r e^{6+\alpha})^{j-k} \\ &\leq e^{\alpha j} \|v^N\|_\alpha \frac{e^{2j}}{1 - \lambda r e^{6+\alpha}} (\eta + \lambda r e^{6+\alpha})^j. \end{aligned} \quad (4.30)$$

We now fix all the parameters but α . Choosing t (and hence r) so large that $\eta = \frac{e^{-4}}{2}$ and, consequently λ such that $\lambda r \leq \frac{1}{2} e^{-2(6+\alpha)}$, we have

$$\|\bar{v}^N(t)\|_\alpha \leq 2e^{-2} \|v^N\|_\alpha. \quad (4.31)$$

We are now in position to comment on the estimates (4.11). In proving estimate (4.31) we have implicitly seen that the probability for a given particle to have an interaction cluster of large size is exponentially small, for λr small. Moreover each cluster is, with large probability, minimal, that is it is constructed by a minimal number of links. Finally the clusters are, with large probability, disjoint, since for an extra link we have to pay a price proportional to N^{-1} . This explains the meaning of estimates (4.11). For further details see (5.3) and (5.24) of [7].

Let us analyze now the term R_J^3 (see (4.17)). Notice that

$$\begin{aligned} \langle (\bar{f}^N(t) * \bar{g}^N(t)^\perp)_I, \varphi_j \rangle &= \sum_{J \subseteq I} (-1)^{|J|} \sum_{S \subseteq I_N/J} \sum_{\substack{S_1 \cdots S_m: \\ \bigcup_i S_i = S \\ S_r \cap S_k = \emptyset, r \neq k}} \sum_{R \subseteq I_N/(I/J)} \sum_{\substack{R_1 \cdots R_m: \\ \bigcup_i R_i = R \\ R_r \cap R_k = \emptyset, r \neq k}} \\ \int dZ_{I \cup S \cup R} f^N \left(Z_{I \cup R} \right) g(Z_{J \cup S}) \mathbb{E}_1 \mathbb{E}_2 [\varphi_j \left((T_1^t Z_{I \cup R})_{I/J}; (T_2^t Z_{J \cup S})_J \right)] \\ &\quad \prod_{i \in I/J} \chi(\text{cl}(i) = R_i) \chi(n(\bar{R}_i) = |R_i|), \\ &\quad \prod_{i \in J} \chi(\text{cl}(i) = S_i) \chi(n(\bar{S}_i) = |S_i|), \end{aligned} \quad (4.32)$$

where $T_i^t, i = 1, 2$ are two identical independent copies of the same process T^t . $\mathbb{E}_i, i = 1, 2$ denote the expectations for the processes and $n = |J|$, $m = |I/J|$.

The difference with the expression (4.15) which defines \bar{v}^N is twofold. In the expansion (4.32) $R \cup J$ and $S \cup (I/J)$ are not disjoint. Furthermore the product of the expectations $\mathbb{E}_1 \mathbb{E}_2$ replaces the expectation \mathbb{E} . Following [7] we can control (see estimate (5.35) for the details) the contributions in (4.32) due to the terms with $R \cup J \cap S \cup (I/J) \neq \emptyset$. This contribution is bounded by C^J/N . An analogous

estimate holds also for the difference between the product of the expectations and the expectation of the product. In conclusion:

$$\|(\bar{f}^N(t) * \bar{g}^N(t)^\perp)_I - \bar{v}_I^N(t)\|_1 \leq \frac{C_2^j}{N}. \quad (4.33)$$

Notice that the constants appearing in (4.11) and (4.33) are independent of t and the parameters of the model. From (4.11), (4.33) and Proposition 3.2:

$$\|v_j^N\|_1 \leq \|\bar{v}_j^N(t)\|_1 + \frac{C_3^j}{N} \quad (4.34)$$

for λ small and a suitable constant $C_3 > 1$. Finally, choosing $\alpha = \log C_3$ we have

$$\|v^N\|_\alpha \leq 2e^{-2}\|v^N\|_\alpha + \frac{1}{N} \quad (4.35)$$

by (4.30). The proof is now complete since $2e^{-2} < 1$. \square

Appendix A

Proof of Proposition 3.1. Since

$$K^t Z_I = \{K^t z_i\}_{i \in I} \quad (A.1)$$

is a collection of independent one-particle stochastic processes, it is enough to prove that, for all $\eta > 0$, there exists t_0 such that, for $t > t_0$, for all $u = u(z)$, $u \in L_1$, satisfying $\int u dz = 0$, then

$$\|\bar{P}^t u\|_1 \leq \eta \|u\|_1. \quad (A.2)$$

Let \bar{h} be the invariant probability measure for the Knudsen process, that is:

$$\begin{aligned} \bar{h}(x, v) &= aH_0(v) \quad \text{for } v > 0, \\ \bar{h}(x, v) &= aH_1(v) \quad \text{for } v < 0 \end{aligned} \quad (A.3)$$

(a being a normalization constant). Then (A.2) will be a consequence of the property:

$$\lim_{t \rightarrow \infty} \|\bar{P}^t h - \bar{h}\|_1 = 0 \quad (A.4)$$

uniformly in all initial probability densities h . Indeed, denoting by u^+ and u^- the positive and negative part of u respectively, setting

$$\alpha = \int u^+ dz = \int u^- dz, \quad (A.5)$$

we have

$$\begin{aligned} \|\bar{P}^t u\|_1 &= \|\bar{P}^t u^+ - \bar{P}^t u^-\|_1 \\ &\leq \alpha \left\| \bar{P}^t \left(\frac{u^+}{\alpha} \right) - \bar{h} \right\|_1 + \alpha \left\| \bar{P}^t \left(\frac{u^-}{\alpha} \right) - \bar{h} \right\|_1 \\ &\leq 2\alpha\eta \leq \eta \|u\|_1 \end{aligned} \quad (A.6)$$

for t sufficiently large.

To prove (A.4) we follow Arkeryd et al. [12] and check that, whenever there is a cutoff on the low velocities, the convergence (A.4) (proved in [12] for general Maxwellians) is indeed uniform in the initial datum h .

Notice that (A.4) cannot hold, in general, uniformly in h . Indeed if h_n is a sequence of initial data approaching $\delta(x - 1/2)\delta(v)$ as $n \rightarrow \infty$, we suddenly realize that the limit (A.4) is achieved with increasing difficulty as $n \rightarrow \infty$.

By formula (2.9) of [12]:

$$\begin{aligned} \|\bar{P}^t h - \bar{h}\|_1 &\leq \int_0^1 dx \int_{x/t}^{(x+1)/t} dv h(x - vt, v) + \int_0^1 dx \int_0^{(x+1)/t} dv H_0(v) \bar{J} \\ &+ \int_0^1 dx \int_{-x/t}^0 dv H_1(v) \bar{J} + \int_0^1 dx \int_{(x+1)/t}^2 dv H_0(v) |\bar{J} + J_-(t - \tau_{x,v})| \\ &+ \int_0^1 dx \int_{-2}^{x/t} dv H_1(v) |\bar{J} - J_+(t - \tau_{x,v})|, \end{aligned} \quad (\text{A.7})$$

where

$$\bar{J} = \frac{1}{\int_1^2 H_0(v) dv + \int_{-2}^{-1} H_1(v) dv} \quad (\text{A.8})$$

is the flux of the stationary measure \bar{h} , $J_-(t)$ and $J_+(t)$ are the fluxes in 0 and 1 respectively and $\tau_{x,v} = -\frac{(1/2)(1 - \text{sgn } v) - x}{v}$ is the time of the free flight from the boundary to the point x .

Note that for $t > 2$ the first three terms in the right-hand side of (A.7) are vanishing. To analyze the last two terms we use the following representation (see Appendix A of [12]) for the fluxes:

$$\begin{aligned} J_{\pm}(t) &= \pm A_{\pm}(t) \pm \int_0^t \beta(d\tau) A_{\pm}(t - \tau) \\ &\pm \int_0^t \beta([0, \tau]) d\tau A_{\pm}(t - \tau), \end{aligned} \quad (\text{A.9})$$

where $\beta(d\tau)$ is a suitable measure on $[0, \infty)$, independent of h , and $A_{\pm}(t)$ are given by:

$$A_{\pm}(t) = \phi_{\pm} + v_{\pm} * \phi_{\mp}(t), \quad (\text{A.10})$$

where

$$\phi_+(t) = \int_0^{1/t} v dv h(1 - vt, v), \quad (\text{A.11})$$

$$\phi_-(t) = - \int_{-1/t}^0 v dv h(-vt, v), \quad (\text{A.12})$$

$$v_{\pm}(t) = \frac{1}{t^3} H_{0,1} \left(\pm \frac{1}{t} \right). \quad (\text{A.13})$$

For $t > 1$, $\phi_{\pm}(t) = 0$ and $A_{\pm}(t) \neq 0$ only for $t < 2$. Therefore, for $t > 2$:

$$\begin{aligned} J_{\pm}(t) &= \pm \int_{t-2}^0 t \beta(d\tau) A_{\pm}(t - \tau) \\ &\pm \int_{t-2}^t \beta([0, \tau]) d\tau A_{\pm}(t - \tau). \end{aligned} \quad (\text{A.14})$$

The last term is bounded from below and above by $\beta([0, t - 2])$ and by $\beta([0, t])$ respectively (because of $\int_0^2 dt A_{\pm}(t) \leq 1$). Since

$$\beta([0, \infty)) = \left[\int_0^{\infty} t N(t) dt \right]^{-1} = \bar{J}, \quad (\text{A.15})$$

where $N = v_+ * v_-$ is the distribution density of the random time of flight from 0 to 0 (or from 1 to 1), we have

$$|J_{\pm}(t) \mp \bar{J}| = \int_{t-2}^0 t \beta(d\tau) A_{\pm}(t - \tau) + \omega(t), \quad (\text{A.16})$$

where $\omega(t)$ does not depend on h and $\omega(t) \rightarrow 0$ for $t \rightarrow \infty$.

Therefore for t large, after a suitable change of variable (see [12] (2.10)), the 4th term in the right-hand side of (A.7) is bounded by

$$\int_0^t d\tau |J_-(t - \tau) + \bar{J}| B_+(\tau), \quad (\text{A.17})$$

where

$$B_+(\tau) = \int_0^{1/\tau} dv H_0(v). \quad (\text{A.18})$$

Finally

$$\begin{aligned} \int_0^1 d\tau |J_-(t - \tau) + \bar{J}| B_+(\tau) &\leq c \int_0^1 d\tau \int_{t-\tau-2}^{t-\tau} d\sigma \beta(\sigma) A_+(t - \tau - \sigma) + c \int_0^1 d\tau \omega(t - \tau) \\ &\leq c(\beta[0, t - 3]) + c \int_0^1 d\tau \omega(t - \tau) \end{aligned} \quad (\text{A.19})$$

and the right-hand side of (A.19) goes to zero and is independent of h . The last term in (A.7) is of the same nature and can be handled analogously. \square

Appendix B

Proof of Proposition 3.2. To prove Proposition 3.2 we use the standard series expansion technique for the short time validity proof (see [10, 11]). The main difference here is that, due to the diffusive boundary conditions, we do not have the conservation of the L_{∞} norm with respect to the free flow and this makes the proof a little more tricky.

We fix a time τ and r the smallest integer for which $r \geq 4\tau + 1$ (see (3.10)). In the sequel we shall assume that $t \leq \tau$.

We first derive an L_∞ estimate for $g_j^N(t)$ which satisfies the following integrated version of the BBKGY hierarchy associated to Eq. (2.4):

$$g_j^N(Z_j, t) = (P_j^t g_j)(Z_j) + \varepsilon(N-j) \int_0^t dt_1 P_j^{(t-t_1)} C_{j,j+1} g_{j+1}^N(Z_j, t_1), \quad (\text{B.1})$$

where P_j^t is the semigroup associated to the j -particle stochastic process $T^t Z_j$ (see (2.3)) and

$$\begin{aligned} C_{j,j+1} g_{j+1}^N(x_1 v_1 \cdots x_j v_j) &= \sum_{i=1}^j \int dv_{j+1} |v_i - v_{j+1}| \chi(i, j) \\ &\quad \times \{g_{j+1}^N(x_1 v_1, \dots, x_i v'_i, \dots, x_j v_j, x_i v'_{j+1}) \\ &\quad - g_{j+1}^N(x_1 v_1, \dots, x_i v_i, \dots, x_j v_j, x_i v_{j+1})\}. \end{aligned} \quad (\text{B.2})$$

Moreover

$$g_j(z_1 \cdots z_j) = \prod_{i=1}^j g(z_i). \quad (\text{B.3})$$

For the Markov semigroup P_j^t we have the following representation (see (3.12)):

$$P_j^t g_j = \mathbb{E}_\Omega [P_j^t(\omega) g_j], \quad (\text{B.4})$$

where

$$\langle P_j^t(\omega) g_j, \varphi_j \rangle = \mathbb{E}_B \left(\int \varphi_j(T_\omega^t Z_j) g_j(Z_j) dZ_j \right). \quad (\text{B.5})$$

On the other hand we have

$$\begin{aligned} P_j^t(\omega) g_j(Z_j) &= g_j(T_\omega^{-t} Z_j) \chi(t < \tau(Z_j)) \\ &\quad + (P_j^{t-\tau(Z_j)}(\omega)) g_j(T_\omega^{-\tau(Z_j)} Z_j) \chi(t > \tau(Z_j)), \end{aligned} \quad (\text{B.6})$$

where

$$\tau(Z_j) = \sup\{s \geq 0 | T_\omega^{-s} Z_j \in A_j^0 \equiv (0, 1)^j \times \mathcal{V}^N\} \quad (\text{B.7})$$

is the (deterministic, once ω is fixed) time of the first collision with the boundary in the past.

Notice that for $Z_j \in A_j^0$ the process $T_\omega^s Z_j$ is deterministic for a short positive or negative time (before the first collision with the boundary in the future or in the past respectively).

Since $T^{-\tau(Z_j)}(Z_j) \in \mathcal{F}_j^{(i)}$ for some $i \in I_j$, (B.6) becomes

$$\begin{aligned} P_j^t(\omega) g_j(Z_j) &= g_j(T_\omega^{-t} Z_j) \chi(t < \tau(Z_j)) \\ &\quad + \chi(t > \tau(Z_j)) H_a(v_i) \int dv'_i v'_i P_j^{t-\tau(Z_j)}(\omega) g_j((T_\omega^{-\tau(Z_j)} Z_j)'), \end{aligned} \quad (\text{B.8})$$

where $(T^{-\tau(Z_j)} Z_j)'$ is the configuration on the boundary in which the outgoing velocity v_i is replaced by the ingoing one v'_i , and $a = 0, 1$, depending whether particle i is in 0 or 1 respectively. The expression (B.8) can be iterated developing

again $P_j^{-\tau(Z_j)}(\omega)$, so that we arrive to write:

$$P_j^t(\omega)g_j(Z_j) = \sum_{k \geq 0} P_j^t(\omega; k)g_j(Z_j), \quad (\text{B.9})$$

where $P_j^t(\omega; k)$ is the contribution to $P_j^t(\omega)$ due to the term with k collisions with the boundary. It is easily seen by (B.8) that

$$\|P_j^t(\omega; k)g_j\|_\infty \leq M^k \|g_j\|_\infty \quad (\text{B.10})$$

for some positive constant M .

Coming back to (B.1) we have

$$\begin{aligned} g_j^N(Z_j, t) &= \sum_{n \geq 0} \varepsilon^n (N-j)(N-j+1) \cdots (N-j+n-1) \\ &\quad \times \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n P_j^{(t-t_1)} C_{j,j+1} P_{j+1}^{(t_1-t_2)} \cdots C_{j+n-1, j+n} P_{j+n}^{t_n} g_{j+n}(Z_j) \\ &= \mathbb{E}_\Omega \sum_{n \geq 0} \varepsilon^n (N-j)(N-j+1) \cdots (N-j+n-1) \sum_{k_1 \cdots k_n} \\ &\quad \times \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n P_j^{(t-t_1)}(k_1, \omega) C_{j, j+1} P_{j+1}^{(t_1-t_2)}(k_2, \omega) \cdots \\ &\quad \times C_{j+n-1, j+n} P_{j+n}^{t_n}(k_n, \omega) g_{j+n}(Z_j). \end{aligned} \quad (\text{B.11})$$

Notice that the operator $C_{j,j+1}$ acts by joining a new particle, say $j+1$, in the same position of one of the group I_j , without changing any position of the particles of such a group. As an important consequence we realize that the right-hand side of (B.11) is vanishing whenever $\sum_{i=1}^n k_i$, that is the total number of collisions with the boundary, is too large. Actually $\sum_{k_1 \cdots k_n}$ is subjected to the constraint

$$\sum k_i \leq (j+n)(2t+1). \quad (\text{B.12})$$

Moreover, we also have the obvious estimate

$$\|C_{j, j+1} g_{j+1}\|_\infty \leq 2j \|g_{j+1}\|_\infty. \quad (\text{B.13})$$

Therefore, from (B.11), reminding that $r \geq 4t+1$:

$$\begin{aligned} \|g_j^N(t)\|_\infty &\leq \sum_{n \geq 0} \lambda^n \sum_{\substack{k_1 \cdots k_n \\ \sum k_i \leq (j+n)(2t+1)}} \frac{(2t)^n}{n!} j(j+1) \cdots (j+n-1) M^{(j+n)r} \|g_{j+n}\|_\infty \\ &\leq (2Me)^r \|g\|_\infty^j \sum_{n \geq 0} ((eM)^r r \lambda)^n \|g\|_\infty^n. \end{aligned} \quad (\text{B.14})$$

Thus for $(eM)^r r \lambda \|g\|_\infty < \frac{1}{2}$:

$$\|g_j^N(t)\|_\infty \leq 2((2eM)^r \|g\|_\infty)^j. \quad (\text{B.15})$$

We now prove, by using the bound (B.15), the convergence of $g_j^N(t)$ to g_j in the limit $N \rightarrow \infty$. Putting

$$\Delta_j^N(t) = g_j^N(t) - g_j. \quad (\text{B.16})$$

after some elementary algebra, we obtain

$$\begin{aligned} \Delta_j^N(Z_j, t) &= \varepsilon(N-j) \int_0^t dt_1 \bar{P}_j^{(t-t_1)} C_{j,j+1} \Delta_{j+1}^N(Z_j, t_1) \\ &\quad + \varepsilon \int_0^t dt_1 \bar{P}_j^{(t-t_1)} G_j g_j^N(t_1), \end{aligned} \quad (\text{B.17})$$

where $\bar{P}_j^t = \bar{P}_{I_j}^t$, is, as usual, the Knudsen semigroup and

$$G_j g_j(Z_j) = \sum_{i < h} |v_i - v_h| \chi(i, h) \delta(x_i - x_h) (g_j(Z_j(i, h)') - g_j(Z_j)) \quad (\text{B.18})$$

(see (2.4) to recall the notation).

Iterating (B.17) we obtain

$$\begin{aligned} \Delta_j^N(t) &= \sum_{n \geq 0} \varepsilon^n (N-j)(N-j+1) \cdots (N-j+n) \\ &\quad \times \varepsilon \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \bar{P}_j^{(t-t_1)} C_{j,j+1} \bar{P}_{j+1}^{t_1-t_2} \cdots C_{j+n-1, j+n} \\ &\quad \times \int_0^{t_n} \bar{P}_{j+n}^{t_n-t_{n+1}} G_{j+n} g_{j+n}^N(t_{n+1}) dt_{n+1}. \end{aligned} \quad (\text{B.19})$$

Thus to estimate $\Delta_j^N(t)$ we are led to control:

$$I_j(t) \equiv \int_0^t \bar{P}_j^{t-s} G_j g_j^N(s). \quad (\text{B.20})$$

For \bar{P}^t we have an analogous representation as (B.8) in which T_ω^t is replaced by the free stream S^t (i.e. $S^t(x_1 v_1 \cdots x_j v_j) = (x_1 - v_1 t, v_1, \dots, x_j - v_j t, v_j)$):

$$\begin{aligned} \bar{P}_j^t G_j g_j(Z_j) &= (G_j g_j)(S^{-t} Z_j) \chi(t < \tau(Z_j)) \\ &\quad + \chi(t > \tau(Z_j)) H_a(v_i) \int dv'_i v'_i \bar{P}_j^{t-\tau(Z_j)} (G_j g_j)((S^{\tau(Z_j)} Z_j)'). \end{aligned} \quad (\text{B.21})$$

Using the obvious inequality

$$\int_0^t \delta(x_i - v_i s - x_h + v_h s) |v_i - v_h| ds \leq 1 \quad (\text{B.22})$$

by (B.21) we obtain

$$\|I_j(t)\|_\infty \leq j^2 M^{rj} \sup_{s \leq t} \|g_j^N(s)\|_\infty. \quad (\text{B.23})$$

The term j^2 in (B.23) arises from the sum appearing in the definition of G_j while the factor M^{rj} is due to the collisions with the boundary, whose maximum number is bounded by rj .

Inserting estimate (B.15) in (B.23), we have

$$\sup_{s \leq t} \|I_{j+n}(s)\|_\infty \leq 2(j+n)^2 ((2Me)^{2r} \|g\|_\infty)^{j+n}. \quad (\text{B.24})$$

By (B.19) we can estimate Δ_j^N as g_j^N and, thanks to (B.24), we finally obtain

$$\begin{aligned} \|\Delta_j^N(t)\|_\infty &\leq \varepsilon(2(Me)^r)^j \sum_{n \geq 0} ((Me)^r r \lambda)^n \sup_{s \leq t} \|I_{j+n}(s)\|_\infty \\ &\leq \varepsilon 2j^2 ((2Me)^{3r} \|g\|_\infty)^j \sum_{n \geq 0} ((2Me)^{3r} r \lambda \|g\|_\infty)^n n^2 \\ &\leq \varepsilon 4j^2 ((2Me)^{3r} \|g\|_\infty)^j \end{aligned} \quad (\text{B.25})$$

for $r\lambda\|g\|_\infty M^{3r}$ sufficiently small. Hence the proof is achieved. \square

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