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Uniform Boundedness of the Solutions for a One-Dimensional Isentropic Model System of Compressible Viscous Gas

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Abstract: This paper studies an initial boundary value problem for a one-dimensional isentropic model system of compressible viscous gas with large external forces, represented by $v_t - u_x = 0$, $u_t + (av^{-\gamma})_x = \mu(u_x/v)_x + f(\int_0^x vdx, t)$, with $(v(x,0), u(x,0)) = (v_0(x), u_0(x))$, $u(0,t) = u(1,t) = 0$. Especially, the uniform boundedness of the solution in time is investigated. It is proved that for arbitrary large initial data and external forces, the problem uniquely has an uniformly bounded, global-in-time solution with also uniformly positive mass density, provided the adiabatic constant $\gamma(> 1)$ is suitably close to 1. The proof is based on L^2-energy estimates and a technique used in [9].

1. Introduction

In this paper we consider the one-dimensional motion of a general viscous isentropic gas in a bounded region, with an external force. In the Lagrangian mass coordinate, such a motion is described by the following system of equations:

$$v_t - u_x = 0\,, \tag{1.1}$$

$$u_t + p(v)_x = \mu\left(\frac{u_x}{v}\right)_x + f\left(\int_0^x vdx, t\right)\,, \tag{1.2}$$

where v, u, p, μ and f in the equations are the specific volume, the velocity, the pressure, the viscosity coefficient, and the external force of the fluid, respectively. We will assume that the equation of state, i.e., the function p is given by

$$p(v) = av^{-\gamma} \quad (a > 0,\ \gamma > 1 \text{ are the constants})\,, \tag{1.3}$$

and that the viscosity coefficient is a positive constant. After normalization, we may assume without loss of generality that the fluid occupies the interval $(0, 1)$, whose total mass is equal to 1. So we shall consider this problem in a fixed domain Q defined by

$$Q = \{(x,t) \mid 0 < x < 1,\ t > 0\}\,, \tag{1.4}$$

together with the initial conditions

$$v(x,0) = v_0(x), \qquad u(x,0) = u_0(x) \quad \text{on } 0 < x < 1\,, \tag{1.5}$$

and with the boundary conditions

$$u(0,t) = u(1,t) = 0 \quad \text{on } t > 0\,, \tag{1.6}$$

where the above initial data satisfy

$$v_0 \in H^1(0,1), \qquad u_0 \in H_0^1(0,1)\,, \tag{1.7}$$

$$C_0^{-1} \leqq v_0(x) \leqq C_0 \quad \text{for some constant } C_0 > 1\,, \tag{1.8}$$

and

$$\int_0^1 v_0(x)\,dx = 1\,. \tag{1.9}$$

Furthermore, for the external force $f = f(\xi,t)$, $\xi = \int_0^x v dx$, we suppose that

$$f, f_\xi \quad \text{and} \quad f_t \in L^\infty((0,1)\times(0,\infty))\,. \tag{1.10}$$

We are interested in the existence of an uniformly bounded global solution with respect to time t. Here and throughout this paper, the term "uniformly bounded, global-in-time solution" means the time-global solution which is uniformly bounded and its density also being uniformly positive with respect to t.

In the case $f \equiv 0$, the existence and uniqueness of the uniformly bounded, global-in-time solution have been obtained by a number of authors including Kanel' [4], Itaya [3], Kazhikhov [5], Kazhikhov and Shelukhin [8], Kazhikhov and Nikolaev [6,7], etc. under various conditions on the initial data, the equation of state p, and so on. Among them, Kazhikhov's result [5] shows that for arbitrary large initial data, our problem with $f \equiv 0$ has a unique uniformly bounded, global-in-time solution. If the external force vanishes sufficiently fast as time tends to infinity, we can extend their results to obtain uniform estimates or the asymptotic behavior of the solution. However this assumption is too restrictive to cover physically meaningful cases, such as time periodic external forces or time independent ones. From this point of view, Beirão da Veiga [1] proved the following result. For suitably small f, if some norm of the initial data is bounded by some constant which is determined by the L^∞-norm of f, then a uniformly bounded, global-in-time solution uniquely exists. Since the constant mentioned above tends to infinity as the L^∞-norm of f tends to 0, there is no gap between his result and Kazhikhov's one. His result also shows that for any fixed initial data, if the external force is sufficiently small, then the uniformly bounded, global-in-time solution uniquely exists. However it does not cover Matsumura and Nishida's result [9]: when the gas is assumed to be isothermal, namely the equation of state is given by $p = a/v$, then there exists a unique, uniformly bounded, global-in-time solution for an arbitrary large external force and large initial data. From this point, our interest in the present work is to make up for the difference between them. To do so, regarding γ as a parameter, we shall get the sufficient condition on the external force f so as to have uniform estimates on the solution, and study precisely how this condition depends on γ. Of course we expect that when γ tends to 1, our goal will be achieved.

In what follows, we denote the norm in L^∞, L^2 and H^1 by $|\cdot|_\infty$, $\|\cdot\|$ and $\|\cdot\|_1$, respectively. The following is our main theorem.

Theorem 1.1. *Assume* (1.7)–(1.10), *and* $1 < \gamma \leqq 2$. *Then there exists a constant* $C(\gamma)$, *which tends to* ∞ *as* γ *tends to* 1, *such that if* $E_1(0) < \frac{\mu}{4}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}$ *and* $|f|_\infty \leqq C(\gamma)$, *then the initial-boundary value problem* (1.1)–(1.3) *with* (1.5), (1.6) *has a unique, uniformly bounded, global-in-time solution* (v,u) *satisfying*

$$C^{-1} \leqq v(x,t) \leqq C \qquad \forall\ (x,t) \in Q\,, \tag{1.11}$$

and

$$\sup_{t \geqq 0} \|(v,u)(t)\|_1 \leqq C\,, \tag{1.12}$$

where $E_1(0)$ *shall be defined in* (2.8), *and* $C(>1)$ *is a constant depending only on* $a, \mu, \gamma, C_0, \|(v_0,u_0)\|_1$, *and* $|f|_\infty$.

Remark 1.1. The above constant $C(\gamma)$ can be chosen to satisfy $C(\gamma) \geqq C(\log(\gamma-1)^{-1})^\beta$ as $\gamma \to 1$ for any β satisfying $0 < \beta < 1$.

Remark 1.2. Theorem 1.1 shows that for any fixed initial data and external force, there exists a unique, uniformly bounded, global-in-time solution, provided the adiabatic constant γ is suitably close to 1.

Finally, we shall mention the asymptotic behavior of the solution to our problem. Let us decompose the external force $f = f(\xi,t)$ into a non-stationary part and a stationary part as

$$f(\xi,t) = f_0(\xi,t) + f_\infty(\xi)\,, \tag{1.13}$$

where $f_0(\xi,t)$ is a non-stationary part and $f_\infty(\xi)$ is a stationary part of the external force. Let $(\eta(\xi),0)$ be the stationary solution to our problem in the Eulerian coordinate, then it must satisfy

$$\frac{(a\eta^\gamma)_\xi}{\eta} = f_\infty(\xi)\,, \tag{1.14}$$

$$\int_0^1 \eta(\xi)d\xi = 1\,. \tag{1.15}$$

For this stationary problem, we have obtained in [2, 11] a sufficient condition that ensures the unique existence of the solution to (1.14) and (1.15), which is expressed as

$$\max_{w\in[0,1]} F(w) - \min_{w\in[0,1]} F(w) < \frac{a\gamma}{\gamma-1}\,, \tag{1.16}$$

where $F(w)$ is defined by $F(w) = \int_0^w f_\infty(\xi)d\xi$. By comparing the order of $C(\gamma)$ with the right-hand side of (1.16) as $\gamma \to 1$, it is easy to see that there exists a constant $\gamma_0 > 1$ such that if $1 < \gamma \leqq \gamma_0$ and $|f|_\infty < C(\gamma)$, then the condition (1.16) is satisfied. Therefore we have the following theorem

Theorem 1.2. *Assume the same hypotheses as in Theorem* 1.1. *Let* $(v,u)(x,t)$ *be the unique, uniformly bounded, global-in-time solution to* (1.1)–(1.3) *with* (1.5), (1.6), *and* $V(\xi)$ *be defined by*

$$V(\xi) = \frac{1}{\eta(\xi)}\,. \tag{1.17}$$

Then there exist constants $\gamma_0 > 1$, $\delta > 0$ *and* $C > 0$ *which depend only on the given data such that if* $1 < \gamma \leqq \gamma_0$, $E_1(0) < \frac{\mu}{4}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}$, *and* $|f|_\infty \leqq C(\gamma)$, *then the following estimate is satisfied for all* $t \geqq 0$:

$$\left\| v(\,\cdot\,,t) - V\left(\int_0^{\cdot} v(x',t)\,dx'\right)\right\|^2 + \|u(\,\cdot\,,t)\|^2 \leqq Ce^{-\delta t}\left(1 + \int_0^t e^{\delta s}|f_0(\,\cdot\,,s)|_\infty^2 ds\right). \tag{1.18}$$

Remark 1.3. Under the assumption that there exists a strictly positive solution η of (1.14) and (1.15), Zlotnik [12] has obtained the asymptotic behavior of the solution, for f nonincreasing with respect to ξ. Yanagi [11] showed in a Lagrangian mass coordinate the stability of the solution around the strictly positive stationary one, which was also assumed to exist, for f with f_ξ and itself being suitably small. Recently, Straškraba [10] considered the case that zero density appears on a set of positive measure, and generalized Zlotnik's result without assuming the existence of the strictly positive stationary solution, for f which does not depend on t and is nonincreasing with respect to ξ.

The proof of Theorem 1.1 is done in Sect. 2 by using energy estimates and the technique found in [9]. Remark 1.1 is obtained by direct calculations in Sect. 3. In Sect. 4, we shall show Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we shall have some estimates for the solution to our problem (1.1)–(1.3) with the initial and the boundary conditions given by (1.5), (1.6). In what follows, we shall denote the letters $C_1, C_2, \ldots$ by constants depending only on the given data.

Let us begin with the following easy result. Integrating (1.1) over [0,1] gives

$$\int_0^1 v(x,t)\,dx = 1, \quad \forall\, t \geqq 0\,. \tag{2.1}$$

Multiplying (1.2) by u and integrating it over $[0,1]$ yields

$$\frac{d}{dt}\int_0^1 \left\{\frac{1}{2}u^2 + \Phi(v)\right\} dx + \mu\int_0^1 \frac{u_x^2}{v}\,dx = \int_0^1 uf\,dx\,, \tag{2.2}$$

where Φ is defined by $\Phi(v) = \frac{a}{\gamma-1}(v^{-\gamma+1} - 1) + a(v-1)(\geqq 0)$. Using the relation $u = \int_0^x u_x\,dx$, we have the estimate $|u|_\infty \leqq \left(\int_0^1 \frac{u_x^2}{v}\,dx\right)^{\frac{1}{2}}$, then the right-hand side of (2.2) is estimated as

$$\int_0^1 uf\,dx \leqq |u|_\infty |f|_\infty \leqq \frac{\mu}{2}\int_0^1 \frac{u_x^2}{v}\,dx + \frac{1}{2\mu}|f|_\infty^2\,, \tag{2.3}$$

from which one gets

$$\frac{d}{dt}\int_0^1 \left\{\frac{1}{2}u^2 + \Phi(v)\right\} dx + \frac{\mu}{2}\int_0^1 \frac{u_x^2}{v}\,dx \leqq \frac{1}{2\mu}|f|_\infty^2\,. \tag{2.4}$$

Multiplying (1.2) by $\dfrac{v_x}{v}$ and integrating it over $[0,1]$ gives

$$\frac{d}{dt}\int_0^1 \left\{\frac{\mu}{2}\left(\frac{v_x}{v}\right)^2 - \frac{uv_x}{v}\right\} dx + a\gamma\int_0^1 \frac{v_x^2}{v^{\gamma+2}}\,dx = \int_0^1 \frac{u_x^2}{v}\,dx - \int_0^1 \frac{v_x}{v} f\,dx\,. \tag{2.5}$$

As the last term in the right-hand side of (2.5) is bounded by $|f|_\infty \left(\int_0^1 \frac{v_x^2}{v^3}\,dx\right)^{\frac{1}{2}}$, we have

$$\frac{d}{dt}\int_0^1 \left\{\frac{\mu}{2}\left(\frac{v_x}{v}\right)^2 - \frac{uv_x}{v}\right\} dx + a\gamma\int_0^1 \frac{v_x^2}{v^{\gamma+2}}\,dx \leqq \int_0^1 \frac{u_x^2}{v}\,dx + |f|_\infty \left(\int_0^1 \frac{v_x^2}{v^3}\,dx\right)^{\frac{1}{2}}. \tag{2.6}$$

Multiplying (2.6) by $\dfrac{\mu}{4}$, adding it with (2.4), one shows that

$$\frac{d}{dt}E_1^2(t) + E_2^2(t) \leqq \frac{1}{2\mu}|f|_\infty^2 + \frac{\mu}{4}|f|_\infty \left(\int_0^1 \frac{v_x^2}{v^3}\,dx\right)^{\frac{1}{2}}, \tag{2.7}$$

where $E_1^2(t)$ and $E_2^2(t)$ are defined by

$$E_1^2(t) = \int_0^1 \left\{\frac{1}{2}u^2 + \frac{\mu^2}{8}\left(\frac{v_x}{v}\right)^2 - \frac{\mu u v_x}{4v} + \Phi(v)\right\} dx\,, \tag{2.8}$$

$$E_2^2(t) = \frac{\mu}{4}\int_0^1 \frac{u_x^2}{v}\,dx + \frac{a\mu\gamma}{4}\int_0^1 \frac{v_x^2}{v^{\gamma+2}}\,dx\,. \tag{2.9}$$

Since the absolute value of the term $\dfrac{\mu u v_x}{4v}$ is bounded by $\frac{1}{4}u^2 + \frac{\mu^2}{16}\left(\frac{v_x}{v}\right)^2$, $E_1^2(t)$ can be estimated as

$$\begin{aligned}\frac{1}{2}\int_0^1 &\left\{\frac{1}{2}u^2 + \frac{\mu^2}{8}\left(\frac{v_x}{v}\right)^2\right\} dx + \int_0^1 \Phi(v)\,dx \\ &\leqq E_1^2(t) \leqq \frac{3}{2}\int_0^1 \left\{\frac{1}{2}u^2 + \frac{\mu^2}{8}\left(\frac{v_x}{v}\right)^2\right\} dx + \int_0^1 \Phi(v)\,dx\,. \end{aligned}\tag{2.10}$$

Now we would like to estimate $E_2^2(t)$ from below and $|f|_\infty \left(\int_0^1 \frac{v_x^2}{v^3}\,dx\right)^{\frac{1}{2}}$ from above. To do so, we shall use some methods found in [9]. Let X and Y be defined by

$$X = \int_0^1 \frac{v_x^2}{v^2}\,dx, \qquad Y = \int_0^1 \frac{v_x^2}{v^{\gamma+2}}\,dx\,. \tag{2.11}$$

Using Hölder's inequality, one has

$$\int_0^1 \frac{v_x^2}{v^3}\,dx \leqq X^{\frac{\gamma-1}{\gamma}} Y^{\frac{1}{\gamma}}\,. \tag{2.12}$$

Then it follows from (2.9)–(2.12) that

$$\begin{aligned} |f|_\infty \left(\int_0^1 \frac{v_x^2}{v^3}\,dx\right)^{\frac{1}{2}} &\leqq |f|_\infty X^{\frac{\gamma-1}{2\gamma}} Y^{\frac{1}{2\gamma}} \leqq \frac{\varepsilon}{2\gamma} Y + \frac{2\gamma-1}{2\gamma}\varepsilon^{-\frac{1}{2\gamma-1}} |f|_\infty^{\frac{2\gamma}{2\gamma-1}} X^{\frac{\gamma-1}{2\gamma-1}} \\ &\leqq \frac{2\varepsilon}{a\mu\gamma^2} E_2^2(t) + \frac{2\gamma-1}{2\gamma}\left(\frac{16}{\mu^2}\right)^{\frac{\gamma-1}{2\gamma-1}} \varepsilon^{-\frac{1}{2\gamma-1}} |f|_\infty^{\frac{2\gamma}{2\gamma-1}} E_1^{\frac{2(\gamma-1)}{2\gamma-1}}, \end{aligned} \tag{2.13}$$

for any $\varepsilon > 0$. If we determine ε that satisfies $\frac{\mu}{4}\frac{2\varepsilon}{a\mu\gamma^2} = \frac{1}{2}$; i.e., $\varepsilon = a\gamma^2$, then (2.7) is reduced to

$$\frac{d}{dt}E_1^2(t) + \frac{1}{2}E_2^2(t) \leqq \frac{1}{2\mu}|f|_\infty^2 + C_1 |f|_\infty^{\frac{2\gamma}{2\gamma-1}} E_1^{\frac{2(\gamma-1)}{2\gamma-1}}\,, \tag{2.14}$$

where $C_1 = \frac{\mu}{4}\frac{2\gamma-1}{2\gamma}\left(\frac{16}{\mu^2}\right)^{\frac{\gamma-1}{2\gamma-1}} (a\gamma^2)^{-\frac{1}{2\gamma-1}}$.

Next, it easily follows from (2.1) that there exists a point $x_0(t) \in [0,1]$ such that $v(x_0(t),t) = 1$. Therefore we have

$$|\log v| \leqq \left|\int_{x_0}^{x} \frac{v_x}{v}\,dx\right| \leqq \left(\int_0^1 \frac{v_x^2}{v^3}\,dx\right)^{\frac{1}{2}}\,, \tag{2.15}$$

from which one obtains the following relation between X and Y:

$$X \leqq |v|_\infty^\gamma Y \leqq Y \exp\left(\gamma X^{\frac{\gamma-1}{2\gamma}} Y^{\frac{1}{2\gamma}}\right)\,. \tag{2.16}$$

In order to proceed with this relation, we use the following lemma, without proof.

Lemma 2.1. *Let $g(x)$ be a function in $C([0,\infty))$ satisfying $g(0) = 0$, and be monotone increasing on some interval $[0, A_0]$. Let A be an arbitrary number satisfying $0 < A \leqq A_0$. Then the following inequality is valid for all $B \geqq 0$:*

$$AB \leqq \begin{cases} \int_0^A g(x)\,dx + \int_0^B g^{-1}(x)\,dx & \text{for } 0 \leqq B \leqq g(A_0)\,, \\ \int_0^A g(x)\,dx + A_0 B - \int_0^{A_0} g(x)\,dx & \text{for } B \geqq g(A_0)\,. \end{cases} \tag{2.17}$$

Putting $A = X^{\frac{\gamma-1}{2\gamma}}$, $A_0 = \left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma}}$, $B = Y^{\frac{1}{2\gamma}}$ and $g(x) = \frac{1}{\gamma-1}\frac{x^{\frac{\gamma+1}{\gamma-1}}}{x^{\frac{2\gamma}{\gamma-1}}+1}$ into (2.17), one shows from (2.16) that

$$\frac{X}{\sqrt{X+1}} \leqq \begin{cases} Y\exp\left(\gamma\int_0^{Y^{\frac{1}{2\gamma}}} g^{-1}(\xi)d\xi\right) & \text{for } 0 \leqq Y \leqq \alpha(\gamma)\,, \\ Y\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}\exp\left(\gamma\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma}} Y^{\frac{1}{2\gamma}}\right) & \text{for } Y \geqq \alpha(\gamma)\,, \end{cases} \tag{2.18}$$

provided that $X \leqq \frac{\gamma+1}{\gamma-1}$, where $\alpha(\gamma)$ is defined by $\alpha(\gamma) = \left(\frac{1}{2\gamma}\right)^{2\gamma}\left(\frac{\gamma+1}{\gamma-1}\right)^{\gamma+1}$.

Let us consider the function

$$G(y) = \begin{cases} y\exp\left(\gamma\int_0^{y^{\frac{1}{2\gamma}}} g^{-1}(\xi)d\xi\right) & \text{for } 0 \leqq y \leqq \alpha(\gamma)\,, \\ y\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}\exp\left(\gamma\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma}} y^{\frac{1}{2\gamma}}\right) & \text{for } y \geqq \alpha(\gamma)\,. \end{cases} \tag{2.19}$$

Since the function $G(y)$ is a monotone increasing one with respect to y, there exists the inverse function $y = G^{-1}(x)$, which has a following property:

$$\begin{aligned} H(x) &\equiv \frac{G^{-1}(x)}{x} \\ &= \begin{cases} \exp\left(-\gamma\int_0^{G^{-1}(x)^{\frac{1}{2\gamma}}} g^{-1}(\xi)d\xi\right) & \text{for } 0 \leqq x \leqq G(\alpha(\gamma)) \\ \left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}\exp\left(-\gamma\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma}} G^{-1}(x)^{\frac{1}{2\gamma}}\right) & \text{for } x \geqq G(\alpha(\gamma)) \end{cases} \\ &\leqq 1\,, \end{aligned} \tag{2.20}$$

and $H(x)$ is a decreasing function of x.

We are now in a position to estimate $E_2^2(t)$ from below. Assuming $X \leqq \frac{\gamma+1}{\gamma-1}$, we have from (2.9) and (2.18),

$$G^{-1}\left(\frac{X}{\sqrt{X+1}}\right) \leqq Y \leqq \frac{4}{a\mu\gamma}E_2^2\,, \tag{2.21}$$

or equivalently

$$H\left(\frac{X}{\sqrt{X+1}}\right)\frac{X}{\sqrt{X+1}} \leqq \frac{4}{a\mu\gamma}E_2^2\,. \tag{2.22}$$

Using the monotonicity of the functions $H(x)$ and $\frac{x}{\sqrt{x+1}}$, and the relation $X \leqq C_2E_1^2$ ($C_2 = 16/\mu^2$) shown in (2.10), one obtains from (2.22),

$$H\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)\frac{X}{\sqrt{C_2E_1^2+1}} \leqq \frac{4}{a\mu\gamma}E_2^2\,, \tag{2.23}$$

namely

$$G^{-1}\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)\frac{X}{C_2E_1^2} \leqq \frac{4}{a\mu\gamma}E_2^2 . \tag{2.24}$$

Next we shall estimate $\Phi(v)$. For the sake of the point $x_0(t)$, it follows that

$$\begin{aligned}\frac{a}{\gamma-1}(v^{-\gamma+1}-1) &= \frac{a}{\gamma-1}\int_{x_0}^{x}\frac{\partial}{\partial x}(v^{-\gamma+1}-1)\,dx \\ &= -a\int_{x_0}^{x}\frac{v_x}{v^\gamma}\,dx \leqq a\int_0^1\frac{|v_x|}{v^\gamma}\,dx .\end{aligned} \tag{2.25}$$

Thus

$$\int_0^1\Phi(v)\,dx \leqq a\int_0^1\frac{|v_x|}{v^\gamma}\,dx \leqq a\left(\int_0^1\frac{v_x^2}{v^{\gamma+2}}\,dx\right)^{\frac{1}{2}}\left(\int_0^1 v^{2-\gamma}\,dx\right)^{\frac{1}{2}} . \tag{2.26}$$

As we are interested in γ which is near 1, we may assume $1<\gamma\leqq 2$, so it is easily verified that the integration $\int_0^1 v^{2-\gamma}\,dx$ is less than or equal to 1. Using (2.9), one gets

$$\int_0^1\Phi(v)\,dx \leqq a\left(\int_0^1\frac{v_x^2}{v^{\gamma+2}}\,dx\right)^{\frac{1}{2}} \leqq 2\sqrt{\frac{a}{\mu\gamma}}E_2 . \tag{2.27}$$

Therefore it follows from (2.27) and the property of $H(x)$ that

$$H\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)\int_0^1\Phi(v)\,dx \leqq 2\sqrt{\frac{a}{\mu\gamma}}E_2 , \tag{2.28}$$

namely

$$G^{-1}\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)\frac{1}{C_2E_1^2}\int_0^1\Phi(v)\,dx \leqq 2\sqrt{\frac{a}{\mu\gamma}}E_2 . \tag{2.29}$$

Similar consideration as above yields

$$G^{-1}\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)\frac{1}{C_2E_1^2}\int_0^1 u^2\,dx \leqq \frac{4}{\mu}E_2^2 . \tag{2.30}$$

Multiplying (2.24) by $\frac{3\mu^2}{16}$, (2.30) by $\frac{3}{4}$ and adding the results together with (2.29) imply

$$G^{-1}\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right) \leqq \frac{12}{\mu^2}\left(\frac{\mu}{a\gamma}+\frac{4}{\mu}\right)E_2^2+\frac{32}{\mu^2}\sqrt{\frac{a}{\mu\gamma}}E_2 , \tag{2.31}$$

where we have used (2.10). As the last term in the right-hand side of (2.31) is bounded by $\varepsilon+\dfrac{256a}{\varepsilon\mu^5\gamma}E_2^2$, easy calculation shows that

$$\frac{\varepsilon}{1+\varepsilon}\left(G^{-1}\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)-\varepsilon\right) \leqq C_3E_2^2 , \tag{2.32}$$

for any $\varepsilon > 0$ and $C_3 = \max\left(\frac{12}{\mu^2}\left(\frac{\mu}{\gamma}+\frac{4}{\mu}\right), \frac{256a}{\mu^5\gamma}\right)$. Putting $\varepsilon = \frac{1}{2}G^{-1}\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)$ into (2.32), and substituting it into (2.14), we derive

$$\frac{d}{dt}E_1^2(t) + \frac{1}{4C_3}\frac{G^{-1}\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)^2}{2+G^{-1}\left(\frac{C_2E_1^2}{\sqrt{C_2E_1^2+1}}\right)} \leqq \frac{1}{2\mu}|f|_\infty^2 + C_1|f|_\infty^{\frac{2\gamma}{2\gamma-1}}E_1^{\frac{2(\gamma-1)}{2\gamma-1}}. \tag{2.33}$$

If $E_1(0)$ and $|f|_\infty$ are sufficiently small so as to satisfy

$$E_1(0) < \left(\frac{1}{C_2}\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}, \tag{2.34}$$

and

$$\frac{1}{2\mu}|f|_\infty^2 + C_1C_2^{-\frac{\gamma-1}{2\gamma-1}}|f|_\infty^{\frac{2\gamma}{2\gamma-1}}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma-1}} < \frac{1}{4C_3}\frac{G^{-1}\left(\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\right)^2}{2+G^{-1}\left(\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\right)}, \tag{2.35}$$

then (2.33) shows that $E_1(t) < \left(\frac{1}{C_2}\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}$ for all $t > 0$, therefore $X < \frac{\gamma+1}{\gamma-1}$. One of the sufficient conditions on f that satisfies (2.35) is

$$|f|_\infty \leqq \min\left\{\begin{array}{l}\left(\frac{\mu}{4C_3}\frac{G^{-1}\left(\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\right)^2}{2+G^{-1}\left(\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\right)}\right)^{1/2}, \\ C_2^{\frac{\gamma-1}{2\gamma}}\left(\frac{1}{8C_1C_3}\frac{G^{-1}\left(\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\right)^2}{2+G^{-1}\left(\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\right)}\right)^{\frac{2\gamma-1}{2\gamma}}\left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{\gamma-1}{2\gamma}}.\end{array}\right. \tag{2.36}$$

We have already got the following result.

Proposition 2.1. *Let the assumptions in Theorem 1.1 be satisfied. If the initial conditions and the external force satisfy* (2.34) *and* (2.35), *then the following estimates are valid*:

$$C^{-1} \leqq v(x,t) \leqq C \quad \forall\ (x,t) \in Q, \tag{2.37}$$

and

$$\sup_{t\geqq 0}\left(\|v(t)\|_1 + \|u(t)\|\right) \leqq C, \tag{2.38}$$

where C is a positive constant depending only on $a, \mu, \gamma, C_0, \|(v_0,u_0)\|_1$, and $|f|_\infty$.

Let $C(\gamma)$ be defined by the right-hand side of (2.36). Then the proof of Theorem 1.1 shall be completed if we estimate $\|u_x(t)\|$. Multiplying (1.2) by $-u_{xx}$

and integrating it over $[0,1]$ yields

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2 dx + \mu \int_0^1 \frac{u_{xx}^2}{v} dx = -a\gamma \int_0^1 \frac{v_x u_{xx}}{v^{\gamma+1}} dx + \mu \int_0^1 \frac{u_x v_x u_{xx}}{v^2} dx - \int_0^1 u_{xx} f \, dx \,. \quad (2.39)$$

Using Proposition 2.1, we can estimate each term in the right-hand side of (2.39) as

$$\left| a\gamma \int_0^1 \frac{v_x u_{xx}}{v^{\gamma+1}} dx \right| \leqq \varepsilon \int_0^1 u_{xx}^2 \, dx + \frac{C}{\varepsilon} \,, \quad (2.40)$$

$$\left| \mu \int_0^1 \frac{u_x v_x u_{xx}}{v^2} dx \right| \leqq \varepsilon \int_0^1 u_{xx}^2 \, dx + \frac{C}{\varepsilon} \int_0^1 v_x^2 u_x^2 \, dx \,, \quad (2.41)$$

$$\left| \int_0^1 u_{xx} f dx \right| \leqq \varepsilon \int_0^1 u_{xx}^2 \, dx + \frac{C}{\varepsilon} \,, \quad (2.42)$$

for any $\varepsilon > 0$. Since u satisfies the boundary conditions (1.6), there exists a point $x_1(t) \in (0,1)$ such that $u_x(x_1(t),t) = 0$. Using this point, we have the relation $u_x^2 = \int_{x_1}^x \frac{\partial}{\partial x}(u_x^2)\, dx = 2\int_{x_1}^x u_x u_{xx} \, dx$, which gives

$$|u_x|_\infty^2 \leqq \varepsilon^2 \int_0^1 u_{xx}^2 \, dx + \frac{1}{\varepsilon^2} \int_0^1 u_x^2 \, dx \,, \quad (2.43)$$

for any $\varepsilon > 0$. Substituting (2.43) into the last term in the right-hand side of (2.41) implies

$$\left| \mu \int_0^1 \frac{u_x v_x u_{xx}}{v^2} dx \right| \leqq C \left(\varepsilon \int_0^1 u_{xx}^2 \, dx + \frac{1}{\varepsilon^3} \int_0^1 u_x^2 \, dx \right) . \quad (2.44)$$

By choosing ε sufficiently small, we obtain from (2.39), (2.40), (2.42) and (2.44),

$$\frac{d}{dt}\int_0^1 u_x^2 \, dx + \int_0^1 u_{xx}^2 \, dx \leqq C \left(1 + \int_0^1 u_x^2 \, dx \right) . \quad (2.45)$$

It easily follows from (2.7) and (2.45) that

$$\frac{d}{dt}\left(E_1^2(t) + \int_0^1 u_x^2 \, dx \right) + \left(E_1^2(t) + \int_0^1 u_x^2 \, dx \right) \leqq C \,, \quad (2.46)$$

from which we conclude

$$E_1^2(t) + \int_0^1 u_x^2 \, dx \leqq C \,. \quad (2.47)$$

This completes the proof of Theorem 1.1.

3. Estimation of $C(\gamma)$

In this section, we shall estimate $C(\gamma)$, which is defined by the right-hand side of (2.36), as γ tends to 1. By the definition of $C(\gamma)$, it is easy to see that

$$C(\gamma) \cong G^{-1}\left(\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}} \right)^{\frac{1}{2}} \quad (\gamma \to 1) \,. \quad (3.1)$$

Since

$$G(\alpha(\gamma)) = \left(\frac{1}{2\gamma}\right)^{2\gamma+\frac{1}{2}} (\gamma+1)^{\gamma+1} \left(\frac{1}{\gamma-1}\right)^{\gamma+\frac{1}{2}} \exp\left(\frac{\gamma}{2}\frac{\gamma+1}{\gamma-1}\right)$$

$$\gg (\gamma-1)^{-\frac{1}{2}} \left(\cong \frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\right) \quad (\gamma \to 1), \tag{3.2}$$

it is enough to consider the function $G(y)$ only for $0 \leqq y \leqq \alpha(\gamma)$. Changing the variable $\eta = g^{-1}(\xi)$ in (2.19) yields

$$G(y) = y \exp\left(\gamma \int_0^{g^{-1}(y^{\frac{1}{2\gamma}})} \frac{1}{(\gamma-1)^2} \frac{\eta^{\frac{\gamma+1}{\gamma-1}} \left\{(\gamma+1) - (\gamma-1)\eta^{\frac{2\gamma}{\gamma-1}}\right\}}{\left(\eta^{\frac{2\gamma}{\gamma-1}}+1\right)^2} d\eta \right). \tag{3.3}$$

Another transformation $\tau = \eta^{\frac{2\gamma}{\gamma-1}}$ in (3.3) implies

$$G(y) = y \exp\left(\frac{1}{2(\gamma-1)} \int_0^{g^{-1}(y^{\frac{1}{2\gamma}})^{\frac{2\gamma}{\gamma-1}}} \frac{(\gamma+1)-(\gamma-1)\tau}{(\tau+1)^2} d\tau \right)$$

$$= y \exp\left(\left[-\frac{1}{2}\log(\tau+1) - \frac{\gamma}{\gamma-1}(\tau+1)^{-1} \right]_0^{g^{-1}(y^{\frac{1}{2\gamma}})^{\frac{2\gamma}{\gamma-1}}} \right)$$

$$= y \left(g^{-1}(y^{\frac{1}{2\gamma}})^{\frac{2\gamma}{\gamma-1}} + 1 \right)^{-\frac{1}{2}} \exp\left(\frac{\gamma}{\gamma-1} \left\{ 1 - \left(g^{-1}(y^{\frac{1}{2\gamma}})^{\frac{2\gamma}{\gamma-1}} + 1 \right)^{-1} \right\} \right). \tag{3.4}$$

Now we will show

$$g^{-1}\left(y(\gamma)^{\frac{1}{2\gamma}}\right)^{\frac{2\gamma}{\gamma-1}} \leqq (\gamma-1)\log R \quad (\gamma \to 1), \tag{3.5}$$

or equivalently

$$g^{-1}\left(y(\gamma)^{\frac{1}{2\gamma}}\right) \leqq (\gamma-1)^{\frac{\gamma-1}{2\gamma}} (\log R)^{\frac{\gamma-1}{2\gamma}} \quad (\gamma \to 1), \tag{3.6}$$

where $y(\gamma)$ is defined by $y(\gamma) = G^{-1}\left(\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\right)$, and R is defined by

$$R = (\gamma-1)^{-\frac{1}{2}} \left(\log(\gamma-1)^{-1}\right)^{-2\beta}, \tag{3.7}$$

for β with $0 < \beta < 1$. We note that for γ sufficiently close to 1, the right-hand side of (3.6) is less than $\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma}}$, therefore (3.6) is equivalent to

$$y(\gamma) \leqq g\left((\gamma-1)^{\frac{\gamma-1}{2\gamma}} (\log R)^{\frac{\gamma-1}{2\gamma}}\right)^{2\gamma} \quad (\gamma \to 1), \tag{3.8}$$

which is followed by the following relations:

$$
\begin{aligned}
&G\left(g\left((\gamma-1)^{\frac{\gamma-1}{2\gamma}}(\log R)^{\frac{\gamma-1}{2\gamma}}\right)^{2\gamma}\right)\\
&=\left(\frac{1}{\gamma-1}\right)^{\gamma-1}\frac{(\log R)^{\gamma+1}}{((\gamma-1)\log R+1)^{2\gamma}}((\gamma-1)\log R+1)^{-\frac{1}{2}}\\
&\quad\times\exp\left(\frac{\gamma}{\gamma-1}[1-\{(\gamma-1)\log R+1\}^{-1}]\right)\\
&\geqq C(\log R)^{\gamma+1}R^{\frac{\gamma}{(\gamma-1)\log R+1}}\\
&=C(\log R)^{\gamma+1}RR^{-\frac{(\gamma-1)(\log R-1)}{(\gamma-1)\log R+1}}\\
&\geqq C(\log R)^{\gamma+1}R\\
&=C(\log((\gamma-1)^{-\frac{1}{2}}(\log(\gamma-1)^{-1})^{-2\beta}))^{\gamma+1}(\gamma-1)^{-\frac{1}{2}}(\log(\gamma-1)^{-1})^{-2\beta}\\
&\geqq C(\gamma-1)^{-\frac{1}{2}}(\log(\gamma-1)^{-1})^{\gamma+1-2\beta}\\
&\geqq\frac{\gamma+1}{\sqrt{2\gamma(\gamma-1)}}\quad(\gamma\to1),
\end{aligned}
\tag{3.9}
$$

where we have used $\left(\frac{1}{\gamma-1}\right)^{\gamma-1}\to1$, $(\gamma-1)\log R\to0$ and $R^{-\frac{(\gamma-1)(\log R-1)}{(\gamma-1)\log R+1}}\to1$ as $\gamma\to1$. Therefore we have from (3.4),

$$G(y(\gamma))\leqq Cy(\gamma)(\gamma-1)^{-\frac{1}{2}}(\log(\gamma-1)^{-1})^{-2\beta}, \tag{3.10}$$

for any γ close to 1. Putting $y(\gamma)=G^{-1}\left(\frac{\gamma-1}{\sqrt{2\gamma(\gamma-1)}}\right)$ into (3.10) implies

$$\frac{\gamma-1}{\sqrt{2\gamma(\gamma-1)}}\leqq Cy(\gamma)(\gamma-1)^{-\frac{1}{2}}(\log(\gamma-1)^{-1})^{-2\beta}, \tag{3.11}$$

namely,

$$C(\log(\gamma-1)^{-1})^{2\beta}\leqq y(\gamma). \tag{3.12}$$

It follows from (3.1) and (3.12) that

$$C(\gamma)\geqq C(\log(\gamma-1)^{-1})^{\beta}\quad(\gamma\to1), \tag{3.13}$$

for any β with $0<\beta<1$. This completes the proof of Remark 1.1.

We note that (3.8) gives the estimation of $C(\gamma)$ from above, as follows:

$$C(\gamma)\leqq C\log(\gamma-1)^{-1}\quad(\gamma\to1), \tag{3.14}$$

which shows that the estimation of $C(\gamma)$ from below, i.e., (3.13) is almost optimal.

4. Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. In what follows, the letter C denotes the universal constant which depends only on the given data. We note that by the definition of $V(\xi)$, it follows from (1.14) and (1.15) that

$$V\left(\frac{a}{V^\gamma}\right)_\xi = f_\infty(\xi), \tag{4.1}$$

$$\int_0^1 \frac{v(x,t)}{V(\xi(x,t))}dx = 1. \tag{4.2}$$

Subtracting (4.1) from (1.2) makes

$$u_t + \left(\frac{a}{v^\gamma}\right)_x - V\left(\frac{a}{V^\gamma}\right)_\xi = \mu\left(\frac{u_x}{v}\right)_x + f_0(\xi,t). \tag{4.3}$$

Noting that $\left(\frac{a}{V^\gamma}\right)_x = \left(\frac{a}{V^\gamma}\right)_\xi \xi_x = \left(\frac{a}{V^\gamma}\right)_\xi v$, (4.3) is rewritten as

$$u_t + \left(\frac{a}{v^\gamma}\right)_x - \left(\frac{a}{V^\gamma}\right)_x + \left(\frac{a}{V^\gamma}\right)_\xi (v - V) = \mu\left(\frac{u_x}{v}\right)_x + f_0(\xi,t). \tag{4.4}$$

Multiplying (4.4) by $u(= \xi_t)$ and integrating it over $[0,1]$ imply

$$\frac{d}{dt}\int_0^1 \left\{\frac{1}{2}u^2 + \Psi(v,V)\right\} dx + \mu\int_0^1 \frac{u_x^2}{v}dx = \int_0^1 f_0 u\, dx, \tag{4.5}$$

where Ψ is defined by $\Psi(v,V) = \frac{a}{\gamma-1}\{v^{-\gamma+1} - \gamma V^{-\gamma+1} + (\gamma-1)vV^{-\gamma}\}$ $(\geqq 0)$. Using the same estimation as (2.3), we have from (4.5),

$$\frac{d}{dt}\int_0^1 \left\{\frac{1}{2}u^2 + \Psi(v,V)\right\} dx + \frac{\mu}{2}\int_0^1 \frac{u_x^2}{v}dx \leqq C|f_0(\,\cdot\,,t)|_\infty^2. \tag{4.6}$$

Multiplying (4.3) by v $(= \xi_x)$ implies

$$vu_t + \frac{a\gamma}{\gamma-1}(v^{-\gamma+1} - V^{-\gamma+1})_x = \mu v\left(\frac{u_x}{v}\right)_x + f_0 v. \tag{4.7}$$

Multiplying (4.7) by $\int_0^x \left(\frac{u}{V} - 1\right) dx'$ and integrating it over $[0,1]$ with respect to x give

$$\begin{aligned}
&\int_0^1 vu_t dx\int_0^x \left(\frac{v}{V} - 1\right) dx' + \frac{a\gamma}{\gamma-1}\int_0^1 (V^{-\gamma+1} - v^{-\gamma+1})\left(\frac{v}{V} - 1\right) dx \\
&\quad + \mu\int_0^1 v_x\frac{u_x}{v}dx\int_0^x \left(\frac{v}{V} - 1\right) dx' + \mu\int_0^1 u_x\left(\frac{v}{V} - 1\right) dx \\
&= \int_0^1 f_0 v dx\int_0^x \left(\frac{v}{V} - 1\right) dx',
\end{aligned} \tag{4.8}$$

where we have used (4.2). The calculation of the first term in the left-hand side of (4.8) proceeds as follows:

$$\int_0^1 vu_t dx \int_0^x \left(\frac{v}{V}-1\right)dx'$$
$$= \frac{d}{dt}\int_0^1 vudx\int_0^x\left(\frac{v}{V}-1\right)dx' - \int_0^1 uu_x dx \int_0^x \left(\frac{v}{V}-1\right)dx'$$
$$-\int_0^1 vudx\int_0^x\left(\frac{u_x}{V}-\frac{vuV_\xi}{V^2}\right)dx'. \tag{4.9}$$

Combining (4.8) and (4.9), one obtains

$$\frac{d}{dt}\int_0^1 vudx\int_0^x\left(\frac{v}{V}-1\right)dx' + \frac{a\gamma}{\gamma-1}\int_0^1 (V^{-\gamma+1}-v^{-\gamma+1})\left(\frac{v}{V}-1\right)dx$$
$$= \int_0^1 uu_x dx\int_0^x\left(\frac{v}{V}-1\right)dx' + \int_0^1 vudx\int_0^x\left(\frac{u_x}{V}-\frac{vuV_\xi}{V^2}\right)dx'$$
$$-\mu\int_0^1 v_x\frac{u_x}{v}dx\int_0^x\left(\frac{v}{V}-1\right)dx' - \mu\int_0^1 u_x\left(\frac{v}{V}-1\right)dx$$
$$+\int_0^1 f_0 vdx\int_0^x\left(\frac{v}{V}-1\right)dx'. \tag{4.10}$$

By using Theorem 1.1, each term in (4.10) is easily estimated as follows:

$$\frac{a\gamma}{\gamma-1}\int_0^1 (V^{-\gamma+1}-v^{-\gamma+1})\left(\frac{v}{V}-1\right)dx \geqq C\|v-V\|^2, \tag{4.11}$$

$$\left|\int_0^1 uu_x dx\int_0^x\left(\frac{v}{V}-1\right)dx'\right| \leqq C\|u_x\|\|v-V\| \leqq \varepsilon\|v-V\|^2 + C\|u_x\|^2, \tag{4.12}$$

$$\left|\int_0^1 vudx\int_0^x\left(\frac{u_x}{V}-\frac{vuV_\xi}{V^2}\right)dx'\right| \leqq C\|u\|_1^2 \leqq C\|u_x\|^2, \tag{4.13}$$

$$\mu\left|\int_0^1 v_x\frac{u_x}{v}dx\int_0^x\left(\frac{v}{V}-1\right)dx'\right| \leqq C\|v-V\|\|u_x\|\|v_x\|$$
$$\leqq C\|v-V\|\|u_x\| \leqq \varepsilon\|v-V\|^2 + C\|u_x\|^2, \tag{4.14}$$

$$\mu\left|\int_0^1 u_x\left(\frac{v}{V}-1\right)dx\right| \leqq C\|u_x\|\|v-V\| \leqq \varepsilon\|v-V\|^2 + C\|u_x\|^2, \tag{4.15}$$

$$\left|\int_0^1 f_0 vdx\int_0^x\left(\frac{v}{V}-1\right)dx'\right| \leqq C|f(\,\cdot\,,t)|_\infty\|v-V\| \leqq \varepsilon\|v-V\|^2 + C|f(\,\cdot\,,t)|_\infty^2, \tag{4.16}$$

for any small $\varepsilon > 0$. Therefore, it easily follows from (4.10)–(4.16) that

$$\frac{d}{dt}\int_0^1 vudx\int_0^x \left(\frac{v}{V}-1\right)dx' + \nu\|v-V\|^2 \leqq C(\|u_x\|^2 + |f(\,\cdot\,,t)|_\infty^2), \tag{4.17}$$

for some $\nu > 0$. Multiplying (4.17) by θ (> 0) and adding it with (4.6) yield

$$\frac{d}{dt}E_3^2(t) + E_4^2(t) \leqq C|f(\,\cdot\,,t)|_\infty^2, \tag{4.18}$$

where $E_3^2(t)$ and $E_4^2(t)$ are defined by

$$E_3^2(t) = \int_0^1 \left\{\frac{1}{2}u^2 + \Psi(v,V) + \theta vu\int_0^x \left(\frac{v}{V}-1\right)dx'\right\}dx, \tag{4.19}$$

$$E_4^2(t) = C(1-\theta)\|u_x\|^2 + \theta\nu\|v-V\|^2. \tag{4.20}$$

We note that

$$\left|\theta vu\int_0^x \left(\frac{v}{V}-1\right)dx'\right| \leqq C\theta\|u\|^2 + C\theta\|v-V\|^2, \tag{4.21}$$

that $\Psi(v,V)$ is equivalent to $\|v-V\|^2$, and that $\|u\| \leqq C\|u_x\|$, so by choosing $\theta > 0$ sufficiently small, we conclude from (4.18),

$$\frac{d}{dt}E_5^2(t) + \delta E_5^2(t) \leqq C|f(\,\cdot\,,t)|_\infty^2, \tag{4.22}$$

for some $\delta > 0$, where $E_5^2(t)$ is defined by

$$E_5^2(t) = \|u\|^2 + \|v-V\|^2. \tag{4.23}$$

The differential inequality (4.22) immediately leads to Theorem 1.2.

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