Commun. Math. Phys. 172, 13–38 (1995)

Communications in Mathematical Physics

Statistical Properties of Shocks in Burgers Turbulence★

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Received: 13 September 1993

Abstract: We consider the statistical properties of solutions of Burgers' equation in the limit of vanishing viscosity, $\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}(\frac{1}{2}u(x,t)^2)=0$, with Gaussian white-noise initial data. This system was originally proposed by Burgers[1] as a crude model of hydrodynamic turbulence, and more recently by Zel'dovich *et al.*[12] to describe the evolution of gravitational matter at large spatio-temporal scales, with shocks playing the role of mass clusters. We present here a rigorous proof of the scaling relation $P(s)\propto s^{1/2}, s\ll 1$, where $P(s)$ is the cumulative probability distribution of shock strengths. We also show that the set of spatial locations of shocks is discrete, i.e. has no accumulation points; and establish an upper bound on the tails of the shock-strength distribution, namely $1-P(s)\leqq \exp\{-Cs^3\}$for $s\gg 1$. Our method draws on a remarkable connection existing between the structure of Burgers turbulence and classical probabilistic work on the convex envelope of Brownian motion and related diffusion processes.

1. Introduction

The study of Burgers' equation with random initial data

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t}+\dfrac{\partial}{\partial x}\left(\dfrac{u(x,t)^2}{2}\right)=\nu\dfrac{\partial^2 u(x,t)}{\partial x^2}, \\ u(x,t=0)=u_0(x) \end{cases} \tag{1}$$

where $u_0(x)$ is a Gaussian white noise; i.e. $\langle u_0(x)\rangle=0$; $\langle u_o(x)u_0(y)\rangle=\delta(x-y)$ originated in the classical work of Burgers[1] as a simplified model of hydrodynamic turbulence. It is now widely recognized that this model, sometimes called "Burgers turbulence" (BT), lacks basic features of Navier-Stokes turbulence such as vorticity stretching, incompressibility, etc.; in fact, the statistical fluctuations of

★ Inadvertently the sequel to this article, "Statistical Properties of Shocks in Burgers Turbulence, II. Tail Probabilities for Velocities, Shock-Strengths and Rarefaction Intervals" has already appeared in an earlier issue of Commun. Math. Phys. (Commun. Math. Phys. **169**, 45–59 (1995).

$u(x,t)$ are entirely determined from the initial conditions and not, as in the latter case, by chaotic nonlinear dynamics arising from the equation itself. However, BT is a prototypical nonlinear wave equation in conservation form describing interesting physics of a different kind. Following the works of Zel'dovich[12] and others[7] there has been increased interest in BT in cosmology, as a possible model to describe the distribution of gravitational matter in the universe. According to the socalled "adhesion model," cold gravitational matter evolves from an initial epoch in which mass is distributed homogeneously in space and particles are assumed to have random, weakly correlated velocities. As gravitational matter evolves in time, it aggregates into clumps (galaxies, superclusters). Two particles or clumps of particles coming together at a given instant collide completely inelastically and turn into a larger clump with mass and momentum conserved. In the idealized case of one space dimension, such evolution was shown[12] to correspond to a distribution of Eulerian velocities given by the inviscid Burgers equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u(x,t)^2}{2}\right) = 0 \tag{2}$$

with $u(x,t=0) = u_0(x)$, where this equation is understood as the $\nu \to 0$ limit of (1). A connection between the evolution of gravitational matter in three space dimensions and a suitable generalization of (2) ("potential" Burgers turbulence) has also been established[12]. In the present paper we will limit the discussion to the scalar wave equation (2).

Viewing (1) as a "cartoon" for hydrodynamic turbulence, Burgers[1] undertook the calculation of the statistical moments and correlations of $u(x,t)$. He showed that in the inviscid limit (2), the solution consists, for each realization of $u_0(x)$, of a "sawtooth profile" or sequence of linear rarefaction waves separated by shocks. Using the fact that $\{u_0(x),\, x \leqq x_0\}$ and $\{u_0(x),\, x > x_0\}$ are independent random processes for all x_0, he obtained analytical expressions, in terms of solutions of certain boundary-value problems, for the probability distributions of intervals between shocks, shock strengths, etc. From these expressions he derived formulas for the statistical correlations of solutions of (2), such as $\langle u(x,t)\, u(x',t)\rangle$. The results obtained by Burgers give a great deal of information on the structure of BT, leaving few questions unanswered. One such open question pertains to the properties of the shock strength distribution in the limits of very large and very small shock strengths. In view of the connection of BT with Zeldovich's cosmological model, in which the shocks represent high-density regions of gravitational matter, this question has more than merely historical interest.

In 1979, Kida[8] attempted to characterize the distribution of shock strengths. Denote by $p_t(s), 0 < s < +\infty$, the shock-strength probability density function, i.e.,

$$\Pr\,\{S < s\} = \int_0^s p_t(s')\, ds' \,,$$

where S represents the strength of, say, the first shock of $u(x,t)$ to the right of $x = 0$. Kida found that

$$p_t(s) \cong 0.37 t^{1/2} s^{-1/2} \,, \quad s \ll 1 \tag{3}$$

matched numerical experiments very well. Relation (3) is equivalent to $\int_0^s p_t(s')\, ds' \propto t^{1/6} s^{1/2}$, $s \ll 1$ and hence to

$$N(s) \propto t^{1/2} s^{1/2} \,, \quad s \ll 1 \,, \tag{4}$$

where $N(s)$ represents the number of shocks per unit length in a single realization of $u(x,t)$ having strength of order s (say, between s and $2s$).

More recently, She, Aurell and Frisch[13] considered the inviscid Burgers equation with more general initial data having non-trivial correlations and postulated a general scaling law: for $t = 1$,

$$N(s) \propto s^{1-h}, \quad s \ll 1 \tag{5}$$

for solutions with initial data satisfying

$$\left\langle \left| \int_{x_1}^{x_2} u_0(x)ds \right|^2 \right\rangle \propto |(x_2 - x_1|^{2h}, \quad |x_1 - x_2| \ll 1 . \tag{6}$$

She et al. made a comparison between the sawtooth profiles corresponding to $h = 3/2$ (Brownian motion) and $h = 1/2$ (white-noise) based on Monte Carlo simulations. Their interesting study shows that in the former case, there are infinitely many shocks per unit volume and the exponent in the scaling law (5) is -0.5 ± 0.005, whereas in the latter case, there are finitely many shocks per unit volume. The numerically determined scaling exponent in this case is close to 0.5 but the data is much less definite.

Sinai[14] gave a rigorous proof of (5) in the case $u_0(x)$ = Brownian motion, which corresponds to $h = 3/2$ in (6). The corresponding exponent in (5) is $1 - h = -1/2$; implying that the number of shocks of size s increases as $s \to 0$. The corresponding set of shock locations has accumulation points. In the case of $u_0(x)$ = white-noise, the situation is different. From Burgers' results on the distribution of the separations between successive shocks (rarefaction intervals) and the Markovian nature of the solution $u(x,t)$ as a function of x (see Sect. 2 below), the set of shock locations is expected to be countable and discrete. These works therefore indicate that the structure of shocks depends crucially on the similarity properties of the random initial data. We believe that white noise initial data is probably the most relevant from the point of view of the cosmological model, since it arises naturally as the coarse-grained limit of a wide class of spatially homogeneous initial "velocities" with finite correlation length (see Sect. 6)

The purpose of this paper is to prove rigorously the aforementioned properties of the shock-strength distribution in BT with white-noise initial data, and, in particular, the scaling law (4). We shall establish

Theorem 1. *For fixed $t > 0$, with probability one, the set of points x at which $u(x,t)$ has jump discontinuities is countable, without accumulation points.* $\square$

Theorem 2. *Let S be the strength of the first shock to the right of $x = 0$ at $t = 1$. There exist numerical constants C_1 and C_2 such that*

$$C_1\sqrt{s} \leqq \Pr\{S < s\} \leqq C_2\sqrt{s}, \quad s \leqq 1 . \quad \square \tag{7}$$

These theorems imply the asymptotic result on the density of small shocks (4), with $\propto$ interpreted as "bounded from above and from below by numerical constants," by an application of Birkhoff's Ergodic Theorem (see also Eq. (14) below).

To derive these results on the structure of shocks of BT, we draw on work done by probabilists on the convex hull of Brownian motion. This problem was carefully studied, most notably by Groenboom[6] and Pitman[11]. To see the connection between these questions, consider the Hopf-Cole solution[5] of (2),

$$u(x,t) = \frac{d}{dx}\left\{ \inf_y \left(\beta(y) + \frac{|x-y|^2}{2t} \right) \right\}, \tag{8}$$

where $\beta(y) = \int_0^y u_0(y')\,dy'$ is Brownian motion pinned at $y = 0$. This formula leads, via Legendre transformation, to the consideration of the convex envelope of the graph of the random function

$$F(y) = \beta(y) + \frac{1}{2}y^2, \quad -\infty < y < +\infty. \tag{9}$$

The local structures of the convex hull of $F(y)$ and the convex hull of standard Brownian motion are similar. In fact, the measures induced by $F(y)$ and $\beta(y)$ on path space are mutually absolutely continuous by Girsanov's theorem. Using this fact and the results of Pitman and Groenboom on the convex hull of Brownian motion, we can establish that the set of shock locations of $u(x,t)$ for fixed t is countable and has no accumulation points. This is done in Sect. 3. Chorin[2] gave an earlier proof of Theorem 1 by different methods.

The proof of Theorem 2 also uses the similarity between the structures of $F(y)$ and $\beta(y)$ over finite intervals. From the Hopf-Cole formula (8), the statistics of small shocks are related to the small-scale oscillations of the convex hull of $F(y)$. Since $\beta(y)$ is rapidly oscillating at all scales but not $\frac{1}{2}y^2$, only the oscillations of Brownian paths contribute, to leading order, to the statistics of small shocks. In order to make this precise and to obtain (7), we use a time-inversion technique, due to J.M. Pitman. With this method the problem is reduced to the study of the asymptotic ($t \to \infty$) behavior of a suitable diffusion process. This program, which is the main part of the paper, is carried out in Sect. 4.

Finally, we prove a theorem which characterizes the tails of the probability distributions of $u(x,t)$, the length of rarefaction intervals, and the shock-strength distribution.

Theorem 3. *There exists a numerical constant* $C_3 > 0$ *such that, for fixed* t,

(i) $$\Pr\{|u(x,t)| > u\} \leqq \exp\{-C_3 t u^3\}, \quad u \geqq 1,$$

(ii) $$\Pr\{S > s\} \leqq \exp\{-C_3 t s^3\}, \quad s \geqq 1,$$

and

(iii) $$\Pr\{\delta x > x\} \leqq \exp\{-C_3 t^{-2} x^3\}, \quad x \geqq 1,$$

where δx *represents the length of an interval between two consecutive shocks.* □

The proof of this theorem, given in Sect. 5, relies on elementary bounds on rare events for Gaussian distribution. The tail distributions for the three random variables of Theorem 3 are determined by the probability that $F(y) = \beta(y) + \frac{1}{2}y^2$ takes small values as $|y| \to +\infty$. Contrary to the case of Theorem 2, we have not been able to prove the corresponding lower bounds for these probabilities. However, we believe that similar lower bounds should exist, i.e. that these upper bounds on the tails are sharp. We present a heuristic argument supporting this conjecture for the tails of the shock-strength distribution at the end of Sect. 5. Our conclusions are presented in Sect. 6.

2. The Hopf–Cole Solution

The "integrated" form of (2) is obtained by introducing the potential

$$\Psi(x,t) = \int_0^x u(x',t)dx' \,. \tag{10}$$

This function satisfies the Hamilton–Jacobi equation

$$\begin{cases} \dfrac{\partial \Psi(x,t)}{\partial t} + \dfrac{1}{2}\left(\dfrac{\partial \Psi(x,t)}{\partial x}\right)^2 = 0 \\ \Psi(x,0) = \beta(x) \,. \end{cases} \tag{11}$$

An explicit solution for (11) can be obtained[1,5] by the least-action principle for Lagrangian trajectories (action $= \int_0^t |\dot{x}(s)|^2 ds$),

$$\Psi(x,t) = \inf_y \left[\beta(y) + \frac{(x-y)^2}{2t}\right] . \tag{12}$$

Using the statistical self-similarity of Brownian motion, $\beta(\delta t)/\delta^{1/2} \approx \beta(t)$ (we use "$\approx$" to denote statistically equivalent quantities), and (11) or (12), we find that

$$\Psi(x,t) \approx t^{1/3}\Psi\left(\frac{x}{t^{2/3}},1\right) , \tag{13}$$

and thus

$$u(x,t) \approx \frac{1}{t^{1/3}} u\left(\frac{x}{t^{2/3}},1\right) . \tag{14}$$

For our purposes, it suffices therefore to consider the statistical properties of BT at a fixed time, say $t=1$. We set $\Psi(x) = \Psi(x,1)$ and $u(x) \equiv u(x,1)$. By definition, we have

$$\begin{aligned} \Psi(x) &= \inf_y \left[\frac{|x-y|^2}{2} + \beta(y)\right] \\ &= \frac{x^2}{2} - \sup_y [xy - F(y)] \\ &= \frac{x^2}{2} - F^*(x) , \end{aligned}$$

with

$$F(y) = \frac{1}{2}y^2 + \beta(y)$$

and

$$F^*(x) = \sup_y (xy - F(y)) .$$

The last function is the Legendre transform of $F(y)$. In particular, it is convex. Therefore it is differentiable almost everywhere with respect to Lebesgue measure and its derivative is a monotone, non-decreasing function with jump discontinuities contained in a set of Lebesgue measure 0. We normalize $dF^*(x)/dx$ so that it is right-continuous. From (11), we have

$$u(x) = x - \frac{dF^*(x)}{dx} . \tag{15}$$

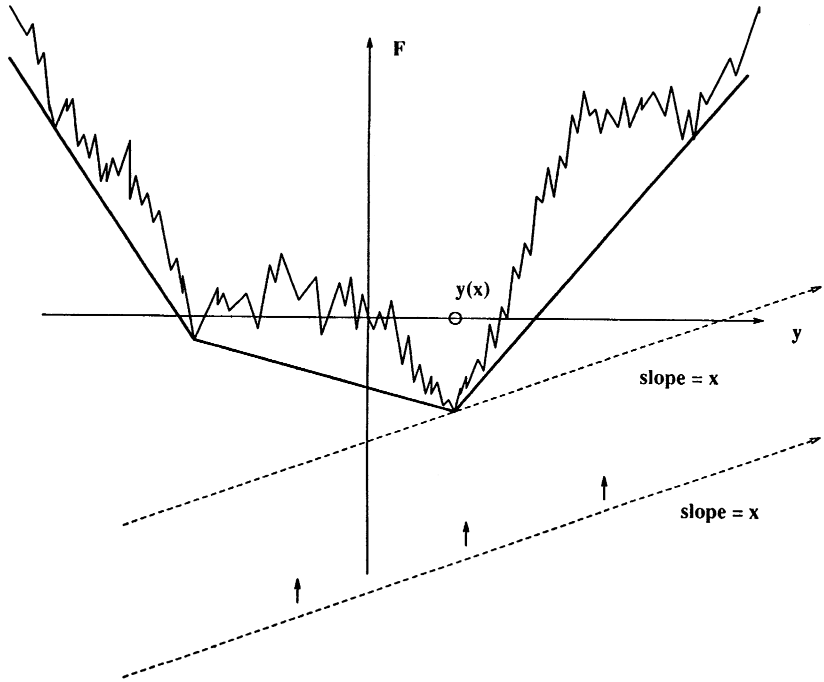

Fig. 1. The curve $F(y) = (1/2)y^2 + \beta(y)$ and its convex envelope, $F^{**}(y)$. For a given number $x, y(x) = dF^*(x)/dx$ is the abscissa of the point of contact of the tangent line with slope x

We can characterize dF^*/dx geometrically by introducing the Legendre transform of $F^*(x), F^{**}(y)$, which is the convex envelope of $F(y)$; see Fig. 1. By elementary convex analysis, dF^*/dx and dF^{**}/dy are inverses of each other. Therefore, dF^*/dx has the following geometric interpretation: given a realization of $\beta(y)$ and of the corresponding curve $F(y)$, consider a line of slope x which does not intersect the graph of $F(y)$ (such a line exists because $\lim_{|y|\to\infty} F(y) = +\infty$). Continuously "raise" this line, keeping the slope always equal to x, until it touches the graph of $F(y)$ for the first time. Then $dF^*(x)/dx = y(x)$, where $y(x)$ is the abscissa of the contact point, if this point is unique, and otherwise $y(x)$ is the largest abscissa corresponding to a contact point. It can be shown using the properties of Brownian motion (cf. Sect. 3 and the Appendix) that, for a given x, $y(x)$ is actually unique with probability 1.

From Eq. (2) we conclude that $u(x)$ is a stationary, or statistically translation invariant, stochastic process. Moreover, for all x and h,

$$u(x+h)[u_0(\,\cdot\,)] = u(x)[u_0(\,\cdot\, + h)]\,,$$

where the square brackets emphasize the dependence of $u(x)$ on the initial condition: translating $u(\,\cdot\,)$ by h is equivalent to computing $u(\,\cdot\,)$ after translating $u_0(\,\cdot\,)$ by h. Since $u_0(\,\cdot\,)$ is ergodic under spatial translations, we conclude that $u(x)$ is ergodic under spatial translations as well. This means that spatial averages of functions of $u(x)$ can be equated with their ensemble averages. In particular, the equivalence between (4) and the conclusion of Theorems 1 and 2 is established.

So far the properties of $u(x)$ derived here essentially follow from the stationarity and ergodicity of $u(x)$. To say more we will use the following

Theorem (Millar). *Let $X(t)$, $t \in I$ (=interval) be a Markov process with continuous paths, and let $\varphi(x,t)$ be a continuous function such that*

$$\mathrm{Pr.}\left[\inf_{t\in I} \varphi(x(t),t) > -\infty\right] = 1 .$$

Define

$$t^* = \sup\{t : \varphi(X(t),t) = \inf_{s\in I} \varphi(X(s),s)\}$$

and

$$x^* = X(t^*) .$$

Then, conditional on (X^*, t^*) the random processes

$$\{X(t),\ t \in I,\ t < t^*\}$$

and

$$\{X(s+t^*),\ s+t^* \in I;\ s > 0\},$$

and statistically independent Markov processes.

In the present situation, $dF^*(x)/dx = y(x)$ is the last "time" that the process $F(y) - y \cdot x$ $(-\infty < y < +\infty)$ reaches its minimum value. By Millar's theorem, $\{F(y); y < y(x)\}$ and $\{F(y),\ y > y(x)\}$ are statistically independent given $(y(x), F(y(x)))$. Notice that, given $y(x)$, $\{F^{**}(y),\ y > y(x)\}$ depends only on $\{F(y),\ y > y(x)\}$ and $\{F^{**}(y),\ y < y(x)\}$ depends only on $\{F(y),\ y < y(x)\}$. Therefore $\{F^{**}(y),\ y < y(x)\}$ and $\{F^{**}(y),\ y > y(x)\}$ are also independent, given $(y(x), F(y(x)))$. This implies that $\{dF^{**}(y)/dy,\ y < y(x)\}$ and $\{dF^{**}(y)/dy,\ y > y(x)\}$ are also independent given $(y(x), F(y(x)))$. Moreover, since $dF(y)/dy$ is an incremental quantity, these processes are independent of $F(y(x))$ as well. The fact that dF^*/dx and dF^{**}/dy are reciprocal functions implies that $\{dF^*(x')/dx',\ x' < x\}$ and $\{dF^*(x')/dx,\ x' > x\}$ are independent given $y(x) = dF^*(x)/dx$ and, hence, given $u(x) = x - y(x)$. This establishes that $u(x)$ is a Markov process.

3. Discrete Structure of the Set of Shock Locations

We will show first that the convex hull of $F(y) = \frac{1}{2}y^2 + \beta(y), -\infty < y < +\infty$, is a piecewise linear function such that $dF^{**}(y)/dy$ has countably many, isolated discontinuities with probability 1.

Let N be a large positive number, and let $\beta_N(y)$ be defined by

$$\beta_N(y) = \begin{cases} \beta(y) & 0 \leqq y \leqq N \\ +\infty & y < 0 \text{ or } y > N . \end{cases}$$

Let us denote by $\beta_N^{**}(y)$ its convex hull. The study of $\beta_N^{**}(y)$ can be done exactly like in Pitman[11], working instead on a finite interval.

Proposition 1. *With probability* 1, *$\beta_N^{**}(y)$, $0 \leqq y \leqq N$, is piecewise linear, and the set of points $\{y_\nu\}$, where $d\beta^{**}/dy$ has jump discontinuities is countable. Moreover, $\{y_\nu\}$ has accumulation points at $y = 0$ and $y = N$ only; i.e. for any numbers a,b* *such that* $0 < a < b < N$, *the number of points of discontinuity of* $d\beta^{**}(y)/dy$ *in* $[a,b]$ *is finite.*

For a sketch of the proof of this proposition, see the Appendix.

Next, we consider the function

$$F_{N,+}(y) = \frac{1}{2}y^2 + \beta(y), \quad 0 \leqq y \leqq N.$$

By Girsanov's theorem[9], the measures induced in path-space by β_N and $F_{N,+}$ are mutually absolutely continuous. Therefore, we conclude that the convex hull of $F_{N,+}(y)$, i.e. $F^{**}_{N,+}(y)$, enjoys the same property regarding the set of discontinuities of $dF^{**}_{N,+}(y)/dy$: there exist countably many points of discontinuity accumulating only at $y = 0$ and $y = N$. Define

$$F_{+}(y) = \frac{1}{2}y^2 + \beta(y), \quad 0 \leqq y < \infty,$$

and denote by F^{**}_{+} its convex envelope. Because $\lim_{y\to\infty} dF^{**}(y)/dy = +\infty$ with probability 1, we have

$$\lim_{N\to\infty} F^{**}_{N,+}(y) = F^{**}_{+}(y),$$

and even more is true: given any positive number a, there exists $N_0 = N_0(a)$ such that if $N > N_0$,

$$F^{**}_{N,+}(y) = F^{**}_{+}(y), \quad 0 \leqq y \leqq a.$$

This leads us to conclude that the derivative of $F^{**}_{+}(y)$ has countably many points of discontinuity and that these points accumulate only near $y = 0$.

The same argument applies to

$$F_{-}(y) = \frac{1}{2}y^2 + \beta(y), \quad y < 0,$$

and its convex hull, $F^{**}_{-}(y)$.

Finally, we observe that $F^{**}(y)$ is the convex envelope of the function which is equal to $F^{**}_{-}(y)$ for $y < 0$ and $F^{**}_{+}(y)$ for $y > 0$. Hence dF^{**}/dy has countably many points of discontinuity and $F^{**}(y)$ is piecewise linear. By the above arguments, the points of discontinuity of $dF^{**}(y)/dy$ can accumulate only at $y = 0$. However, note that we have $F^{**}(0) < 0$ with probability 1. Therefore, 0 must be contained in an interval with endpoints y^-, y^+ such that $F^{**}(y)$ is linear for $y^- < y < y^+$ and y^-, y^+ are points of discontinuity of $dF^{**}_{-}(y)/dy$ and dF^{**}_{+}/dy, respectively. Hence discontinuities of dF^{**}/dy cannot accumulate at $y = 0$. We have shown that *the set of discontinuities of* dF^{**}/dy *is countable without accumulation points and* $F^{**}(y)$ *is piecewise linear*. Note also that this implies that $y(x)$ is unique with probability 1, for any given x.

We turn next to the study of the discontinuities of dF^*/dx. Let $\{y_k\} = \{\cdots < y_{-2} < y_{-1} < y_0 < y_1 < \cdots\}$ represent the discontinuity set of dF^{**}/dy. From the above analysis, we conclude that dF^*/dx is piecewise constant and given by

$$\frac{dF^*(x)}{dx} = y_k, \quad \text{if } x_k \leqq x < x_{k+1},$$

where

$$x_k = \frac{dF^{**}}{dy}(y_k).$$

(This property is self-evident in view of the geometric interpretation of $dF^*/dx = y(x)$, cf. Fig. 1).

We thus see that the set of discontinuities of $dF^*(x)/dx$, $\{x_k\}$, is countable (it coincides with the range of dF^{**}/dy). In addition to this, since dF^{**}/dy increases from $-\infty$ to $+\infty$ and $\{y_k\}$ has no accumulation points, $\{x_k\}$ cannot have accumulation points either. Recalling that we have $u(x) = x - dF^*(x)/dx$, this proves Theorem 1.

Actually, the above discussion tells us more: the sequence (x_k, y_k), $k = 0, \pm 1, \pm 2, \ldots$ is a Markov process. This property is the basis for Burgers' quadrature formulas: the statistics of the increments of this point process $(x_{k+1} - x_k, y_{k+1} - y_k)$ depend only on (x_k, y_k) but not on the "past" $\{(x_j, y_j), j < k\}$. Nevertheless, it is worthwhile to emphasize that $\{x_k\}$ and $\{y_k\}$ are not *individually* Markovian, so the statistics of successive increments in x_k or y_k are given by complicated relations[1].

The spatial homogeneity of $u(x)$ proved in Sect. 2 implies that the sequences $\{x_{k+1} - x_k\}$ and $\{y_{k+1} - y_k\}$ are stationary. Moreover, as shown by Burgers,

$$\langle x_{k+1} - x_k \rangle = \langle y_{k+1} - y_k \rangle = 0.95 .$$

However, the probability distribution of rarefaction intervals $\{x_{k+1} - x_k\}$ and shock strengths $\{y_{k+1} - y_k\}$ are quite different. In the proof of Theorem 2, we will use the following result about the distribution of $x_{k+1} - x_k = \delta x$, established by Burgers [1, Sect. 47, p. 135 and Sect. 49, p. 139].

Proposition 2. *The random variable δx has a continuous probability density function $\phi(x), 0 \leqq x < +\infty$, such that* $\lim_{x \to 0} \phi(x) = \phi_0 > 0$.

In particular, we have

$$\Pr\{\delta x < x\} = \int_0^x \phi(x')\, dx' \leqq 2\phi_0 x ,$$

for x sufficiently small. (Burgers found that $\phi_0 \approx 3.4$.)

4. Statistics of Small Shocks

The problem of estimating $P\{S < s\}$ for small s is tantamount to calculating the probability that the convex hull of $F(y) = \frac{1}{2}y^2 + \beta(y)$ has a small linear segment to the right of $y(0)$, where $F(y(0)) = \min F(y)$. We begin by formulating this problem in terms of a suitable diffusion process describing $F(y)$ for $y > y(0)$. The technique that we use follows closely Pitman's approach[11] for characterizing the convex hull of Brownian motion.

Before entering the unavoidable technical details, we give a basic idea of the procedure. The problem of interest can be formulated as the study of the "slope process," $G(z) = (F(z + y(0)) - F(y(0)))/z$, for $z > 0$. Recall that we are interested in the statistics of small shocks corresponding to minima of $G(z)$ for $z \ll 1$. By making a change of variables $z = 1/t$, the problem reduces to the study of the long-time recurrence properties of a suitable diffusion process. It turns out that this process is transient, i.e., it tends to infinity with t, so recurrence becomes increasingly unlikely as $t \longrightarrow +\infty$. Pitman[11] introduced this technique with $F(y) = \beta(y)$ and showed that the recurrence probabilities are precisely of the order of $t^{-1/2}$. By changing back to the original variables $z = 1/t$, we obtain a probability of the order $s^{1/2}$ for the minimum of $G(z)$ for $z < s$, which is consistent with the desired scaling result. Thus, the proof is essentially an adaptation of Pitman's method for the case of $F(y) = \beta(y) + \frac{1}{2}y^2$.

4.1. Conditioned diffusion Processes. Let $X(t)$, $t > 0$, be a Markov process with continuous paths taking values on the real numbers, and let $P(x,s;y,t)$ be the corresponding transition probabilities. We denote by $P_{xs}(\,\cdot\,)$ the probability measure on path space corresponding to paths originating at the point x at time s, so that

$$\begin{aligned} P_{xs}[X(t) \in A] &= P[X(t) \in A | X(s) = x] \\ &= \int_A P(x,s;y,t)\,dy\,. \end{aligned}$$

Let τ_R denote the first time that a path exits the interval $0 < x < R$. For an arbitrary set E of paths, measurable with respect to the σ-algebra of events occurring before time $t(t > s)$, we define

$$P^{(R)}_{x,s}(E) = \frac{P_{x,s}[X(\,\cdot\,) \in E;\ X(\tau_R) = R]}{P_{x,s}[X(\tau_R) = R]}\,.$$

Clearly, $P^{(R)}_{x,s}(\,\cdot\,)$ defines a measure on path-space, corresponding to conditioning the Markov process $X(t)$ to exit the interval $0 < x < R$ through R. It can be shown that $P^{(R)}_{x,s}$ generates a Markov process with transition probabilities

$$P^{(R)}(x,s;y,t) = \frac{P_{x,s}(X(t) = y; \tau_R > t)P_{y,t}(X(\tau_R) = R)}{P_{x,s}(X(\tau_R) = R)}\,. \tag{16}$$

We set

$$\widetilde{P}(x,s;y,t) = \lim_{R\to\infty} P^{(R)}(x,s;y,t)$$

whenever this limit exists. Formally, we have

$$\widetilde{P}(x,s;y,t) = \left[\lim_{R\to\infty} \frac{P_{y,t}(X(\tau_R) = R)}{P_{x,s}(X(\tau_R) = R)}\right] \times P_{x,s}\left[X(t) = y;\ \tau_0 > t\right]\,, \tag{17}$$

where τ_0 is the first time that a path hits 0. If the limit exists (as is the case for Brownian motion and for diffusion processes with bounded drift), the measure induced by $\widetilde{P}(x,s;y,t)$ on path-space, henceforth denoted by $\{\widetilde{P}_{x,s}(\,\cdot\,)\}, x > 0, s > 0$, corresponds to the *probability distribution of* $X(t)$, *conditional on* $X(t) > 0$ *for all* t.[15]

Example. If $X(t) = \beta(t)$, $t > 0$ is standard Brownian motion, then according to a classical result of Williams[15], $\widetilde{P}_{x,0} = \widetilde{P}_x$ is the measure associated with the Markov process with infinitesimal generator

$$\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\,.$$

Thus the conditional Brownian motion $\tilde{\beta}(t), t > 0$ is distributed like a Bessel-3 process. This result follows from (17) and the formula

$$P_x[\beta(\tau_R) = R] = \frac{x}{R}, \tag{18}$$

which is a consequence of the spatial homogeneity of Brownian walks. Probabilistically, a Bessel-3 process is equivalent to the radial component of a three-dimensional random walk, and is therefore transient: with probability 1,

$$\lim_{t\to\infty} \tilde{\beta}(t) = +\infty. \tag{19}$$

We state the following proposition, which will be used in the sequel:

Proposition 3. *Let $\tilde{\beta}(t)$ be conditioned Brownian motion. Then, for $t \gg 1$,*

(i)

$$\sup_{y \leqq x} \tilde{P}_y[\tilde{\beta}(s) \leqq x \textit{ for some } s > t] \propto \frac{x}{\sqrt{t}}. \tag{20}$$

(ii) *Let*

$$t_* = \sup\{t > 0 : \ \tilde{\beta}(t) = \inf_{s>0} \tilde{\beta}(s)\}.$$

Then,

$$\tilde{P}_x[t_* > t] \propto \frac{x}{\sqrt{t}}. \tag{21}$$

For a proof, see Williams[15].

4.2. Conditioned process associated with shocks in BT. For a given real number u, we set

$$X_u(t) \equiv \beta(t) + \frac{1}{2t} - u, \ \ 0 < t < \infty.$$

This is a Markov process; the associated family of probability measures is to be denoted by $\{P_{x,s}\}$. Notice that $P_{x,s}\{X_u(\tau_R) = R\} > 0$ for all x, s and R. We consider the associated conditional Markov process $\tilde{X}_u(t)$, which is constructed from $X_u(t)$ as indicated in the previous paragraph. We define

$$x_u^* = \inf_{t>0} \tilde{X}_u(t)$$

and

$$t_u^* = \sup\{t : \tilde{X}_u(t) = x_u^*\}.$$

Lemma 4. *Let S represent the strength of the first shock of $u(x)$ to the right of $x = 0$. Then*

$$\text{Pr.}\{S < s\} = E\left\{\tilde{P}\left\{t^*_{u(0)} > \frac{1}{s}\right\}\right\}, \tag{22}$$

and if x_1 denotes the location of the shock,

$$\text{Pr.}\{x_1 > \alpha\} = E\{\tilde{P}\{x^*_{u(0)} > \alpha\}\}. \tag{23}$$

Here $E\{\ \cdot\ \}$ represents the expectation value with respect to the distribution of $u(0)$.

Proof. Let x be an arbitrary point on the real line. Consider a realization of $F(y) = \frac{1}{2}y^2 + \beta(y)$, $-\infty < y < +\infty$. We assume first that $y(x) > 0$. Later this assumption will be removed. We wish to characterize the strength and position of the shock to the left of x. By definition, the position of this shock is $x + \delta x$, where

$$\begin{aligned}\delta x &= \inf_{z>0} \frac{F(z + y(x)) - F(y(x)) - z \cdot x}{z} \\ &= \inf_{z>0} \frac{\beta(z + y(x)) - \beta(y(x)) - z(x - y(x)) + \frac{1}{2}z^2}{z} \\ &\approx \inf_{z>0} \frac{\beta(z) + \frac{1}{2}z^2 - z \cdot u(x)}{z}, \end{aligned} \tag{24}$$

where we set $\beta(z) \approx \beta(z + y(x)) - \beta(y(x))$ (with slight abuse of notation). The fraction in (24) is positive for all $z > 0$, by definition of $y(x)$. Moreover, the process

$$G(z) \equiv \frac{\beta(z) + \frac{1}{2}z^2 - z \cdot u(x)}{z} \tag{25}$$

is independent of $\{F(y),\ y < y(x)\}$, given $u(x)$, by Millar's theorem. We introduce the variable

$$t = \frac{1}{z} \tag{26}$$

(which is not to be confused with the physical time in the evolution of u). The process $\beta(z)$ in (25) is a Brownian path. Therefore $t\beta(1/t)$ is also a Brownian path. It follows that

$$\begin{cases} G(\frac{1}{t}) \approx \beta(t) + \frac{1}{2t} - u(x), \\ \text{conditioned on remaining positive for all times } t > 0\,, \end{cases}$$

or, more precisely,

$$G\left(\frac{1}{t}\right) \approx \tilde{X}_{u(x)}(t), \quad t > 0\,.$$

Therefore,

$$\delta x \approx x^*_{u(x)}$$

and

$$S_x \approx \frac{1}{t^*_{u(x)}},$$

where S_x denotes the strength of the first shock to the right of x. The second relation is a consequence of the fact that $S_x = z^* =$ point where the minimum of $G(z)$ is achieved, taking account the time inversion (26). Hence, the statement of this lemma follows if we assume that $y(x) > 0$.

However, this condition is not really necessary – it appears in the above argument only because we normalized the integral of $u_0(y)$ to vanish at $y = 0$. To remove it, note that

$$\Pr[S < s] = \Pr[S_x < s]$$

by stationarity. Therefore,

$$\begin{aligned}\Pr[S < s] &= \Pr[S_x < s,\ u(x) < x] + \Pr[u(x) > x] \\ &= \Pr[S_x < s;\ u(x) < x] + \Pr[u(0) > x] \\ &= E\{u(x) < x;\ \Pr\{S_x < s | u(x)\}\} + \Pr[u(0) > x] \\ &= E\left\{u(x) < x;\ \widetilde{P}\left[t^*_{u(x)} > \frac{1}{s}\right]\right\} + \mathrm{P}[u(0) > x],\end{aligned}$$

since $u(x) > x$ is equivalent to $y(x) > 0$. Hence, by stationarity again,

$$\mathrm{P}\{S < s\} = E\left\{u(0) < x;\ \widetilde{\mathrm{P}}\left(t^*_{u(0)} > \frac{1}{s}\right)\right\} = \mathrm{P}\{u(0) > x\}.$$

Letting $x \to \infty$, we obtain

$$\Pr\{S < s\} = E\left\{\widetilde{\mathrm{P}}\left(t^*_{u(0)} > \frac{1}{s}\right)\right\},$$

which gives (22). The argument for δx is similar. This concludes the proof of the lemma.

In the sequel we will make use of the following estimate on the moments of $\widetilde{X}_{u(0)}(t)$.

Lemma 5. *For all $\gamma > 0$, there exist $C_\gamma > 0$ such that for all $t > 0$,*

$$E[(\widetilde{X}_{u(0)}(t))^\gamma] \leqq C_\gamma t^{\frac{\gamma}{2}}. \tag{27}$$

The proof of this lemma is given in the Appendix.

4.3. Proof of the Asymptotic Estimates for the Probability of small Shocks. Let A be a subset of part-space which is measurable with respect to the σ-algebra of events up to some finite time. Then

$$\begin{aligned}&\widetilde{P}_{x,s}[\widetilde{X}(\,\cdot\,) \in A] \\ &= \lim_{R\to\infty} \frac{P_{x,s}(X(\,\cdot\,) \in A,\ X(\tau_R) = R)}{P_{x,s}(X(\tau_R) = R)}.\end{aligned} \tag{28}$$

We note that, for $t > s$, and $X(s) = x$,

$$\begin{aligned}X(t) &= x - \frac{1}{2s} + \frac{1}{2t} + \beta(t-s) \\ &\leqq x + \beta(t-s) \equiv \beta_{x,s}(t).\end{aligned}$$

Therefore, if $X(\tau_R) = R$ then $\beta_{x,s}(\,\cdot\,)$ also exits $(0,R)$ through R and

$$\begin{aligned}&P_{x,s}(X(\,\cdot\,) \in A;\ X(\tau_R) = R) \\ &\leqq P_{x,s}(X(\,\cdot\,) \in A;\ \beta_{x,s}(\widetilde{\tau}_R) = R),\end{aligned} \tag{29}$$

where $\widetilde{\tau}_R$ is the first time $\beta_{x,s}(\,\cdot\,)$ exits $(0,R)$. Moreover, since

$$X(t) \geqq x - \frac{1}{2s} + \beta(t-s) \equiv \beta_{x-\frac{1}{2s},s}(t),$$

if $x > \frac{1}{2s}$, then $\beta_{x-\frac{1}{2s}}(\tilde{\tau}_R) = R$ implies $X(\tau_R) = R$, and thus

$$P_{x,s}(X(\tau_R) = R) \geqq P\,[\beta_{x-\frac{1}{2s},s}(\tilde{\tau}_R) = R]\,. \tag{30}$$

Using estimates (29) and (30) in (28), we obtain that, for $x > \frac{1}{2s}$,

$$\tilde{P}_{x,s}[\tilde{X}(\,\cdot\,) \in A] \leqq \overline{\lim_{R\uparrow\infty}} \frac{P_{x,s}[X(\,\cdot\,) \in A,\ \beta_{x,s}(\tilde{\tau}_R) + R]}{P\,[\beta_{x-\frac{1}{2s},s}(\tilde{\tau}_R) = R]}$$

$$= \left(\overline{\lim_{R\uparrow\infty}}\ \frac{P\,[\beta_{x,s}(\tilde{\tau}_R) = R]}{P\,[\beta_{x-\frac{1}{2s},s}(\tilde{\tau}_R) = R]}\right) \times \left(\overline{\lim_{R\uparrow\infty}}\ \frac{P\,[X(\,\cdot\,) \in A;\ \beta_{x,s}(\tilde{\tau}_R) = R]}{P\,[\beta_{x,s}(\tilde{\tau}_R) = R]}\right).$$

Recalling that, if $X(s) = x$,

$$X(t) = \beta_{x,s}(t) + \frac{1}{2t} - \frac{1}{2s}\,,$$

and using the explicit formula for exit probabilities (18), we obtain

$$\tilde{P}_{x,s}[\tilde{X}(\,\cdot\,) \in A] \leqq \left(\frac{x}{x - 1/2s}\right)\ P\left[\tilde{\beta}_{x,s}(\,\cdot\,) + \frac{1}{2(\,\cdot\,)} - \frac{1}{2s} \in A\right], \tag{31}$$

where $\tilde{\beta}_{x,s}$ is conditional Brownian motion starting at x at time s. This inequality is valid for all measurable sets on path-space as well, by a standard measure-theoretic extension argument. In particular, if $x > 1/s$, we have

$$\frac{x}{x - \frac{1}{2s}} \leqq 2\,;$$

and thus

Proposition 6. *If $x > 1/s$, for all measurable sets A,*

$$\tilde{P}_{x,s}[\tilde{X}(\,\cdot\,) \in A] \leqq 2P\left[\tilde{\beta}_{x,s}(\,\cdot\,) + \frac{1}{2(\,\cdot\,)} - \frac{1}{2s} \in A\right]. \tag{32}$$

a similar argument gives a lower bound on $\tilde{P}_{x,s}(\tilde{X}(\,\cdot\,) \in A)$:

Proposition 7. *If $x > \frac{1}{s}$, for all measurable sets A in path-space,*

$$\tilde{P}_{x,s}(\tilde{X}(\,\cdot\,) \in A) \geqq \frac{1}{2}\,P\left[\tilde{\beta}_{x,s}(\,\cdot\,) + \frac{1}{2(\,\cdot\,)} - \frac{1}{2s} \in A\right]. \tag{33}$$

Propositions 6 and 7 will be used to obtain the desired estimates on the probabilities of small shocks. To do this, we consider the set

$$A = \{\tilde{X}(\,\cdot\,) : t^* > t\}\,.$$

Consider first the upper bound in (7). We define

$$\tilde{X}_{-1} = 0,\ \ \tilde{X}_n = \tilde{X}(2^n),\ \ n = 0, 1, 2, 3 \ldots\,.$$

Then,

$$\Pr.\{t^*_{u(0)} > t\} = \Pr.\{t^*_{u(0)} > t;\ \widetilde{X}_n > 2^{-n} \text{ for some } n\} + \Pr.\{t^*_{u(0)} > t;\ \widetilde{X}_n \leqq 2^{-n} \text{ for all } n\}.$$

Clearly, if $\widetilde{X}_n \leqq 2^{-n}$ for all n, then $x^*_{u(0)} = \inf_t \widetilde{X}_{u(0)}(t) = 0$. But, by Proposition 2, this happens with zero probability. Therefore,

$$\begin{aligned}\Pr.\{t^*_{u(0)} > t\} &= \Pr.\{t^*_{u(0)} > t;\ \widetilde{X}_n > 2^{-n} \text{ for some } n\}\\ &= \sum_{n=0}^{\infty} \Pr.\{\widetilde{X}_j \leqq 2^{-j};\ j \leqq n-1;\ \widetilde{X}_n > 2^{-n};\ t^*_{u(0)} > t\}\\ &\leqq \sum_{n=0}^{N} \Pr.\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n > 2^{-n};\ t^*_{u(0)} > t\} + \Pr.\{\widetilde{X}_N \leqq 2^{-N}\}. \end{aligned} \tag{34}$$

Here the integer N is chosen so that

$$2^{N+1} \leqq t \leqq 2^{N+2}. \tag{35}$$

We estimate each term in (34). By Proposition 6,

$$\begin{aligned}&\Pr.\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n > 2^{-n};\ t^*_{u(0)} > t\}\\ &= E\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n > 2^{-n};\ \widetilde{P}_{\widetilde{X}_n, 2^n}[t^* > t]\}\\ &\leqq E\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n > 2^{-n};\ \widetilde{P}_{\widetilde{X}_n, 2^n}(\widetilde{X}(s) \leqq \widetilde{X}_n \text{ for some } s > t)\}\\ &\leqq 2E\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n > 2^{-n};\ P(\widetilde{\beta}_{\widetilde{X}_n, 2^n}(s) + \frac{1}{2s} - \frac{1}{2^{n+1}}\\ &\leqq \widetilde{X}_n \text{ for some } s > t)\}, \end{aligned} \tag{36}$$

where we used the fact that the expectation is taken only with respect to paths such that $\widetilde{X}_n > 2^{-n}$. Moreover, using (20), we have

$$\begin{aligned}&P\left[\widetilde{\beta}_{\widetilde{X}_n, 2^n}(s) + \frac{1}{2s} - \frac{1}{2^{n+1}} \leqq \widetilde{X}_n;\ \text{for some } s > t\right]\\ &\leqq P[\widetilde{\beta}_{\widetilde{X}_n, 2^n}(s) \leqq 2\widetilde{X}_n;\ \text{for some } s > t]\\ &\leqq \frac{C\widetilde{X}_n}{\sqrt{t - 2^n}}. \end{aligned} \tag{37}$$

We conclude from (36) and (37) that

$$\begin{aligned}&\Pr.\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n > 2^{-n};\ t^*_{u(0)} > t\}\\ &\leqq \frac{C}{\sqrt{t - 2^n}} E\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n\}. \end{aligned}$$

Applying Hölder's inequality to this last expectation, we obtain

$$\begin{aligned}&E\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n\}\\ &\leqq (P[\widetilde{X}_{n-1} \leqq 2^{-(n-1)}])^{1-\frac{1}{\gamma}} (E\{(\widetilde{X}_n)^\gamma\})^{\frac{1}{\gamma}}, \end{aligned}$$

for any $\gamma > 1$. Hence, taking into account that N satisfies (35),

$$\Pr\{\widetilde{X}_{n-1} \leqq 2^{-(n-1)};\ \widetilde{X}_n > 2^{-n};\ t^*_{u(0)} > t\}$$
$$\leqq \frac{C}{\sqrt{t}}(P[\widetilde{X}_{n-1} \leqq 2^{-(n-1)}])^{1-\frac{1}{\gamma}}(E\{(\widetilde{X}_n)^\gamma\})^{\frac{1}{\gamma}}\,.$$

By Lemma 5, the last expression can be bounded by

$$\frac{C}{\sqrt{t}}(P[\widetilde{X}_{n-1} \leqq 2^{-(n-1)}])^{1-\frac{1}{\gamma}}2^{n/2}\,. \tag{38}$$

We note also that,

$$\mathrm{Pr.}[\widetilde{X}_n \leqq 2^{-n}] \leqq \mathrm{Pr.}[x^*_{u(0)} \leqq 2^{-n}] \leqq 2\phi_0 \times 2^{-n}\,, \tag{39}$$

for n sufficiently large, by Proposition 2. All this gives, using (34),

$$\mathrm{Pr.}\{t^*_{u(0)} > t\} \leqq \frac{C}{\sqrt{t}}\sum_{n=0}^{N}(2\phi_0)^{1-\frac{1}{\gamma}}\frac{2^{n/2}}{2^n(1-\frac{1}{\gamma})} + (2\phi_0)2^{-N}$$
$$\leqq \frac{C}{\sqrt{t}}\sum_{n=0}^{N}\frac{1}{2^{n(1-\frac{1}{\gamma}-\frac{1}{2})}}z + \frac{C'}{t}\,.$$

For $\gamma > 2$, the geometric series converges and hence

$$\mathrm{Pr.}\{t^*_{u(0)} > t\} \leqq \frac{C''}{\sqrt{t}}, \quad t > 1\,,$$

where C'' is a numerical constant. Therefore, recalling lemma 4, we have

$$\mathrm{Pr.}\{S > s\} \leqq C''\sqrt{s}, \quad s < 1,$$

as desired.

Finally, we prove the lower bound in (33). Notice that

$$\mathrm{Pr.}\{t^*_{u(0)} > t\} \geqq E\{\widetilde{X}_1 > 2;\ \widetilde{P}_{\widetilde{X}_1,2}(t^* > t)\}\,. \tag{40}$$

Let t_1^* denote the last time that the infimum of $\widetilde{\beta}_{\widetilde{X}_1-\frac{1}{4},2}(s) + \frac{1}{2s}$ is achieved, and let t_2^* denote the last time that the infimum of $\widetilde{\beta}_{\widetilde{X}_1-\frac{1}{4},2}(s)$ is achieved. We claim that

$$t_2^* < t_1^*\,. \tag{41}$$

In fact, omitting the starting point/time, we have, by definition,

$$\widetilde{\beta}(t_2^*) + \frac{1}{2t_2^*} \geqq \widetilde{\beta}(t_1^*) + \frac{1}{2t_1^*}\,,$$

or

$$\widetilde{\beta}(t_2^*) - \widetilde{\beta}(t_1^*) \geqq \frac{1}{2t_1^*} - \frac{1}{2t_2^*}\,.$$

But the left-hand side is negative and hence $1/(2t_1^*) - 1/(2t_2^*) < 0$, hence (59) follows.

Returning to (40), and applying Proposition 7 and (41), we obtain

$$\begin{aligned}\Pr.\{t^*_{u(0)} > t\} &\geqq E\{\widetilde{X}_1 > 2;\ \widetilde{P}_{\widetilde{X}_1,2}(t^* > t)\} \\ &\geqq \frac{1}{2}E\{\widetilde{X}_1 > 2;\ P(t^*_1 > t)\} \\ &\geqq \frac{1}{2}E\{\widetilde{X}_1 > 2;\ P\{t^*_2 > t\}\}.\end{aligned}$$

By Proposition 3, (Eq. (21)), we have

$$P\{t^*_2 > t\} \geqq \frac{C(\widetilde{X}_1 - \frac{1}{4})}{\sqrt{t - \frac{1}{2}}},$$

where C is a numerical constant, and hence

$$\begin{aligned}\Pr.\{t^*_{u(0)} > t\} &\geqq \frac{C}{\sqrt{t - \frac{1}{2}}}\Pr.[\widetilde{X}_1 > 2] \\ &\geqq \frac{C}{\sqrt{t - \frac{1}{2}}}\Pr.[x^*_{u(0)} > 2] \geqq \frac{C''}{\sqrt{t}},\quad \text{for } t > 1,\end{aligned}$$

where C' is another numerical constant. Finally, since $S_1 \approx (t^*_{u(0)})^{-1}$, we conclude that

$$\Pr.\{S < s\} \geqq C'\sqrt{s},\quad \text{for } s < 1,$$

and the lower bound in (7) is proved.

5. Tail Probabilities for u(x) and the Distribution of Large Shocks and Rarefaction Intervals.

In this section we prove the upper bounds on the tail probabilities of $u(x), \delta x$ and S stated in Theorem 3.

5.1. Tail Probabilities for $u(x)$. We are interested in the distribution $u(x)$ for fixed x. Due to the dissipation caused by shocks, we expect that the tails of the probability distribution of $u(x)$ should decay rapidly. In fact, we claim that there exists a positive constant C such that, if $u > 1$,

$$\text{Prob.}\{|u(x)| > u\} \leqq \exp\{-Cu^3\}. \tag{42}$$

Notice that restoring the time-dependence by means of the scaling relation (14) yields the estimate

$$\text{Prob.}\{|u(x,t)| > u\} \leqq \exp\{-Ctu^3\} \tag{43}$$

for $ut^{1/3} > 1$. In particular, the probability that $|u(x,t)|$ will take a value larger than a prescribed positive number u decays exponentially as $t \to \infty$.

Let us show (42). Since $u(x)$ is stationary, we consider $x = 0$, hence

$$u(0) = -y(0).$$

where $y(0)$ is such that

$$F(y(0)) = \beta(y(0)) + \frac{1}{2}(y(0))^2 = \inf_y \left(\beta(y) + \frac{1}{2}y^2\right).$$

We can assume without any loss in generality that $y(0) > 0$ (symmetry of $F(y)$). Clearly, $F(y(0)) < 0$ with probability 1. Therefore, $|u(0)| > u$ implies that

$$\inf_{y>u} \left(\beta(y) + \frac{1}{2}y^2\right) < 0,$$

so, in particular,

$$\inf_{y>u} \left(\beta(y) + \frac{1}{2}y\,u\right) < 0.$$

This implies also that

$$\inf_{y>u} \left(\frac{\beta(y)}{y}\right) + \frac{1}{2}u < 0. \tag{44}$$

Observe that if $z \equiv 1/y$, then $\beta(y)/y = z\beta(\frac{1}{z}) \approx \beta(z)$, by Brownian time inversion. Thus (44) becomes

$$\inf_{z<1/u} (\beta(z)) < -\frac{1}{2}u.$$

Therefore, we conclude that

$$\begin{aligned}\Pr\{|u(x)| > u\} &\leqq 2\int_{\frac{1}{2}u}^{+\infty} e^{-\frac{us^2}{2}} \frac{u^{1/2}ds}{\sqrt{2\pi}} \\ &= 2\int_{\frac{1}{2}u^{3/2}}^{+\infty} e^{-s^2/2} \frac{ds}{\sqrt{2\pi}} \leqq \text{const.}\, e^{-\frac{u^3}{8}}\end{aligned}$$

This proves our claim.

5.2. Rarefaction Intervals. We estimate the tail probabilities of rarefaction intervals δx, using the previous estimate (42). To fix ideas, let δx represent the rarefaction interval containing $x = 0$. The function $u(x)$ is linear with slope 1 in this interval. Therefore, if $\delta x > l$, where l is a positive number, then

$$u(l) = u(0) + l,$$

so we must have either

$$u(0) < -\frac{l}{2},$$

or

$$u(l) > \frac{l}{2}.$$

This implies that

$$\Pr\{\delta x > l\} \leqq \Pr\left\{u(0) < -\frac{l}{2}\right\} + \Pr\left\{u(l) > \frac{l}{2}\right\} \leqq 2\Pr\left\{|u(0)| > \frac{l}{2}\right\},$$

where we used the stationarity of the function u. From the estimate (42), we conclude that

$$\Pr\{\delta x > l\} \leqq Ce^{-l^3/64}.$$

The statement (iii) in Theorem 3 is obtained by restoring time units.

5.3. Tail Probabilities for Large Shocks. Let S represent the strength of the first shock to the right of $x = 0$. Our goal here is to show that there exists a constant C such that, for $s > 1$,

$$\text{Prob.}\{S > s\} \leqq \exp\{-Cs^3\}. \tag{45}$$

The dimensional result (ii), stated for arbitrary times $t > 0$, is recovered using the scaling relation (14).

Let (x_0, x_1) denote the rarefaction interval containing $x = 0$. Then

$$u(x_0^+) = u(0) - |x_0| = u(0) + x_0$$

and

$$u(x_1^-) = u(0) + x_1.$$

Therefore,

$$|u(x_0^+)| \leqq |u(0)| + \delta x,$$

and, similarly,

$$|u(x_0^-)| \leqq |u(0)| + \delta x.$$

This shows that, for all integers k,

$$\Pr\{|u(x_k^\pm)| > u\} \leqq \Pr\left\{|u(0)| > \frac{u}{2}\right\} + \Pr\left\{\delta x > \frac{u}{2}\right\} \leqq e^{-Cu^3}, \quad u \geqq 1,$$

where C is a numerical constant. In words, the tail probabilities of the values of $u(x)$ *at a shock*, either to the right or to the left of the shock, are bounded by $\exp\{-C\,u^3\}$. But then, since the shock strength at $x = x_k$ is

$$S_k = u(x_k^-) - u(x_k^+), \quad k = 0, \pm 1, \pm 2, \dots,$$

we have,

$$\Pr\{S_k > s\} \leqq \Pr\left\{|u(x_k^-)| > \frac{s}{2}\right\} + \Pr\left\{|u(x_k^+)| > \frac{s}{2}\right\} \leqq 2e^{-Cs^3},$$

for some numerical constant C. This concludes the proof of Theorem 3.

The question of whether there exist similar lower bounds for the tails of the shock-strength distribution is natural. We have not found an elementary proof of such a lower bound but nevertheless conjecture that the inequality

$$\Pr\{S > s\} \geqq \exp\{-C s^3\}, \quad s \gg 1,$$

should also hold for suitable C. Let us give a heuristic justification for this conjecture, based on the structure of the convex hull of Brownian motion over a finite interval. (Proposition 8 of the Appendix). Accordingly, the asymptotic behavior of the probability distribution of $y_1 - y(0)$ for Brownian motion in the finite interval $[0, Y_0]$ is

$$\Pr\{y_1 - y(0) > \theta\} \approx E\{e^{-\frac{1}{2}a_1^2\theta}\}, \quad \theta \ll 1,$$

where

$$a_1 = \frac{\beta(Y_0) - \beta(y(0))}{Y_0 - y(0)} \,.$$

If we replace, self-consistently, $\beta(y)$ by $F(y) = \beta(y) + \frac{1}{2}y^2$ in the last relation and take $Y_0 - y(0) \approx \theta$, we obtain $a_1 \approx \theta$ and have

$$\Pr\{y_1 - y(0) > \theta\} \approx e^{-C\theta^3} \,,$$

which would be the derived asymptotics. This argument is far from being rigorous, but we believe that a mathematical justification could be made using the theory of Large Deviations[3].

6. Conclusions

We have shown that the statistical properties of Burgers' equation with white-noise Gaussian initial data can be studied rigorously and in detail using a probabilistic approach. Specifically, we exploited a remarkable connection existing between this problem and the work of Groenboom and Pitman, who showed that the convex envelope of Brownian graphs could be characterized using the technique of time-reversal and conditional diffusions.

The main features of BT which emerge from this study are

(i) The set of shock locations is discrete, in contrast with the case of initial data such as Brownian motion. For fixed time, the solution is a Markov process as a function of the spatial variable.

(ii) The tails of $u(x,t)$ for fixed x and t, the rarefaction intervals and the shock-strength distribution decay at least like $exp\ (-C\,u^3)$. Note that this is much narrower than Gaussian.

(iii) The $s^{1/2}$ asymptotics for the small shock-strength distribution can be obtained rigorously by the technique of time-reversal of Pitman[11]. The analysis clearly shows that this scaling is determined essentially by the consecutive peaks, or oscillations of Brownian paths. The additional drift term coming from the Burgers nonlinearity is irrelevant in the small-shock regime.

The assumption that $u_0(x)$ is a Gaussian white noise, as opposed to a generic stationary random function with finite correlation length, plays undoubtedly a crucial role in the analysis. This notwithstanding, we believe, with She et al[13], that the properties (i), (ii) and (iii) apply, in a suitable sense, to solutions of Burgers' equations with initial data in the universality class of stationary processes $u_0(x)$ with short-range spatial correlation, to which the Central limit theorem

$$\frac{1}{\sqrt{N}} \int_0^{Nx} u_0(y)\,dy \approx \beta(x), \quad N \gg 1 \,,$$

applies. This class includes a variety of strongly mixing processes, such as finite-state, stationary Markov processes. In this broader context, the statement is that properties (i), (ii) and (iii) hold *asymptotically.* Let δ represent the ratio between the correlation length of $u_0(x)$ and the "integral" length scale at which physical observations are made. If $\delta \ll 1$ it makes sense to consider the rescaled functions

$$u_\delta(x,t) = \frac{1}{\delta^{\frac{1}{3}}} u\left(\frac{x}{\delta^{\frac{2}{3}}}, \frac{t}{\delta}\right).$$

It is not hard to see that, as $\delta \to 0$, the sequence $u_\delta(x,t)$ converges in distribution to the solution of BT with white-noise initial data. As an example, let us take $u_0(x)$ to be the Ornstein-Uhlenbeck process. From the discussions above, we have the following picture for u. On one hand, in light of the works of Sinai[14] and She *et al*,[13] the set of shock locations of u at a fixed time is not expected to be discrete, since the Ornstein-Uhlenbeck process behaves like Brownian motion at small scales. On the other hand, if we look at the long-time, large-scale features, and if we pick out shocks of amplitude greater than $\delta^{1/3}$, then these shocks tend to be discrete and obey the $s^{\frac{1}{2}}$ law.

Finally, in hindsight, the proofs of (i), (ii) and (iii) suggest that these features are valid not only for BT, (i.e. for (2)) but also in the case of general scalar conservation laws

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}[f(u(x,t))] = 0\,, \tag{46}$$

where $f(u)$ is convex and grows sufficiently rapidly as $|u| \to \infty$. In particular, the present results suggest that the statistical properties of the set of small shocks in hyperbolic conservation laws with short-range correlated initial data are primarily due to multiple shock interactions through inelastic collisions, and the specific form of the flux function giving the wave speed is irrelevant. On the other hand, the tails of the distribution of large shocks depend strongly on the form of the nonlinearity as suggested by the analysis of Sect. 5. These tails are associated to the probabilities of rare events for the Gaussian distribution. A more detailed analysis of (46) will be taken up in future work.

Note added in proof: Recently, Vergassola et al. [16] obtained scaling results for $P(S)$, $S \gg 1$, for general Gaussian initial data. These results agree with ours in the case $u_0 =$ white noise.

Acknowledgements. We thank U. Frisch, C.D. Levermore, A.J. Majda, H.P. McKean, G.C. Papanicolaou, J.M. Pitman, A.-S. Snitzman, M. Vergassola and M. Yor for fruitful discussions on this subject, and A. Chorin for pointing out reference 2. We are grateful to the Universities of Paris VI (Analyse Numérique), Paris VII (Mathematiques Appliqueés) and to the Forschung Institut für Mathematik-ETH Zentrum, for their warm hospitality during the completion of this work. We also gratefully acknowledge support from NSF (DMS-9207085, DMS-9100383), U.S. Army (ARO-DAAL-03-92-G0011), and the U.S. Air Force (AFOSR-90-0090).

Appendix: Time Reversal, Convex Hull of Brownian Motion and Moment Estimates for $\widetilde{X}$

In this section we present some technical results which were used in Sects. 3 and 4, concerning the convex envelope of Brownian motion and the behavior of certain conditional diffusions. The first result is a characterization of the convex hull of a Brownian graph over a finite time interval used in Sect. 3. Here we follow the method of J. Pitman[9] for Brownian motion on an infinite interval, without substantial modifications.

Proposition 8. *Let $\beta(y)$, $0 \leqq y \leqq Y_0$, be standard Brownian motion, and let $\beta^{**}(y)$ be the convex hull of this graph. Then $\beta^{**}(y)$ is a polygonal curve and*

$d\beta^{**}(y)/dy$ has countably many jump discontinuities, which accumulate only near $y = 0$ and $y = Y_0$. Let $\{y_k\}$, $k = 0, \pm 1, \pm 2, \dots$ denote the set of discontinuities, with $\cdots < y_k < y_{k+1} < \cdots$, and let

$$x_k = \frac{\beta^{**}(y_k) - \beta^{**}(y_{k-1})}{y_k - y_{k-1}}, \quad k = 0, \pm 1, \pm 2, \dots ,$$

be the successive slopes of $\beta^{**}(y)$. Then,
(i) The process (x_k, y_k) has independent increments;
(ii) Given $(x_{k-1}, y_{k-1})\beta(Y_0)$ and $\beta(y_{k-1})$, the increment $\delta x_k = x_k - x_{k-1}$ is distributed uniformly in the interval

$$0 < \delta x_k < \frac{\beta(Y_0) - \beta(y_{k-1})}{Y_0 - y_{k-1}} - x_{k-1} .$$

(iii) Set $a_{k-1} = \frac{\beta(Y_0)-\beta(y_{k-1})}{Y_0 - y_{k-1}} - x_{k-1}$.
Then, given $(x_{k-1}, y_{k-1}), \beta(Y_0)$ and $\beta(y_{k-1})$, the increment $y_k - y_{k-1}$ is distributed like

$$\left[\frac{1}{Y_0 - y_{k-1}} + \frac{1}{\theta}\right]^{-1} ,$$

where θ is a random variable with density

$$\frac{a_{k-1}}{\sqrt{2\pi\theta}} e^{-\frac{a^2_{k-1}\theta}{2}}, \quad \theta > 0 .$$

Proof. For a given $x > 0$, let $y(x)$ be the last time that the process

$$\beta(y) - y \cdot x, \quad 0 \leqq y \leqq Y_0 ,$$

reaches its minimum. Clearly, $0 < y(x) < Y_0$ with probability 1, since

$$\underline{\lim}_{y \to 0} \beta(y)/y = -\infty \text{ and } \overline{\lim}_{y \to Y_0} (\beta(y) - \beta(Y_0))/(y - Y_0) = +\infty .$$

By Millar's theorem $\{\beta(y), y < y(x)\}$ and $\{\beta(y), y > y(x)\}$ are independent given $(y(x), \beta(y(x)))$. For $z > 0$, the function

$$G_1(z) = \frac{\beta(z + y(x)) - \beta(y(x))}{z}, \quad 0 < z < Y_0 - y(x)$$

represents the slope of the chord passing through the points $(y(x), \beta(y(x)))$ and $(y(x) + z, \beta(y(x) + z))$. Due to the definition of $y(x)$, the function

$$G(t) = G_1(z) - x = \frac{\beta(z + y(x)) - \beta(y(x))}{z} - x$$

is positive for $0 < z < Y_0 - y(x)$. Let $t = 1/z$; i.e. we make a time inversion. Then, we have

$$\frac{1}{Y_0 - y(x)} < t < +\infty .$$

The process

$$\widetilde{X}(t) = G\left(\frac{1}{t}\right) = t\left[\beta\left(\frac{1}{t} + y(x)\right) - \beta(y(x))\right] - x$$

satisfies

$$\widetilde{X}\left[\frac{1}{Y_0 - y(x)}\right] = \frac{\beta(Y_0) - \beta(y(x))}{Y_0 - y(x)} - x \equiv a(x)\,.$$

Moreover, $\widetilde{X}(t)$ is a Brownian motion conditioned on not hitting $\widetilde{X} = 0$, because if β is standard Brownian motion the increments $t[\beta(\frac{1}{t} + y(x)) - \beta(y(x))]$ are again Brownian. Therefore, by William's characterization of conditional Brownian motion[13], $\widetilde{X}(t)$ is a Bessel-3 process. In particular,

$$\delta x^* = \inf_{t > (Y - y(x))^{-1}} (\widetilde{X}(t))$$

is uniformly distributed on the interval

$$0 < \delta x^* < \widetilde{X}\left(\frac{1}{Y_0 - y(x)}\right) = a(x)\,.$$

The results of Williams also yield a characterization of the time interval $t^* - \frac{1}{Y_0 - y(x)}$ after which $\widetilde{X}(t)$ reaches its minimum[13]. Accordingly, $t^* - (Y_0 - y(x))^{-1}$ is distributed like the first passage time of standard Brownian motion through $\widetilde{X} = a(x)$. Hence[9] $[t^* - (Y_0 - y(x))^{-1}]^{-1} \equiv \theta$ has density

$$\frac{a(x)}{\sqrt{2\pi\theta}} \exp\left\{-\frac{a(x)^2\theta}{2}\right\}, \quad \theta > 0\,,$$

so that

$$t^* \approx \frac{1}{\theta} + \frac{1}{(Y_0 - y(x))}\,.$$

Therefore, the chord passing through $(y(x), \beta(y(x)))$ and $(z + y(x), \beta(z + y(x)))$ with smallest slope corresponds to $z^* = \frac{1}{t^*}$, where

$$z^* \approx \left[\frac{1}{\theta} + \frac{1}{(Y_0 - y(x))}\right]^{-1}\,.$$

Due to the Markovian structure of $\beta^{**}(y)$; this process can be repeated with the new values $x' = x + \delta x^*$ and $y' = y(x) + z^*$. Successive applications of this method yields a sequence of $\{y_k\}$ of discontinuities of $d\beta^{**}(y)/dy$ and a sequence of slopes $\{x_k\}$ for $y > y(x)$. Since the starting slope x was chosen arbitrarily, we have shown that the set of discontinuities of $d\beta/dy$ is discrete. The characterization of the distributions of (x_k, y_k) also follows. □

Remark. The method of proof is due to Pitman[9], we have included the present case ($Y_0 < +\infty$) just for the sake of completeness. Notice that if $x < 0$, then, as $Y_0 \to \infty, y(x)$ converges to the abscissa of the contact point of the full Brownian graph with the supporting line of the slope x and $a(x)$ converges to $-x$ so the results of references 4 and 9 are recovered. □

The next result presented here is the

Proof of Lemma 5. We prove here the estimate

$$E\,\{(\widetilde{X}_{u(0)}(t))^\gamma\} \leqq C_\gamma t^{\gamma/2}, \quad t > 1 \tag{A.1}$$

for all $\gamma > 0$, where C_γ is a numerical constant which is independent of t. For this purpose, we shall use the equivalence of the measures in path-space corresponding to $\beta(y) + \frac{1}{2}y^2 = F(y)$ and $\beta(y)$, for $|y| \leqq Y_0 < +\infty$. The estimate will be established by proving that

$$\mathrm{Pr.}\{\widetilde{X}_{u(0)}(t) > \alpha\} \leqq \exp\left\{-C\left(\frac{\alpha^2}{t}\right)\right\} \tag{A.2}$$

for $\frac{\alpha^2}{t} > 1$, where C is a numerical constant. Clearly (A.1) follows from (A.2) and the formula

$$E((\widetilde{X}_{u(0)}(t))^\gamma) = \gamma\int_0^\infty \alpha^{\gamma-1}\mathrm{P}(\widetilde{X}_{u(0)}(t) > \alpha)d\alpha\,.$$

Let Y_0 be a large positive number. Then, for all α,

$$\mathrm{Pr.}[\widetilde{X}_{u(0)}(t) > \alpha] \leqq \mathrm{Pr.}\{\widetilde{X}_{u(0)}(t) > \alpha;\ |y(0)| \leqq Y_0\} + \exp\{-CY_0^3\}\,; \tag{A.3}$$

where we used the estimate (19). Suppose that $\frac{1}{t} < Y_0$. Then,

$$\begin{aligned}&\mathrm{Pr}\{\widetilde{X}_{u(0)}(t) > \alpha;\ |y(0)| \leqq Y_0\}\\ &\quad\leqq \mathrm{Pr}\left\{\frac{1}{t}[F(y_* + t) - F(y_*)] > \alpha;\ |y_*| \leqq Y_0\right\},\end{aligned}$$

where y_* is such that $\min\limits_{|y|<2Y_0} F(y) = F(y_*)$.

By Girsanov's theorem[7]

$$\begin{aligned}&\mathrm{Pr}\left\{t\left[F\left(y_* + \frac{1}{t}\right) - F(y_*)\right] > \alpha;\ |y_*| \leqq Y_0\right\}\\ &\quad= E\left\{t\left[\beta\left(z_* + \frac{1}{t}\right) - \beta(z_*)\right] > \alpha;\ |z_*| < Y_0;\ M(\beta, Y_0)\right\},\end{aligned}$$

where z_* is such that $\min\limits_{|y|<2Y_0} \beta(y) = \beta(z_*)$, and

$$\begin{aligned}M(\beta, Y_0) &= \exp\left\{\int_{-2Y_0}^{2Y_0} y\,d\beta(y) - \frac{1}{2}\int_{2Y_0}^{2Y_0} y^2\,dy\right\}\\ &= \exp\left\{\int_{-2Y_0}^{2Y_0} y\,d\beta(y)\right\} \times \exp\left(-\frac{8}{3}Y_0^3\right).\end{aligned}$$

Using the Cauchy–Schwartz inequality

$$\begin{aligned}&\mathrm{Pr.}\left\{t\left[F\left(y_* + \frac{1}{t}\right) - F(y_*)\right] > \alpha;\ |y_*| < 2Y_0\right\}\\ &\left(\mathrm{Pr.}\left\{t\left[\beta\left(z_* + \frac{1}{t}\right) - \beta(z_*)\right] > \alpha;\ |z_*| < 2Y_0\right\}\right)^{1/2} \times (E[M(\beta, Y_0)^2])^{1/2}\\ &= \left(\mathrm{Pr.}\left\{t\left[\beta\left(z_* + \frac{1}{t}\right) - \beta(z_*)\right] > \alpha;\ |z_*| < 2Y_0\right\}\right)^{\frac{1}{2}} \times e^{8/3Y_0^3}\,.\end{aligned} \tag{A.4}$$

By previous results, $\frac{1}{t}(\beta(z_* + \frac{1}{t}) - \beta(z_*))$, is a Bessel-3 process starting at time $t_0 = \frac{1}{2Y_0 - z_*}$ at position $\frac{\beta(2Y_0)-\beta(z_*)}{2Y_0 - z_*} \equiv a_0$.

We note that, since $|z_*| < 2Y_0$,

$$t_0 < \frac{1}{Y_0}$$

and

$$a_0 < \frac{2 \max\limits_{|s|<2Y_0} |\beta(s)|}{Y_0} .$$

By standard estimates,

$$\Pr.[\tilde{\beta}_{a,t_0}(t) > \alpha] \leqq e^{-\frac{(\alpha-a)^2}{2(t-t_0)}}$$

and

$$\Pr.\left[a_0 > \frac{\alpha}{2}\right] \leqq e^{-C\alpha^2 Y_0} ,$$

for some constant C. Therefore, if $t > \frac{2}{Y_0}$,

$$\begin{aligned}
&\Pr.\left\{\frac{1}{t}\left(\beta\left(z_* + \frac{1}{t}\right) - \beta(z_*)\right) > \alpha,\ |z_*| < Y_0\right\} \\
&\leqq \Pr.\left\{\frac{1}{t}\left(\beta\left(z_* + \frac{1}{t}\right) - \beta(z_*)\right) > \alpha;\ a_0 < \frac{\alpha}{2};\ |z_*| < Y_0\right\} + \Pr.\left\{a_0 > \frac{\alpha}{2}\right\} \\
&\leqq e^{-C_1\frac{\alpha^2}{t}} + e^{-C_2\alpha^2 Y_0} , \qquad \text{(A.5)}
\end{aligned}$$

where we used the fact that $\frac{1}{t}(\beta(z_* + \frac{1}{t}) - \beta(z_*))$ is a Bessel-3 process starting at $a_0 < \alpha/2$ at time $t_0 < \frac{1}{Y_0}$. Putting together the estimates (A.3), (A.4), and (A.5) for $t > \frac{2}{Y_0}$ we have

$$\Pr.\{\widetilde{X}_{u(0)}(t) > \alpha\} \leqq 2\left(e^{\frac{8}{3}Y_0^3 - C_1\frac{\alpha^2}{2t}} + e^{\frac{8}{3}Y_0^3 - C_2\frac{\alpha^2 Y_0}{2}} + e^{-C_3 Y_0^3}\right), \qquad \text{(A.6)}$$

C_1, C_2, C_3 being numerical constants. Let us choose Y_0 such that

$$Y_0^3 = C_4(\alpha^2/t) . \qquad \text{(A.7)}$$

Note that this choice of Y_0 implies that, for $t > 1$,

$$t > t^{1/3} = \frac{C_4^{1/3}\alpha^{2/3}}{Y_0} ,$$

so the condition $t > \frac{2}{Y_0}$ holds if $C_4^{1/3}\alpha^{2/3} > 2$. Substituting (A.7) into (A.6) we obtain

$$\begin{aligned}
&\Pr.\{\widetilde{X}_{u(0)}(t) > \alpha\} \\
&\leqq e^{\frac{8}{3}C_4\frac{\alpha^2}{t} - C_1\frac{\alpha^2}{t}} + e^{\frac{8}{3}C_4\frac{\alpha^2}{t} - C_2 C_4^{\frac{1}{3}}\alpha^2\left(\frac{\alpha^2}{t}\right)^{\frac{1}{3}}} + e^{-C_3 C_4\frac{\alpha^2}{t}} \\
&= e^{-\left(C_1 - \frac{8}{3}C_4\right)\frac{\alpha^2}{t}} + e^{-[C_2 C_4^{\frac{1}{3}}\left(\frac{\alpha^2}{t}\right)^{\frac{1}{3}}t - \frac{8}{3}C_4]\frac{\alpha^2}{t}} + e^{-C_3 C_4\frac{\alpha^2}{t}} .
\end{aligned}$$

Therefore, if we choose C_4 such that $C_1 - 8/3C_4 > 0$, then for $\alpha > \alpha_0, t > 1$ and

$$\frac{\alpha^2}{t} > \frac{1}{2}\left(\frac{8}{3}\right)^3 C_3^2 C_3^{-3} ,$$

we obtain the final estimate

$$\mathrm{Pr.}\{\widetilde{X}_{u(0)}(t) > \alpha\} \leqq e^{-C\left(\frac{\alpha^2}{t}\right)} ,$$

as claimed. This concludes the proof of Lemma 5.

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Communicated by T. Spencer