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Markov Partitions and Feigenbaum-like Mappings

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Abstract: We construct a Markov partition for a Feigenbaum-like mapping. We prove that this Markov partition has bounded nearby geometry property similar to that for a geometrically finite one-dimensional mappings [8]. Using this property, we give a simple proof that any two Feigenbaum-like mappings are topologically conjugate and the conjugacy is quasisymmetric.

Contents

0. Introduction

Markov process has been introduced by Sinai and Bowen, etc. in the study of dynamical systems in the 1960's. Sinai [16] and Bowen [2] constructed a Markov partition for a hyperbolic dynamical system. Using Markov partitions, they related hyperbolic dynamical systems with symbolic dynamical systems. Thus hyperbolic dynamical systems can be studied topologically through symbolic dynamical systems rather easily. Indeed, to construct a Markov partition for a dynamical system is quite important in the study of dynamical systems. In this note, I shall give a construction of a partition for a Feigenbaum-like mapping. I shall prove that this partition has all but finiteness properties as those of a Markov partition for a hyperbolic dynamical system. It will be called an (infinite) induced Markov partition. A Feigenbaum-like mapping is definitely not hyperbolic for its critical orbit is recurrent. However, from the construction and properties of this induced Markov partition, one can study topologically and geometrically a Feigenbaum-like mapping

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by using methods in hyperbolic dynamical systems. Moreover, using the properties of induced Markov mappings, I give a simple proof that any two Feigenbaum-like mappings are topologically conjugate and the conjugacy is quasisymmetric in this note. The same idea has been successfully used to give a construction of a certain Markov partition of the Julia set of the Feigenbaum quadratic polynomial. This partition enables us to prove a long standing conjecture that the Julia set of the Feigenbaum quadratic polynomial is locally connected. The proof of this theorem and its generalization will be written in a forthcoming paper [9].

This paper is organized as follows: I shall introduce notations and review some known results in Sect. 1. In Sect. 2, I shall construct the induced Markov map from a Feigenbaum-like mapping and prove one of the main results, i.e., this Markov map has bounded nearby geometry. In the last section, I shall apply the property of bounded nearby geometry to prove another main result, i.e., any two Feigenbaum-like mappings are topologically conjugate and the conjugacy between them is quasisymmetric.

1. Infinitely Renormalizable Unimodal Mappings

Suppose I is the interval $[-1,1]$. A continuous function f from I into itself is called a unimodal mapping if $f(x)=h(-|x|^{\gamma})$ for some real number $\gamma>1$ and some homeomorphism h from $[-1,0]$ onto $[-1,h(0)]$. The Schwarzian derivative $S(g)$ of a C^3-diffeomorphism g from an interval onto another interval is, by definition,

$$S(g)=\frac{g'''}{g'}-\frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

A unimodal mapping $f(x)=h(-|x|^{\gamma})$ is called a S-unimodal if h is a C^3-diffeomorphism from $[-1,0]$ onto $[-1,h(0)]$ and satisfies that $S(h)(x)\leqq 0$ for all x in $[-1,0]$.

A S-unimodal mapping f is said to be renormalizable if there is a subinterval $J=[-a,a]$ of I for some $a>0$ and an integer $n\geqq 2$ such that $f^{\circ n}$ are monotone when restricted on $[-a,0]$ and on $[0,a]$ and $f^{\circ n}(J)\subset J$ and $f^{\circ i}(J)\cap J=\emptyset, 0<i<n$. One can normalize J to I by a linear transformation $\alpha(x)=px$ such that $\mathscr{R}(f)=\alpha^{-1}\circ f^{\circ n}\circ\alpha$ is a unimodal mapping again (see below). To fix notations, I always assume that $n\geqq 2$ is the smallest such integer and J is the biggest such interval. Thus one can say that f is once n-renormalizable and $\mathscr{R}(f)$ is the renormalization of f.

Suppose f is a once n_1-renormalizable S-unimodal mapping. If $\mathscr{R}(f)$ is once n_2-renormalizable, then f is said to be twice (n_1,n_2)-renormalizable. Further, f is k-times $(n_1,n_2,\dots,n_k)$-renormalizable if $\mathscr{R}^{\circ i}(f)$ is n_{i+1}-renormalizable for $0\leqq i<k$ and is infinitely $(n_1,n_2,\dots,n_k,\dots)$-renormalizable if $\mathscr{R}^{\circ i}(f)$ is n_{i+1}-renormalizable for every integer $i\geqq 0$. A S-unimodal mapping f is infinitely $(n_1,n_2,\dots,n_k,\dots)$-renormalizable if and only if there is a sequence $\{I_k=[-a_k,a_k]\}_{k=1}^{\infty}$ of nested intervals so that $f^{\circ m_k}$ is monotone when restricted on $[-a_k,0]$ and on $[0,a_k]$, $f^{\circ i}(I_k)\cap I_k^{\circ}=\emptyset$ for $0<i<m_k$, and $f^{\circ m_k}(I_k)\subset I_k$, where $m_k=\prod_{i=1}^{k}n_i$. An infinitely $(n_1,n_2,\dots,n_k,\dots)$-renormalizable S-unimodal mapping f is said to be of bounded type if $\{n_k\}_{k=1}^{\infty}$ is a bounded sequence, otherwise f is said to be of unbounded type. In particular, if all $n_k=2$, then f is called a Feigenbaum–Coullet–Tresser-like mapping, in short, Feigenbaum-like mapping [3–5].

Suppose $f(x)=h(-|x|^{\gamma})$ is an infinitely $(n_1,n_2,\dots,n_k,\dots)$-renormalizable S-unimodal mapping and $m_k=\prod_{i=1}^{k}n_i$. Let $\mathscr{R}^{\circ k}(f)=\alpha_k^{-1}\circ f^{\circ m_k}\circ\alpha_k$ be the

k^{th}-renormalization of f, where $\alpha_k(x) = -p_k x$ is the linear rescale from $I = [-1,1]$ to $I_k = [-a_k, a_k]$, where $a_k = |p_k|$ and $I_k = [-a_k, a_k]$ is the maximal interval containing 0 such that

a. $f^{\circ m_k}$ is monotone when restricted on $[-a_k, 0]$ and on $[0, a_k]$,
b. $f^{\circ m_k}(I_k) \subset I_k$, and $f^{\circ i}(I_k) \cap I_k^{\circ} = \emptyset, 0 < i < m_k$,
c. $f^{\circ m_k}$ has exactly two fixed points p_k and q_k in I_k which are also periodic points of f of period m_k.

Suppose $c(i) = f^{\circ i}(0)$ is the i^{th} critical value of f. For each $k > 0$, let $I_k(i) = f^{\circ i}(I_k)$ and $p_k(i) = f^{\circ i}(p_k)$ be the images of I_k and p_k under the i^{th}-iterate of f. Then $I_k(i)$ is an interval bounded by $p_k(i)$ and $c(i)$ for $0 < i < m_k$. Note that $I_k(0) = I_k$ is an interval bounded by $-p_k$ and p_k and $I_k(m_k)$ is an interval bounded by p_k and $c(m_k)$. The mapping $f|I_k$ is fold from I_k onto $I_k(1)$ and all other mappings $f|I_k(i)$ from $I_k(i)$ to $I_k(i+1)$ are homeomorphisms for $1 \leqq i < m_k$ (see Fig. 1). Hence the k^{th}-renormalization can be written into a form $\mathcal{R}^{\circ k}(f)(x) = h_k(-|x|^{\gamma})$ as a S-unimodal mapping, where $h_k = \alpha_k^{-1} \circ f^{\circ(m_k-1)} \circ h \circ \tilde{\alpha}_k$ is a diffeomorphism from I into $h_k(I)$, where $\tilde{\alpha}_k(x) = |p_k|^{\gamma} x$.

The nonlinearity $N(g)$ of a C^2-diffeomorphism g from an interval onto another interval is, by definition, $N(g) = g''/g'$. The a priori real bounds for the nonlinearities of renormalizations of f have been found in [18] (see also [6]). These a priori real bounds depend only on the power law $-|x|^{\gamma}$.

Lemma 1 (Bounded and eventually universally bounded). *There is a universal constant $C(\gamma) > 0$ and a sequence $\{C(k,\gamma)\}_{k=1}^{+\infty}$ of positive real numbers such that $C(k,\gamma) \to C(\gamma)$ as $k \to \infty$ and*

$$\max_{x \in h_k([-1,0])} |N(h_k^{-1})(x)| \leqq C(k,\gamma).$$

Remark 1. For a C^2-diffeomorphism h, $N(h^{-1})(x) = -h''(y)/(h'(y))^2$, where $x = h(y)$.

The proof of Lemma 1 can be found in [12, 18]. The next two lemmas are actually two steps in the proof of Lemma 1. I would like to highlight them. I shall first state a well-known result (see [12, 18, etc.]) to estimate the nonlinearity of a C^3-diffeomorphism.

Lemma 2 (C^3-Koebe distortion lemma). *Suppose g is a C^3 function on an open interval $J = (a,b)$ and $S(g)(x) \geqq 0$ for all x in J. Then*

$$|N(g)(x)| \leqq \frac{2}{d(x, \partial J)}$$

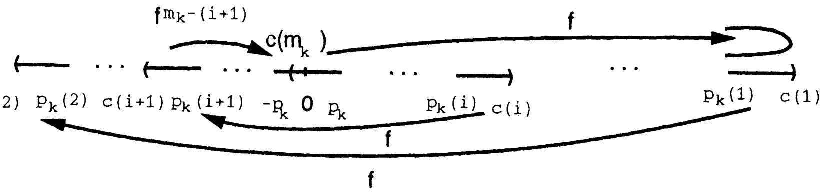

Fig. 1.

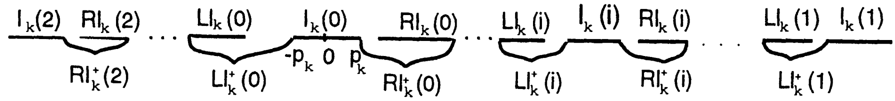

Fig. 2.

for any x in J, where $d(x, \partial J) = \min\{|x-a|, |x-b|\}$ is the distance between x and the boundary of J.

The second one is about Koebe space around every interval $I_k(i)$ (see [12, 18]). Let $\xi_k = \{I_k(i)\}_{i=0}^{m_k-1}$ for $k = 1, 2, \ldots$ be the hierarchical system induced from f. For each interval $I_k(i)$, use $LI_k(i)$ and $RI_k(i)$ to denote the intervals in ξ_k adjacent to $I_k(i)$ and in the left and right sides of $I_k(i)$, respectively, (there is only $LI_k(1)$ or $RI_k(2)$ in ξ_k). Let $LI_k^+(i)$ be the smallest interval containing $LI_k(i)$ and the left end-point of $I_k(i)$ and let $RI_k^+(i)$ be the smallest interval containing $RI_k(i)$ and the right end-point of $I_k(i)$ for $i = 0$ or $3 \leqq i < m_k$. Let $LI_k^+(2) = [-1, c(2)]$ and $RI_k^+(1) = [c(1), 1]$ (see Fig. 2).

Lemma 3. *There is a universal constant $C(\gamma) > 0$ and a sequence $\{C(k,\gamma)\}_{k=1}^{+\infty}$ of positive real numbers such that $C(k,\gamma) \to C(\gamma)$ as $k \to \infty$ and*

$$\min\{|LI_k^+(0)|, |RI_k^+(0)|\} \geqq C(k,\gamma)|I_k(0)|\,.$$

2. Markov Maps Induced From Feigenbaum-like Mappings

Suppose $f(x) = h(-|x|^\gamma)$ is a Feigenbaum-like mapping. The hierarchical system $\xi_k = \{I_k(i)\}_{i=0}^{m_k-1}$ for $k = 1, 2, \ldots$ of f is quite simple. For each $k > 0$, the interval $I_k(0)$ is bounded by a periodic point p_k of f of period 2^{k-1} and $-p_k$. The mapping $f^{\circ 2^k}|I_k(0)$ has two fixed points p_k and p_{k+1}. Every interval $I_k(i)$ in ξ_k contains only two intervals $I_{k+1}(i)$ and $I_{k+1}(2^k+i)$ in ξ_{k+1} which have a common endpoint $p_{k+1}(i)$ for $0 \leqq i < 2^k$.

Using the sequence of nested intervals $\{I_k(0)\}_{k=1}^{\infty}$, I construct a partition in $I = [-1, 1]$. Let P_{-0} and P_0 be the closures of the left and right connected components of $I \backslash I_1(0)$. Inductively, let P_{-k} and P_k be the closures of the left and right connected components of $I_k(0) \backslash I_{k+1}(0)$. Finally set $P_\infty = \{0\}$. The collection $\beta_0 = \{P_{-0}, P_0, P_{-1}, P_1, \ldots, P_{-k}, P_k, \ldots, P_\infty\}$ forms a partition of $I = [-1, 1]$ (see Fig. 3), that is, P_i and P_j have disjoint interiors for $i \neq j$ and $I = P_\infty \cup \bigcup_{k=1}^{\infty}(P_{-k} \cup P_k)$. Let F be the function defined as $F(0) = 0$ and

$$F(x) = \begin{cases} f(x), & x \in P_{-0} \cup P_0; \\ f^{\circ 2}(x), & x \in P_{-1} \cup P_1; \\ \vdots & \\ f^{\circ 2^i}(x), & x \in P_{-i} \cup P_i; \\ \vdots & \end{cases} .$$

Then F is continuous on I (see Fig. 3).

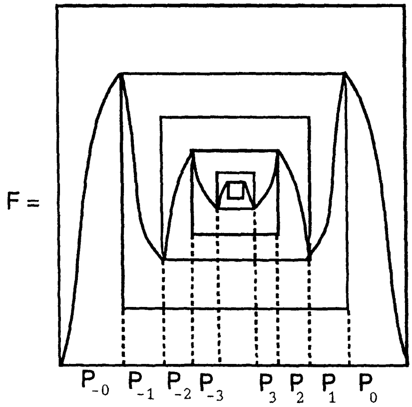

Fig. 3.

Lemma 4. *For every even integer* $k = 2n \geqq 0$, $F(P_{\pm k}) = \bigcup_{i=k}^{\infty} P_{-i} \cup \bigcup_{i=k+1}^{\infty} P_i$, *and for every odd integer* $k = 2n+1 > 0$, $F(P_{\pm k}) = \bigcup_{i=k+1}^{\infty} P_{-i} \cup \bigcup_{i=k}^{\infty} P_i$.

Proof. It can be seen from Fig. 3.

From Lemma 4, the mapping F and the partition β_0 satisfy the Markov property in the sense that the image of every element in the partition β_0 is the union of some intervals in the partition β_0. Thus I call F the induced Markov mapping from f.

Let $g_{\pm i} = (F|P_{\pm i})^{-1}$ be the inverse branches of F for $1 \leqq i < \infty$. Suppose $w = i_0 i_1 \cdots i_{k-1}$ is a finite sequence of $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{-0\}$. It is said to be admissible if the range P_{i_l} of g_{i_l} is contained in the domain $F_{i_{l-1}}(P_{i_{l-1}})$ of $g_{i_{l-1}}$ for $l = 1, \ldots, k-1$. For an admissible sequence $w = i_0 i_1 \cdots i_{k-1}$, define the composition $g_w = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{k-1}}$. Use $D(g_w)$ to denote the domain of g_w and $|D(g_w)|$ to denote the length of the iterval $D(g_w)$.

Definition 1. *We say the induced Markov map F from f has bounded nearby geometry if there is a constant* $C = C(f) > 0$ *such that*

(i) $C^{-1} \leqq |P_k|/|\bigcup_{i=k+1}^{\infty} P_i| \leqq C$ and $C^{-1} \leqq |P_{-k}|/|\bigcup_{i=k+1}^{\infty} P_{-i}| \leqq C$ for all $k \geqq 0$, and

(ii) $|N(g_w)(x)| \leqq C/|D(g_w)|$ for all x in $D(g_w)$ and all finite admissible sequence w of $\overline{Z}$.

Remark 2. Condition **(ii)** implies that the distortion $|\log(|g_w(x)|/|g_w(y)|)|$ of g_w at any x and y in $D(g_w)$ is bounded by C. Condition **(i)** is an analogy to bounded nearby geometry defined in [8] for geometrically finite one-dimensional mappings.

Theorem 1. *Suppose* $f(x) = h(-|x|^{\gamma})$ *is a Feigenbaum-like S-unimodal mapping. Then the induced Markov mapping F from f has bounded nearby geometry.*

Before proving Theorem 1, I shall prove some useful lemmas.

Lemma 5. *Suppose h from $[-1,0]$ to $\mathbf{R}^1$ is a C^3 orientation-preserving diffeomorphism and $S(h)(x) \leqq 0$ for all x in $[-1,0]$. Suppose ϕ is a linear fractional transformation satisfying that*

(a) $\phi(a) = h(a)$ for $a = 0$ and -1, and
(b) $N(h^{-1})(-1) \geqq N(\phi^{-1})(-1)$.

Then $\phi(x) \leqq h(x)$ for all x in $[-1,0]$.

Proof. Let $Z = h^{-1} \circ \phi$. Then $Z(a) = a$ for $a = 0$ and -1 and for x in $[-1,0]$,

$$S(Z)(x) = (\phi'(x))^2 \cdot (S(h^{-1}))(\phi(x)) \geqq 0 .$$

The goal is to show that $Z(x) \leqq x$ for x in $[-1,0]$.

Using **(b)**, one can get $N(Z)(-1) \geqq 0$. This implies that $Z''(-1) \geqq 0$, and moreover,

$$Z(x) \geqq F(x) = -1 + Z'(-1)(x+1)$$

for small $x+1 \geqq 0$. Thus $Z(x) \geqq F(x)$ for all x in $[-1,0]$ since $S(Z)(x) \geqq 0$ for all x in $[-1,0]$. In particular, $Z(0) \geqq F(0)$. Hence $Z'(-1) \leqq 1$. Therefore, $Z(x) \leqq x$ for all x in $[-1,0]$ because $S(Z)(x) \geqq 0$ for all x in $[-1,0]$. So $\phi(x) \leqq h(x)$ for all x in $[-1,0]$.

Let $SF(\gamma, C)$ be the subspace of S-unimodal mappings $f(x) = h(-|x|^\gamma)$ so that $\min_{x\in h([-1,0])} (N(h^{-1})(x)) \geqq -C$.

Lemma 6. *There is a constant $C_1 = C_1(\gamma, C) > 0$ such that $f(0) \geqq C_1$ for every infinitely renormalizable mapping in $SF(\gamma, C)$.*

Proof. Suppose $f(x) = h(-|x|^\gamma)$ is a mapping in $SF(\gamma, C)$. Since h is a C^3 orientation-preserving diffeomorphism and $S(h)(x) \leqq 0$ for all x in $[-1,0]$, one can compare h with some linear fractional transformation $\phi(x) = (ax+b)/(cx+d)$. Let ϕ be the linear fractional trnsformation satisfying that **(a)** $\phi(a) = h(a)$ for $a = -1$ and 0, and **(b)** $N(\phi^{-1})(-1) = -C$. Then

$$\phi(x) = \frac{x+1}{\frac{C}{2}(x+1) + \frac{1}{f(0)+1} - \frac{C}{2}} - 1 .$$

From Lemma 5, $\phi(x) \leqq h(x)$ for all x in $[-1,0]$.

Suppose, at the moment, $c = f(0) > 0$ is a variable. Let $C_1 = C_1(\gamma, C) > 0$ be the smallest solution of $\phi(-|c|^\gamma) = 0$. Then for $0 < c < C_1$, $f^{\circ 2}(0) \geqq \phi^{\circ 2}(0) = \phi(-|c|^\gamma) > 0$. This says that $f^{\circ 2}$ has an attractive fixed point and thus is not once renormalizable. Hence $f(0) > C_1$ if f is infinitely renormalizable.

For a S-unimodal mapping $f(x) = h(-|x|^\gamma)$ with $f(0) > 0$, let q_f be the fixed point of f in (0,1).

Lemma 7. *There is a constant $C_2 = C_2(\gamma, C) > 0$ such that $q_f \geqq C_2$ for all infinitely renormalizable f is $SF(\gamma, C)$.*

Proof. Let

$$\phi_0(x) = \frac{x+1}{\frac{C}{2}(x+1) + \frac{1}{C_1+1} - \frac{C}{2}} - 1$$

and C_2 be the fixed point of $\phi_0(-|x|^\gamma)$ in (0,1). Take

$$\phi(x) = \frac{x+1}{\frac{C}{2}(x+1) + \frac{1}{f(0)+1} - \frac{C}{2}} - 1 \, .$$

Then the fixed point q' of $\phi(-|x|^\gamma)$ in (0,1) is greater than C_2 since $f(0) > C_1$. But $q_f \geqq q' > C_2$.

Proof of Theorem 1. Suppose $\mathscr{R}^{\circ k}(f) = h_k(-|x|^\gamma)$ is the k^{th}-renormalization of f. It is the rescale of $f^{\circ 2^k}|I_k(0)$. From Lemma 1, there is a constant $C = C(f) > 0$ such that $\max_{x \in h_k([-1,0])} |N(h_k^{-1})(x)| \leqq C$ for all $k \geqq 0$ (set $h_0 = h$). Now Lemma 7 says that there is a constant $C_2 = C_2(\gamma, C) > 0$ so that

$$\frac{|I_{k+1}(0)|}{|I_k(0)|} \geqq C_2$$

for all $k \geqq 0$ (set $I_0(0) = I$).

Following Lemma 3, there is a constant $C_3 = C_3(f) > 0$ such that

$$\frac{|I_{k+1}(2^k)|}{|I_{k+1}(0)|} \geqq C_3$$

for all $k \geqq 0$ since $I_{k+1}(2^k)$ is either $LI_{k+1}^+(0)$ or $RI_{k+1}^+(0)$. This implies that

$$\frac{|I_{k+1}(0)|}{|I_k(0)|} \leqq C_4 = \frac{1}{2C_3 + 1}$$

for all $k \geqq 0$.

Now take $C_5 = \max\{C_2^{-1}, C_4\}$. Then

$$C_5^{-1} \leqq \frac{|I_{k+1}(0)|}{|I_k(0)|} \leqq C_5$$

for all $k \geqq 0$.

Since $I_k(0) \backslash I_{k+1}(0)$ is the closure of $\bigcup_{i=k+1}^{\infty}(P_{-i} \cup P_i)$, and $|P_{-k}| = |P_k|$, **(i)** of Definition 1 is verified.

Now let me prove **(ii)** of Definition 1. For an integer $i \neq 0$, g_i can be extended to the interval $\Omega_{|i|} = I_{|i|-1}(2^{|i|-1}) \cup D(g_i) \cup I_{|i|}(2^{|i|})$ as a C^3-diffeomorphism and $S(g_i)(x) \geqq 0$ for all x in $\Omega_{|i|}$. For g_0 and g_{-0}, without loss of generality, we may assume that they can be extended to the interval $\Omega_0 = (-\infty, -1] \cup D(g_0) \cup I_1(1)$ and $S(g_0)(x) \geqq 0$ and $S(g_0)(x) \geqq 0$ for all x in Ω_0.

Suppose $w = i_0 i_1 \cdots i_{k-1}$ is an admissible sequence of $\overline{Z} = \mathbf{Z} \cup \{-0\}$ and $g_w = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{k-1}}$. By the definition of an admissible sequence, one can check that

$$|i_0| \leqq |i_i| \leqq \cdots \leqq |i_{k-1}| \, .$$

Hence g_w can be extended to the domain $\Omega_{|i_{k-1}|}$ as a C^3-diffeomorphism and $S(g_w)(x) \geqq 0$ for all x in $\Omega_{|i_{k-1}|}$. We note that $D(g_w) = D(g_{i_{k-1}})$ and the intervals $\Omega_{|i|}$ are nested for $|i| = 0, 1, \ldots$. Then **(ii)** of Definition 1 follows now from Lemmas 2 and 3.

Remark 3. In [10], there is a more general discussion of the induced Markov map from an infinitely renormalizable S-unimodal mapping.

3. Conjugacies Between Feigenbaum-like Mappings

It is known that two Feigenbaum-like S-unimodal mappings are topologically conjugate. The proof of this depends on two deep facts, kneading theory developed by Milnor and Thurston [14] and non-wandering interval theorem proved by Guckenheimer [7] and de Melo and van Strien [13]. Using Markov partitions constructed in this note, I can set a topological model for all Feigenbaum-like mappings as Sinai and Bowen [2, 16] did for hyperbolic dynamical systems. Applying Theorem 1, I give a simple proof of that two Feigenbaum-like S-unimodal mappings are topologically conjugate.

Suppose f is a Feigenbaum-like mapping and $\beta_0 = \{P_{\pm k}\}_{k=0}^{\infty} \cup \{P_\infty\}$ is the induced partition and F is the induced Markov map. Let $A = (a_{ij})$ be the bi-infinite matrix so that $a_{ij} = 1$ if $P_j \subset F(P_i)$ and $a_{ij} = 0$ otherwise. From the construction of F, we can see that **1.** for $i = \pm 2n, n \geqq 0, a_{ij} = 1$ if and only if $|j| > |i|$ or $j = -2n$; **2.** $i = \pm(2n+1) > 0$, $n > 0$, $a_{ij} = 1$ if and only if $|j| > |i|$ or $j = 2n+1$. Now consider the symbolic space $\Sigma_A = \{w = i_0 i_1 \cdots i_k i_{k+1} \cdots | i_k \in \overline{Z} \cup \{\infty\}, a_{i_k i_{k+1}} = 1, k = 0, 1, \ldots\}$ with product topology and the shift map $\sigma_A(w) = i_1 \cdots i_k i_{k+1} \cdots$ if $w = i_0 i_1 \cdots i_k i_{k+1} \cdots$.

A sequence $w_k = i_0 i_1 \cdots i_k$ of $\overline{\mathbf{Z}}$ is admissible if $a_{i_l i_{l+1}} = 1$ for $0 \leqq l < k$. For an admissible sequence $w_k = i_0 i_1 \cdots i_k$, define $g_{w_k} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_k}$ and $P_{w_k} = g_{w_k}(F(P_{i_k}))$.

Lemma 8. *Suppose f is a Feigenbaum-like S-unimodal mapping and F is the induced Markov map. Then F is semi-conjugate to σ_A, this means that there is a continuous surjective map H from Σ_A to I such that $F \circ H = H \circ \sigma_A$.*

Proof. For any $w = i_0 i_1 \cdots i_k i_{k+1} \cdots$ in Σ_A, $w_k = i_0 \cdots i_k$ is admissible for every $k \geqq 0$. Applying Theorem 1, $\bigcap_{k=0}^{\infty} P_{w_k}$ contains only one point x_w. Set $H(w) = x_w$. It is a continuous map from Σ_A to I from Theorem 1. Since $\bigcup_{w_k} P_{w_k} = I$, where w_k runs over all admissible sequences of length $k+1$, H is surjective. Moreover, every point x has at most two preimages in Σ_A under H and only a boundary point x of P_{w_k} for some admissible w_k has two preimages. Now it is easy to see $F \circ H(w) = H \circ \sigma_A(w)$.

Theorem 2. *Any two Feigenbaum-like mappings f and g are topologically conjugate.*

Proof. Suppose F and G are the induced Markov maps from f and g. Let H_1 and H_2 be the semi-conjugacies from F and G to σ_A. From the proof of Lemma 8, $H = H_1 \circ H_2^{-1}$ can be defined as a homeomorphism of I and $F \circ H = H \circ G$. Hence F and G are topologically conjugate. Furthermore, H is also the conjugacy between f and g.

A homeomorphism H of $I = [-1, 1]$ is said to be quasisymmetric [1] if there is a constant $C > 0$ so that for any x and y in I,

$$C^{-1} \leqq \frac{|H(x) - H(z)|}{|H(z) - H(y)|} \leqq C\,,$$

where $z = (x+y)/2$. Furthermore, I can use a similar method to that in [8] to prove

Theorem 3. *Suppose f and g are two Feignbaum-like mappings and H is the conjugacy between them. Then H is quasisymmetric.*

Proof. Suppose $\beta_{0,f} = \{P_{\pm k,f}\}_{k=0}^{\infty} \cup \{P_{\infty,f}\}$ is the induced Markov partition from f and $\beta_{0,g} = \{P_{\pm k,g}\}_{k=0}^{\infty} \cup \{P_{\infty,g}\}$ is the induced Markov partition from g. Let $\beta_{k,f} = \{P_{w_k,f} | w_k$ be an admissible sequence of $\overline{Z} \cup \{\infty\}$ of length $k+1\}$ and $\beta_{k,g} = \{P_{w_k,g} | w_k$ is an admissible sequence of $\overline{Z} \cup \{\infty\}$ of length $k+1\}$. They are called induced k^{th}-partitions of $I = [-1,1]$ from f and g. From Theorem 1, there is a constant $C > 0$ so that

$$C^{-1} \leqq \frac{|P_{\pm l w_k, f}|}{|\bigcup_{i=l+1}^{\infty} P_{\pm i w_k, f}|} \leqq C$$

and

$$C^{-1} \leqq \frac{|P_{\pm l w_k, g}|}{|\bigcup_{i=l+1}^{\infty} P_{\pm i w_k, g}|} \leqq C$$

for all l in $\overline{Z}$ and all admissible sequences w_k of $\overline{Z}$ of length $k+1$. This exactly means that the hierarchical system $\{\beta_{k,f}\}_{k=0}^{\infty}$ and $\{\beta_{k,g}\}_{k=0}^{\infty}$ satisfy a similar property to bounded nearby geometry defined in [8] for a geometrically finite one-dimensional mapping. Now using a similar argument to the proof of Theorem B in [8], one can prove that H is quasisymmetric. However, for the sake of completeness of this note, I shall write down the proof in more details.

I first construct a little different sequence of nested partitions $\{\eta_{k,f}\}_{k=0}^{\infty}$ of $I = [-1, 1]$ from $\beta_{0,f}$. Let $\eta_{0,f}$ consist of one interval I. Cut I into three intervals $L_0 = P_{-0,f}$, $M_0 = cl\left(\bigcup_{i=1}^{\infty}(P_{-i,f} \cup P_{i,f})\right)$, where cl means closure, and $R_0 = P_{0,f}$ (see Fig. 4). Then $\eta_{1,f} = \{L_0, M_0, R_0\}$.

The map F is a diffeomorphism when restricted on L_0 or R_0 and $F(L_0) = F(R_0) = L_0 \cup M_0$. Cut L_0 (respectively, R_0) into two intervals L_0L_0 and L_0M_0 (respectively, R_0L_0 and R_0M_0) which are preimages of L_0 and M_0 under $F|L_0$ (respectively, $F|R_0$). And cut M_0 into three intervals $L_1 = P_{-1,f}$, $M_1 = cl\left(\bigcup_{i=2}^{\infty}(P_{-i,f} \cup P_{i,f})\right)$, and $R_1 = P_{1,f}$ (see Fig. 5). Then

$$\eta_{2,f} = \{L_0L_0, L_0M_0, L_1, M_1, R_1, R_0L_0, R_0M_0\}\,.$$

Now I shall define $\eta_{n,f}$ for $n \geqq 3$ inductively. Suppose $\eta_{n,f}$ has been defined for some $n \geqq 2$ and contains $L_{n-1} = P_{-(n-1),f}$, $M_{n-1} = cl\left(\bigcup_{i=n}^{\infty}(P_{-i,f} \cup P_{i,f})\right)$, and $R_{n-1} = P_{n-1,f}$. Cut M_{n-1} into three intervals $L_n = P_{-n,f}$, $M_n = cl\left(\bigcup_{i=n+1}^{\infty}(P_{-i,f} \cup P_{i,f})\right)$, and $R_n = P_{n,f}$. For an interval $J \neq M_{n-1}$ in $\eta_{n,f}$, there

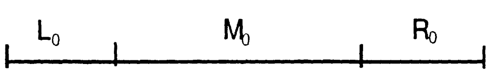

Fig. 4.

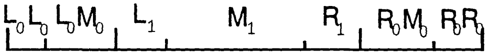

Fig. 5.

is the maximum integer $i \geqq 1$ such that $F^{\circ i}|J$ is a diffeomorphism. Then $F^{\circ i}(J)$ is either **1)** M_{n-i-1} or **2)** $L_{n-i} \cup M_{n-i}$ or **3)** $R_{n-i} \cup M_{n-i}$. In case **1)**, cut J into three intervals $\mathscr{J} = \{JL_{n-i}, JM_{n-i}, JR_{n-i}\}$ which are the preimages of L_{n-i}, M_{n-i} and R_{n-i} under $F^{\circ i}|J$. In case **2)**, cut J into two intervals $\mathscr{J} = \{JL_{n-i}, JM_{n-i}\}$ which are the preimages of L_{n-i} and M_{n-i} under $F^{\circ i}|J$. In case **3)**, cut J into two intervals $\mathscr{J} = \{JM_{n-i}, JR_{n-i},\}$ which are the preimages of M_{n-i} and R_{n-i} under $F^{\circ i}|J$. Then

$$\eta_{n+1,f} = \{\mathscr{J} | J \in \eta_{n,f}\} \cup \{L_n, M_n, R_n\}.$$

Therefore I have defined a sequence $\eta_f = \{\eta_{n,f}\}_{n=0}^{\infty}$ of nested partitions from $\beta_{0,f}$. Similarly, one can define a sequence $\eta_g = \{\eta_{n,g}\}_{n=0}^{\infty}$ of nested partitions from $\beta_{0,g}$.

From the construction of η_f and Theorem 1, η_f has bounded and bounded nearby geometry which is defined in [8]. More precisely, there is a constant $C > 0$ such that (**BG**) (*bounded geometry*): for any intervals $J \subset T$ with J in η_{n+1} and T in η_n, $n \geqq 0$, $|J|/|T| \geqq C$, and (**BNG**) (*bounded nearby geometry*): for any intervals J_1 and J_2 in η_n with a common endpoint, $n \geqq 1$, $|J_1|/|J_2| \geqq C$.

The statement (**BG**) follows from Theorem 1 and the construction of η_f directly. To prove the statement (**BNG**), one need to check when the common endpoint point q of J_1 and J_2 is a preimage of a fixed point p_k of F under F. In this case, let $J_{1,i} = F^{\circ i}(J_1)$ and $J_{2,i} = F^{\circ i}(J_2)$ for $i \geqq 0$. Then there is the biggest integer $j \geqq 0$ such that $F^{\circ j}|J_1 \cup J_2$ is a diffeomorphism. So $F^{\circ j}(q) = p_{\pm k}$. Therefore there exists another integer $m \geqq j$ such that both of $F^{\circ m}|J_1$ and $F^{\circ(m+1)}|J_2$ are diffeomorphisms and $J_{1,m} = J_{2,m+1} = P_k$ or P_{-k}. This implies that $J_{1,l} = J_{2,l+1}$ for all $j < l \leqq m$. In particular, $J_{1,j} = J_{2,j+1}$. So $J_{1,j} = F(J_{2,j})$. From Theorem 1, there is a constant $C_0 > 0$ such that $C_0^{-1} \leqq F'(x) \leqq C_0$ for all x in $P_{\pm k}$ and $k \geqq 0$. Hence

$$C_0^{-1} \leqq \frac{|J_{1,j}|}{|J_{2,j}|} \leqq C_0 .$$

Applying Theorem 1 again, there is a constant $C_1 > 0$ such that

$$C_1^{-1} \leqq \frac{|J_1|}{|J_2|} \leqq C_1 .$$

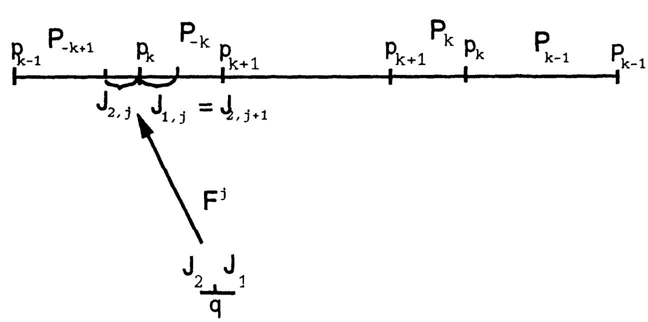

Fig. 6.

Similarly, one can prove that η_g has bounded and bounded nearby geometry ((**BG**) and (**BNG**)) too.

Now I use the property of bounded and bounded nearby geometry ((**BG**) and (**BNG**)) of η_f and η_g to prove that h is quasisymmetric (refer to [8]). For any $x < y$ in I, let $z = (x + y)/2$ be the midpoint between x and y and $N > 0$ be the smallest integer such that there is an interval J in $\eta_{K,f}$ contained in $[x, y]$. Let $\tilde{J}$ be the interval in $\eta_{N-1,f}$ containing J. Then the union of $\tilde{J}$ and one JJ of its adjacent intervals in $\eta_{N-1,f}$ contains $[x, y]$ (see Figs. 7, 8 and 9). Because of bounded and bounded nearby geometry ((**BG**) and (**BNG**)) of η_g (and refer to Figs. 7, 8 and 9), there is a constant $C_2 > 0$ such that

$$\frac{|H(J)|}{|H([x, z])|} \geqq C_2 \quad \text{and} \quad \frac{|H(J)|}{|H([z, y])|} \geqq C_2 .$$

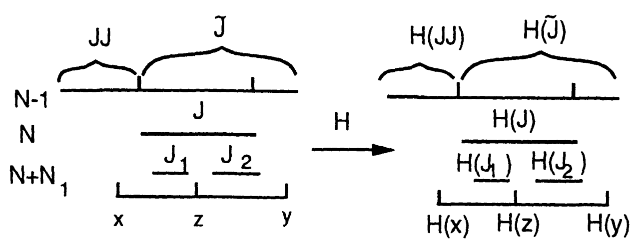

Fig. 7.

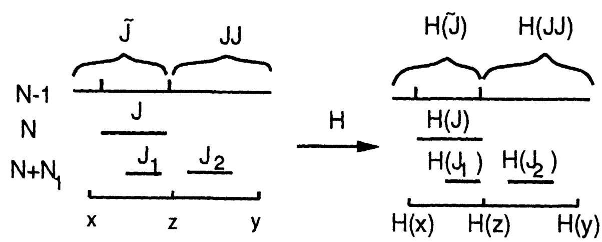

Fig. 8.

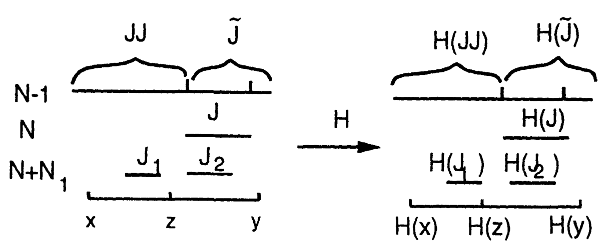

Fig. 9.

Since η_f has bounded geometry (**BG**), the maximum length of intervals in $\eta_{n,f}$ tends to zero exponentially, that is, there is a constant $C_3 > 0$ and $0 < \lambda < 1$ such that

$$\max_{J \in \eta_{n,f}} |J| \leqq C_3 \lambda^n$$

for all $n \geqq 0$. Thus there is a constant integer $N_1 > 0$ (does not depend on N) such that there are intervals J_1 and J_2 in η_{N+N_1} contained in $[x, z]$ and $[z, y]$, respectively. This implies that $H(J_1)$ and $H(J_2)$ are contained in $H([x, z])$ and $H([z, y])$ respectively because H is the conjugacy.

Because of bounded and bounded nearby geometry ((**BG**) and (**BNG**)) of η_g again, there is a constant $C_4 > 0$ (see Figs. 7, 8 and 9) such that

$$C_4^{-1} \leqq \frac{|H(x) - H(z)|}{|H(z) - H(y)|} \leqq C_4 ,$$

which means that H is quasisymmetric.

Remark 5. The quasisymmetric property of a conjugacy is first studied in [18] for complex quadratic-like Feigenbaum-like mappings by using the complex method. The proof of Theorem 3 here is for more general unimodal mappings and is a real method developed from [8, 10]. A different approach to the proof in the general case was tried in [15]. This theorem can be also proven for infinitely renormalizable S-unimodal mappings of bounded type (see, for example, [10]). However, for infinitely renormalizable S-unimodal mappings of unbounded type, it is still an open question. In [11, 17], a result about infinitely renormalizable quadratic polynomials of unbounded type has been announced recently.

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