Commun. Math. Phys. 166, 633–643 (1995)

Communications in
Mathematical
Physics

The Dirac Operator and Gravitation

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Received: 1 December 1993/in revised form: 29 March 1994

Abstract: We give a brute-force proof of the fact, announced by Alain Connes, that the Wodzicki residue of the inverse square of the Dirac operator is proportional to the Einstein-Hilbert action of general relativity. We show that this also holds for twisted (e.g. by electrodynamics) Dirac operators, and more generally, for Dirac operators pertaining to Clifford connections of general Clifford bundles.

Recently Connes' non-commutative geometry turned out – besides its yielding a fascinating reinterpretation of the full standard model of elementary particles – to be relevant to gravitation. Indeed, on the one hand, Alain Connes made the challenging observation[1] that the Wodzicki[2] residue of the inverse square of the (Atiyah-Singer-Lichnérowicz) Dirac operator yields the Einstein-Hilbert action of general relativity. And moreover he worked out a quantal form of the Polyakov action of strings [2] which reproduces it in the usual case of a Riemann surface, but also makes sense for conformal 4-manifolds, then yielding a conformally invariant action hoped to be connected with gravitation[3].

In this paper we are concerned with the Wodzicki residue of D^{-2}. We first compute this object for the pure Dirac operator (built with the spin connection of a riemannian spin manifold, cf. (1) below): we then obtain, as announed by Connes, a multiple of the scalar curvature (Theorem [1] below). Our proof is a brute-force computation performed in arbitrary coordinate patches.

Now, since the Einstein-Hilbert action and the action of the standard model are both obtained by algorithms based on the Dixmier trace, one naturally wishes to obtain these two actions within a single procedure. Along this line the first natural object to investigate is the Wodzicki residue of $\mathbb{D}^{-2}$, $\mathbb{D}$ the compound Dirac operator

[1] Unpublished, but mentioned verbally in different talks
[2] The Wodzicki residue is a (in fact the unique, thus canonical) trace on the pseudo-differential operators (concentrated on pseudo-differential operators of order – the dimension of the manifold)
[3] The work of the Zürich group on gravitation in non-commutative geometry is based on a different approach related to the Yang-Mills agorithm [6]

built with the tensor product of the spin connection σ_μ and the electrodynamics $U(1)$-connection a_μ. But computation of this object (Proposition [2] below) yields the same result as that obtained in Theorem [1]: the connection a_μ drops out of the calculation. In fact, since our calculation is based on the Lichnérowicz formula for the square or the Dirac operator holding in the case of general Dirac operators stemming from Clifford connections on Clifford bundles, our result naturally generalizes to this frame (Proposition [3] below).

We thus conclude that the present algorithms of non-commutative geometry yielding the respective lagrangians of the microworld and the cosmos seem (superficially) to tend to repel each other: whilst a_μ drops out of the Wodzicki residue of the inverse square of the compound Dirac operator, σ_μ drops out of the non-commutative Yang-Mills algorithm[4].

0. Setting and Notation

In what follows $\mathbf{M}$ is a 4-dimensional oriented riemannian spin manifold with riemannian metric g (yielding the norm $\|\ \|$ and the volume-element dv). We recall that the Dirac operator D is locally given as follows in terms of an orthonormal section e_i (with dual section θ^k) of the frame bundle of $\mathbf{M}$: one has

$$\begin{cases} D = i\gamma^i \tilde{\nabla}_i = i\gamma^i(e_i + \sigma_i) \\ \text{with} \quad \sigma_i(x) = \frac{1}{4}\gamma_{ij,k}(x)\gamma^j\gamma^k = \frac{1}{8}\gamma_{ij,k}(x)\left[\gamma^j\gamma^k - \gamma^k\gamma^j\right] \end{cases}, \tag{1}$$

where the $\gamma_{ij,k}$ represent the Levi-Civita connection ∇ with spin connection $\tilde{\nabla}$, specifically:

$$\begin{cases} \gamma_{ij,k} = -\gamma_{ik,j} = \frac{1}{2}\left[c_{ij,k} + c_{ki,j} + c_{kj,i}\right], \quad i,j,k = 1,\dots,4. \\ \text{with} \quad c_{ij}^k = \theta^k([e_i, e_j]) \end{cases} \tag{2}$$

Here the γ^i are constant self-adjoint Dirac matrices s.t. $\gamma^i\gamma^j + \gamma^j\gamma^i = \delta^{ij}$. In terms of local coordinates x^μ inducing the alternative vierbein $\partial_\mu = S^i_\mu(x)e_i$ (with dual vierbein dx^μ) we have $\gamma^i e_i = \gamma^\mu \partial_\mu$, the γ^μ being now x-dependent Dirac matrices s.t. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = g^{\mu\nu}$ (we use latin sub-(super-)scripts for the basic e_i and greek sub-(super-)scripts for the basis ∂_μ, the type of sub-(super-)scripts specifying the type of Dirac matrices). The specification of the Dirac operator in the greek basis is as follows: one has

$$\begin{cases} D = i\gamma^\mu \tilde{\nabla}_\mu = i\gamma^\mu(\partial_\mu + \sigma_\mu) \\ \text{with} \quad \sigma_\mu(x) = S^i_\mu(x)\sigma_i(x) \end{cases}. \tag{1a}$$

In what follows the notation D^{-1} refers to an inverse modulo smoothing operators.

We first establish ([1], see also [3, p. 322]):

1. Theorem. *The value of the Wodzicki residue* [4, 4a] *on the inverse square of the Dirac operator, namely:*

$$I = 4\,\mathrm{Tr}_\omega\{\sigma_{-4}(x,\xi)\} = 4(2\pi)^{-4} \int\limits_{\xi \in S^3} \mathrm{tr}\{\sigma_{-4}(x,\xi)\} d^3\xi dv\,, \tag{3}$$

[4] Indeed σ_μ drops out of the commutators $[D, a]$, $a \in C^\infty(\mathbf{M})$

*(*tr *the normalized Clifford trace) where:*

$$\sigma_{-4}(x,\xi) = \textit{part of order } -4 \textit{ of the total symbol } \sigma(x,\xi) \textit{ of } D^{-2}\,, \tag{4}$$

coincides up to a constant with the Hilbert-Einstein action $\int \mathscr{L}_g dv$ *of general relativity, where:*

$$\mathscr{L}_g = R_{\mu\nu} \wedge *(dx^\mu \wedge dx^\nu) \tag{5}$$

(specifically

$$\mathscr{L}_g = \tfrac{1}{2} R_{ikmn}(dx^m \wedge dx^n, dx^i \wedge dx^k) = (g^{im}g^{nk} - g^{in}g^{mk})R_{ikmn} = s\,, \tag{5a}$$

s the scalar curvature). One has $I = -\dfrac{1}{24\pi^2}\int \mathscr{L}_g dv.$

We recall the Lichnérowicz formula for the square of the Dirac operator:

$$\begin{aligned} D^2 &= -g^{\mu\nu}(\tilde{\nabla}_\mu \tilde{\nabla}_\nu - \Gamma^\alpha_{\mu\nu}\tilde{\nabla}_\alpha) + \tfrac{1}{4}s \\ &= -g^{\mu\nu}[\partial^x_\mu \partial^x_\nu + 2\sigma_\mu \cdot \partial^x_\nu - \Gamma^\alpha_{\mu\nu}\partial^x_\alpha + \partial^x_\mu \sigma_\nu + \sigma_\mu\sigma_\nu - \Gamma^\alpha_{\mu\nu}\sigma_\alpha] + \tfrac{1}{4}s\,. \end{aligned} \tag{6}$$

Our computations are based on the algorithm yielding the principal symbol of a product of pseudo-differential operators in terms of the principal symbols of the factors, namely, with the shorthand $\partial^\alpha_\xi = \partial^\alpha/\partial\xi_\alpha$, $\partial^x_\alpha = \partial_\alpha/\partial x^\alpha$:

$$\sigma^{PQ}(x,\xi) = \Sigma_\alpha \frac{(-i)^{|\alpha|}}{\alpha!}\,\partial^\alpha_\xi \sigma^P(x,\xi)\cdot \partial^x_\alpha \sigma^Q(x,\xi)\,. \tag{7}$$

We need to compute the total symbol $\sigma(x,\xi)$ of D^{-2} up to order -4, with D^2 the following sum of terms $(D^2)_k$ of order k:

$$D^2 = (D^2)_2 + (D^2)_1 + (D^2)_0\,,$$

$$\begin{cases} (D^2)_2 = -g^{\mu\nu}\partial^x_\mu\partial^x_\nu \\ (D^2)_1 = -g^{\mu\nu}(2\sigma_\mu \cdot \partial^x_\nu - \Gamma^\alpha_{\mu\nu}\partial^x_\alpha) \\ (D^2)_0 = -g^{\mu\nu}(\partial^x_\mu\sigma_\nu + \sigma_\mu\sigma_\nu - \Gamma^\alpha_{\mu\nu}\sigma_\alpha) + \tfrac{1}{4}s \end{cases} \tag{6a}$$

with the respective symbols:

$$\begin{cases} \sigma_2(x,\xi) = g^{\mu\nu}(x)\xi_\mu\xi_\nu \\ \sigma_1(x,\xi) = ig^{\mu\nu}(x)\,[\Gamma^\alpha_{\mu\nu}(x)\xi_\alpha - 2\sigma_\mu(x)\xi_\nu] \\ \sigma_0(x) = -g^{\mu\nu}(x)\,(\partial^x_\mu\sigma_\nu + \sigma_\mu\sigma_\nu - \Gamma^\alpha_{\mu\nu}\sigma_\alpha)\,(x) + \tfrac{1}{4}s(x) \end{cases} \tag{8}$$

abbreviated as follows, using the shorthand $\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}$:

$$\begin{cases} \sigma_2(x,\xi) = \|\xi\|^2 \\ \sigma_1(x,\xi) = i(\Gamma^\mu - 2\sigma^\mu)\,(x)\xi_\mu \\ \sigma_0(x) = -(\partial^{x\mu}\sigma_\mu + \sigma^\mu\sigma_\mu - \Gamma^\mu\sigma_\mu)\,(x) + \tfrac{1}{4}s(x) \end{cases} \tag{8a}$$

We want to compute a parametrix D^{-2} of D^2 up to order -4 using the above recipe: this amounts to computing the parts σ_{-k}, $k = 2,3,4$, in the expansion of the full symbol σ of D^{-2} into terms of decreasing order:

$$\sigma^{D^{-2}} = \sigma = \sigma_{-2} + \sigma_{-3} + \sigma_{-4} + \text{terms of order } \leq -5\,. \tag{9}$$

Application of (7) with $P = D^2$ and $Q = D^{-2}$ yields in the respective orders $0, -1, -2$ the recurrence relations:

$$\sigma_2\sigma_{-2} = 1\,, \tag{10}$$

$$\sigma_2\sigma_{-3} + \sigma_1\sigma_{-2} - i\partial_\xi^\mu\sigma_2\cdot\partial_\mu^x\sigma_{-2} = 0\,, \tag{11}$$

$$\begin{aligned}&\sigma_2\sigma_{-4} + \sigma_1\sigma_{-3} + \sigma_0\sigma_{-2} - i\partial_\xi^\mu\sigma_2\cdot\partial_\mu^x\sigma_{-3}\\ &\quad - i\partial_\xi^\mu\sigma_1\cdot\partial_\mu^x\sigma_{-2} - \tfrac{1}{2}\,\partial_\xi^{\mu\nu}\sigma_2\cdot\partial_{\mu\nu}^x\sigma_{-2} = 0\,,\end{aligned} \tag{12}$$

where the relevant terms are read off the following tabulation of products $\partial_\xi^\alpha\sigma_p(x,\xi)\cdot\partial_\alpha^x\sigma_q(x,\xi)$:

$|\alpha| = 0$

	σ_2	σ_1	σ_0
σ_{-2}	0	-1	-2
σ_{-3}	-1	-2	-3
σ_{-4}	-2	-3	-4

$|\alpha| = 1$

	σ_2	σ_1	σ_0
σ_{-2}	-1	-2	-3
σ_{-3}	-2	-3	-4
σ_{-4}	-3	-4	-5

$|\alpha| = 2$

	σ_2	σ_1	σ_0
σ_{-2}	-2	-3	-4
σ_{-2}	-3	-4	-5
σ_{-4}	-4	-5	-6.

With the σ_k as in (8a), this reads:

$$\sigma_{-2} = \|\xi\|^{-2}\,, \tag{10a}$$

$$\|\xi\|^2\sigma_{-3} + i\|\xi\|^2(\Gamma^\mu - 2\sigma^\mu)\xi_\mu - i\partial_\xi^\mu\|\xi\|^2\cdot\partial_\mu^x\|\xi\|^{-2} = 0\,, \tag{11a}$$

$$\begin{aligned}&\|\xi\|^2\sigma_{-4} + i(\Gamma^\mu - 2\sigma^\mu)\xi_\mu\sigma_{-3} - \|\xi\|^{-2}[(\partial^{x\mu}\sigma_\mu + \sigma^\mu\sigma_\mu - \Gamma^\mu\sigma_\mu)\\ &\qquad + \tfrac{1}{4}\,\|\xi\|^{-2}s(x) - i\partial_\xi^\mu\|\xi\|^2\cdot\partial_\mu^x\sigma_{-3} + (\Gamma^\nu - 2\sigma^\nu)\delta_\nu^\mu\cdot\partial_\mu^x\|\xi\|^{-2}\\ &\qquad - \tfrac{1}{2}\,\partial_\xi^{\mu\nu}\|\xi\|^2\cdot\partial_{\mu\nu}^x\|\xi\|^{-2} = 0\,.\end{aligned} \tag{12a}$$

With

$$\left.\begin{aligned}&\partial_\xi^\mu\|\xi\|^2 = 2\xi^\mu,\ \partial_\xi^\mu\|\xi\|^{-2} = -2\|\xi\|^{-4}\xi^\mu,\ \partial_\xi^\mu\|\xi\|^{-4} = -4\|\xi\|^{-6}\xi^\mu\,,\\ &\partial_\xi^\mu\|\xi\|^{-6} = -6\|\xi\|^{-8}\xi^\mu\,,\\ &\partial_\mu^x\|\xi\|^2 = \xi^\alpha\xi^\beta\partial_\mu^x g^{\alpha\beta},\ \partial_\mu^x\|\xi\|^{-2} = -\|\xi\|^{-4}\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta}\,,\\ &\partial_\mu^x\|\xi\|^{-6} = -3\|\xi\|^{-8}\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta}\,,\\ &\partial_\xi^{\mu\nu}\|\xi\|^2 = 2g^{\mu\nu},\\ &\partial_{\mu\nu}^x\|\xi\|^{-2} = -\|\xi\|^{-4}\xi_\alpha\xi_\beta\partial_{\mu\nu}^x g^{\alpha\beta} + 2\|\xi\|^{-6}\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta}\xi_\gamma\xi_\delta\partial_\nu^x g^{\gamma\delta}\,,\end{aligned}\right\} \tag{13}$$

we get

$$i\sigma_{-3} = \|\xi\|^{-4}\xi_\mu(\Gamma^\mu - 2\sigma^\mu) + 2\|\xi\|^{-6}\xi^\mu\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta} \tag{11b}$$

and

$$
\begin{aligned}
\sigma_{-4} = & -i\|\xi\|^{-2}(\Gamma^\mu - 2\sigma^\mu)\xi_\mu \sigma_{-3} + \|\xi\|^{-4}(\partial^{x\mu}\sigma_\mu + \sigma^\mu\sigma_\mu - \Gamma^\mu\sigma_\mu) \\
& - \tfrac{1}{4}\|\xi\|^{-4}s(x) + 2i\|\xi\|^{-2}\xi^\mu \cdot \partial^x_\mu \sigma_{-3} + \|\xi\|^{-6}\xi_\alpha\xi_\beta(\Gamma^\mu - 2\sigma^\mu)\partial^x_\mu g^{\alpha\beta} \\
& + \tfrac{1}{2}\|\xi\|^{-2}\partial_\xi^{\mu\nu}\|\xi\|^2 \cdot \partial^x_{\mu\nu}\|\xi\|^{-2} \\
= & -\|\xi\|^{-6}\xi_\mu\xi_\nu(\Gamma^\mu - 2\sigma^\mu)(\Gamma^\nu - 2\sigma^\nu) \\
& - 2\|\xi\|^{-8}\xi^\mu\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu - 2\sigma^\nu)\partial^x_\mu g^{\alpha\beta} + \|\xi\|^{-4}(\partial^{x\mu}\sigma_\mu + \sigma^\mu\sigma_\mu - \Gamma^\mu\sigma_\mu) \\
& - \tfrac{1}{4}\|\xi\|^{-4}s(x) - 2i\|\xi\|^{-2}\xi^\mu \cdot \partial^x_\mu\sigma_{-3} + \|\xi\|^{-6}\xi_\alpha\xi_\beta(\Gamma^\mu - 2\sigma^\mu)\partial^x_\mu g^{\alpha\beta} \\
& - \|\xi\|^{-6}\xi_\alpha\xi_\beta g^{\mu\nu}\partial^x_{\mu\nu}g^{\alpha\beta} + 2\|\xi\|^{-8}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta g^{\mu\nu}\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta} \,. \qquad (12b)
\end{aligned}
$$

Regrouping the terms and inserting

$$
\begin{aligned}
i\partial^x_\mu\sigma_{-3} = & -2\|\xi\|^{-6}\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu - 2\sigma^\nu)\partial^x_\mu g^{\alpha\beta} + \|\xi\|^{-4}\xi_\nu\partial^x_\mu(\Gamma^\nu - 2\sigma^\nu) \\
& - 6\|\xi\|^{-8}\xi^\nu\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta} + 2\|\xi\|^{-6}\xi_\alpha\xi_\gamma\xi_\delta\partial^x_\mu g^{\nu\alpha}\partial^x_\nu g^{\gamma\delta} \\
& + 2\|\xi\|^{-6}\xi^\nu\xi_\gamma\xi_\delta\partial^x_{\mu\nu}g^{\gamma\delta} \,, \qquad (14)
\end{aligned}
$$

we get for σ_{-4} the sum of terms:

$$
\begin{aligned}
A &= -\|\xi\|^{-6}\xi_\mu\xi_\nu\Gamma^\mu\Gamma^\nu + \|\xi\|^{-4}[g_{\mu\nu} - 4\|\xi\|^{-2}\xi_\mu\xi_\nu]\,[\sigma^\mu\sigma^\nu - \Gamma^\nu\sigma^\nu] \,, \\
B &= \|\xi\|^{-4}\partial^{x\mu}\sigma_\mu - \tfrac{1}{4}\|\xi\|^{-4}s \,, \\
C &= -6\|\xi\|^{-8}\xi^\mu\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu - 2\sigma^\nu)\partial^x_\mu g^{\alpha\beta} \,, \\
D &= 2\|\xi\|^{-6}\xi^\mu\xi_\nu\partial^x_\mu(\Gamma^\nu - 2\sigma^\nu) \,, \\
E &= -12\|\xi\|^{-10}\xi^\mu\xi^\nu\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta} \,, \\
F &= 4\|\xi\|^{-8}\xi^\mu\xi_\alpha\xi_\gamma\xi_\delta\partial^x_\mu g^{\nu\alpha}\partial^x_\nu g^{\gamma\delta} \,, \\
G &= \|\xi\|^{-6}\xi_\alpha\xi_\beta(\Gamma^\mu - 2\sigma^\mu)\partial^x_\mu g^{\alpha\beta} \,, \\
H &= 4\|\xi\|^{-8}\xi^\mu\xi^\nu\xi_\gamma\xi_\delta\partial^x_{\mu\nu}g^{\gamma\delta} \,, \\
K &= -\|\xi\|^{-6}\xi_\alpha\xi_\beta g^{\mu\nu}\partial^x_{\mu\nu}g^{\alpha\beta} \,, \\
L &= 2\|\xi\|^{-8}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta g^{\mu\nu}\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta} \,. \qquad (15)
\end{aligned}
$$

We have to take the Clifford trace and integrate over the sphere S^3 (commuting procedures). Owing to (1a), all the terms linear in σ_μ vanish under the Clifford trace. We proceed to the integration over S^3, using the following facts: we have, using the shorthand $\int = \frac{1}{2\pi^2} \int_{\xi \in S^3} d^3v$:

$$
\left\{
\begin{aligned}
&\int \xi^\mu\xi^\nu = \frac{1}{4}\,[^{\mu\nu}] \\
&\int \xi^\mu\xi^\nu\xi^\alpha\xi^\beta = \frac{1}{3\cdot 2^3}\,[^{\mu\nu\alpha\beta}] \\
&\int \xi^\mu\xi^\nu\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta = \frac{1}{3\cdot 2^6}\,[^{\mu\nu\alpha\beta\gamma\delta}]
\end{aligned}
\right. \,, \qquad (16)
$$

where $[^{\mu\nu\cdots\gamma\delta}]$ stands for the sum of products of $g^{\alpha\beta}$ determined by all "pairings" of $\mu\nu\ldots\gamma\delta$. Averaging over S^3 the terms surviving in (15) yields (we now write ∂_μ instead of ∂^x_μ without risking confusion, and use the fact that one has, $\cong$ indicating equivalence when multiplied with an expression symmetric in $\alpha\beta$, and γ,δ):

$$[^{\mu\nu}{}_{\alpha\beta\gamma\delta}] \cong g^{\mu\nu}[_{\alpha\beta\gamma\delta}] + 2\delta^\mu_\alpha[^\nu{}_{\beta\gamma\delta}] + 2\delta^\mu_\gamma[^\nu{}_{\alpha\beta\delta}]\,, \tag{17}$$

we get the terms[5]:

$$\begin{aligned}
A &\to -\frac{1}{4}[_{\mu\nu}]\Gamma^\mu\Gamma^\nu = -\frac{1}{4}g_{\mu\nu}\Gamma^\mu\Gamma^\nu\,,\\
B &\to -\frac{1}{4}s\,,\\
C &\to -6\frac{1}{3\cdot 2^3}[^\mu{}_{\nu\alpha\beta}]\Gamma^\nu\partial_\mu g^{\alpha\beta} = -\frac{1}{4}\Gamma^\mu g_{\alpha\beta}\partial_\mu g^{\alpha\beta} - \frac{1}{2}\Gamma^\nu g_{\nu\beta}\partial_\mu g^{\mu\beta}\,,\\
D &\to 2\frac{1}{4}[^\mu{}_\nu]\partial_\mu\Gamma^\nu = \frac{1}{2}\delta^\mu_\nu\partial_\mu\Gamma^\nu = \frac{1}{2}\partial_\mu\Gamma^\mu\,,\\
E &\to -12\frac{1}{3\cdot 2^6}[^{\mu\nu}{}_{\alpha\beta\gamma\delta}]\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta}\\
&= -\frac{1}{16}g^{\mu\nu}g_{\alpha\beta}g_{\gamma\delta}\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta} - \frac{1}{8}g^{\mu\nu}g_{\alpha\gamma}g_{\beta\delta}\partial_\nu g^{\alpha\beta}\partial_\mu g^{\gamma\delta}\\
&\quad -\frac{1}{8}g_{\gamma\delta}\partial_\mu g^{\mu\nu}\partial_\nu g^{\gamma\delta} - \frac{1}{4}g_{\beta\delta}\partial_\mu g^{\mu\beta}\partial_\nu g^{\nu\delta}\\
&\quad -\frac{1}{4}g_{\beta\delta}\partial_\mu g^{\nu\beta}\partial_\nu g^{\mu\delta} - \frac{1}{8}g_{\alpha\beta}\partial_\mu g^{\alpha\beta}\partial_\nu g^{\mu\nu}\,,\\
F &\to 4\frac{1}{3\cdot 2^3}[^\mu{}_{\alpha\gamma\delta}]\partial_\mu g^{\nu\alpha}\partial_\nu g^{\gamma\delta}\\
&= \frac{1}{6}g_{\gamma\delta}\partial_\mu g^{\mu\nu}\partial_\nu g^{\gamma\delta} + \frac{1}{3}g_{\alpha\delta}\partial_\mu g^{\nu\alpha}\partial_\nu g^{\mu\delta}\,,\\
G &\to \frac{1}{4}[_{\alpha\beta}]\Gamma^\mu\partial_\mu g^{\alpha\beta} = \frac{1}{4}\Gamma^\mu g_{\alpha\beta}\partial_\mu g^{\alpha\beta}\,,\\
H &\to 4\frac{1}{3\cdot 2^3}[^{\mu\nu}{}_{\gamma\delta}]\partial_{\mu\nu}g^{\gamma\delta} = \frac{1}{6}[g^{\mu\nu}g_{\gamma\delta} + 2\delta^\mu_\gamma\delta^\nu_\delta]\partial_{\mu\nu}g^{\gamma\delta}\,,\\
K &\to -\frac{1}{4}[_{\alpha\beta}]g^{\mu\nu}\partial_{\mu\nu}g^{\alpha\beta} = -\frac{1}{4}g_{\alpha\beta}g^{\mu\nu}\partial_{\mu\nu}g^{\alpha\beta} = \frac{1}{4}g^{\alpha\beta}g^{\mu\nu}\partial_{\mu\nu}g_{\alpha\beta}\,,\\
L &\to 2\frac{1}{3\cdot 2^3}g^{\mu\nu}[_{\alpha\beta\gamma\delta}]\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta}\\
&= \frac{1}{12}g^{\mu\nu}[g_{\alpha\beta}g_{\gamma\delta} + 2g_{\alpha\gamma}g_{\beta\delta}]\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta}\,.
\end{aligned} \tag{18}$$

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[5] A line-by-line account of these calculations is available as the Marseille research report CPT-93/P.2970

We now convert into the following expressions in which the partial derivatives act on the $g^{\alpha\beta}$ with upper indices:

$$\begin{cases} U = g^{\alpha\beta} g^{\gamma\delta} \partial_{\alpha\beta} g_{\gamma\delta} \\ X = g^{\alpha\beta} g^{\gamma\delta} \partial_{\alpha\gamma} g_{\beta\delta} \\ hhG = g^{\alpha\beta} g^{\gamma\delta} g^{\sigma\tau} \partial_\alpha g_{\beta\sigma} \partial_\gamma g_{\delta\tau} \\ ghH = g^{\alpha\beta} g^{\gamma\delta} g^{\sigma\tau} \partial_\alpha g_{\beta\sigma} \partial_\tau g_{\gamma\delta} \\ gg\Delta = g^{\alpha\beta} g^{\gamma\delta} g^{\sigma\tau} \partial_\sigma g_{\alpha\beta} \partial_\tau g_{\gamma\delta} \\ GHH = g^{\mu\sigma} g^{\alpha\eta} g^{\xi\beta} \partial_\mu g_{\eta\xi} \partial_\alpha g_{\beta\sigma} \\ \Delta GG = g^{\mu\sigma} g^{\alpha\eta} g^{\xi\beta} \partial_\mu g_{\eta\xi} \partial_\sigma g_{\alpha\beta} , \end{cases} \tag{19}$$

where g, G, h, H, Δ, stand for contractions between the following kinds of pairs of indices: indices of letters g (same of different), indices of letters g and ∂ (nearby or remote), indices of letters ∂ and ∂.

Using the facts:

$$\Gamma^\mu = g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} = g^{\alpha\beta} g^{\mu\sigma} \Gamma_{\alpha\beta,\sigma} = g^{\alpha\beta} g^{\mu\sigma} \left[\partial_\alpha g_{\beta\sigma} - \tfrac{1}{2} \partial_\sigma g_{\alpha\beta}\right] , \tag{20}$$

and:

$$\partial_\mu g^{\alpha\beta} = -g^{\alpha\sigma} g^{\beta\tau} \partial_\mu g_{\sigma\tau} , \qquad g_{\alpha\gamma} \partial_\mu g^{\alpha\beta} = -g^{\alpha\beta} \partial_\mu g_{\alpha\gamma} , \tag{21}$$

we get:

$$A \to -\tfrac{1}{4} hhG + \tfrac{1}{4} ghH - \tfrac{1}{16} gg\Delta , \tag{22}$$

$$C \to \tfrac{1}{2} hhG - \tfrac{1}{8} gg\Delta , \tag{23}$$

$$D \to \tfrac{1}{2} X - \tfrac{1}{4} U - \tfrac{1}{2} hhG + \tfrac{1}{4} ghH - \tfrac{1}{2} GHH + \tfrac{1}{4} \Delta GG , \tag{24}$$

$$E \to -\tfrac{1}{4} ghH - \tfrac{1}{4} hhG - \tfrac{1}{16} gg\Delta - \tfrac{1}{8} \Delta GG - \tfrac{1}{4} GHH , \tag{25}$$

$$F \to \tfrac{1}{6} ghH + \tfrac{1}{3} GHH , \tag{26}$$

$$G \to -\tfrac{1}{4} ghH + \tfrac{1}{8} gg\Delta , \tag{27}$$

$$H \to -\tfrac{1}{6} U - \tfrac{1}{3} X - \tfrac{1}{3} \Delta GG + \tfrac{1}{3} GHH + \tfrac{1}{3} hhG , \tag{28}$$

$$K \to \tfrac{1}{4} U - \tfrac{1}{2} \Delta GG , \tag{29}$$

$$L \to \tfrac{1}{12} gg\Delta + \tfrac{1}{6} \Delta GG . \tag{30}$$

The sum of those terms amounts to:

$$-\tfrac{1}{6} \left[U - X + hhG - ghH + \tfrac{1}{4} gg\Delta + \tfrac{1}{2} GHH + \tfrac{3}{4} \Delta GG\right] = \tfrac{1}{6} s , \tag{31}$$

which, added to the contribution $-\frac{1}{4} s$ of the term B, yields Theorem [1].

We now investigate the case of twisted Dirac operators [5]. With $\mathbf{M}$ a compact riemannian manifold, denoting by $\mathbb{C}l_M$ the set of smooth sections of the vector bundle with fibre over $x \in \mathbf{M}$ the Clifford algebra over the cotangent space of x (a $\mathbb{Z}/2$-graded complex algebra and $C^\infty(\mathbf{M})$-module), we now consider an additional smooth vector bundle $\mathscr{V}$ over $\mathbf{M}$ (with $C^\infty(\mathbf{M})$-module of smooth sections W), equipped with a connection $\nabla^{\mathscr{V}}$, with corresponding curvature-tensor $R^{\mathscr{V}}$. We consider the tensor-product vector bundle $S \otimes \mathscr{V}$ [with $C^\infty(\mathbf{M})$-module of smooth sections $\mathbb{S}_{\mathbf{M}} \otimes w$] which becomes a *Clifford bundle* via the definition:

$$c(a) = \gamma(a) \otimes \mathrm{id}_w\,, \qquad a \in \mathbb{C}l_{\mathbf{M}}\,, \tag{32}$$

and which we equip with the *compound connection*:

$$\bar{\nabla}_\xi = \tilde{\nabla}_\xi \otimes \mathrm{id}_w + \mathrm{id}_{\mathbb{S}_M} \otimes \nabla^{\mathscr{V}}_\xi\,, \qquad \xi, \eta \in \chi(\mathbf{M})\,, \tag{33}$$

the latter becoming a *Clifford connection* in the sense that:

$$[\bar{\nabla}_\xi, c(a)] = c(\nabla_\xi a)\,, \qquad a \in \mathbb{C}l_{\mathbf{M}},\ \xi \in \chi(\mathbf{M})\,, \tag{34}$$

where ∇ is the connection of $\mathbb{C}l_{\mathbf{M}}$ induced by the Levi-Civita connection of $\mathbf{M}$. The corresponding *twisted Dirac operator* $\mathbb{D}$, and *twisted connection-laplacian* $\underline{\Delta}$ are then respectively locally specified as follows:

$$\mathbb{D} = ic(dx^\mu)\bar{\nabla}_\mu\,, \tag{35}$$

and

$$\underline{\Delta} = -g^{\mu\nu}(\bar{\nabla}_\mu \bar{\nabla}_\nu - \Gamma^\alpha_{\mu\nu}\bar{\nabla}_\alpha)\,. \tag{36}$$

We have the following Lichnérowicz formulae for the square of the twisted Dirac operator:

$$\begin{aligned} \mathbb{D}^2 &= \underline{\Delta} - \tfrac{1}{2}\mathbf{R}^{\mathscr{V}}(\partial_\mu, \partial_\nu)c(dx^\mu)c(dx^\nu) + \tfrac{1}{4}\,s \\ &= \underline{\Delta} - \tfrac{1}{2}\,\gamma(dx^\mu)\gamma(dx^\nu) \otimes \mathbb{R}^{\mathscr{V}}(\partial_\mu, \partial_\nu) + \tfrac{1}{4}\,s \otimes \mathrm{id}_w\,. \end{aligned} \tag{31}$$

2. Proposition. *The Wodzicki residue of* $\mathbb{D}^{-2}$ *coincides with that of* D^{-2}*, thus also yields a multiple of the Einstein-Hilbert action.*

Proof. Let $a_\mu dx^\mu$ be the connection one-form of the connection $\nabla^{\mathscr{V}}$: the connection one-form of the compound connection (33) then reads:

$$\bar{\sigma}_\mu = \sigma_\mu \otimes \mathrm{id}_w + \mathrm{id}_{\mathbb{S}_M} \otimes a_\mu = \text{“}\sigma_\mu + a_\mu\text{”}\,, \tag{33a}$$

the computation of the Wodzicki residue of $\mathbb{D}^{-2}$ then is obtained from that of D^{-2} through the changes:

$$\begin{cases} \sigma_\mu \to \bar{\sigma}_\mu = \sigma_\mu + a_\mu \\ s \to \mathbb{S} = s - 2R^{\mathscr{V}}_{\mu\nu}\gamma^\mu\gamma^\nu \end{cases}, \tag{38}$$

with corresponding replacements $A \to \mathbb{A}$ through $L \to \mathbb{L}$ which we now compute. The term $\mathbb{A}$, obtained from A through the change $\sigma_\mu \to \bar{\sigma}_\mu$, vanishes as the latter in the integration over S^3. As for the other terms, we have the following changes, leading to the indicated results after taking the Clifford trace and integrating over S^3:

$$\begin{aligned}
\mathbb{B} - B &= \|\xi\|^{-4}\partial^{x\mu}a_\mu + \frac{1}{2}\,\|\xi\|^{-4}R^{\mathscr{V}}_{\mu\nu}\gamma^\mu\gamma^\nu \to \partial^\mu a_\mu\,, \\
\mathbb{C} - C &= 12\|\xi\|^{-8}\xi^\mu\xi_\nu\xi_\alpha\xi_\beta a^\nu\partial^x_\mu g^{\alpha\beta} \to 12\,\frac{1}{3\cdot 2^3}\,[^\mu{}_{\nu\alpha\beta}]a^\nu\partial_\mu g^{\alpha\beta} \\
&= \frac{1}{2}\,a^\mu g_{\alpha\beta}\partial_\mu g^{\alpha\beta} + a^\nu g_{\nu\beta}\partial_\alpha g^{\alpha\beta}\,, \\
\mathbb{D} - D &= -4\|\xi\|^{-6}\xi^\mu\xi_\nu\partial^x_\mu a^\nu \to -\delta^\mu_\nu\partial_\mu a^\nu = -\partial_\mu a^\mu = -\partial_\mu(g^{\mu\nu}a_\nu) \\
&= -g^{\mu\nu}\partial_\mu a_\nu - a_\nu\partial_\mu g^{\mu\nu} = -\partial^\mu a_\mu - a_\nu\partial_\mu g^{\mu\nu}\,, \\
\mathbb{E} - E &= 0\,, \\
\mathbb{F} - F &= 0\,, \\
\mathbb{G} - G &= -2\|\xi\|^{-6}\xi_\alpha\xi_\beta a^\mu\partial^x_\mu g^{\alpha\beta} \to -\frac{1}{2}\,a^\mu g_{\alpha\beta}\partial_\mu g^{\alpha\beta}\,, \\
\mathbb{H} - H &= 0\,, \\
\mathbb{K} - K &= 0\,, \\
\mathbb{L} - L &= 0\,,
\end{aligned} \tag{39}$$

adding up to zero.

In fact the above result can be generalized further to the (generalized twisted) Dirac operators pertaining to Clifford connections of general Clifford bundles [5]. Let $\mathscr{E}$ be a $\mathbb{Z}/2$-graded vector bundle over $\mathbf{M}$, with $C^\infty(\mathbf{M})$-module of smooth sections $\mathbf{E}$: $\mathscr{E}$ is called a *Clifford bundle* whenever there is a homomorphism of $\mathbb{Z}/2$-graded complex algebras $c\colon \mathbb{C}l_M \to \mathrm{End}_{C^\infty(\mathbf{M})}\,\mathbf{E}$. Furthermore a connection $\bar\nabla$ of $\mathscr{E}$ is called a *Clifford connection* whenever all $\bar\nabla_\xi$, $\xi \in \chi(\mathbf{M})$, are even, and one has

$$[\bar\nabla_\xi, c(a)] = c(\nabla_\xi a)\,, \qquad a \in \mathbb{C}l_{\mathbf{M}},\ \xi \in \chi(\mathbf{M}) \tag{34a}$$

(generalizing (34)). Those elements then specify as follows a generalized Dirac operator $\mathbb{D}_{\bar\nabla}$:

$$\mathbb{D}_{\bar\nabla} = ic(dx^\mu)\bar\nabla_\mu \tag{35a}$$

(generalizing (35)) giving rise to the generalized Lichnérowicz formula:

$$\mathbb{D}^2_{\bar\nabla} = \underline{\Delta} - \tfrac{1}{2}\,F^{\mathscr{E}/S}(\partial_\mu, \partial_\nu)c(dx^\mu)c(dx^\nu) + \tfrac{1}{4}\,s\,, \tag{37a}$$

where $F^{\mathscr{E}/S}$ is the so-called *twisting curvature* of the bundle $\mathscr{E}$ [5, Proposition 3.43]. The replacement $R^{\mathscr{V}} \to F^{E/S}$ then leaves the above calculation unchanged, to the effect that one has:

3. Proposition. *The Wodzicki residue of* $\mathbb{D}^{-2}_{\bar\nabla}$ *still yields a multiple of the Einstein-Hilbert action.*

Appendix. The Einstein-Hilbert Action. Scalar Curvature

With $\mathbf{M}$ a 4-dimensional riemannian manifold with metric g (inducing the volume-element dv and the scalar product $(\cdot,\cdot)$ on the tenors), the Levi-Civita connection ∇

is defined as follows in terms of local coordinates:

$$\nabla\partial_i = \Gamma^k_{ij}\partial_k dx^j \quad \text{with} \quad \begin{cases} \Gamma^k_{ij} = g^{km}\Gamma_{ij,m} \\ \Gamma_{ij,m} = \frac{1}{2}\left[\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right] \end{cases} . \tag{A.1}$$

The corresponding curvature ∇^2 is the two-form with value endomorphisms of the tangent bundle of $\mathbf{M}$ locally given by the matrix:

$$R^j_k = d\omega^j_k + \omega^j_s \wedge \omega^s_k = \frac{1}{2} R^j_{kmn} dx^m \wedge dx^n , \tag{A.2}$$

explicitly given by:

$$R^i_{kmn} = \partial_m \Gamma^i_{nk} - \partial_n \Gamma^i_{mk} + \Gamma^i_{ms}\Gamma^s_{nk} - \Gamma^i_{ns}\Gamma^s_{mk} , \tag{A.3}$$

alternatively:

$$\begin{aligned} R_{jkmn} = g_{ij}R^j_{kmn} &= \frac{1}{2}\,\partial_m(\partial_k g_{nj} - \partial_j g_{nk}) - \frac{1}{2}\,\partial_n(\partial_k g_{mj} - \partial_j g_{mk}) \\ &\quad + g_{st}\Gamma^s_{nj}\Gamma_{mk} - g_{st}\Gamma^s_{mj}\Gamma^t_{nk} \\ &= \frac{1}{2}\,\partial_m(\partial_k g_{nj} - \partial_j g_{nk}) - g_{st}\Gamma^s_{mj}\Gamma^t_{nk} - (m \leftrightarrow n) . \end{aligned} \tag{A.4}$$

The corresponding scalar curvature is

$$s = R^{mn}_{mn} = g^{mj}g^{nk}R_{jkmn} = -(g^{mi}g^{nk} - g^{ni}g^{mk})\left[\partial_{mi}g_{nk} + g_{st}\Gamma^s_{mi}\Gamma^t_{nk}\right] . \tag{A.5}$$

In terms of the shorthand (21) one has:

$$s = X - U - hhG + ghH - \frac{1}{4}\,gg\Delta - \frac{1}{2}\,GHH + \frac{3}{4}\,\Delta GG . \tag{A.6}$$

The *Einstein-Hilbert action density* is by definition:

$$L_g = R_{ik} \wedge *(dx^i \wedge dx^k) = \frac{1}{2}\,R_{ikmn}(dx^m \wedge dx^n) * (dx^i \wedge dx^k) , \tag{A.7}$$

alternatively:

$$L_g = \mathscr{L}_g dv , \tag{A.8}$$

with the lagrangian density:

$$\begin{aligned} \mathscr{L}_g &= \frac{1}{2}\,R_{ikmn}g(dx^m \wedge dx^n, dx^i \wedge dx^k) = \frac{1}{2}\,(g^{im}g^{nk} - g^{in}g^{mk})R_{ikmn} \\ &= g^{im}g^{nk}R_{ikmn} = R^{mn}_{mn} = s . \end{aligned} \tag{A.9}$$

Note. After completion of this work, we had the visit in Marseille of Markus Walze and Wolfgang Kalau from Mainz (R.F.A.), who reported about an analogous calculation (using normal coordinates) leading to the same results.

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Communicated by A. Connes