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An Investigation of the Limiting Behavior of Particle-Like Solutions to the Einstein-Yang/Mills Equations and a New Black Hole Solution

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Abstract: A mathematical investigation of the limiting behavior of particle-like solutions of Einstein-Yang-Mills equations leads to a discovery of a new type of black hole solution.

1. Introduction

In the paper [4], we proved the existence of a countably infinite number of smooth, static, spherically symmetric solutions of the Einstein-Yang/Mills equations (EYM) with $SU(2)$ gauge group (first observed by Bartnik and McKinnon in [1]). These solutions are indexed by a bounded real parameter λ_n. Our first objective here is to study the limiting solution corresponding to the parameter value $\bar{\lambda}$, where

$$\bar{\lambda} = \lim \lambda_n ,$$

and to describe some of the rather interesting mathematical properties of this solution. In particular, we prove that this solution is the first "crashing" solution, (in the sense that a metric coefficient becomes singular) and that this solution crashes at $r = 1$. Next we show that this degenerate orbit admits (at least) one pseudo-continuation (PC) defined for all $r > 1$. The concatenation of the $\bar{\lambda}$-orbit, defined for $r < 1$, and the "PC orbit" defined for $r > 1$, $(w(r), w'(r), A(r))$, satisfies, (for some subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$),

$$\lim_{j\to\infty} (w_{n_j}(r, \lambda_{n_j}), w'_{n_j}(r, \lambda_{n_j}), A_{n_j}(r, \lambda_{n_j})) = (w((r), w'(r), A(r)) .$$

In addition, $\lim\limits_{r\nearrow 1}(A(r,\bar{\lambda}), A'(r,\bar{\lambda}), w(r,\bar{\lambda})) = (0,0,0) = \lim\limits_{r\searrow 1}(A(r), A'(r), w(r))$, but neither $\lim\limits_{r\searrow 1} w'(r)$, nor $\lim\limits_{r\nearrow 1} w'(r,\bar{\lambda})$ exists. Now although the λ_n-orbits are all particle-like solutions of the EYM equations, the PC orbit in $r > 1$ can be interpreted as a

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new type of black hole solution with event horizon at $r = 1$. This new black hole solution is *very* different from those black hole solutions obtained rigorously in [5]; see also [2, 6, 7, 9, 10] for numerical results, and see too the related paper [8].

The EYM equations, for static, spherically symmetric solutions with $SU(2)$ gauge group reduces to the following system of ordinary differential equations cf. [1, 3–5]),

$$r^2 A w'' + r(1-A) - \left[\frac{(1-w^2)^2}{r}\right] w' + w(1-w^2) = 0\,, \tag{1.1}$$

$$rA' + (1+2w'^2)A = 1 - \frac{(1-w^2)^2}{r^2}\,, \tag{1.2}$$

$$2rAT' = \left[\frac{(1-w^2)^2}{r} + (1-2w'^2)A - 1\right] T\,. \tag{1.3}$$

Here the unknowns A and T are metric coefficients, where the metric is given by

$$ds^2 = -T^{-2}(r)dt^2 + \frac{1}{A(r)}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,,$$

and $w(r)$ is the unknown $su(2)$ connection coefficient. Since (1.1) and (1.2) do not involve T, we solve these first, and then use (1.3) to obtain T.

Equations (1.1), (1.2) are considered together with the following initial conditioins:

$$A(0) = 1\,, \qquad w(0) = 1\,, \qquad w'(0) = 0\,, \qquad w''(0) = -\lambda < 0\,.$$

All solutions are parametrized by λ, and in [4], we proved the existence of a sequence $\{\lambda_n\} \subset (0,2)$, for which the corresponding solutions are non-singular, and

$$\lim_{r\to\infty} \theta(r,\lambda_n) = -n\pi\,, \qquad n = 1, 2, \ldots\,.$$

Here $\theta(r,\lambda)$ is defined by $\theta(r,\lambda) = \mathrm{Tan}^{-1}(w'(r,\lambda)/w(r,\lambda))$, if $r > 0$, and $\theta(0,\lambda) = 0$. Moreover, $A(r,\lambda_n) > 0$ for all $r > 0$ and $\lim_{r\to\infty} A(r,\lambda_n) = 1$. On the other hand, we showed in [3], that if $\lambda > 2$, there is an $\bar{r} = \bar{r}(\lambda)$ such that

$$\lim_{r\nearrow\bar{r}} A(r,\lambda) = 0\,, \tag{1.4}$$

and both

$$\lim_{r\nearrow\bar{r}} w(r) = \bar{w} > 0\,, \quad \text{and} \quad \lim_{r\nearrow\bar{r}} w'(r) = -\infty$$

hold; that is, if $\lambda > 2$, the λ-orbit crashes; (see also [2]).

In this paper, we shall show that the $\bar{\lambda}$-orbit is the first crashing orbit, and crashes at $\bar{r} = 1$, in the sense that

$$\lim_{r\nearrow 1} A(r,\bar{\lambda}) = 0\,,$$

but for $\lambda < \bar{\lambda}$, $A(r,\lambda) > 0$ for all $r > 0$, provided that $w^2(r,\lambda) \le 1$. Moreover we shall show that

$$\lim_{r\nearrow 1} w(r,\bar{\lambda}) = 0\,,$$

and $w'(r,\bar{\lambda})$ is unbounded near $r = 1$. In [4] we proved that a limit of non-crashing orbit segments having uniformly bounded rotation converges to a non-crashing orbit of bounded rotation. In the case considered here, the λ_n-orbit has rotation $-n\pi$, and hence the set of λ_n-orbits has unbounded rotation.

Next, we shall show that for the PC orbit, the corresponding metric coefficient $T^{-2}(r,\bar{\lambda})$ can be chosen so as to satisfy $T^{-2}(1,\bar{\lambda}) = T^{-2}(1)$. Furthermore, the PC orbit has infinite rotation as $r \searrow 1$, in the sense that for any $\varepsilon > 0$,

$$\lim_{r \searrow 1} [\theta(1+\varepsilon) - \theta(r)] = -\infty ,$$

where $\theta(r) = \mathrm{Tan}^{-1}(w'(r)/w(r))$. The proof of this statement is based on the fact that the $\bar{\lambda}$-orbit crashes at $r = 1$, and relies on a technique introduced in [4]. We prove too that the PC orbit is a connecting orbit" in the sense that

$$\lim_{r \to \infty} (w(r), w'(r), A(r)) = (\pm 1, 0, 1) .$$

Thus the PC orbit can be interpreted as a black hole solution with event horizon at $r = 1$.

It is interesting to note that the black hole PC solution is the limit of the λ_n-orbits. On the other hand, in Sect. 6 we shall show how the black hole solution enables us to obtain information about the λ_n-orbits. In particular, we use this black hole solution to prove that for any $\varepsilon > 0$, there is a constant $c = c(\varepsilon) > 0$, such that if $r > 1 + \varepsilon$, each λ_n-orbit has rotation bounded below by $-c(\varepsilon)$. Thus for large n, "most" of the rotation takes place before r exceeds $1 + \varepsilon$.

The plan of the paper is as follows. The next section recalls some crucial facts from [3, 4]. In Sect. 3 we show that the $\bar{\lambda}$-orbit is the first crashing orbit. In Sect. 4 we construct the PC orbit and study some of its properties. Section 5 is a fairly technical section where we derive certain properties of the $\bar{\lambda}$-orbit near $r = 1$. The last section consits of some concluding remarks, together with a short discussion of some open problems. In particular we use the PC orbit to prove that the λ_n-orbits have uniformly bounded rotation if $r > 1 + \varepsilon$. We also construct the Einstein metric for the black hole solution in the region $r > 1$.

2. Preliminaries

Static, spherically solutions of the EYM equations with $SU(2)$ gauge group correspond to solutions of the following system of ordinary differential equations, (see [1, 3, 4]):

$$rA' + 2w'^2 A = \Phi/r , \tag{2.1}$$

$$r^2 A w'' + \Phi w' + w(1 - w^2) = 0 , \tag{2.2}$$

where

$$\Phi = \Phi(r, A, w) = r(1 - A) - \frac{(1 - w^2)^2}{r} . \tag{2.3}$$

It is useful to define

$$v = Aw' , \tag{2.4}$$

and from (2.1) and (2.2) we see that v satisfies the equation

$$v' + \frac{2w'^2 v}{r} + \frac{w(1 - w^2)}{r^2} = 0 . \tag{2.5}$$

If we consider regular solutions of (2.1), (2.2); i.e., smooth solutions defined for all $r \geq 0$, then the following initial conditions are required to hold:

$$A(0) = 1 , \quad w(0) = 1 , \quad w'(0) = 0 , \quad w''(0) = -\lambda < 0 . \tag{2.6}$$

Any solution to (2.1), (2.2), (2.6) is uniquely determined by λ; (see [3]), i.e. there is a unique solution

$$(A(r,\lambda), w(r,\lambda), w'(r,\lambda), r), \tag{2.7}$$

defined on an interval $0 < r < s(\lambda)$. We shall refer to the solution (2.7) as the λ-orbit.

Now we define the region $\Gamma \subset \mathbb{R}^4$ by

$$\Gamma = \{(A, w, w', r) : A > 0,\ w^2 < 1,\ (w, w') \neq (0,0),\ r > 0\};$$

as in [4], we shall only be concerned with orbits in Γ. We denote by $r_e(\lambda)$, the smallest $r > 0$ for which the λ-orbit exits Γ; $r_e(\lambda) = +\infty$ if the λ-orbit stays in Λ for all $r > 0$. If the λ-orbits exits Γ through $A = 0$, we say that the λ-orbit *crashes*. In [3, Theorem 4.1], we proved that if $\lambda \geq 2$, then the λ-orbit crashes; (see too [2]). Furthermore, we showed in [5, Lemma 3.3], that if an orbit crashes, then it crashes for $r \leq 1$. If we define $\theta(r, \lambda)$ by $\theta(0, \lambda) = 0$, and for $r > 0$,

$$\theta(r,\lambda) = \mathrm{Tan}^{-1}(w'(r,\lambda)/w(r,\lambda)),$$

then the *rotation number* of the λ-orbit, $\Omega(\lambda)$, is defined as

$$\Omega(\lambda) = -\frac{1}{\pi}\,\theta(r_e(\lambda), \lambda).$$

In [4, Theorem 3.7], we proved that there is an increasing sequence $0 < \lambda_1 < \ldots < 2$, such that

$$\Omega(\lambda_n) = n. \tag{2.8}$$

An orbit for which $\Omega(\lambda) = k$, will be called a *k-connecter*. By construction each λ_k is the smallest k-connecter.

Since the sequence $\{\lambda_k\}$ is increasing and bounded, it has a limit; thus set

$$\bar{\lambda} = \lim_{k\to\infty} \lambda_k. \tag{2.9}$$

It is a major purpose of this paper to investigate the properties of this $\bar{\lambda}$-orbit. In the next section we shall prove that the $\bar{\lambda}$-orbit is the first crashing orbit, and in Sect. 5 we shall investigate the interesting behavior of this orbit.

In order to study the $\bar{\lambda}$-orbit, we will need the following two results which were proved in [4, Lemmas 4.1 and 4.2]. Before stating the results, we introduce a definition; namely if $P \in \mathbb{R}^4$ lies in Γ, we call P a *good point.*

Lemma 2.1. *Suppose $\Sigma_n = \{(w_n(r), w_n'(r), A_n(r), r) : a_n \leq r \leq b_n\}, n = 1, 2, \ldots,$ is a sequence of orbit segments in Γ such that for all n, $w_n(b_n) = 0$, $w_n'(b_n) \neq 0$, $w'(a_n) = 0$, and $(w_n(r), w_n'(r))$ lies in[1] $\bar{Q}_2$ (resp. $\bar{Q}_4$), for $a_n \leq r \leq b_n$. If the right-hand endpoints $((0, w_n'(b_n), A_n(b_n), b_n)$ converge to a good point P, then the backwards orbit through P, defined for $r < b \equiv \lim b_n$, reaches the hyperplane $w' = 0$ at a point $Q \in \Gamma$, and this orbit segment lies in Γ.*

We shall need a similar result for quadrants Q_1 and Q_3.

Lemma 2.2. *If $Q = (\bar{w}, 0, \bar{A}, \bar{b}) \in \Gamma$, then the backwards orbit through Q, reaches the hyperplane $w = 0$ at a point $P \in \Gamma$, and this orbit segment lies in Γ. In particular if $\Sigma_n = \{(w_n(r), w_n'(r), A_n(r), r) : a_n \leq r \leq b_n\}$ is a sequence of orbit segments in Γ such that for all n, $w_n'(b_n) = 0$, $w_n(b_n) \neq 0$, $w_n(b_n) = 0$, and $(w_n(r), w_n'(r))$ lies in $\bar{Q}_1$ (resp. $\bar{Q}_3$), for $a_n \leq r \leq b_n$. If the right-hand endpoints converge to a good*

[1] Q_1 is the first quadrant in the $w - w'$, plane, etc.

point Q, then these orbit segments converge uniformly to the orbit segment from P, to Q, and this orbit segment lies in Γ.

The principal consequence of these two lemmas, is the following result [4, Corollary 3.3].

Theorem 2.3. *Suppose that $\lambda_n \to \bar{\lambda}$, and that*

$$\Lambda_n = \{w(r,\lambda_n), w'(r,\lambda_n), A(r,\lambda_n), r) : 0 \le r \le r_n\},$$

is a sequence of orbit segments in Γ, where $r_n = r_e(\lambda_n)$, and $\Omega(\lambda_n) \le N$. Then the $\bar{\lambda}$-orbit lies in Γ for $0 \le r \le r_e(\bar{\lambda})$, and $\Omega(\bar{\lambda}) \le N$.

3. The First Crashing Orbit

In this section we shall prove that the $\bar{\lambda}$-orbit is the first crashing orbit.

Theorem 3.1. *The $\bar{\lambda}$-orbit crashes; i.e., there is an $\bar{r} \le 1$ such that* $\lim_{r\to\bar{r}} A(r,\bar{\lambda}) = 0$. *Moreover, if $\lambda < \bar{\lambda}$, then the λ-orbit doesn't crash; i.e., $A(r,\lambda) > 0$ for all r, $0 \le r \le r_e(\lambda)$.*

Proof. We first show that if the λ'-orbit crashes then $\lambda' \ge \bar{\lambda}$. To see this, note that from Theorem 2.3, the set of λ-orbits, with $\lambda < \lambda'$ cannot have bounded rotation. Since the $\lambda = 0$ orbit has zero rotation, it follows from [4, Corollary 3.6], that for every integer $k > 0$ there is a λ'_k, $0 < \lambda'_k < \lambda'$, such that $\Omega(\lambda'_k) = k$. Thus by definition of λ_k, $\lambda_k < \lambda'$, and so $\bar{\lambda} \le \lambda'$.

We now show that the $\bar{\lambda}$-orbit crashes by eliminating all other possibilities:
1. The $\bar{\lambda}$-orbit cannot exit the region Γ through $w^2 = 1$. (If it did, then by "continuous dependence on initial conditions," the same would be true for the λ_k-orbit if k is large, and this is obviously false.)
2. The $\bar{\lambda}$-orbit cannot exit the region Γ through $(w, w') = (0,0)$. (The point $(0,0)$ is a "rest point" of the system (1.1), (1.2), and cannot be reached in finite r; see [4, Remark 4].)
3. The $\bar{\lambda}$-orbit cannot be a connecting orbit. (If the $\bar{\lambda}$-orbit were a k-connecter, then as was proved in [4, Proposition 3.4], if λ is near $\bar{\lambda}$, $\Omega(\lambda) < k+1$, and this contradicts (2.8).)
4. The $\bar{\lambda}$-orbit canot stay in the region Γ for all $r > 0$, (i.e., $r_e(\bar{\lambda}) = \infty$), with $\Omega(\bar{\lambda}) < \infty$. (In [4, Proposition 2.10], we proved that orbits with finite rotation staying in Γ for all $r > 0$, must be connecting orbits. This possibility was eliminated in Case 3.)
5. The $\bar{\lambda}$-orbit cannot stay in the region Γ for all $r > 0$ (i.e., $r_e(\bar{\lambda}) = \infty$, and $\Omega(\bar{\lambda}) = \infty$).

To eliminate this last possibility requires some work. Thus, assume

$$\Omega(\bar{\lambda}) = +\infty, \tag{3.1}$$

and

$$r_e(\bar{\lambda}) = +\infty. \tag{3.2}$$

We shall show that these lead to a contradiction, in a series of steps:

Step 1: We define a function H which is a "Lyapunov function" for large r; i.e., $H'(r) > 0$.

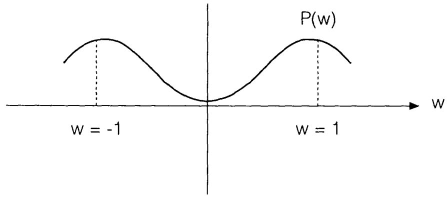

Fig. 1.

Step 2: We show that for large r, $H(r) < \frac{1}{4}$.

Step 3: We show that there exists a $c > 0$ such that H increases by at least c for every rotation of the orbit. This implies that the $\bar{\lambda}$-orbit cannot satisfy (3.1), since this violates Step 2.

We now proceed with the details.

We define the function $H(r)$ by

$$H(r) = P(w(r)) + \frac{r^2 A(r) w'^2(r)}{2}, \tag{3.3}$$

where

$$P(w) = \frac{w^2}{2} - \frac{w^4}{4}, \tag{3.4}$$

so that $P'(w) = w(1-w^2)$; cf. Fig. 1. Then using (2.1) and (2.2), an easy calculation gives

$$H'(r) = w'^2 \left[-\frac{\Phi}{2} + rA - rAw'^2 \right], \tag{3.5}$$

so that

$$H'(r) \geq rw'^2 \left[\tfrac{3}{2} A - \tfrac{1}{2} - Aw'^2 \right]. \tag{3.6}$$

To show that $H' > 0$, we show $A \to 1$ and $Aw'^2 \to 0$ as $r \to \infty$.

Lemma 3.2. $A(r) \to 1$ as $r \to \infty$.

Proof. We define the function $\mu(r)$ by (cf. [3]),

$$\mu(r) = r(1 - A(r)).$$

Now in [2] it was proved that there is a constant $c > 0$ such that $\mu(r) < c$ for all $r > 0$. Thus since $A(r) = 1 - \frac{\mu(r)}{r}$, we see that $A(r) \to 1$ as $r \to \infty$. $\square$

Lemma 3.3. $Aw'^2 \to 0$ *as* $r \to \infty$.

Proof. If $v = Aw'$, then (2.5) shows that when $v' = 0$, $|v|^3 = |w(1-w^2)A^2/2r| \leq \frac{1}{2r}$. Thus if $r_n < r_{n+1}$ are two consecutive "Neumann" times ($w'(r') = 0 = w'(r'')$, $w(r) \neq 0$ if $r' < r < r''$), then we see $|v(r)|^3 \leq \frac{1}{2r_n}$ on this interval, and as $\Omega(\bar{\lambda}) = \infty$, $r_n \to \infty$, so that $v(r) \to 0$ as $r \to \infty$. Thus since $Aw'^2 = v^2/A$, we see that in view of the last lemma, $Aw'^2 \to 0$ as $r \to \infty$. $\square$

In view of the last two lemmas, we see that we can find $\tilde{r} > 0$ such that if $r > \tilde{r}$,

$$\tfrac{3}{2} A(r) - \tfrac{1}{2} - (Aw'^2)(r) > \tfrac{1}{2},$$

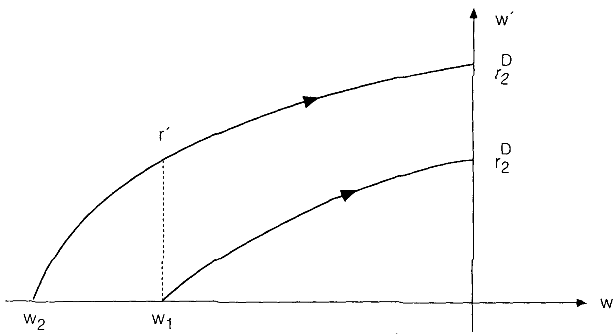

Fig. 2.

and thus using (3.6), we have

$$H'(r) > \tfrac{1}{2} r w'^2(r), \quad \text{if} \quad r > \tilde{r}. \tag{3.8}$$

This inequality yields *Step 1.*

To prove Step 2, we have the following lemma.

Lemma 3.3. $H(r) < \frac{1}{4}$ *for* $> \tilde{r}$.

Proof. Let $r > \tilde{r}$, and let $r_1 > r$ be such that $w'(r_1) = 0$. Then $H(r) > H(r_1) = P(w(r_1)) < \frac{1}{4}$. $\square$

Now let $r \leq r_1 < r_2 < \ldots$ be a sequence of "Neumann" times satisfying $w'(r_k) = 0$, and

$$-1 < w(r_k) < 0, \qquad w'(r) > 0 \quad \text{if} \quad r_k < r < r_k^D, \tag{3.9}$$

where r_k^D is the next "Dirichlet" time; i.e., $r_k^D > r_k$, and (see Fig. 2)

$$w(r_k^D) = 0, \quad \text{and if} \quad r_k < r < r_k^D, \qquad w(r) < 0. \tag{3.10}$$

Then *Step 3* is a consequence of the next lemma.

Lemma 3.4. *There exists* $c > 0$ *such that for* $k = 1, 2, \ldots,$

$$H(r_k) - H(r_k^D) \geq c. \tag{3.11}$$

Proof. For ease in notation, set

$$w_k = w(r_k), \quad \text{and} \quad w_k^D = w(_k^D).$$

Now since $H'(r) > 0$ if $r > \tilde{r}$, we have $P(w_{k+1}) > P(w_k)$, and thus

$$w_k > w_{k+1}, \qquad k = 1, 2, \ldots. \tag{3.12}$$

Now let $w(r)$ be such that $w_1 = w(r_1) < w(r) \leq 0$, where $r < r_1^D$. Then since $H' > 0$, we have

$$P(w_1) < P(w) + \tfrac{1}{2} r^2 A(r) w'(r)^2,$$

or

$$\tfrac{1}{2} r w'(r) \geq \sqrt{\frac{P(w_1) - P(w)}{A}} > \sqrt{P(w_1) - P(w)}, \tag{3.13}$$

since $A < 1$. Hence from (3.8) and (3.13),

$$H(r_1^D) - H(r_1) = \int_{r_1}^{r_1^D} H'(r)dr > \int_{r_1}^{r_1^D} \tfrac{1}{2} rw'^2(r)dr$$
$$= \int_{w_1}^{0} \tfrac{1}{2} rw'\, dw > \int_{w_1}^{0} \sqrt{P(w_1) - P(w)}\, dw \equiv c > 0\,;$$

that is,

$$H(r_1^D) - H(r_1) > c > 0\,.$$

Now from (3.12), $w_2 > w_1$, so there is an $r' > r_2$ for which $w(r') = w(r_1)$. Then

$$H(r_2^D) - H(r_2) \geq H(r_2^D) - H(r')$$
$$= \int_{r'}^{r_1^D} H'(r)dr$$
$$\geq \int_{w_1}^{0} \frac{rw'}{2}\, dw$$
$$\geq \int_{w_1}^{0} \sqrt{P(w_1) - P(w)}\, dw \equiv c\,.$$

Continuing in this way, we see that (3.11) holds. □

Now in view of (3.11), it follows that after a finite number of rotations that $H(r_n) > \frac{1}{4}$, and this contradicts Lemma 3.3. The proof of Theorem 3.1 is complete.

As a consequence of the method of proof of Theorem 3.1, we have the following corollary.

Corollary 3.4. *There is no solution of* (2.1), (2.2) *which has infinite rotation, and stays in the region Γ for all $r > 0$.*

4. The Pseudo-Continuation of the $\bar{\lambda}$-Orbit

It was shown in [4] that there is an increasing sequence $\lambda_1 < \lambda_2 < \ldots < 2$ such that $\Omega(\lambda_n) = n$; i.e., the λ_n-orbit is an n-connector. Let $b > 1$, and let Λ_n denote the following set of orbit segments defined form $0 \leq r \leq b$:

$$\Lambda_n = \{(w(r, \lambda_n), w'(r, \lambda_n), A(r, \lambda_n), r) : \ 0 \leq r \leq b\}\,.$$

From [4, Proposition 3.2], we can find a subsequence $\{\lambda_{n_k}\} \subset \{\lambda_n\}$, such that the right-hand endpoints $P_{n_k} = (w(b, \lambda_{n_k}), w'(b, \lambda_{n_k}), A(b, \lambda_{n_k}), b)$ converge to $P \in \Gamma$; i.e.,

$$P_{n_k} \to P = (\bar{w}, \bar{w}', \bar{A}, b) \in \Gamma\,;$$

that is, P is a good point; namely $\bar{w}^2 < 1$, $|\bar{w}'| < \infty$, $\bar{A} > 0$, $(w, w') \neq (0,0)$, and $1 < b < \infty$. (Informally, P is the point where the $\bar{\lambda}$-orbit would be if it didn't crash.) We consider the orbit $(w(r), w'(r), A(r), r)$ through P, defined for $1 < r < b$, and we call this orbit segment the pseudo-continuation (PC) of the $\bar{\lambda}$-orbit, for reasons which will become clear below. In this section we shall investigate the rather interesting properties of the PC orbit. These are summarized in the following theorem.

Theorem 4.1. *The* PC *orbit satisfies the following:*
1) $\lim_{r\to 1} A(r) = 0$.
2) *The* PC *orbit has infinite rotation; i.e.,* $\lim_{r\to 1}[\theta(b) - \theta(r)] = -\infty$, *where* $\theta(b) = \mathrm{Tan}^{-1}(\bar{w}/\bar{w})$.
3) $\lim_{r\to 1} w(r) = 0$.
4) $\lim_{r\to 1} \Phi(r) = 0$.
5) $w'(r)$ is unbounded near $r = 1$.

Remark. In view of 1, 3, 4, we see that the $\bar{\lambda}$-orbit, and the PC orbit can be concatenated.

In order to prove this theorem we begin with the following lemma, which shows that PC orbit does not crash if $r > 1$. This lemma is the most technical part of this section.

Lemma 4.2. *Given any $\varepsilon > 0$, there is a positive constant $\eta = \eta(\varepsilon)$, independent of n, such that for every $n \in \mathbb{R}_+$,*

$$A(1+\varepsilon, \lambda_n) \geq \eta\,. \tag{4.1}$$

Proof. With μ as defined in (3.3), we note that if

$$\mu(1+\varepsilon) \leq 1 + \frac{\varepsilon}{4}\,, \tag{4.2}$$

then

$$A(1+\varepsilon) = 1 - \frac{\mu(1+\varepsilon)}{1+\varepsilon} \geq 1 - \frac{(1+\varepsilon/4)}{1+\varepsilon} = \frac{3\varepsilon/4}{1+\varepsilon} \equiv \eta_1\,,$$

and this implies (4.1). Thus, it suffices to assume that

$$\mu(1+\varepsilon) > 1 + \frac{\varepsilon}{4}\,. \tag{4.3}$$

Next, if

$$\mu\left(1 + \frac{\varepsilon}{2}\right) \geq 1 + \frac{\varepsilon}{4}\,, \tag{4.4}$$

then for $r \in \left(1 + \frac{\varepsilon}{2}, 1 + \varepsilon\right)$, we have [cf. (3.4)]

$$\Phi(r) = \mu(r) - \frac{(1-w^2)^2}{r} \geq 1 + \frac{\varepsilon}{4} - \frac{1}{1+\frac{\varepsilon}{2}} > \frac{\varepsilon}{4}\,,$$

since $\mu' > 0$. Thus from [4, Proposition 2.7], there exists an $\eta_2 > 0$, depending only on ε, such that

$$A(1+\varepsilon) \geq \eta_2\,. \tag{4.5}$$

Thus, we may assume that $\mu\left(1+\frac{\varepsilon}{2}\right) < 1+\frac{\varepsilon}{4}$, and hence from (4.3), there is as an r_1, $1+\frac{\varepsilon}{2} < r_1 < 1+\varepsilon$ such that

$$\mu(r_1) = 1+\frac{\varepsilon}{4}\,. \tag{4.6}$$

Now we claim that there is a constant $\tau > 0$, depending only on ε such that

$$|w'(r)| \le \tau \quad \text{if} \quad 1+\frac{\varepsilon}{2} \le r \le 1+\varepsilon\,. \tag{4.7}$$

To see this, recall from [3, Proposition 5.1], there is an $L > 0$, independent of n such that

$$(Aw'^2)(r) \le L\,. \tag{4.8}$$

Then if $r \in \left[1+\frac{\varepsilon}{2}, r_1\right]$, since $\mu(r) < \mu(r_1)$,

$$A(r) = 1 - \frac{\mu(r)}{r} > 1 - \frac{1+\varepsilon/4}{1+\varepsilon/2} \equiv \eta_3\,, \tag{4.9}$$

and thus (4.8) implies

$$w'(r)^2 < L/\eta_3\,, \quad \text{if} \quad 1+\frac{\varepsilon}{2} \le r \le r_1\,. \tag{4.10}$$

Suppose now that $r_1 < r \le 1+\varepsilon$; on this interval we will show that

$$w'(r)^2 \le \max\left(\frac{L}{\eta_3}, \frac{16}{\varepsilon^2}\right) \equiv \tau\,. \tag{4.11}$$

Indeed, on the interval $[r_1, 1+\varepsilon]$, $\mu(r) \ge \mu(r_1)$, so

$$\Phi(r) = \mu(r) - \frac{(1-w^2)^2}{r} > 1+\frac{\varepsilon}{4} - \frac{1}{1+\frac{\varepsilon}{2}} > \frac{\varepsilon}{4}\,.$$

Thus from (2.2)

$$r^2 A w'' w' = -\Phi w'^2 - w(1-w^2)w'\,,$$

so that if $|w'| > \frac{4}{\varepsilon}$, since $|w(1-w^2)| < 1$, we see that $w'w'' < 0$; that is, if $r_1 \le r \le 1+\varepsilon$,

$$w'(r)^2 \quad \text{is decreasing, if} \quad |w'(r)| > \frac{4}{\varepsilon}\,. \tag{4.12}$$

To show (4.11), assume first that $L/\eta_3 \ge 16/\varepsilon^2$. Then $w'^2(r) \le L/\eta_3$ on $r_1 \le r \le 1+\varepsilon$; indeed, if there was a first point r_2, $r_1 < r_2 \le 1+\varepsilon$ for which $w'^2(r_2) = L/\eta_3$, then $\frac{d}{dr}(w'^2(r_2)) < 0$, in view of (4.12). Thus no such r_2 can exist. If now $L/\eta_3 < 16/\varepsilon^2$, then $w'^2(r_1) < 16/\varepsilon^2$ so (4.12) implies that $w'^2(r) < 16/\varepsilon^2$ if $r_1 \le r \le 1+\varepsilon$. This shows that (4.11) holds.

Now define η_4 by

$$\eta_4 = \frac{\varepsilon/4}{2(1+\varepsilon)\tau}\,.$$

We shall show that

$$A(r) \geq \min(\eta_3, \eta_4) \quad \text{if} \quad r_1 \leq r \leq 1+\varepsilon\,, \tag{4.13}$$

and this will complete the proof of Lemma 4.2. The proof of (4.13) is similar to the proof of (4.10); in fact, from (2.1), if $r_1 \leq r \leq 1+\varepsilon$,

$$rA'(r) = \frac{\Phi(r)}{r} - 2Aw'^2 > \frac{\varepsilon/4}{1+\varepsilon} - 2A\tau > 0\,,$$

if $A(r) \leq \eta_4$. Thus, if $\eta_3 \geq \eta_4$, then as $A(r) \geq \eta_3$, A cannot get below η_4 on $[r_1, 1+\varepsilon]$. In fact if there were a first point $r_2 > r_1$ for which $A(r_2) = \eta_4$, then $A'(r_2) > 0$, and this is impossible. On the other hand, if $\eta_4 > \eta_3$, then $A(r) > \eta_3$ on $[r_1, 1+\varepsilon]$, because $A'(r) > 0$ if $A(r) \leq \eta_4$ (draw a picture!) Hence (4.12) holds, and this completes the proof of Lemma 4.2. □

Remark. Using the last lemma, together with (4.8), we see that $w'(1+\varepsilon)^2 \leq L/\eta$, and this shows that for any $r > 1$, the set $\{(w(r,\lambda_n), w'(r,\lambda_n), A(r,\lambda_n), r)\}$ has a subsequence which converges to a point in Γ.

Corollary 4.3. *The* PC *orbit crashes at* $r = 1$.

Proof. The last result shows that $A(r) > 0$ if $r > 1$. If $A(1) > 0$, then this would imply, via the standard existence theorems for ordinary differential equations, that the PC orbit continues to $r = 1 - \varepsilon$, for some $\varepsilon > 0$. Moreover, by "continuous dependence,"

$$\lim_{n\to\infty} (w(r,\lambda_n), w'(r,\lambda_n), A(r,\lambda_n)) = (w(r), w'(r), A(r))\,,$$

for $r \geq 1-\varepsilon$. On the other hand, since $\bar{\lambda} = \lim\limits_{n} \lambda_n$, and the solutions form a continuous one-parameter family [4, p. 305], we have

$$\lim_{n\to\infty} (w(r,\lambda_n), w'(r,\lambda_n), A(r,\lambda_n)) = (w(r,\bar{\lambda}), w'(r,\bar{\lambda}), A(r,\bar{\lambda}))\,, \quad \text{for} \quad r \leq 1\,.$$

Thus the PC and $\bar{\lambda}$ orbits coincide for $1-\varepsilon \leq r \leq 1$, and hence the $\bar{\lambda}$-orbit does not crash at $r = 1$. This contradicts Theorem 3.1. □

Note that this corollary proves part 1) in the statement of Theorem 4.1. We now complete the

Proof of Theorem 4.1. We say that a point $(\tilde{w}, \tilde{w}', \tilde{A}, \tilde{r})$ in Γ lies in $\tilde{Q}_i$ $(i = 1,2,3,4)$ provided that $(\tilde{w}, \tilde{w}')$ lies in Q_i.

Now since [cf. (4.1)],

$$P_k = (w(b,\lambda_{n_k}), w'(b,\lambda_{n_k}), A(b,\lambda_{n_k}), b) \to P \in \Gamma\,, \quad \text{as} \quad k \to \infty\,,$$

we may assume, for definiteness, that $P \in \tilde{Q}_1$ (the proofs for the other cases are similar). Note that $\underline{\lim}|\theta(b,\lambda_{n_k})| = \infty$, for otherwise from Theorem 2.3, we would have that the $\bar{\lambda}$-orbit doesn't crash, since it would be a limit of orbits in Γ having uniformly bounded rotation. Thus from Lemma 2.2, the backwards orbit through P reaches a point Q on the hyperplane $w = 0$ without crashing. Now for each k, we can find $r_k < b$ such that $w(r_k, \lambda_{n_k}) = 0$ and $w'(r,\lambda_{n_k}) > 0$ for $r_k < r < b$. The point Q is a limit of right-hand endpoints of orbit segments in Γ, and hence as before $\underline{\lim}|\theta(r_k, \lambda_{n_k}| = \infty$. Thus from Lemma 2.1, the backwards orbit through Q reaches a point P' on the hyperplane $w' = 0$ without crashing. This process can be continued indefinitely because by alternately applying Lemmas 2.1 and 2.2, we always find points on the hyperplane $w = 0$ or $w' = 0$, which are limits of endpoints of orbit segments in Γ having unbounded rotation. This proves part 2) in the statement of Theorem 4.1.

To show that part 3) holds, we define (cf. [3,4]), $\mu(r) = r(1 - A(r))$, and then as $\mu' = 2Aw'^2 + (1 - w^2)^2/r^2$, we have $\mu' > 0$ and, since $A(1) = 0$, $\mu(1) = 1$. Thus $\mu(r) > 1$ if $r > 1$. It follows that [cf. (2.3)]

$$\Phi(r) = \mu(r) - \frac{(1 - w^2(r))^2}{r} > 1 - (1 - w^2(r))^2, \quad \text{if} \quad r > 1. \tag{4.14}$$

Now suppose that

$$\overline{\lim_{r \to 1}}\, w(r) = 4d > 0; \tag{4.15}$$

we will show that this leads to a contradiction. For this, we first note that, from part 2), there are an infinite number of $r > 1$ such that $w(r) = 3d$. Then on the interval $d \le w(r) \le 3d$, (4.14) gives

$$\phi(r) \ge 1 - (1 - d^2)^2 = d^2(2 - d^2) \equiv \eta > 0. \tag{4.16}$$

From this, it follows from [4, Proposition 2.6], that there is a constant $c > 0$ such that

$$|w'(r)| \le c \quad \text{if} \quad d \le w(r) \le 2d.$$

Hence on the interval $d \le w \le 2d$, we have that Δr, the change in r, satisfies $\Delta r \ge \dfrac{d}{c}$, and this implies that for the PC orbit $\Delta\theta = \theta(b) - \theta(1)$ satisfies $|\Delta\theta| \le (b - 1)|d/c$. This contradicts part 2), and hence (4.15) is false; i.e., $\overline{\lim}_{r \to 1} w(r) = 0$. Similarly, we can show that $\underline{\lim}_{r \to 1} w(r) = 0$, and thus $\lim_{r \to 1} w(r) = 0$.

To show $\Phi(r) \to 0$ as $r \searrow 1$, we argue as follows. First, in [3, Proof of Theorem 3.1], it was shown that if $\Phi(\bar r) > 0$ for some orbit, then that orbit cannot crash as $r \nearrow \bar r$. It was also shown in [3, Proposition 5.11 ff.], that if $\Phi(\bar r) < 0$ and the orbit crashes as $r \nearrow \bar r$, then nearby orbits must also crash. Hence if we have an orbit that crashes as $r \nearrow \bar r$ which is a limit of non-crashing orbits, then $\Phi(\bar r) = 0$; (the details are presented in the proof of Theorem 5.2, below). If now we make the substitution $r \to -r$, Eqs. (2.1) and (2.2) are invariant, but $\Phi \to -\Phi$. Thus the PC orbit which crashes as $r \searrow \bar r$, after the transformation gives rise to an orbit which crashes as $-r \nearrow -\bar r$. By the above observation, $-\Phi(\bar r) = 0$.

We now turn to statment 5). Thus, suppose the statement is false, then we can find $m > 0$ such that

$$|w'(r)| \le m \quad \text{if} \quad 1 < r < b. \tag{4.17}$$

Choose r_1, $1 < r_1 < b$ such that $w'(r_1) \ne 0$. Then by the Cauchy mean-value theorem,

$$w'(r_1) = \frac{v(r_1)}{A(r_1)} = \frac{v(r_1) - v(1)}{A(r_1) - A(1)} = \frac{v'(r_2)}{A'(r_2)} \tag{4.18}$$

for some r_2, $1 < r_2 < r_1$. Now from (4.17), Eq. (2.5) and part 3), we see that $v'(1) = 0$. Similarly, (4.17) together with (3.1) shows that $A'(1) = 0$. Thus using (4.18), we have as before

$$w'(r_1) = \frac{v'(r_2) - v'(1)}{A'(r_2) - A(1)} = \frac{v''(r_3)}{A''(r_3)}, \tag{4.19}$$

for some r_3, $1 < r_3 < r_2$. Since (2.5) implies

$$v'' = -2\,\frac{r(Aw'^3)' - Aw'^3}{r^2} - \frac{r^2(1-3w^2)w' - w(1-w^2)(2r)}{r^4}\,,$$

we see that for r near 1, $v''(r) \approx -\frac{1}{r^2}\,w'(r)$. Similarly from (3.1), we have $A''(r) \approx 2/r^2$. We thus conclude from (4.19) that for r_1 near 1,

$$w'(r_1) \approx -\frac{w'(r_3)}{2}\,,$$

or $|w'(r_3)| \approx 2|w'(r_1)|$. Repeating this argument with r_1 replaced by r_3, and continuing, shows that $w'(r)$ is unbounded near $r = 1$. This contradicts (4.17); thus $w'(r)$ is unbounded near $r = 1$. The proof of theorem 4.1 is complete. □

We next consider the behavior of the PC orbit in the far field; i.e., for large r.

Theorem 4.4. *The* PC *orbit satisfies the following:*
(i) $\lim\limits_{r\to\infty} (w(r), w'(r)) = (\pm 1, 0)$.
(ii) $\lim\limits_{r\to\infty} r(1 - A(r)) < \infty$.
(iii) *Given any $\varepsilon > 0$, there is a $k = k(\varepsilon) > 0$ such that $\theta(r) - \theta(1+\varepsilon) \geq -k$.*

(Statement (i) says that the PC orbit is a connecting orbit, and statement (ii) implies that the PC orbit has finite (ADM) mass; cf. [3, 4, 5].)

Proof. From [5, Lemma 3.3], the PC orbit cannot crash for $r > 1$. It cannot exit Γ via $w^2 = 1$, (for otherwise, the same would be true for nearby orbits; in particular for the λ_n-orbits if n is large), nor can it exit Γ via $(w, w') = (0, 0)$, (by [4, Remark 4]). Thus the PC orbit stays in Γ for all $r > 1 + \varepsilon$. From Corollary 3.4, the PC orbit cannot have infinite rotation for $r > 1 + \varepsilon$. The desired result now follows from [4, Proposition 2.10]. □

We now study the berhavior of the metric coefficients $A^{-1}(r)$ and $T^{-2}(r)$ near $r = 1$, for the PC orbit, [cf. (1.3)]. To this end, we recall from [3] that T satisfies

$$-2\,\frac{T'}{T} = \frac{2w'^2}{r} + \frac{\Phi}{r^2 A}\,. \tag{4.21}$$

If we write (cf. [3–5]),

$$P' = \frac{\Phi}{r^2 A}\,, \qquad Q' = \frac{2w'^2}{r}\,,$$

then (4.2.1) becomes

$$-2\,\frac{T'}{T} = P' + Q'\,, \tag{4.22}$$

and (cf. [5, Eq. (4.6)]),

$$AT^2 = e^k e^{-2Q}\,, \tag{4.23}$$

where k is a constant. In order to study the behavior of A' near $r = 1$, we need the following lemma.

Lemma 4.5. *If an orbit has infinite rotation near $\bar{r}$, and crashes at $\bar{r}$, then*

$$\lim_{r\to\bar{r}} (Aw'^2)(r) = 0\,. \tag{4.24}$$

Corollary 4.6. *For the* PC *orbit,* (4.24) *holds with* $\bar{r} = 1$.

Proof of Lemma 4.5. Let $f = Aw'^2$; then f satisfies the equation

$$rf' + (2rf + \Phi)w'^2 + 2w(1 - w^2)w' = 0\,. \tag{4.25}$$

Now suppose that

$$\overline{\lim_{r\to\bar{r}}}\, f(r) = \sigma > \eta = \underline{\lim_{r\to\bar{r}}}\, f(r)\,.$$

Then we can find an $\varepsilon > 0$, and a sequence $r_n \to \bar{r}$ such that $f(r_n) \geq \sigma - \varepsilon$, $f'(r_n) > 0$ and $|w'(r_n)| \to \infty$ $\left(\text{since } |w'| = \sqrt{\dfrac{\sigma-\varepsilon}{A}}\right)$. Now from Theorem 4.1, $\Phi(r_n) \to 0$ so we can find $c > 0$ such that for n large,

$$2r_n f(r_n) + \Phi(r_n) \geq c > 0\,.$$

Thus for large n, we get the contradiction

$$\begin{aligned} 0 &= r_n^2 f'(r_n) + (2r_n f(r_n) + \Phi(r_n))w'(r_n)^2 + 2w(r_n)(1 - w^2(r_n))w'(r_n) \\ &> r_n^2 f'(r_n) + cw'(r_n)^2 + 2w(r_n)(1 - w^2(r_n))w'(r_n) > 0\,. \end{aligned}$$

Hence $\overline{\lim_{r\to\bar{r}}}\, f(r) = \underline{\lim_{r\to\bar{r}}}\, f(r)$, so that $\lim_{r\to\bar{r}} Aw'^2(r)$ exists. Now since the PC orbit has infinite rotation as $r \to \bar{r}$ (by Theorem 4.1), it follows that $Aw'^2(\varrho_n) = 0$ for infinitely many $\varrho_n \searrow \bar{r}$; thus (4.24) holds. $\square$

Now with the aid of this lemma, we see from (2.1) that

$$\lim_{r\searrow 1} A'(r) = 0\,, \tag{4.25}$$

because $\Phi \to 0$ as $r \searrow 1$.

Finally, we consider the behavior of the metric coefficient $T^{-2}(r)$ near $r = 1$. Using (4.22), if we fix $\tilde{r} > 1$ and take $r > \tilde{r}$, we have

$$\ln \frac{T^{-2}(r)}{T^{-2}(\tilde{r})} = Q(r) + P(r)\,. \tag{4.26}$$

Since $Q - P = lnA$, and $A(\infty) = 1$ (from Theorem 4.4), we see that $Q(\infty) = P(\infty)$. Thus from (4.25) we have

$$T^{-2}(\infty) = T^{-2}(\tilde{r})e^{2Q(\infty)}\,,$$

so if we choose $T(\tilde{r}) = e^{Q(\infty)}$, then we have $T^{-2}(\infty) = 1$. With this choice of $T^{-2}(\tilde{r})$, (4.23) becomes $(AT^2)(r) = e^{2Q(\infty)-2Q(r)}$, and so

$$T^{-2}(r) = A(r)e^{2Q(r)-2Q(\infty)}\,. \tag{4.27}$$

Now as $Q' > 0$, we see that $\lim_{r\searrow 1} Q(r)$ exists, (it may equal $-\infty$), so that (4.27) gives

$$\lim_{r\searrow 1} T^{-2}(r) = 0\,. \tag{4.28}$$

We have thus proved the following result.

Proposition 4.6. *The following properties hold for the metric-coefficients of the* PC *orbit:*

(i) $\lim_{r\searrow 1} A(r) = 0 = \lim_{r\searrow 1} A'(r)$;

(ii) $\lim_{r\searrow 1} T^{-2}(r) = 0$.

5. Properties of the $\bar{\lambda}$-Orbit

In this section we shall derive some properties of the $\bar{\lambda}$-orbit. Recall that in Sect. 3 we have shown that this orbit crashes at some $\bar{r} \leq 1$. We are not able to decide if $\Omega(\bar{\lambda}) = \infty$, or $\Omega(\bar{\lambda}) < \infty$. We thus consider the two possibilities separately, in Theorems 5.1 and 5.2, and although the proofs are different, we obtain similar results.

Theorem 5.1. *Assume $\Omega(\bar{\lambda}) < \infty$; then the following properties hold for the $\bar{\lambda}$-orbit:*

$$\lim_{r \to \bar{r}} \Phi(r, \bar{\lambda}) = 0, \tag{5.1}$$

$$\lim_{r \to \bar{r}} |w'(r, \bar{\lambda})| = \infty, \tag{5.2}$$

$$\lim_{r \to \bar{r}} w(r, \bar{\lambda}) = 0, \tag{5.3}$$

$$\bar{r} = 1. \tag{5.4}$$

Proof. We shall prove these statements in the above order. Since $\bar{\lambda}$ is fixed, for notational convenience, we shall suppress the dependence on $\bar{\lambda}$; this should cause no confusion. Since $\Omega(\bar{\lambda}) < \infty$, it follows that $w'(r)$ is of one sign for r near $\bar{r}$, and hence $w(r)$ has a limit as $r \to \bar{r}$. Since $\Phi(r) = \mu(r) - \dfrac{(1-w^2)^2}{r}$, [cf. (3.7)], where $\mu' > 0$, and $\mu(r)$ is bounded, ([2]), it follows that $\lim_{r \to \bar{r}} \Phi(r)$ exists. Now suppose $\lim_{r \to \bar{r}} \Phi(r) > 0$. Then using (2.1), we see that $A'(r) > 0$ for r near $\bar{r}$, $r < \bar{r}$. Then for some intermediate point ξ, $A(\bar{r}) = A(r) + A'(\xi)(\bar{r} - r) > 0$, and this is impossible, thus $\Phi(\bar{r}) \leq 0$.

Now assume that

$$\Phi(\bar{r}) = -2\delta < 0; \tag{5.5}$$

we shall show that this leads to a contradiction. Thus, if (5.5) holds, and the crash occurs in $\bar{Q}_2 \cup \bar{Q}_4$, then from [3, Proposition 5.11], we obtain the desired contradiction. Hence we may assume that the crash occurs in $Q_1 \cup Q_3$. In this case we consider the function $\psi(r, w) = r - \dfrac{(1-w^2)^2}{r}$, and notice that $\psi(\bar{r}, \bar{w}) = -2\delta$. It follows that we can find an $\varepsilon > 0$ such that $\psi(r, w) < -\delta$ if $|r - \bar{r}| < \varepsilon$ and $|w - \bar{w}| < \varepsilon$. Now for $\lambda < \bar{\lambda}$, $\mu(r, \lambda) < r$, and hence for sufficiently large n,

$$\Phi(r, \lambda_n) = \mu(r, \lambda_n) - \frac{(1 - w^2(r, \lambda_n))^2}{r} < -\delta, \tag{5.6}$$

if $|r - \bar{r}| \leq \varepsilon$ and $|w - \bar{w}| \leq \varepsilon$. Now for definiteness, let's assume $w'(r, \lambda_n) \leq 0$ on this range. Then if for some subsequence $\{\lambda_{n_k}\}$, $\{w'(r, \lambda_{n_k})\}$ is unbounded on either $\bar{r} \leq r \leq \bar{r} + \varepsilon$ or $\bar{w} - \varepsilon \leq w \leq \bar{w}$, then from [3, Lemma 5.13], we obtain a contradiction – viz, $w'(r, \lambda_{n_k}) = -\infty$ for some w on this interval. It follows that we may assume that the set $\{w'(r, \lambda_n)\}$ is bounded on $\bar{w} - \varepsilon \leq w \leq \bar{w}$, and $\bar{r} \leq r \leq \bar{r} + \varepsilon$. That is, on this range, $|w'(r, \lambda_n)| \leq M$. Now choose τ satisfying $0 < \tau \leq \min(\varepsilon/2M, \varepsilon/2)$. Then for large n

$$w(r + \tau, \lambda_n) - w(\bar{r}, \bar{\lambda}) = (w(\bar{r} + \tau, \lambda_n) - w(\bar{r}, \lambda_n)) + (w(\bar{r}, \lambda_n) - w(\bar{r}, \bar{\lambda})) < \varepsilon.$$

Thus, for such n, using (5.6), we obtain

$$A(\bar{r} + \tau, \lambda_n) - A(\bar{r}, \lambda_n) = A'(\xi, \lambda_n) < \frac{-\delta}{(\bar{r} + \varepsilon)^2} \tau.$$

But as $A(\bar{r}, \lambda_n) \to 0$ as $\lambda_n \to \bar{\lambda}$, we see that the last inequality implies that for large n, $A(\bar{r}+\tau, \lambda_n) < 0$, and this is a contradiction. It follows that (5.5) cannot hold, and this proves (5.1).

To prove (5.2), we shall first show that

$$w'(r) \quad \text{is unbounded near } \bar{r}\,. \tag{5.8}$$

Thus, assume that w' is bounded near $\bar{r}$, and hence $v(\bar{r}) = 0$. Then if the crash occurs in $\bar{Q}_2 \cup \bar{Q}_4$, then if $Q' = \dfrac{2w'^2}{r}$, we may write (2.5) as $(e^Q v)' = \dfrac{-w(1-w^2)e^Q}{r^2}$, so $e^Q v$ is monotone, so $v(\bar{r}) = 0$ cannot hold; thus the crash must occur in $Q_1 \cup Q_3$; in particular this rules out the possibility that $\bar{w} = 0$. In this case

$$\lim_{r\to\bar{r}} w'(r) = \lim_{r\to\bar{r}} \frac{v(r)}{A(r)} = \lim_{r\to\bar{r}} \frac{v'(r)}{A'(r)} = \lim_{r\to\bar{r}} \frac{-w(1-w^2)}{\Phi}\,, \tag{5.9}$$

where in the last equality, we have used the fact that w' is bounded near $\bar{r}$, to conclude that both $-2w'^2 v/r \to 0$, and $2w'^2 A \to 0$ as $r \to \bar{r}$. So in view of (5.1), $\lim\limits_{r\to\bar{r}} |w'(r)| = \infty$; thus (5.8) holds. Note that if $\bar{w} \neq 0$, then (5.9) together with (5.1) shows that (5.2) holds. Thus we may assume that $\bar{w} = 0$. If now (5.2) were false, then we could find an $N > 0$ and a sequence $r_n \nearrow \bar{r}$ such that $w'(r_n) < N$, $w''(r_n) = 0$, $w(r_n) \to 0$, and w' has a minimum at r_n. Now an easy calculation shows that $\Phi'(r) = 2(1-w^2)^2/r^2 + 2Aw'^2 + 4w(1-w^2)w'/r$, so that $\Phi'(r_n) \to 2/\bar{r}^2$ as $n \to \infty$. Then we find that at r_n,

$$r^2 A w'''(r_n) = -\Phi' w' - (1-3w^2)w' < 0\,.$$

This shows that w' cannot have a minimum at r_n; thus (5.2) holds.

We turn now to the proof of (5.3). We define $\bar{w}$ by

$$\bar{w} = \lim_{r\nearrow\bar{r}} w(r)\,.$$

Suppose that $\bar{w} > 0$ (the same proof works if $\bar{w} < 0$). Then the crash occurs in $Q_1 \cup Q_4$. We claim first that the crash cannot occur in Q_4. Indeed, since an infinite number of the λ_n orbits reach the hyperplane $w = 0$, we may apply Proposition 5.1.4 in [3] to arrive at a contradiction. Thus we may assume that the crash occurs in Q_1 with $\bar{w} > 0$.

Now choose $\varepsilon > 0$ so that $\bar{w} - 2\varepsilon > 0$. Then for large n, $w(\bar{r}, \lambda_n) > \bar{w} - \varepsilon$. We consider some cases. First, suppose that[2]

$$\varliminf_{n\to\infty} \Phi(r_{\bar{w}-\varepsilon}(\lambda_n), \lambda_n) = -k^2 < 0\,, \tag{5.10}$$

we shall show that this is impossible.

Now as in the proof above of (5.1),

$$\Phi(r, \lambda_n) \le -k^2/2$$

if $|r - \bar{r}| \le \delta$, and $|w - (\bar{w} - \varepsilon)| \le \delta$, where $\delta < \varepsilon$. Again as in the proof of (5.1), the set $\{w'(r, \lambda_n)\}$ is bounded if these conditions are satisfied; i.e., there is an $M > 0$

[2] $r_a(\lambda)$ is defined by $w(r_a(\lambda), \lambda) = a$

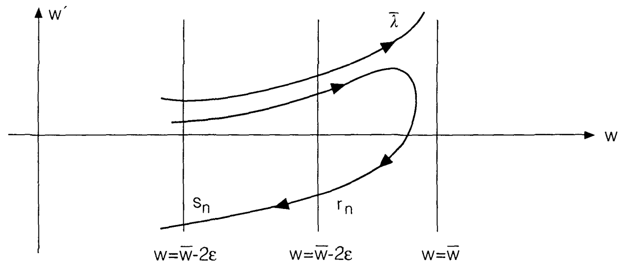

Fig. 3.

such that $|w'(r, \lambda_n)| \le M$ if $|r - \bar{r}| < \delta$ and $|w - (\bar{w} - \varepsilon)| < \delta$. Thus (suppressing the λ_n, for notational convenience)

$$r_{\bar{w}-\varepsilon-\delta} - r_{\bar{w}-2\varepsilon} \ge \frac{\varepsilon - \delta}{M} \equiv \eta > 0 .$$

Hence, using (2.1),

$$\begin{aligned} -A(r_{\bar{w}-\varepsilon}) &\le A(r_{\bar{w}-\varepsilon-\delta}) - A(r_{\bar{w}-\varepsilon}) = A'(\xi)\,(\varepsilon - \delta) \\ &\le -\frac{k}{2}\,\frac{1}{(\bar{r} - \delta)^2}\,(\varepsilon - \delta) \equiv -\tau^2 < 0 , \end{aligned}$$

so $A(r_{\bar{w}-\varepsilon}(\lambda_n)) \ge \tau^2$. Moreover, $|w'(r_{\bar{w}-\varepsilon}(\lambda_n))| \le M$, so if the sequence $\{r_{\bar{w}-\varepsilon}(\lambda_n)\}$ is bounded, then the points $p_n = (w - \varepsilon, w'(r_{\bar{w}-\varepsilon}(\lambda_n)), A(r_{\bar{w}-\varepsilon}(\lambda_n))$ have a convergent subsequence which converges to a point $P \in \Gamma$. Since $\Omega(\lambda_n) < \infty$, it follows from Theorem 3.3, that the $\bar{\lambda}$-orbit doesn't crash, and this contradicts Theorem 3.1. If on the other hand $r_{\bar{w}-\varepsilon}(\lambda_n) \to \infty$, then we can find a subsequence, call it $\{\lambda_n\}$ again, and points $Q_n = (\bar{w} - \varepsilon, w'(2, \lambda_n), A(2, \lambda_n), 2)$ which again converge to a point $P \in \Gamma$ (see Lemma 4.2, and Lemma 3.3 of [5]), so we again obtain a contradiction; thus (5.10) cannot hold, and we have

$$\varliminf_{n\to\infty} \Phi(r_{\bar{w}-\varepsilon}(\lambda_n)) \ge 0 . \tag{5.11}$$

Now suppose

$$\varlimsup_{n\to\infty} \Phi(r_{\bar{w}-\varepsilon}(\lambda_n)) = k^2 > 0 ; \tag{5.12}$$

we shall show that this too is impossible.

If (5.12) holds, then as above, $\Phi(r, \lambda_n) \ge \frac{k^2}{2}$ if $|r - \bar{r}| \le \delta$, and $|w - (\bar{w} - \varepsilon)| \le \delta$, where $\delta < \varepsilon$. Then from [4, Propositions 2.5 and 2.7], we can find constants $\sigma > 0$ and $\tau > 0$ such that $A(r_{\bar{w}-\varepsilon}(\lambda_n)) \ge \tau^2$ and $|w'(r_{\bar{w}-\varepsilon}(\lambda_n)| \le \sigma$, and the desired contradiction is obtained just as before. Hence (5.12) cannot hold, so in view of (5.11), we may assume that

$$\lim_{n\to\infty} \Phi(r_{\bar{w}-\varepsilon}(\lambda_n)) = 0 . \tag{5.13}$$

To obtain the desired contradiction in this remaining case, we define r_n and s_n by

$$\begin{aligned} w(r_n, \lambda_n) &= \bar{w} - \varepsilon , \quad & w'(r_n, \lambda_n) < 0 , \\ w(s_n, \lambda_n) &= \bar{w} - 2\varepsilon , \quad & w'(s_n, \lambda_n) < 0 , \end{aligned}$$

where r_n is the smallest $r > r_{\bar{w}-\varepsilon}(\lambda_n)$, where $w'(r_{\bar{w}-\varepsilon}(\lambda_n), \lambda_n) > 0$, and s_n too is minimal in an analogous sense; see Fig. 3.

We now consider two subcases:

$$\text{a) } \overline{\lim} s_n = \infty\,; \qquad \text{b) } \overline{\lim} s_n = S < \infty\,.$$

Suppose first that we are in case a), then consider the points $Q_n = (\bar{w} - \varepsilon, w'(2, \lambda_n), A(2, \lambda_n), 2)$, which as above, contain a subsequence which converges to a point $P \in \Gamma$, and we obtain a contradiction as above. Thus we may assume that we are in Case b). Since $S \geq s_n > r_n$, if for some subsequence $\{\lambda_{n_k}\}$, the set $\{w'(r, \lambda_{n_k})\}$ were unbounded on the interval $[\bar{w} - 2\varepsilon, \bar{w} - \varepsilon]$, then as in the proof of [4, Proposition 5.14], we obtain the contradiction $w'(r_w(\lambda_n), \lambda_n) = -\infty$ for some n and some w, $\bar{w} - 2\varepsilon \leq w \leq \bar{w} - \varepsilon$. Thus we may assume that the set $\{w'(r, \lambda_n)\}$ is bounded on $[\bar{w} - 2\varepsilon, \bar{w} - \varepsilon]$; i.e., $|w'(r, \lambda_n)| \leq M$ if $\bar{w} - 2\varepsilon \leq w(r, \lambda_n) \leq \bar{w} - \varepsilon$. Suppose first that for each n, we can find τ_n, $\bar{w} - 2\varepsilon \leq w(\tau_n, \lambda_n) \leq \bar{w} - \varepsilon$, such that at τ_n

$$\frac{-2w'^2 v}{\tau_n} - \frac{w(1 - w^2)}{\tau_n^2} \geq \frac{w(1 - w^2)}{2\tau_n^2}\,.$$

Then at τ_n, $-v \geq w(1 - w^2)/4\tau_n^2 w'^2$, so that at τ_n,

$$A(\tau_n, \lambda_n) \geq \frac{-w(1 - w^2)}{4\tau_n w'^2} \geq \frac{c}{4SM^3}\,,$$

where c is chosen such that $w(1 - w^2) \geq c$ on $[\bar{w} - 2\varepsilon, \bar{w} - \varepsilon]$. Thus the points

$$P_n = (w(\tau_n, \lambda_n), w'(\tau_n, \lambda_n), A(\tau_n, \lambda_n), \tau_n)$$

have a subsequence which converges to a point $P \in \Gamma$, and we get the contradiction, as before, that the $\bar{\lambda}$-orbit doesn't crash. It follows that we may assume that

$$\frac{-2w'^2 v}{r} - \frac{w(1 - w^2)}{r^2} \leq \frac{-w(1 - w^2)}{2r^2}$$

for all λ_n, and for all r, $r_n \leq r \leq s_n$. Then

$$\begin{aligned} v(s_n, \lambda_n) - v(r_n, \lambda_n) &= \int_{r_n}^{s_n} v'(r)dr = \int_{r_n}^{s_n} \left(\frac{-2w'^2 v}{r} - \frac{w(1 - w^2)}{r^2} \right) dr \\ &\leq \int_{r_n}^{s_n} \left(-\frac{w(1 - w^2)}{r^2} \right) dr \\ &\leq \frac{-c}{S^2}(s_n - r_n) \leq \frac{-c}{S^2}\frac{\varepsilon}{M} \equiv -k^2\,. \end{aligned}$$

Thus

$$v(s_n, \lambda_n) \leq -k^2\,,$$

and hence $A(s_n, \lambda_n) \geq \dfrac{k^2}{M}$, and as before the sequence

$$P_n = (w(s_n, \lambda_n), w'(s_n, \lambda_n), A(s_n, \lambda_n), s_n)$$

contains a subsequence which converges to a point $P \in \Gamma$, and we get a contradiction as above. The proof of (5.3) is complete. Since (5.1) and (5.3) imply that $\bar{r} = 1$, we see that this finishes the proof of Theorem 5.1. □

We turn now to the case where $\Omega(\bar{\lambda}) = \infty$.

Theorem 5.2. *Assume $\Omega(\bar{\lambda}) = \infty$; then the following properties hold for the $\bar{\lambda}$-orbit:*

$$\lim_{r \to \bar{r}} w(r, \bar{\lambda}) = 0\,, \tag{5.14}$$

$$\lim_{r \to \bar{r}} \Phi(r, \bar{\lambda}) = 0\,, \tag{5.15}$$

$$w'(r, \bar{\lambda}) \text{ is unbounded for } r \text{ near } \bar{r}\,, \tag{5.16}$$

$$\bar{r} = 1\,. \tag{5.17}$$

Proof. As in the proof of the last theorem, we shall suppress the dependence on $\bar{\lambda}$.

We begin with (5.14). Thus, let $\{r_n^N\}$ be an increasing sequence such that $r_n^N < \bar{r}$, $w'(r_n^N) = 0$, and $w(r_n^N) < 0$. Suppose that

$$\underline{\lim} w(_n^N) = -2\gamma < 0\,; \tag{5.18}$$

we will show that this leads to a contradiction. Set

$$\psi(r, A, z) = r(1 - A) - \frac{(1 - z^2)}{r}\,,$$

and let us regard ψ as an abstract (continuous) function of 3 variables; then

$$\psi(\bar{r}, 0, 0) = \bar{r} - \frac{1}{\bar{r}}\,. \tag{5.19}$$

Suppose first that

$$\bar{r} - \frac{1}{\bar{r}} = 3\varepsilon > 0\,. \tag{5.20}$$

Then $\psi(r, 0, z) = r - \dfrac{(1 - z^2)^2}{r} > 2\varepsilon$, if $|z| < 2\delta$, $|r - \bar{r}| < 2\delta$, $0 < A < 2\delta$, for some $\delta < \gamma$. Hence by continuity,

$$\Phi(r, \lambda_n) > \varepsilon\,, \quad \text{if} \quad |w| < 2\delta\,, \quad |r - \bar{r}| < 2\delta\,, \quad 0 < A < 2\delta\,. \tag{5.21}$$

From [4, Proposition 2.6], $w'(r_{\frac{-\delta}{2}}(\lambda_n), \lambda_n)$ is uniformly bounded, and hence $w'(r, \lambda_n)$ is uniformly bounded on $r_{-\delta}(\lambda_n) \le r \le r_{-2\delta}(\lambda_n)$; say $|w'(r, \lambda_n)| \le M$ on this interval. It follows that $|w'(r)| \le M$ on this interval, so that

$$r_{-\delta} - r_{-2\delta} \ge \frac{\delta}{M}\,, \tag{5.22}$$

so after finitely many rotations $r_n^N > 1$, and thus there can be no crash, (crashing orbits must crash for $r \le 1$; see [5, Lemma 3.3]). It follows that (5.22) cannot hold; thus $\bar{r} - \frac{1}{\bar{r}} \le 0$. Now if $\bar{r} - \frac{1}{\bar{r}} = 0$, then $\psi(\bar{r}, 0, z) > 0$ on $-2\gamma < z < -\gamma$, and thus we get the same contradiction as before. Therefore we must have

$$\bar{r} - \frac{1}{\bar{r}} = -3\varepsilon < 0\,, \quad \text{for some} \quad \varepsilon > 0\,.$$

Hence as above,

$$\Phi(r, w, A) < -\varepsilon \quad \text{if} \quad |w| < 2\delta\,, \quad |r - \bar{r}| < 2\delta\,, \quad 0 < A < 2\delta\,,$$

for some $\delta < \gamma$. Now $w'(r)$ must be unbounded on $-\delta \le w \le 0$; otherwise as in (5.22), $r_{-\delta} - r_0 \equiv \eta > 0$, and we again would not have a crash for the $\bar{\lambda}$-orbit. Thus we may apply [3, Lemma 5.13] to obtain the contradiction that some λ_n-orbit crashes. It follows that (5.18) cannot hold; i.e. $\underline{\lim}\, w(r_n^N) = 0$, and by a similar argument $\overline{\lim}\, w(\tilde{r}_n^N) = 0$, where $\tilde{r}_n^N$ is defined by $w'(\tilde{r}_n^N) = 0$, $w(\tilde{r}_n^N) > 0$. This implies that (5.14) holds.

We turn now to the proof of (5.15). First note that in view of (5.14), it follows as in the proof of Theorem 5.1, that $\lim_{r\to\bar{r}} \Phi(r) \equiv \bar{\Phi}$ exists. Suppose that $\bar{\Phi} > 0$; then as above, $\Phi(r) \ge \varepsilon > 0$ for r near $\bar{r}$, and thus the $\bar{\lambda}$-orbit cannot crash, see [4, Proposition 2.6]. Thus $\bar{\Phi} \le 0$. Suppose that

$$\bar{\Phi} = -2b < 0\,, \tag{5.23}$$

we shall show that this leads to a contradiction. Thus if (5.23) holds, then as above, we can find $\delta > 0$ such that for large n, say $n \ge N$,

$$\Phi(r, w(r, \lambda_n), A(r, \lambda_n)) < -L\,, \tag{5.24}$$

if $|w(r, \lambda_n)| < \delta$, $|r - \bar{r}| < \delta$, $0 < A(r, \lambda_n) < \delta$. Now as before, from [3, Lemma 5.13], the functions $w'(r, \lambda_n)$ are uniformly bounded on the above intervals. It follows that $w'(r)$ is bounded on the set $|w| < \delta$, $0 < \bar{r} - r < \delta$. Hence

$$\begin{aligned}\lim_{r\to\bar{r}} w'(r) &= \lim_{r\to\bar{r}} \frac{v(r)}{A(r)} = \lim_{r\to\bar{r}} \frac{v'(r)}{A'(r)} \\ &= \lim_{r\to\bar{r}} \frac{\dfrac{-2w'^2}{r} - \dfrac{w(1-w^2)}{r^2}}{\Phi/r^2 - \dfrac{2aw'^2}{r}} = 0\,,\end{aligned}$$

i.e., $w'(r) \to 0$ as $r \to \bar{r}$. Thus $(w(r), w'(r)) \to (0,0)$ as $r \to \bar{r}$. Hence given any $\varepsilon_1 > 0$, we can find $N_1 > N$ such that if $n \ge N_1$, then $\bar{r}^2 w'(\bar{r}, \lambda_n)^2 + w(\bar{r}, \lambda_n)^2 < \varepsilon_1$. Then using [5, Lemma 3.6], given any $T > 0$, and $\varepsilon > 0$, if ε_1 is sufficiently small, $r^2 w'(r, \lambda_n)^2 + w(r, \lambda_n)^2 < \varepsilon$ if $\bar{r} \le r \le \bar{r} + T$, for all $n \ge N_1$. In particular, choosing $T = \delta$, and taking ε so small that $|w(r, \lambda_n)| < \delta$ on $\bar{r} \le r \le \bar{r} + \delta$, we see that (5.23) holds. Then from (2.1), if $n \ge N_1$, we have, for some intermediate point $\xi = \xi_n$,

$$A(\bar{r} + \delta, \lambda_n) - A(\bar{r}, \lambda_n) = A'(\xi)\delta \le \frac{\Phi(\xi)}{\xi^2}\,\delta \le \frac{-L\delta}{(\bar{r} + \delta)^2}\,.$$

But for n large, $A(\bar{r}, \lambda_n) \to A(\bar{r}, \bar{\lambda}) = 0$, and thus $A(\bar{r} + \delta, \lambda_n) < 0$. This is a contradiction. Hence (5.23) cannot hold and thus $\bar{\Phi} = 0$; this proves (5.15).

Observe that (5.14) and (5.15) imply (5.17). Thus to complete the proof of the theorem, we need only prove (5.16).

Suppose that w' is bounded near $\bar{r}$. Then for $r < \bar{r}$, r near $\bar{r}$, since $v(\bar{r}) = 0$ we have from the Cauchy mean-value theorem,

$$w'(r) = \frac{v(r)}{A(r)} = \frac{v(r) - v(\bar{r})}{A(r) - A(\bar{r})} = \frac{v'(\tilde{r})}{A'(\tilde{r})}\,, \qquad r > \tilde{r} > \bar{r}\,.$$

Now from (2.1), $A'(\bar{r}) = 0$, and from (2.5), $v'(\bar{r}) = 0$, so that again by the Cauchy mean-value theorem,

$$w'(r) = \frac{v'(\tilde{v}) - v'(\bar{r})}{A'(\tilde{r}) - A'(\bar{r})} = \frac{v''(\tilde{\tilde{r}})}{A''(\tilde{\tilde{r}})}\,, \qquad r < \tilde{r} < \tilde{\tilde{r}} < \bar{r}\,.$$

Now as $v' = -2w'^3 A/r - w(1-w^2)/r^2$, we have

$$v'' = -2\,\frac{r(Aw'^3)' - Aw'^3}{r^2} - \frac{r^2(1-3w^2)w' - w'(1-w^2)(2r)}{r^4}\,,$$

and for r near $\bar{r}$, $v''(r) \approx \frac{-1}{r^2}\,w'(r)$. Similarly from (3.1) we find

$$A'' = -2\,\frac{r(Aw'^3)' - (Aw'^2)}{r^2} - \frac{-r^2\Phi' - 2r\Phi}{r^4}\,,$$

so that for r near $\bar{r}$, $A''(r) \approx 2/r^2$. Hence for r near $\bar{r}$, $w'(r) \approx \dfrac{-w'(\tilde{\bar{r}})}{2}$, or $-w'(\tilde{\bar{r}}) \approx 2w'(r)$. Thus if $w'(r) \neq 0$, there exists an $\tilde{\bar{r}}$, $r < \tilde{\bar{r}} < \bar{r}$ with $w'(\tilde{\bar{r}}) \approx -2w'(r)$. Hence by repeatedly applying this result, we get a sequence r_n, with $r < r_1 < r_2 < \ldots < r_k < r$ such that $w'(r_{n+1}) \approx -2w'(r_n)$; i.e., w' is unbounded near $\bar{r}$. Since we can choose r near $\bar{r}$ such that $w'(r) \neq 0$, this contradiction completes the proof of (5.17). The proof of Theorem 5.2 is complete. $\square$

We shall now study the behavior of the metric coefficients $A(r)$ and $T(r)$ near $r = 1$; cf. (4.20). To this end, we first need the following lemma which follows just as in the proof of Lemma 4.4.

Lemma 5.3. *For the $\bar{\lambda}$-orbit, $\lim\limits_{r \nearrow 1} (Aw'^2)(r)$ exists.*

Proposition 5.4. *For the $\bar{\lambda}$-orbit, $\lim\limits_{r \nearrow 1} (Aw'^2)(r) = 0$.*

Proof. From Lemma 4.4, the result holds if $\Omega(\bar{\lambda}) = \infty$. Thus suppose $\Omega(\bar{\lambda}) < \infty$, and assume the result is false; i.e.,

$$\lim_{r \nearrow 1} (Aw'^2)(r) = \sigma > 0\,. \tag{5.25}$$

We shall show that this leads to a contradiction. To this end, for any orbit, we define $P(r)$ by

$$P'(r) = \frac{\Phi(r)}{r^2 A(r)}\,, \qquad P(0) = 0\,, \tag{5.26}$$

and we recall Q is defined by (4.22); i.e.,

$$Q'(r) = \frac{2w'^2(r)}{r}\,, \qquad Q(0) = 0\,.$$

We now again define $f(r) = (Aw'^2)(r)$, and we have

Lemma 5.5. *For r near* 1, $(r < 1)$,

$$[Q'(r) + P'(r)]f(r) \geq \frac{\sigma}{2}\,Q'(r)\,.$$

Proof. We have, for r near 1,

$$\begin{aligned}
[Q'(r) + P'(r)]f(r) &= \left(\frac{2w'^2}{r} + \frac{\Phi}{r^2 A}\right)(Aw'^2)(r) \\
&= \left(2Aw'^2 + \frac{\Phi}{r}\right)\frac{w'^2}{r} \\
&\geq \frac{3}{2}\,\sigma\,\frac{w'^2(r)}{r}
\end{aligned}$$

because $\Phi(r) \to 0$ as $r \nearrow 1$. Thus for r near 1,

$$[Q'(r) + P'(r)]f(r) \geq \frac{3}{4}\sigma\left(\frac{2w'^2}{r}\right) > \frac{\sigma}{2}Q'(r);$$

this proves the lemma. □

We can now complete the proof of Proposition 5.4. Thus, from (4.24), we can write

$$\begin{aligned} 0 &= f' + Q'f + \frac{\Phi}{r^2 A}Aw'^2 + \frac{2w(1-w^2)w'}{r^2} \\ &= f' + (Q' + P')f + \frac{2w(1-w^2)w'}{r^2}. \end{aligned}$$

Thus, for r near 1, say $r_1 < r < 1$, we have from Lemma 5.5,

$$\frac{\sigma}{2}Q'(r) \leq (Q' + P')f = -f' - \frac{2w(1-w^2)w'}{r^2}.$$

Thus, integrating from r_1 to r,

$$\frac{\sigma}{2}(Q(r) - Q(r_1)) \leq f(r_1) - f(r) - \int_{r_1}^{r} \frac{2w(1-w^2)w'}{s^2}\,ds. \tag{5.27}$$

Now from (5.2), $Q(r) \to \infty$ as $r \nearrow 1$, and as $\Omega(\bar{\lambda}) < \infty$, and $\lim |w'| = \infty$,

$$\int_{r_1}^{r} \frac{2w(1-w^2)w'}{s^2}\,ds$$

is a finite number of integrals of the form

$$\sum_{i=1}^{k} \int_{w(r_i)}^{w(r_{i+1})} \frac{2w(1-w^2)}{s^2}\,dw,$$

where $r_{k+1} > r_k > \ldots > r_1$. It follows that the right-hand side of (5.27) is finite as $r \nearrow 1$, while the left-hand side tends to $+\infty$. This contradiction completes the proof of Proposition 5.4. □

With the aid of Lemma 5.5, the next result follows exactly as in Proposition 4.6, part (i).

Proposition 5.6. *The following properties hold for the metric coefficients of the $\bar{\lambda}$-orbit:*

$$\lim_{r \nearrow 1} A(r, \bar{\lambda}) = 0 = \lim_{r \nearrow 1} A'(r, \bar{\lambda}).$$

We now investigate the behavior of the metric coefficient $T^{-2}(r, \bar{\lambda})$, for $r \leq 1$. For this we need the following lemma, (cf. [5]; we include a different proof here for the sake of completeness).

Lemma 5.7. *For any* λ, $T'(r, \lambda) < 0$ *if* $r > 0$.

Proof. From (4.21),

$$-2r^2 A \frac{T'}{T} = 2w'^2 rA + \Phi ,$$

so it suffices to show that $2w'^2 rA + \Phi > 0$. Thus, set $g = 2w'^2 rA + \Phi$, and notice that $g(r) > 0$ if r is large. Furthermore,

$$g' = -\frac{2w'^2}{r} g + 4Aw' + \frac{2(1-w^2)^2}{r^2} .$$

Now if $g(\varepsilon) < 0$ for some $\varepsilon > 0$, let $r_1 = \sup\{r < \varepsilon : g(r) = 0\}$. Then $r_1 > 0$ and $0 = g(r_1) = g(r_1) - g(0) = r_1 g'(\xi) > 0$. This is a contradiction, so $g(r) > 0$ if $r > 0$. □

We now have

Lemma 5.8. *For any* $r \leq 1$,

$$\lim_{\lambda_k \to \bar{k}} T^{-2}(r, \lambda_k) = 0 . \tag{5.28}$$

Prof. Since $T^{-2}(r, \lambda_k) \geq 0$, in order to show (5.28) it suffices to show

$$\overline{\lim_{\lambda_k \to \bar{\lambda}}}\, T^{-2}(r, \lambda_k) = 0 .$$

Thus, suppose there was a subsequence, call it $\{\lambda_k\}$ again, for which $\overline{\lim}\, T^{-2}(r, \lambda_k) \geq b > 0$ for some $r \leq 1$. Then in view of the previous lemma, we have $\overline{\lim}\, T^{-2}(1, \lambda_k) \geq b > 0$. From Proposition 4.6, the PC orbit satisfies $\lim_{r \searrow 1} T^{-1}(r) = 0$, so that $\lim_{\lambda_k \to \bar{\lambda}} T^{-2}(1, \lambda_k) = 0$. This contradiction establishes the result. □

6. Concluding Remarks

We can use the PC orbit to construct a new "black hole" solution to the EYM equations. Thus if $(A(r), w(r), w'(r))$ denotes the PC orbit, we define

$$w_0(r) = \begin{cases} w(r, \bar{\lambda}) , & r > 1 \\ 0 , & r = 1 \\ w(r) , & r > 1 ; \end{cases}$$

then $w_0(r)$ is a continuous function (cf. Theorems 4.1, 5.1, and 5.2). Moreover, if

$$A_0(r) = \begin{cases} A(r, \bar{\lambda}) , & r < 1 \\ 0 , & r = 1 \\ A(r) , & r > 1 , \end{cases}$$

then A_0 is continuous, and in fact, so is A_0'; note that $A_0'(1) = 0$ (cf. Theorems 3.1, 4.1, Proposition 5.6, and Eq. (4.25)). Now if we want $T^{-2}(r, \bar{\lambda})$ to depend continuously on the values of $T^{-2}(r, \bar{\lambda}_k)$, for $r \leq 1$, then we may define $T_0^{-2}(r)$ by

$$T_0^{-2}(r) = \begin{cases} 0\,, & r \leq 1 \\ T^{-2}(r)\,, & r > 1\,, \end{cases}$$

and T_0^{-2} is continuous. The functions $T_0^{-2}(r)$ and $A_0^{-1}(r)$ can be used to define the metric for this black hole solution. Observe that $w_0'(r)$ is unbounded near $r = 1$ (Theorems 4.1, 5.1, and 5.2), and that as $r \searrow 1$, this orbit has infinite rotation (cf. Theorem 4.1, Part 2). Thus this black hole solution is *very* different from the black hole solutions whose existence was proved in [5]; the stability of these latter solutions was investigated numerically in [2, 6–10].

We next note that the existence of a PC orbit gives us a proof that the λ_n-(connecting) orbits have uniformly bounded rotation in $r > 1$. More precisely, we have the following theorem.

Theorem 6.1. *Given any* $\varepsilon > 0$*, there is a constant* $c = c(\varepsilon) > 0$ *such that for all* $r > 1 + \varepsilon$,

$$|\theta(r, \lambda_n) - \theta(1 + \varepsilon, \lambda_n)| \leq c(\varepsilon).$$

Proof. Let $r_2 > r_1 > 1+\varepsilon$, and let $\Delta\theta_n = \theta(r_2, \lambda_n) - \theta(r_1, \lambda_n)$. We claim that $|\Delta\theta_n|$ is uniformly bounded. For suppose not; as before we can find a subsequence, call it $\{\lambda_n\}$ again, such that $\lambda_n \to \bar{\lambda}$ and the corresponding points $P_n = (w(1 + \varepsilon, \lambda_n), w'(1 + \varepsilon, \lambda_n), A(1 + \varepsilon, \lambda_n), 1 + \varepsilon)$ converge to a good point $\bar{P} \in \Gamma$. Now consider the PC orbit through $\bar{P}$. From Theorem 4.4, Part iii), the PC orbit satisfies $|\Delta\theta| \leq N$ for some N. It follows from [4, Proposition 3.4], that for large n, $|\Delta\theta_n| \leq N + 1$; this contradiction completes the proof of the theorem. $\square$

It is interesting to see that the black hole solution, constructed as above, from the PC orbit, can be used to give information on the particle-like λ_n-orbits, as in Theorem 6.1. Conversely, the particle-like λ_n-orbits are used to construct the PC orbit which then yields the black hole solution.

We end this section with a list of some open problems, together with some conjectures.

1. is it true that if $\lambda \geq \bar{\lambda}$, then the λ-orbit crashes? We conjecture the answer is yes. Note that in [3, Theorem 4.1], we have shown that if $\lambda \geq 2$, then the λ-orbit crashes; see also [2].
2. We do not know if the $\bar{\lambda}$-orbit has infinite rotation as $r \nearrow 1$. Again we conjecture that the answer is affirmative.
3. Is there always a unique k-connecting orbit? We believe that the answer is yes.
4. Is the more than one PC orbit? Note that in our construction of the PC orbit, (in Sect. 4), the "starting point" $P \in \Gamma$ (where $r = 1 + \varepsilon$) was the limit of a *subsequence* of points $P_n \in \Gamma$. Thus there is the possibility of having an uncountable number of PC orbits.
5. Is the PC orbit linearly stable? Is it nonlinearly stable? Since it is almost impossible to find the PC orbit numerically, the answer to these questions must be determined analytically.

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