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Vertex Representations for N-Toroidal Lie Algebras and a Generalization of the Virasoro Algebra

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Abstract: Vertex representations are obtained for toroidal Lie algebras for any number of variables. These representations afford representations of certain n-variable generalizations of the Virasoro algebra that are abelian extensions of the Lie algebra of vector fields on a torus.

0. Introduction

In this paper we construct faithful vertex operator representations for the universal central extension τ_n of $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $\mathfrak{g}$ is a simple, simply-laced finite dimensional Lie algebra over $\mathbb{C}$. We call τ_n the n-toroidal Lie algebra. These representations also afford representations for an abelian extension of the Lie algebra of derivations of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$. This latter Lie algebra is a generalization of the Virasoro algebra, and so this whole construction is a generalization of both the Frenkel–Kac and the Segal–Sugawara constructions which are well known for the case $n = 1$.

For a suitable non-degenerate integral lattice Γ and an even integral sublattice Q (cf. Sect. 3), we construct the Fock space $V(\Gamma, \mathfrak{b}) = \mathbb{C}[\Gamma] \otimes S(\mathfrak{b}_-)$, where $\mathfrak{b}$ is a Heisenberg algebra defined by Γ. For each α in Q we define vertex operator $X(\alpha, z)$ (cf. 3.7) such that its Fourier components $X_n(\alpha)$ act on $V(\Gamma, \mathfrak{b})$. Our first result (Theorem 3.14) says that the Lie algebra generated by operators $X_k(\alpha)$ ($\alpha \in Q, (\alpha \,|\, \alpha) = 2$) is isomorphic to $\tau_{[n]}$. We also prove that the "zero moments" (taking $k = 0$ above) generate the Lie algebra $\tau_{[n-1]}$ (Theorem 3.17).

Theorem 3.14 in the case $n = 1$ is due to Frenkel–Kac [FK] and $\tau_{[1]}$ is the non-twisted affine Lie algebra. The case $n = 2$ is due to [MEY]. Our method of proof here differs considerably from that of [MEY]. It is more explicit in the sense that we give operators for every vector of $\tau_{[n]}$ and prove that the necessary commutators hold. For example the vector $h \otimes t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}$ in $\tau_{[n]}$ is represented by the operator $T^h_{r_n}(\delta_{\underline{r}})$ (cf. 3.10) which is not clear from [MEY]. A significant difference is the

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identification of central operators of $\mathcal{T}_{[n]}$. The centre of $\mathcal{T}_{[n]}$ is given by operators $T_k^{\gamma}(\gamma')$ and $X_k(\gamma)$ with the relation

$$(*) \qquad T_k^{\gamma}(\gamma) + kX_k(\gamma) = 0 .$$

Here γ, γ' are vectors of Q with zero norm and they form an $(n-1)$-dimensional vector space over $\mathbb{C}$. In the case $n = 2$, zero norm vectors are *one* dimensional and due to $(*)$, the T operator can be replaced by an X operator. However for general n one needs the new T operators together with the relation $(*)$.

By taking $n = 3$ in Theorem 3.17 we have a new class of representations for $\mathcal{T}_{[2]}$ which are different from the one constructed in [MEY].

An important new development in the present paper is the introduction of a generalization of the Virasoro algebra. Recall that the Virasoro algebra is the universal central extension of the Lie algebra of vector fields on a circle. This amounts to saying that it is the universal central extension of the Lie algebra of derivations of the polynomial ring $\mathbb{C}[t^{\pm 1}]$. In the same way the Lie algebra of vector fields on an n-dimensional complex torus can be identified with the Lie algebra $\operatorname{Der} A$ of derivations of $A = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$. However there is a significant difference when $n > 1$ because in this case $\operatorname{Der} A$ is already centrally closed. Because of this, finding a good generalization of the Virasoro algebra to $n > 1$ has turned out to be difficult, and there have been a number of attempts by physicists to do this. We put forward one solution to this problem here.

Every element of $\operatorname{Der} A$ has an obvious extension to $\mathfrak{g} \otimes A$. It can be further extended uniquely as a derivation to $\mathcal{T}_{[n]}$ where it leaves the centre $\Omega_{[n]}$ invariant (cf. 5.1). For each such element we construct an operator on V whose commutators with the vertex operators of $\mathcal{T}_{[n]}$ yield precisely its action as a derivation (Propositions 5.5 and 5.8). Next we attempt to understand the Lie algebra generated by these operators corresponding to the derivations. We have a complete answer only in the "zero moments" case (Proposition 5.11), where it is seen to be an *abelian* extension of $\operatorname{Der} A$ by $\Omega_{[n]}$, the vector space which is the centre of $\mathcal{T}_{[n]}$. In general the operators also generate an abelian extension of derivations on A; however, an explicit description of the extension Lie algebra remains an open problem. In the $n = 1$ it is Virasoro algebra.

In Sect. 1, we recall the construction of universal central extension of $\mathfrak{g} \otimes A$ from [MEY]. In Sect. 2, we give a presentation for $\mathcal{T}_{[n]}$ generalizing that of [MEY] and also deduce a second presentation starting from a Chevalley $\mathbb{Z}$-basis. The latter is used to prove our theorems in Sect. 3. The technical lemmas on vertex operators that are used in Sect. 3 are established in Sect. 4. In Sect. 5 we define the operators of the generalized Virasoro algebra.

For additional material on recent developments in the theory of toroidal Lie algebas one may consult [BC, EMY, FM, and MS].

1. Construction of Central Extension Recalled

Let $\dot{\mathfrak{g}}$ be a finite dimensional simple Lie algebra over complex numbers $\mathbb{C}$. Let $\dot{\mathfrak{h}}$ be a Cartan subalgebra of $\dot{\mathfrak{g}}$, let $\dot{\Delta}$ denote the corresponding root system, $\dot{\Delta} \subseteq \dot{\mathfrak{h}}^*$, and let $\dot{\Pi} = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be a basis of $\dot{\Delta}$. The Killing form $(\cdot|\cdot)$ is non-degenerate on $\dot{\mathfrak{h}}$, and we will usually identify $\dot{\mathfrak{h}}^*$ with $\dot{\mathfrak{h}}$ by means of it. We assume that $(\cdot|\cdot)$ is so normalized that after this identification long roots have square length equal to 2. For each root $\alpha \in \dot{\Delta}$ the Lie algebra $\dot{\mathfrak{g}}^{\alpha} + [\dot{\mathfrak{g}}^{\alpha}, \dot{\mathfrak{g}}^{-\alpha}] + \dot{\mathfrak{g}}^{-\alpha}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. An $\mathfrak{sl}_2$-triplet for this is a choice of $e_\alpha \in \dot{\mathfrak{g}}^{\alpha}$, $e_{-\alpha} \in \dot{\mathfrak{g}}^{-\alpha}$ for which, with $h_\alpha := [e_\alpha, e_{-\alpha}]$, we have $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, e_{-\alpha}] = -2e_{-\alpha}$. Using our identification of $\dot{\mathfrak{h}}^*$ with $\mathfrak{h}$ we have $[e_\alpha, e_{-\alpha}] = (e_\alpha | e_{-\alpha})\alpha$, $\alpha = \frac{(\alpha|\alpha)}{2} h_\alpha$ and $(e_\alpha | e_\alpha) = \frac{2}{(\alpha|\alpha)}$.

1.1. Central Extension. Let $\mathfrak{g}$ be any Lie algebra. A central extension of $\mathfrak{g}$ is a Lie algebra $\tilde{\mathfrak{g}}$ and a surjective homomorphism $\pi\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$ whose kernel lies in the centre of $\tilde{\mathfrak{g}}$. The pair $(\tilde{\mathfrak{g}}, \pi)$ is called a *universal central extension* (also called a *universal cover* of $\mathfrak{g}$ if for every central extension $(\hat{\mathfrak{g}}, \phi)$ of $\mathfrak{g}$ there is a unique homomorphism $\psi\colon \tilde{\mathfrak{g}} \to \hat{\mathfrak{g}}$ for which $\phi\psi = \pi$.

A Lie algebra $\mathfrak{g}$ is called *perfect* if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. It is known that a perfect Lie algebra admits universal central extension. A good reference to the theory of central extensions is [G].

1.2. Module of Differentials. Let A be any commutative algebra over $\mathbb{C}$. The module of differentials (Ω_A, d) of A is defined in the following way. Let $\{a_i\}$ be any basis for A over $\mathbb{C}$ and let F be the free left A-module on a basis $\{\tilde{d}a_i\}$, where $\{\tilde{d}a_i\}$ is some set of equipotent with $\{a_i\}$. We treat F as a 2-sided A-module by setting $b(\tilde{d}a) = (\tilde{d}a)b$ for all a, b in A. Let $\tilde{d}\colon A \to F$ be the $\mathbb{C}$-linear map $\sum C_i a_i \mapsto \sum C_i \tilde{d}a_i$ and let K be the A-submodule of F generated by the relations $\tilde{d}(ab) - (\tilde{d}a)b - a(\tilde{d}b)$ $a, b \in A$. Then $\Omega_A := F/K$ and the canonical quotient map $a \mapsto \tilde{d}a + K$ is the differential map $d\colon A \to \Omega_A$.

Up to evident isomorphism (Ω_A, d) is characterized by the property that for every A-module M and every derivation $D\colon A \to M$ there is a unique A-module map $f\colon \Omega_A \to M$ such that

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "D"'] & \Omega_A \arrow[d, "f"] \\ & M \end{tikzcd} \qquad \text{commutes}.$$

In this way $\mathrm{Der}_{\mathbb{C}}(A, M) \cong \hom_A(\Omega_A, M)$.

Let $\overline{}\colon \Omega_A \mapsto \Omega_A/dA$ be the canonical linear map. Observe that from $\overline{d(ab)} = 0$ we have $\overline{ad(b)} = -\overline{(da)b} = -\overline{b(da)}$ for all a, b in A.

(1.3). Consider the Lie algebra

$$\mathfrak{u} = A \otimes_C \dot{\mathfrak{g}} \oplus (\Omega_A/dA)$$

with Lie structure

$$[a \otimes X, b \otimes Y] = ab \otimes [X, Y] + \overline{(da)b}(X \mid Y)$$

and

$$\Omega_A/dA \quad \text{is central in } \mathfrak{u}\,.$$

Let $\omega\colon \mathfrak{u} \to A \otimes_C \dot{\mathfrak{g}}$ be the obvious projection with kernel Ω_A/dA.

(1.4) Proposition. ([K], [MEY]) $(\mathfrak{u}, \omega)$ is the universal central extension of $A \otimes_C \mathfrak{g}$. □

Throughout this paper we fix a positive integer n. We are interested in commutative algebra $A = A_{[n]} := \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ the $\mathbb{C}$-algebra of polynomial functions of the torus $\mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$. We also use the symbol $\Omega_{[n]}$ to denote Ω_A/dA.

(1.5) Notation.

$$\bar{r} := (r_1, r_2, \dots, r_n) \in \mathbb{Z}^n\,,$$
$$t^{\bar{r}} := t_1^{r_1} t_2^{r_2} \dots t_n^{r_n} \in A\,,$$
$$D_i(\bar{r}) := t_1^{r_1} t_2^{r_2} \dots t_i^{r_i - 1} \dots t_n^{r_n} dt_i \in \Omega_{[n]}\,.$$

Clearly $\{D_i(\bar{r}), i = 1, 2, \dots, n, \bar{r} \in \mathbb{Z}^n\}$ is a vector space basis for $\Omega_{[n]}$ over $\mathbb{C}$.

(1.6). $\Omega_{[n]}$ is spanned by vectors $\overline{D_i(\bar{r})}$ $(i = 1, 2, \dots, n,\ \bar{r} \in \mathbb{Z}^n)$ which satisfy the single relation

$$0 = \overline{d(t^{\bar{r}})} = \sum_{i=1}^{n} r_i \overline{D_i(\bar{r})}\,.$$

2. Presentations of $\tau_{[n]}$

As in [MEY] we will denote the universal central extension of the Lie algebra $\dot{\mathfrak{g}} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$ by $\tau_{[n]}$ and its centre by $\Omega_{[n]}$. In this section we will give two different presentations for $\tau_{[n]}$. The first presentation is similar in spirit to one given in [MEY]. The second presentation is deduced from a Chevalley $\mathbb{Z}$-basis of $\dot{\mathfrak{g}}$ which is also given in Kassel [K]. This presentation is tailor-made for our theorems in Sect. 3.

Let $A = (A_{ij})_{i,j=0}^{\ell}$ be an indecomposable matrix of affine type X_ℓ^1 $(X = A, \dots, G)$ (non-twisted). Let Q be free $\mathbb{Z}$-module on generators $\alpha_0, \alpha_1, \dots, \alpha_\ell$ with basis Π of Δ. We know that there is a $\mathbb{Z}$-valued symmetric bilinear form $(\cdot|\cdot)$ on Q for which, after suitable choice of indexing, $2(\alpha_i\,|\,\alpha_j)/(\alpha_j\,|\,\alpha_j) = A_{ij}$. Let $\delta = \sum_{i=0}^{\ell} n_i \alpha_i$, $n_i \in \mathbb{Z}_+$, $\gcd(n_0, n_1, \dots, n_\ell) = 1$, be the null root. We assume that the notation is chosen so that $n_0 = 1$, so α_0 is an "extension node" and $\dot{A} := (A_{ij})_{i,j=1}^{\ell}$ is of finite type X_ℓ. We assume that $(\cdot|\cdot)$ is scaled so that $(\alpha_0\,|\,\alpha_0) = 2$. In general, objects associated with $\dot{A}$ carry an over-dot, so for instance $\dot{Q} := \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \dots \oplus \mathbb{Z}\alpha_\ell \subseteq Q, \dot{\Pi} := \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$.

For each $i = 0, 1, 2, \dots, \ell$ let $\alpha_i^\vee := 2\alpha_i/(\alpha_i\,|\,\alpha_i)$. Then $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\}$ forms a basis for the coroot system $\Delta^\vee$ whose cartan matrix is A^T. The null root is $\sum_{i=0}^{\ell} n_i^\vee \alpha_i^\vee = \delta$, where $n_i^\vee := n_i(\alpha\,|\,\alpha)/2$. The fact that $n_i \in \mathbb{Z}$ can be verified by inspection, case by case.

(2.1) Presentation 1. Let $\underline{r} = (r_1, r_2, \dots, r_{n-1}) \in \mathbb{Z}^{n-1}$. We let $\mathfrak{t} = \mathfrak{t}(A_{\text{aff}})$ be the Lie-algebra over $\mathbb{C}$ with the following presentation.

Generators.

$$\delta_{\underline{r}}(\underline{s}), \alpha_i^\vee(\underline{k}), X_{\underline{k}}(\pm\,\alpha_i), \quad (i = 0, 1, 2, \cdots, \ell,\ \underline{s}, \underline{k}, \underline{r} \in \mathbb{Z}^{n-1})\,.$$

Relations.

(TA0) (i) $\delta_{\underline{r}}(\underline{s})$ central;

(ii) $\delta_{\underline{r}}(\underline{s}) + \delta_{\underline{k}}(\underline{s}) = \delta_{\underline{r}+\underline{k}}(\underline{s})$;

(iii) $\delta_{\underline{r}}(\underline{r}) = 0$,

(TA1) $[\alpha_i^\vee(\underline{r}), \alpha_j^\vee(\underline{s})] = (\alpha_i^\vee \mid \alpha_j^\vee)\delta_{\underline{r}}(\underline{r} + \underline{s})$,

(TA2) $[\alpha_i^\vee(\underline{r}), X_{\underline{m}}(\pm\,\alpha_j)] = \pm\,(\alpha_i^\vee\,|\,\alpha_j)X_{\underline{m}+\underline{r}}(\pm\,\alpha_j)$, (TA3) $$[X_{\underline{m}}(\alpha_i), X_{\underline{n}}(-\alpha_j)] = -\delta_{ij}\{\alpha_i^{\vee}(\underline{m}+\underline{n}) + \frac{2}{(\alpha_i \mid \alpha_i)}\delta_{\underline{m}}(\underline{m}+\underline{n})\},$$

(TA4) (i) $[X_{\underline{m}}(\pm\alpha_i), X_{\underline{n}}(\pm\alpha_i)] = 0$;

(ii) $\mathrm{ad}\, X_{\underline{0}}(\alpha_i)^{-A_{ij}+1} X_{\underline{m}}(\alpha_j) = 0$, $\quad i \neq j$;

(iii) $\mathrm{ad}\, X_{\underline{0}}(-\alpha_i)^{-A_{ij}+1} X_{\underline{m}}(-\alpha_j) = 0$, $\quad i \neq j$.

(*2.2*) *Note.* (i) $\mathfrak{t}$ is generated by $X_{\underline{m}}(\pm\alpha_i), \underline{m} \in \mathbb{Z}^{n-1}, i = 0,1,2,\ldots,\ell$.

(ii) TA0 can be replaced by $[\delta_{\underline{r}}(\underline{s}), \delta_{\underline{m}}(\underline{n})] = 0$.

As in the earlier section let $\dot{\mathfrak{g}}$ be finite dimensional Lie-algebra with Cartan matrix $\dot{A}$, Cartan subalgebra $\dot{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \dot{Q}$ and usual set of generators $e_i, f_i, h_i, i = 1,2,\ldots,\ell$. Let ζ be the highest root of $\dot{\Delta}$ relative to $\dot{\Pi} = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ and let $e_0 \in \dot{\mathfrak{g}}^{-\zeta}, f_0 \in \dot{\mathfrak{g}}^{\zeta}$ be chosen so that $\{e_0, f_0, h_0\}$ is an $\mathfrak{sl}_2$-triplet, where $h_0 := \sum_{i=1}^{\ell} n_i^{\vee} h_i$.

We will write $\overline{r} = \underline{r}$ whenever $r_n = 0$.

Then the mapping π

$$\begin{aligned}
\delta_{\underline{r}}(\underline{s}) &\mapsto 0\,, \\
\alpha_i^{\vee}(\underline{k}) &\mapsto t^{\underline{k}} \otimes h_i, \quad & i = 0,1,2,\ldots,\ell\,, \\
X_{\underline{m}}(\alpha_i) &\mapsto t^{\underline{m}} \otimes e_i, \quad & i = 1,2,\ldots,\ell\,, \\
X_{\underline{m}}(-\alpha_i) &\mapsto t^{\underline{m}} \otimes f_i, \quad & i = 1,2,\ldots,\ell\,, \\
X_{\underline{m}}(\alpha_0) &\mapsto t^{\underline{m}} t_n \otimes e_0\,, \\
X_{\underline{m}}(-\alpha_0) &\mapsto t^{\underline{m}} t_n^{-1} \otimes f_0
\end{aligned} \tag{2.3}$$

defines a surjective homomorphism of Lie-algebras $\pi\colon \mathfrak{t} \to \mathfrak{g}_{[n]} = A_{[n]} \otimes \dot{\mathfrak{g}}$.

As in [MEY] we wish to prove that $\mathfrak{t}$ is the universal central extension of $\mathfrak{g}_{[n]}$ in otherwords $\mathfrak{t} \cong \tau_{[n]}$. The proof is similar to one given in [MEY]. So we only introduce the necessary notation and sketch the proof.

We begin by introducing a grading of $\mathfrak{t}$ by $\mathbb{Z}^{n-1} \times Q$ by assigning degrees to the generators as follows.

$$\begin{aligned}
\deg\, \delta_{\underline{r}}(\underline{s}) &= (\underline{s}, 0)\,, \\
\deg\, \alpha_i^{\vee}(\underline{k}) &= (\underline{k}, 0)\,, \\
\deg\, X_k(\pm\alpha_i) &= (\underline{k}, \pm\alpha_i)
\end{aligned} \tag{2.4}$$

($i = 0,1,2,\ldots,\ell$ and for all $\underline{k} \in \mathbb{Z}^{n-1}$). We denote the space of elements of degree $(\underline{k}, \alpha)$ in $\mathfrak{t}$ by $\mathfrak{t}_{\underline{k}}^{\alpha}$.

Define

$$Q_+ := \left(\sum_{i=0}^{\ell} \mathbb{Z}_{\geqq 0}\alpha_i\right) \backslash \{0\}, \quad Q_- := -\,Q_+\,;$$

$$\mathfrak{t}_{\underline{n}}^{\pm} := \sum_{\alpha \in Q_\pm} \mathfrak{t}_{\underline{n}}^{\alpha}, \quad \mathfrak{t}^{\pm} := \sum_{\underline{n} \in \mathbb{Z}^{n-1}} \mathfrak{t}_{\underline{n}}^{\pm}, \quad \mathfrak{t}^{\alpha} := \sum_{\underline{n} \in \mathbb{Z}^{n-1}} \mathfrak{t}_{\underline{n}}^{\alpha}.$$

$S_{\underline{n}}^{+}$:= linear span of all products $[X_{\underline{n}_k}(\beta_k), \ldots, X_{\underline{n}_1}(\beta_1)]$, where $\beta_1, \beta_2, \ldots, \beta_k \in \Pi$, $\underline{n}_1, \underline{n}_2, \ldots, \underline{n}_k \in \mathbb{Z}^{n-1}$, and $\sum \underline{n}_i = \underline{n}$;

$$\mathcal{S}^+ := \sum_{\underline{n} \in \mathbb{Z}^{n-1}} \mathcal{S}^+_{\underline{n}} \text{ and similarly } \mathcal{S}^-_{\underline{n}}, \text{ and } \mathcal{S}^- ;$$

$$\mathcal{S}^0_{\underline{n}} := \text{linear span of } \{\alpha_i^\vee(\underline{n}), \delta_{\underline{r}}(\underline{n})\} ;$$

$$\mathcal{S}^0 := \sum_{\underline{n} \in \mathbb{Z}^{n-1}} \mathcal{S}^0_{\underline{n}}, \quad \mathcal{S} := \mathcal{S}^+ + \mathcal{S}^0 + \mathcal{S}^- .$$

(2.5) Lemma.

$$\text{(i)} \quad \mathfrak{t} = \mathcal{S}, \quad \mathfrak{t}^\pm_{\underline{n}} = \mathcal{S}^\pm_{\underline{n}}, \quad \text{and} \quad \mathfrak{t}^\pm = \mathcal{S}^\pm_- ,$$

$$\text{(ii)} \quad \mathfrak{t}_{\underline{n}} = \mathfrak{t}^-_{\underline{n}} + \mathfrak{t}^0_{\underline{n}} + \mathfrak{t}^+_{\underline{n}}, \quad \mathfrak{t} = \mathfrak{t}^- + \mathfrak{t}^0 + \mathfrak{t}^+ .$$

The proof is similar to Lemma (3.1) of [MEY].

(2.6) Proposition.

$$\dim t^\alpha_{\underline{n}} = \begin{cases} 1, & \textit{if } \alpha \in \Delta^{re} ; \\ 0, & \textit{if } \alpha \notin \Delta . \end{cases}$$

(2.7) Corollary.

$$\operatorname{ad} X_{\underline{n}}(\alpha_i)^{-A_{ij}+1} X_{\underline{m}}(\alpha_j) = 0 ,$$
$$\operatorname{ad} X_{\underline{n}}(-\alpha_i)^{-A_{ij}+1} X_{\underline{m}}(-\alpha_j) = 0 ,$$

for all $i \neq j$, $\underline{m}, \underline{n} \in \mathbb{Z}^{n-1}$.

Proofs are similar to Proposition 3.2 and Corollary 3.3 of [MEY].

(2.8) Proposition. (ξ, Π) *is the universal central extension of* $\mathfrak{g}_{[n]}$.

Proof. The proof is again similar to the proof of Proposition (3.5) of [MEY]. We will only indicate the proof when it is different from [MEY].

As in [MEY], kernel Π is central in $\mathfrak{t}$. To prove the extension is universal we construct a mapping ψ from $\mathfrak{t}$ to $\tau_{[n]}$. Explicitly

$$\begin{aligned}
\delta_{\underline{r}}(\underline{s}) &\mapsto \sum r_i \overline{D_i(\underline{s})} , \\
X_{\underline{m}}(\alpha_i) &\mapsto t^{\underline{m}} \otimes e_i, \quad i = 1, 2, \ldots, \ell , \\
X_{\underline{m}}(-\alpha_i) &\mapsto -t^{\underline{m}} \otimes f_i, \quad i = 1, 2, \ldots, \ell , \\
X_{\underline{m}}(\alpha_0) &\mapsto t^{\underline{m}} t_n \otimes e_0 , \\
X_{\underline{m}}(-\alpha_0) &\mapsto -t^{\underline{m}} t_{\underline{n}}^{-1} \otimes f_0 , \\
\alpha_i^\vee(\underline{k}) &\mapsto t^{\underline{k}} \otimes h_i, \quad i = 1, 2, \ldots, \ell , \\
\alpha_0^\vee(\underline{k}) &\mapsto t^{\underline{k}} \otimes h_0 + \overline{t^{\underline{k}} t_n^{-1} dt_n} .
\end{aligned}$$

In order to complete the proof of Proposition 2.8, we have to prove that the elements on the right-hand side satisfy relations TA.

The relations TA0, TA2, TA4 are essentially trivial using the definition of Sect. 1:

$$\text{(TA1)} \quad [t^{\underline{k}} \otimes h_i, t^{\underline{m}} \otimes h_j] = \overline{d(t^{\underline{k}}) t^{\underline{m}}}(h_i | h_j) = \sum k_i \overline{D_i(\underline{k} + \underline{m})}(h_i | h_j) .$$

(TA3) We need only consider the case $i=j$ since $(e_i\,|\,f_j)=0$ if $i\neq j$. Suppose $i=j\neq 0$. Then

$$[t^{\underline{m}}\otimes e_i, -t^{\underline{n}}\otimes f_i] = -\ t^{\underline{m}+\underline{n}}\otimes h_i - \overline{d(t^{\underline{m}})t^{\underline{n}}}(e_i\,|\,f_i)$$
$$= -\ t^{\underline{m}+\underline{n}}\otimes h_i - \sum_{i=1}^{n-1} m_i\overline{D_i(\underline{m}+\underline{n})}(e_i\,|\,f_i)\,.$$

If $i=j=0$ then

$$[t^{\underline{m}}t_n\otimes e_0, -t^{\underline{n}}t_n^{-1}\otimes f_0] = -t^{\underline{m}+\underline{n}}\otimes h_0 - \overline{d(t^{\underline{m}}t_n)(t^{\underline{n}}t_n^{-1})}(e_0\,|\,f_0) =: a\,.$$

Consider

$$\begin{aligned} d(t^{\underline{m}}t_n)t^{\underline{n}}t_n^{-1} &= \sum_{i=1}^{n-1} m_iD_i(\underline{m}_1)t^{\underline{n}}t_n^{-1} \\ &\quad + D_n(\underline{m}_1)t^{\underline{n}}t_n^{-1}, \quad \text{where} \quad \bar{m}_1 := (\underline{m},1) \\ &= \sum_{i=1}^{n-1} m_iD_i(\underline{m}+\underline{n}) + D_n(\underline{m}+\underline{n})\,. \end{aligned}$$

Hence

$$a = -\psi(\alpha_0^\vee(\underline{m}+\underline{n}) + \frac{2}{(\alpha_0\,|\,\alpha_0)}\delta_{\underline{m}}(\underline{m}+\underline{n})\,.$$

This completes TA3. □

(2.9) Proposition.

(1) $\psi^{-1}(t^{\underline{m}}t_n^r)\in t_{\underline{m}}^{r\delta}, \qquad \underline{m}\in\mathbb{Z}^{n-1}, r\in\mathbb{Z}.$

(2) $\dim t_{\underline{m}}^{r\delta} = \begin{cases} \ell+n-1, & (\underline{m},r\delta)\neq(0,0) \\ \ell+n, & (\underline{m},r\delta) = (0,0)\,. \end{cases}$

Proof. Clear, since ψ is a graded isomorphism. □

(2.10) Presentation II. As earlier let $\mathfrak{g}$ be a simple finite dimensional Lie-algebra over $\mathbb{C}$ and let $\dot{\Delta}$ be its roots. Let $\dot{Q}$ be the root lattice and let $\epsilon:\dot{Q}\times\dot{Q}\to\{1,-1\}$ be a two-cocycle on $\dot{Q}$ satisfying

$$\begin{aligned} &(1)\ \epsilon(\alpha,\alpha)=(-1)^{\frac{(\alpha|\alpha)}{2}}\,, \\ &(2)\ \epsilon(\alpha,\beta)\epsilon(\beta,\alpha)=(-1)^{(\alpha|\beta)}\,, \\ &(3)\ \epsilon(\alpha+\beta,\gamma)=\epsilon(\alpha,\gamma)\epsilon(\beta,\gamma)\,, \\ &(4)\ \epsilon(\alpha,\beta+\gamma)=\epsilon(\alpha,\beta)\epsilon(\alpha,\gamma)\,. \end{aligned} \tag{2.11}$$

Such an ϵ is known to exist. In fact it is easy to construct in view of (3) (see [FK]).

We note that $\epsilon(\alpha,\alpha)=\epsilon(\alpha,-\alpha)=-1$ whenever $(\alpha,\alpha)=2$.

Let

$$\dot{\mathfrak{g}} = \bigoplus_{\alpha\in\dot{\Delta}}\dot{\mathfrak{g}}_\alpha + \dot{\mathfrak{h}}$$

be the root space decomposition. Choose a Chevalley basis $X_\alpha, \alpha \in \dot{\Delta}$ such that

$$\begin{aligned}
&(1)\ [X_\alpha, X_\beta] = \begin{cases} \epsilon(\alpha,\beta)X_{\alpha+\beta}\ , & \alpha+\beta \in \dot{\Delta} \\ 0, & \alpha+\beta \notin \dot{\Delta} \cup \{0\}\ , \\ \epsilon(\alpha,-\alpha)\alpha^\vee, & \alpha+\beta = 0 \end{cases} \\
&(2)\ [h, X_\alpha] = \alpha(h)X_\alpha\ , \\
&(3)\ [h_i, h_j] = 0\ .
\end{aligned} \tag{2.12}$$

It is well known that (2.12) defines the finite dimensional simple Lie algebra $\dot{\mathfrak{g}}$.

(2.13) Proposition. [K] *$\tau_{[n]}$ has the following presentation as a Lie algebra. Generators $a \otimes X_\alpha, b \otimes h, \Omega_{[n]}$, where $\alpha \in \dot{\Delta},\ h \in \dot{\mathfrak{h}},\ a, b \in A$.*
Relations

(R1) $(\lambda a + \mu b) \otimes X_\alpha = \lambda(a \otimes X_\alpha) + \mu(b \otimes X_\alpha)$.

(R2) $[a \otimes X_\alpha, b \otimes X_\beta] = \begin{cases} \epsilon(\alpha,\beta)ab \otimes X_{\alpha+\beta}, & \text{if } \alpha+\beta \in \dot{\Delta} \\ 0, & \text{if } \alpha+\beta \notin \dot{\Delta} \cup \{0\} \\ \epsilon(\alpha,-\alpha)\left(ab \otimes \alpha^\vee + \overline{(da)b}\right), & \text{if } \alpha+\beta = 0. \end{cases}$

(R3) $[a \otimes h, b \otimes X_\alpha] = (h \mid \alpha)ab \otimes X_\alpha$.

(R4) $[a \otimes h_i, b \otimes h_j] = (h_i \mid h_j)\overline{(da)b}$.

(R5) *$\Omega_{[n]}$ is central.*

Proof. Follows from (1.3) and (2.12).

(2.14) Remark.
(1) Proposition 2.13 is true for any commutative associative algebra A.
(2) In view of R1, in R2, R3 and R4, a and b in Proposition 2.13 can be replaced by any monomials in A.

3. Vertex Operators and the Main Theorems

In this section we recall the basic construction [FK] of vertex operators acting on Fock space. We will state our main result and give its proof assuming some lemmas whose proofs will be given in Sect. 4.

Henceforth we will assume that $\dot{\mathfrak{g}}$ is a finite dimensional simply-laced simple Lie algebra. Let $\dot{A}$ be the corresponding Cartan matrix of order ℓ. Let A_{aff} be the affine Cartan matrix of the corresponding non-twisted affine Lie algebra of $\dot{\mathfrak{g}}$. Since $\dot{\mathfrak{g}}$ is simply laced, $\dot{A}$ and A_{aff} are symmetric matrices and of order ℓ and $\ell+1$ respectively.

(3.1). Let $A = (A_{ij})$ be a square matrix of order $\ell + n - 1$ such that removal of any $n-2$ rows and the corresponding columns in the last $n-1$ rows and $n-1$ columns determines an affine matrix A_{aff}. Further assume $A_{\ell+i,\ell+j} = 2$ for $i, j = 1, 2, \ldots, n-1$. Observe that rank of $A =$ rank of $A_{\text{aff}} = \ell$.

Let Γ be a free $\mathbb{Z}$-module on generators $\alpha_1, \alpha_2, \ldots, \alpha_\ell, \alpha_{\ell+1}, \ldots, \alpha_{\ell+n-1}$, $d_1, d_2, \ldots, d_{n-1}$ and let $(\cdot|\cdot)$ be a $\mathbb{Z}$-valued symmetric bilinear form on Γ such that

$$(1)\ (\alpha_i \mid \alpha_j) = A_{i,j},$$

$$(2)\ (d_i \mid \alpha_j) = \begin{cases} 0, & \text{if } j = 1, 2, \ldots, \ell; \\ \delta_{i,j-\ell}, & \text{otherwise;} \end{cases}$$

$$(3)\ (d_i \mid d_j) = 0. \tag{3.2}$$

Let Q be a sublattice of Γ with generators $\alpha_1, \alpha_2, \ldots, \alpha_{\ell+n-1}$. Observe that Γ is a non-degenerate integral lattice and Q is even integral lattice in the sense that $(\alpha \mid \alpha)$ is even integer for α in Q. We will identify $\dot{Q}$ the root lattice of $\dot{\mathfrak{g}}$ with $\bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$ inside Γ.

Define $\delta_1, \delta_2, \ldots, \delta_{n-1}$ in Γ such that $\delta_i = \zeta + \alpha_{\ell+i}$, where ζ is the highest root of $\dot{\mathfrak{g}}$. Clearly $(\delta_i \mid d_j) = \delta_{ij}$.

The ... Γ can also be viewed as a free $\mathbb{Z}$-module on generators $\alpha_1, \alpha_2, \ldots, \alpha_\ell$, $\delta_1, \delta_2, \ldots, \delta_{n-1}$, $d_1, d_2, \ldots, d_{n-1}$ and the matrix of the non-degenerate symmetric bilinear form on Γ defined in (3.2) is given below with respect to the new basis,

$$\begin{pmatrix} \dot{A} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

We briefly review the construction of Fock space and the vertex operators that act on it. The theory is due to Kac–Frenkel [FK]. For further details one may also consult [FLM, GO and MP]. We closely follow the notation from [MEY].

Let $\mathfrak{p} = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$ and define a Heisenberg algebra

$$\hat{\mathfrak{b}} := \bigoplus_{k \in \mathbb{Z}} \mathfrak{p}(k) \oplus \mathbb{C}c,$$

where each $\mathfrak{p}(k)$ is an isomorphic copy of $\mathfrak{p}$ and the isomorphism is by $\alpha \mapsto \alpha(k)$. The Lie algebra structure on $\hat{\mathfrak{b}}$ is defined by

$$[\alpha(k), \beta(m)] = k(\alpha \mid \beta)\delta_{k+m,0}\, c.$$

Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ and let $\hat{\mathfrak{a}} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}(k) \oplus \mathbb{C}c$.

Define $\mathfrak{b} := \sum_{k \neq 0} \mathfrak{p}(k) \oplus \mathbb{C}c \subseteq \hat{b}$, and $\mathfrak{b}_\pm = \sum_{k \gtrless 0} \mathfrak{p}(k)$.

Similarly define $\mathfrak{a}, a_\pm$ by replacing $\mathfrak{p}$ by $\mathfrak{h}$.

The Fock space representation of $\mathfrak{b}$ is the symmetric algebra $S(\mathfrak{b}_-)$ of $\mathfrak{b}_-$ together with action of $\mathfrak{b}$ on $S(\mathfrak{b}_-)$ defined by:

c acts as 1;
$a(-m)$ acts as multiplication by $a(-m), m > 0$;
$a(m)$ acts as the unique derivation on $S(\mathfrak{b}_-)$ for which $b(-n) \to \delta_{m,-n} m(a \mid b)$ for all $a, b \in \dot{\mathfrak{t}},\ m, n > 0$;

$S(\mathfrak{b}_-)$ affords an irreducible representation of $\mathfrak{b}$. However, $S(\mathfrak{a}_-)$ does not afford an irreducible representation of $\mathfrak{a}$ since the form $(\cdot \mid \cdot)$ is degenerate on $\mathfrak{h}$.

(3.3) Definition. *A vector δ in Q is called a null root if $\delta = m_1\delta_1 + m_2\delta_2 + \cdots + m_{n-1}\delta_{n-1}$ for some integers m_i.*

Clearly $(\delta \mid \delta) = 0$. Conversely any $\alpha \in Q$ such that $(\alpha \mid \alpha) = 0$ is necessarily a null root.

(3.4). We first extend ϵ defined in (2.11) to $\epsilon : Q \times Q \to \pm 1$ by $\epsilon(\alpha, \delta) = 1$ for any $\alpha \in Q$ and δ a null root. We will further extend ϵ to a bimultiplicative map $\epsilon : Q \times \Gamma \to \pm 1$ in any convenient way.

For each γ in Γ let e^γ be a symbol and form the vector space $\mathbb{C}[\Gamma]$ with basis $\{e^\gamma\}$ over $\mathbb{C}$. In particular $\mathbb{C}[\Gamma]$ contains the subspace $\mathbb{C}[Q] = \sum_{\alpha \in Q} \mathbb{C}e^\alpha$. Following Borcherds [B] we define a twisted group algebra on $\mathbb{C}[Q]$ by

$$e^\alpha e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta}$$

and make $\mathbb{C}[\Gamma]$ a $\mathbb{C}[Q]$-module by defining

$$e^\alpha e^\gamma = \epsilon(\alpha, \gamma) e^{\alpha+\gamma} \quad (\alpha, \beta \in Q,\ \gamma \in \Gamma).$$

Let $M \subseteq S(\mathfrak{b}_-)$ be any $\mathfrak{a}_-$ submodule (with respect to the Fock space action). We define

$$V(\Gamma, M) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} M .$$

Of particular interest in the sequel will be $V(\Gamma, S(\mathfrak{a}_-))$ and $V(\Gamma, S(\mathfrak{b}_-))$ which we simply denote by $V(\Gamma)$ and $V(\Gamma, \mathfrak{b})$ respectively. We extend the action of $\mathfrak{a}$ on M to $\hat{\mathfrak{a}}$ on $V(\Gamma, M)$ by

$$\begin{aligned} a(m)(e^\gamma \otimes u) &= e^\gamma \otimes a(m)u, & m \neq 0 , \\ a(0)(e^\gamma \otimes u) &= (a \mid \gamma) e^\gamma \otimes u, & m = 0 . \end{aligned}$$

(3.5) Gradation. Let $\{a^i\}, \{a_i\}$ be a dual basis of $\mathfrak{p}$. The Segal operators L_n are defined on $V(\Gamma)$ by

$$L_n := -\frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_i : a_i(n-j) a^i(j) : ,$$

where : : is the normal ordering defined by

$$: a(j)a(k) : := \begin{cases} a(j)a(k), & j \leqq k \ ; \\ a(k)a(j), & j > k . \end{cases}$$

The following are well known: (see [GO, KMPS, LFM and KF]).

(1) $[a(m), b(n)] = (a \mid b) m \delta_{m,-n}$;

(2) $[L_m, a(n)] = n a(m+n)$;

(3) $[L_m, L_n] = (n-m) L_{m+n} + \delta_{m+n,0} \dfrac{m(m^2-1)}{12} (\dim \mathfrak{p}) c.$ (3.6)

Using the first of these and the simple computation

$$L_0(e^\mu \otimes 1) = (-1/2)(\mu \mid \mu) e^\mu \otimes 1 ,$$

it is easy to see that

$$\begin{aligned} &L_0(e^\mu \otimes a_1(-n_1) \cdots a_k(-n_k)) \\ &\quad = -\left(\frac{(\mu \mid \mu)}{2} + n_1 + n_2 + \cdots + n_k \right) e^\mu \otimes a_1(-n_1) \cdots a_k(-n_k) . \end{aligned}$$

We use this to define a $\mathbb{Z}$-grading on $V(\Gamma, \mathfrak{b})$:

$$\begin{aligned} V^m &:= \{ v \in V(\Gamma, \mathfrak{b}) \mid L_0 v = -mv \}, \\ V(\Gamma, \mathfrak{b}) &:= \bigoplus_{m \in \mathbb{Z}} V^m . \end{aligned}$$

$V(\Gamma, \mathfrak{b})$ is also graded by Γ with $V^\alpha = e^\alpha \otimes F$ for $\alpha \in \Gamma$.
Hence $V(\Gamma, \mathfrak{b})$ is graded by $\mathbb{Z} \times \Gamma$.

(3.7) Vertex Operators. Let z be a complex valued variable and let $\alpha \in Q$. Define

$$T_\pm(\alpha, z) := -\sum_{n \gtrless 0} \frac{1}{n} \alpha(n) z^{-n} .$$

Then the vertex operator for α in Q is defined as

$$X(\alpha, z) := z^{\frac{(\alpha|\alpha)}{2}} \exp T(\alpha, z) ,$$

where $\exp T(\alpha, z) = \exp T_-(\alpha, z) e^\alpha z^{\alpha(0)} \exp T_+(\alpha, z)$ and the operator $z^{\alpha(0)}$ is defined by

$$z^{\alpha(0)} (e^\gamma \otimes u) = z^{(\alpha|\gamma)} e^\gamma \otimes u .$$

Strictly, for each $z \in \mathbb{C}^\times$ the operator $X(\alpha, z)$ maps $V(\Gamma, M)$ into the space $\mathbb{C}(\Gamma) \otimes \widehat{S(\mathfrak{b}_-)}$, where

$$\widehat{S(\mathfrak{b}_-)} = \prod_n S(\mathfrak{b}_-)^n$$

is the completion to formal series $S(\mathfrak{b}_-)$. However $X(\alpha, z)$ can be formally expanded in powers of z to give

$$X(\alpha, z) = \sum_{n \in \mathbb{Z}} X_n(\alpha) z^{-n}$$

and the "moments"

$$X_n(\alpha) = \frac{1}{2\pi i} \int X(\alpha, z) z^n \frac{dz}{z}$$

are indeed operators on $V(\Gamma, M)$. Moreover, for all v in M, $X_n(\alpha)(e^\gamma \otimes v) = e^{\gamma+\alpha} \otimes v'$ is obtained from v by applying some polynomial expressions in the operator $a(m), a \in \mathfrak{h}$, $m \in \mathbb{Z} \backslash \{0\}$.

The basic commutation relations for operators $\alpha(k)$ and $X_m(\alpha)$ on $V(\Gamma, \mathfrak{b})$ are given by ([KF, GO, MP]).

$$\begin{aligned} &(1)\ [\alpha(k), X_n(\beta)] = (\alpha|\beta) X_{n+k}(\beta); \\ &(2)\ [X_m(\alpha), X_n(\beta)] = \begin{cases} 0 & \text{if } (\alpha|\beta) \geqq 0; \\ \epsilon(\alpha, \beta) X_{m+n}(\alpha+\beta) & \text{if } (\alpha|\beta) = -1; \\ -\alpha^\vee(m+n) - m\delta_{m+n,0} c & \text{if } \beta = -\alpha, (\alpha|\alpha) = 2. \end{cases} \end{aligned} \tag{3.8}$$

(3.9) Lemma. *Let $\alpha, \beta \in Q$ with $(\alpha|\alpha) = 2 = (\beta|\beta) = -(\alpha|\beta)$. Then*

$$[X_m(\alpha), X_n(\beta)] = \epsilon(\alpha, \beta)\{m X_{m+n}(\alpha+\beta) + \sum_{k \in \mathbb{Z}} \alpha(k) X_{m+n-k}(\alpha+\beta)\} .$$

(3.10) Definition. *Let $h \in \mathfrak{t}, a \in \mathbb{Z}$, and let γ be a null root*

$$T_a^h(\gamma) = \frac{1}{2\pi i} \int : h(z) X(\gamma, z) : \frac{dz}{z} z^a ,$$

where the normal ordering symbol is defined:

$$: h(z) X(\gamma, z) : := h^-(z) X(\gamma, z) + X(\gamma, z) h^+(z) ,$$

where

$$h^+(z) := \sum_{0 \leqq n \in \mathbb{Z}} h(n) z^{-n} ,$$

$$h^-(z) := \sum_{0 > n \in \mathbb{Z}} h(n) z^{-n} ,$$

$$h(z) := h^-(z) + h^+(z) .$$

Normal ordering is necessary for the components of $: h(z) X(\gamma, z) :$ *to be well defined.*

Note. There is no need for normal order in Lemma 3.9 as the operators $\alpha(k)$ and $X_{m+n-k}(\alpha+\beta)$ commute. This follows from 3.8(1) since $(\alpha | \alpha+\beta) = 0$.

(3.11) Lemma. *Let* $a \in \mathfrak{p}$, *let* γ *be a null root, and let* $m, n \in \mathbb{Z}$. *Then*

(1) $[L_n, X_m(\gamma)] = (m+n) X_{m+n}(\gamma)$;
(2) $[L_n, a(m)] = m a(m+n)$;
(3) $[L_n, T^a_m(\gamma)] = m T^a_{m+n}(\gamma)$.

Proof.
(1) is well known (Proposition 2.7, [FK]).
(2) has already been noted in (3.6).
(3) follows from (2) and (3).

(3.12) Corollary. *With the gradation defined in (3.5),*

$$X_m(\gamma) \ \textit{has degree}\ (m, \gamma) ;$$
$$a(m) \ \textit{has degree}\ (m, 0) ;$$
$$T^h_m(\gamma) \ \textit{has degree}\ (m, \gamma) .$$

Proof. Taking $n = 0$ in Lemma 3.11 the first component of the gradation follows. The second component follows from the definition of operators.

Before we state our main theorem we will prove one important relation. First some notation:

$$\underline{r} = (r_1, r_2, \ldots, r_{n-1}) \in \mathbb{Z}^{n-1}, \quad r_n \in \mathbb{Z} ,$$

$$\delta_{\underline{r}} := r_1 \delta_1 + r_2 \delta_2 + \cdots + r_{n-1} \delta_{n-1} .$$

(3.13) Lemma.

$$T^{\delta_{\underline{r}}}_{r_n}(\delta_{\underline{r}}) + r_n X_{r_n}(\delta_{\underline{r}}) = 0 .$$

Proof. From Lemma 3.9, for $\alpha, \beta \in Q$ such that $(\alpha|\alpha) = 2 = (\beta|\beta) = -(\alpha|\beta)$ we have

$$\begin{aligned} 0 &= [X_m(\alpha), X_n(\beta)] + [X_n(\beta), X_m(\alpha)] \\ &= \epsilon(\alpha, \beta) \left\{ (m+n) X_{m+n}(\alpha+\beta) + \sum_{k \in \mathbb{Z}} : (\alpha+\beta)(k) X_{m+n-k}(\alpha+\beta) : \right\} , \end{aligned}$$

where we have used $(\alpha|\beta) = -2, \epsilon(\alpha, \beta) = \epsilon(\beta, \alpha)$.

Now choose $\alpha = \alpha_0 + \delta_{\underline{r}}$, $\beta = -\alpha_0$ for some $\alpha_0 \in \dot{Q}$ such that $(\alpha_0|\alpha_0) = 2$. Choosing m, n such that $m + n = r_n$ we obtain the desired identity. □

Note. In fact if $\alpha, \beta \in Q$ such that $(\alpha|\alpha) = 2 = (\beta \mid \beta) = -(\alpha \mid \beta)$. Then $\alpha = \alpha_0 + \delta$ and $\beta = -\alpha_0 + \delta'$ for some $\alpha_0 \in \dot{Q}$ such that $(\alpha_0|\alpha_0) = 2$ and δ, δ' null roots.

(3.14) Theorem. *The Lie algebra $\mathfrak{k}$ of operators on $V(\Gamma, \mathfrak{b})$ generated by the operators $X_m(\alpha + \delta)$, where $\alpha \in \dot{\Delta}, \delta$ is a null root, $m \in \mathbb{Z}$, is isomorphic to $\tau_{[n]}$ the universal central extension of $\dot{\mathfrak{g}} \otimes A_{[n]}$.*

To prove the theorem we need the following lemmas whose proof will be given in Sect. 4.

Lemma A. *Let $h, g \in \dot{\mathfrak{h}}$, and let δ, δ' be null roots. Then*

$$[T_a^h(\delta), T_b^g(\delta')] = (h|g)(T_{a+b}^{\delta}(\delta + \delta') + aX_{a+b}(\delta + \delta'))\,.$$

Lemma B. *Let $h \in \dot{\mathfrak{h}}, \alpha \in \dot{\Delta}$, and let δ, δ' be null roots. Then*

$$[T_a^h(\delta), X_b(\alpha + \delta')] = (h|\alpha)X_{a+b}(\alpha + \delta + \delta')\,.$$

Lemma C. *Let $\mathfrak{Z}$ be the $\mathbb{C}$-linear span of central operators $T_a^{\delta}(\delta'), X_b(\delta'')$, where δ, δ', δ'' are null roots and $a, b \in \mathbb{Z}$. Then all the relations in $\mathfrak{Z}$ as operators on $V(\Gamma, \mathfrak{b})$ are linearly generated by the relations*

$$T_a^{\delta}(\delta) + aX_a(\delta) = 0\,. \tag{3.15}$$

(3.16) Lemma. *There is a natural vector space isomorphism between $\Omega_{[n]}$ and $\mathfrak{Z}$ given by*

$$\begin{aligned} \overline{D_i(\bar{r})} &\mapsto T_{r_n}^{\delta_i}(\delta_{\underline{r}}), \quad \text{if } 1 \leqq i \leqq n-1\,, \\ \overline{D_n(\bar{r})} &\mapsto X_{r_n}(\delta_{\underline{r}})\,, \end{aligned}$$

where $\bar{r} = (\underline{r}, r_n) \in \mathbb{Z}^n$.

Proof. The proof is clear by noting that the only relation (1.3) in $\Omega_{[n]}$ is translated to the only relation (3.15) of $\mathfrak{Z}$ by the above map.

Proof of Theorem (3.14).

Claim 1. $T_a^h(\delta), T_a^{\delta'}(\delta), X_a(\delta) \in \mathfrak{k}$ *for all $h \in \dot{h}$ and δ, δ' null roots.*

(1) $$T_a^h(\delta) \text{ is linear in } h\,.$$

Choosing $\alpha = \alpha_i + \delta, \beta = -\alpha_i + \delta', \alpha_i \in \Pi$ in Lemma 3.9 we have

(2) $$\begin{aligned} &\epsilon(\alpha_i, -\alpha_i)[X_m(\alpha_i + \delta), X_n(-\alpha_i + \delta')] \\ &= (mX_{m+n}(\delta + \delta') + T_{m+n}^{\alpha_i+\delta}(\delta + \delta')) \in \mathfrak{k}\,. \end{aligned}$$

By choosing $m = 0$ and $\delta = 0$ in (C2) we have $T_n^{\alpha_i}(\delta') \in \mathfrak{k}$. By (C1) it follows that

(3) $$T_n^h(\delta) \in \mathfrak{k} \quad (h \in \dot{\mathfrak{h}},\ \delta \text{ a null root},\ n \in \mathbb{Z})\,.$$

By choosing $m = 0$ in (2) we have

(4) $$T_n^{\alpha_i+\delta}(\delta + \delta') \in \mathfrak{k}\,.$$

But by (3) we have $T_n^{\alpha_i}(\delta+\delta') \in \mathfrak{k}$ (by replacing h by α_i and δ by $\delta+\delta'$). Subtracting from (4) we have

$$T_n^{\delta}(\delta+\delta') \in \mathfrak{k}\,. \tag{5}$$

By taking $m+n$ in place of n in (4) and subtracting it from (2) we have

$$mX_{m+n}(\delta+\delta') \in \mathfrak{k}\,. \tag{6}$$

Hence Claim 1 follows from (3), (5) and (6).

Claim 2. For all null roots δ, δ'', the operators $X_a(\delta), T_a^{\delta}(\delta'')$ are central in $\mathfrak{k}$.

By Lemma 3.8(1) and 3.8(2) we have for $\alpha \in Q$, δ' null,

$$[\delta(k), X_n(\alpha+\delta')] = 0$$

and

$$[X_a(\delta), X_n(\alpha+\delta')] = 0\,,$$

since $(\delta, \alpha+\delta') = 0$. Thus $X_a(\delta)$ is central.

By Definition (3.10) of $T_a^{\delta}(\delta'')$ and what we have just seen it follows that $T_a^{\delta}(\delta'')$ commutes with generators of $\mathfrak{k}$. This completes the proof of Claim 2.

We shall describe a map linear ψ from $\tau_{[n]}$ to $\mathfrak{k}$ and then prove that it is a Lie algebra isomorphism using Proposition 2.13.

Recall the notation (1.5), $\bar{r}, t^{\bar{r}}$.

Write

$$\bar{r} = (\underline{r}, r_n), \quad \text{where} \quad \underline{r} \in \mathbb{Z}^{n-1},\ r_n \in \mathbb{Z}\,.$$

Explicitly ψ is given

$$\begin{aligned} t^{\bar{r}} \otimes X_\alpha &\mapsto X_{r_n}(\alpha+\delta_{\underline{r}})\,; \\ t^{\bar{r}} \otimes h &\mapsto T^h_{r_n}(\delta_{\underline{r}})\,, \\ \overline{D_i(\bar{r})} &\mapsto T^{\delta_i}_{r_n}(\delta_{\underline{r}}), \quad 1 \leqq i \leqq n-1\,, \\ \overline{D_n(\bar{r})} &\mapsto X_{r_n}(\delta_{\underline{r}})\,. \end{aligned}$$

To conclude the proof of Theorem 3.14 it is sufficient to verify the relations (R) in Proposition 2.13 are satisfied for the right-hand side. First we will prove another claim.

Claim 3. Let $\underline{r}, \underline{s} \in \mathbb{Z}^{n-1}, r_n, s_n \in \mathbb{Z}$ and let $\bar{r} = (\underline{r}, r_n)$ and $\bar{s} = (\underline{s}, s_n)$. Then

$$\psi(d(t^{\bar{r}})t^{\bar{s}}) = T^{\delta_{\underline{r}}}_{r_n+s_n}(\delta_{\underline{r}}+\delta_{\underline{s}}) + r_n X_{r_n+s_n}(\delta_{\underline{r}}+\delta_{\underline{s}})\,.$$

In fact

$$\begin{aligned} \psi(d(t^{\bar{r}})t^{\bar{s}}) &= \psi\left(\sum_{i=1}^{n} r_i D_i(t^{\bar{r}+\bar{s}})\right) \\ &= \sum_{i=1}^{n-1} r_i T^{\delta_i}_{r_n}(\delta_{\underline{r}}+\delta_{\underline{s}}) + r_n X_{r_n}(\delta_{\underline{r}}+\delta_{\underline{s}})\,, \end{aligned}$$

which proves the claim.

We will now verify the relations (R). (R1) is trivial. (R2) follows from (3.8), Lemma 3.9 and Claim 3. (R3) follows from Lemma A and Claim 3. (R5) follows from Claim 2 and Lemma 3.16. (R4) follows from Lemma A and Claim 3.

In view of the degrees of the operators (3.12) and dimensions of root spaces (Proposition (2.6)) we see that $\ker\psi \subset \sum_{\underline{m}\in\mathbb{Z}^{n-1}}(\tau_{[n]})^0_{\underline{m}}$ from which it follows that

$$\ker\psi \subset \text{centre}(\tau_{[n]}) = \Omega_{[n]}\,.$$

But $\psi \mid \Omega_{[n]}$ is injective by Lemma C. Then ψ is injective.

(3.17) Theorem. *The Lie algebra $\mathfrak{k}_0$ of operators on $V(\Gamma,\ \underline{b})$ generated by the operators $X_0(\alpha+\delta)$, $(\alpha\in\dot{\Delta},\ \delta$ null root) is isomorphic to $\tau_{[n-1]}$, the universal central extension of $\dot{\mathfrak{g}}\otimes A_{[n-1]}$.*

Proof.

Claim 1. $T_0^h(\delta)$, $T_0^{\delta'}(\delta)\in\mathfrak{k}_0$. Choose $\alpha=\alpha_i+\delta$, $\beta=-\alpha_i+\delta'$, $\alpha_i\in\Pi$ in Lemma 3.9. Then we have

$$\text{(D1)}\qquad \epsilon(\alpha_i,-\alpha_i)[X_0(\alpha_i+\delta),X_0(-\alpha_i+\delta')] = T_0^{\alpha_i+\delta}(\delta+\delta')\in\mathfrak{k}_0\,.$$

Choosing $\delta=0$, we obtain $T_0^{\alpha_i}(\delta')\in\mathfrak{k}_0$ for $\alpha_i\in\Pi$.

Since $T_0^h(\delta')$ is linear in h it follows that

$$\text{(D2)}\qquad T_0^h(\delta)\in\mathfrak{k}_0\quad (h\in\dot{\mathfrak{h}},\delta \text{ a null root})\,.$$

In particular

$$\text{(D3)}\qquad T_0^{-\alpha_i}(\delta+\delta')\in\mathfrak{k}_0\,.$$

Adding (D1) *and* (D3) *we have*

$$T_0^{\delta}(\delta+\delta')\in\mathfrak{k}_0\,.$$

This completes the claim.

Note that the central operator $X_0(\delta)$ cannot be obtained in this manner. In fact it is not an element of $\mathfrak{k}_0$.

Claim 2. $\mathfrak{k}_0$ is the $\mathbb{C}$ linear span of operators $\{X_0(\alpha+\delta),T_0^h(\delta),T_0^\delta(\delta')\}$ for $\alpha\in\dot{\Delta}$, δ, δ' null roots, $h\in\dot{\mathfrak{h}}$.

It is easy to see from Lemmas A, B and 3.19 that the above linear span closed under Lie-brackets. Hence it is equal to $\mathfrak{k}_0$ by Claim 1.

Consider the map ψ_0 from $\tau_{[n-1]}$ to $\mathfrak{k}_0$,

$$\begin{aligned} X_\alpha\otimes t^{\underline{r}} &\mapsto X_0(\alpha+\delta_{\underline{r}})\,,\\ h\otimes t^{\underline{r}} &\mapsto T_0^h(\delta_{\underline{r}})\,,\\ \overline{D_i(\underline{r})} &\mapsto T_0^{\delta_i}(\delta_{\underline{r}})\,. \end{aligned}$$

As in the proof of Theorem 3.14, one can prove that ψ_0 defines a Lie algebra isomorphism.

(3.18) Remark. By taking $n=3$ in Theorem 3.17, $V(\Gamma,\underline{b})$ provides a new class of representations for $\tau_{[2]}$ which are different from the representations constructed in [MEY].

4. Proofs of Lemmas A, B, and C

In this section we prove Lemmas A, B and C stated in Sect. 3.

(*4.1*) *Notation.* For $h, g \in \mathfrak{t}, \alpha \in Q$ and z, w complex variables, define the following functions:

$$h^+(z) = \sum_{0 \leqq n \in \mathbb{Z}} h(n) z^{-n} ,$$

$$h^-(z) = \sum_{0 > n \in \mathbb{Z}} h(n) z^{-n} ,$$

$$h(z) = h^+(z) + h^-(z) ,$$

$$: h(z)g(w) : = h^-(z)g(w) + g(w)h^+(z)$$

$$= \sum_{m<n} h(m)g(n)z^{-m}w^{-n} + \sum_{m \geqq n} g(n)h(m)z^{-m}w^{-n}.$$

(4.2) Lemma. *Let $h, g \in \mathfrak{t}, \alpha \in Q$ and let γ, γ' be null roots.*
For $| w | < | z |$

(1) $[h^+(z), g^-(w)] = (h \mid g)\dfrac{zw}{(z-w)^2}$.

(2) $[h^\pm(z), g^\pm(w)] = 0$.

(3) $[h^+(z), X(\alpha, w)] = (h|\alpha)X(\alpha, w)\dfrac{z}{(z-w)}$.

(4) $[X(\alpha, z), h^-(w)] = -(h|\alpha)X(\alpha, z)\dfrac{w}{z-w}$.

(5) $[X(\gamma', z), X(\gamma, w)] = 0$.

(6) $h(z)g(w) = : h(z)g(w) : + (g|h)\dfrac{zw}{(z-w)^2}$.

(7) $h(z)X(\alpha, w) = : h(z)X(\alpha, w) : + (h|\alpha)X(\alpha, w)\dfrac{z}{z-w}$.

For $|z|<|w|$

(8) $X(\alpha, w)h(z) = : h(z)X(\alpha, w) : + (h|\alpha)X(\alpha, w)\dfrac{z}{z-w}$.

Proof. (2) is trivial. (5) follows from 3.8 (2) as $(\gamma|\gamma') = 0$. (1), (3), and (4) follow from 3.8 and 3.6(1). (6) follows from (1) and (2). (7) follows from (3). □

(4.3) Lemma. *Let $h, g \in \mathfrak{t}$, and let γ be a null root, $|w|<|z|$. Then*

$$h^+(z) : g(w)X(\gamma, w) : = \; : g(w)X(\gamma, w) : h^+(z) + (h|\gamma) : g(w)X(\gamma, w) : \frac{z}{z-w} + (h|g)X(\gamma, w)\frac{zw}{(z-w)^2} .$$

Proof.

$$
\begin{aligned}
&h^+(z) : g(w)X(\gamma, w) : \\
&\quad = h^+(z)\{(g^-(w)X(\gamma, w) + X(\gamma, w)g^+(w)\} \\
&\quad = \left\{g^-(w)h^+(z) + (h\,|\,g)\frac{zw}{(z-w)^2}\right\} X(\gamma, w) + h^+(z)X(\gamma, w)g^+(w)
\end{aligned}
$$

(from Lemma 4.2(1))

$$
\begin{aligned}
&= g^-(w)\{X(\gamma, w)h^+(z) + (h\,|\,\gamma)X(\gamma, w)\}\frac{z}{z-w} \\
&\quad + \left\{X(\gamma, w)h^+(z) + (h\,|\,\gamma)X(\gamma, w)\frac{z}{z-w}\right\} g^+(w) \\
&\quad + (h\,|\,g)X(\gamma, w)\frac{zw}{(z-w)^2}
\end{aligned}
$$

(from Lemma 4.1(3)). Now rewriting the terms in normal ordering Lemma 4.3 follows. □

(4.4) Lemma. *Let $g \in \mathfrak{t}, \gamma, \gamma'$ be null roots and let $|w|<|z|$. Then*

$$
X(\gamma', z) : g(w)X(\gamma, w) : = : g(w)X(\gamma, w)X(\gamma', z) : -(\gamma'\,|\,g)X(\gamma', z)X(\gamma, w)\frac{w}{z-w}.
$$

Proof.

$$
\begin{aligned}
X(\gamma', z) : g(w)X(\gamma, w) : &= X(\gamma', z)\{g^-(w)X(\gamma, w) + X(\gamma, w)g^+(w)\} \\
&= \left(g^-(w)X(\gamma', z) - (g\,|\,\gamma')X(\gamma', z)\frac{w}{z-w}\right) X(\gamma, w) \\
&\quad + X(\gamma', z)X(\gamma, w)g^+(w)\,,
\end{aligned}
$$

using Lemma 4.2(4).

Now Lemma 4.4 follows from rewriting the terms in normal ordering.

(4.5) Lemma. *Let $h, g \in \mathfrak{t}$, and let γ, γ' be null roots. Then*

(1)
$$
\begin{aligned}
: h(z)X(\gamma', z) :: g(w)X(\gamma, w) : &= : h(z)g(w)X(\gamma', z)X(\gamma, w) : \\
&\quad + (h\,|\,\gamma) : g(w)X(\gamma', z)X(\gamma, w) : \frac{z}{z-w} \\
&\quad - (\gamma'\,|\,g) : h(z)X(\gamma', z)X(\gamma, w) : \frac{w}{z-w} \\
&\quad + ((h\,|\,g) - (h\,|\,\gamma)(g\,|\,\gamma'))X(\gamma', z)X(\gamma\,|\,w)\frac{zw}{(z-w)^2}, \quad |w|<|z|;
\end{aligned}
$$

(2)
$$
: g(w)X(\gamma, w) :: h(z)X(\gamma', z) :
$$

is the same as the right-hand side of (1), $|z|<|w|$.

Proof.

$$
\begin{aligned}
&: h(z)X(\gamma', z) :: g(w)X(\gamma, w) : \\
&= \{h^-(z)X(\gamma', z) + X(\gamma', z)h^+(z)\} : g(w)X(\gamma, w) : \\
&= h^-(z)X(\gamma', z) : g(w)X(\gamma, w) : + X(\gamma', z) : g(w)X(\gamma, w) : h^+(z) \\
&\quad + X(\gamma', z)\left((h|\gamma) : g(w)X(\gamma, w) : \frac{z}{z-w} + (h|g)X(\gamma, w)\frac{zw}{(z-w)^2}\right)
\end{aligned}
$$

(from Lemma 4.3)

$$
\begin{aligned}
&= h^-(z)\left(: g(w)X(\gamma', z)X(\gamma, w) : -(\gamma'|g)X(\gamma', z)X(\gamma, w)\frac{w}{z-w}\right) \\
&\quad + : g(w)X(\gamma', z)X(\gamma, w) : h^+(z) - (\gamma'|g)X(\gamma', z)X(\gamma, w)h^+(z)\frac{w}{z-w} \\
&\quad + (h|\gamma) : g(w)X(\gamma', z)X(\gamma, w) : \frac{z}{z-w} \\
&\quad - (h|\gamma)(\gamma'|g)X(\gamma', z)X(\gamma, w)\frac{wz}{(z-w)^2} \\
&\quad + (h|g)X(\gamma', z)X(\gamma, w)\frac{zw}{(z-w)^2}
\end{aligned}
$$

(from Lemma 4.4).

Now rewriting the normal ordering, Lemma 4.5 follows.

(2) Similar. □

(4.6) Lemma. *Let $\alpha \in Q, a \in \mathbb{Z}$. Then*

$$
\frac{d}{dz}(z^a X(\alpha, z)) = z^{a-1}X(\alpha, z)\left\{\alpha(z) + \frac{(\alpha|\alpha)}{2} + a\right\}.
$$

Proof. This follows from differentiating $z^a X(\alpha, z)$. Also see Sect. 2.8 of [FK] for a sketch of the proof. □

(4.7) Proposition. *Let $h, g \in \mathfrak{t}$, let γ, γ' be null roots, and let $a, b \in \mathbb{Z}$. Then*

$$
\begin{aligned}
[T_a^h(\gamma'), T_b^g(\gamma)] &= (h|\gamma)T_{a+b}^g(\gamma + \gamma') - (\gamma'|g)T_{a+b}^h(\gamma + \gamma') \\
&\quad + ((h|g) - (\gamma'|g)(h|\gamma))\left\{T_{a+b}^{\gamma'}(\gamma + \gamma') + aX_{a+b}(\gamma + \gamma')\right\}.
\end{aligned}
$$

Proof. Recall

$$
\begin{aligned}
T_a^h(\gamma') &= \frac{1}{2\pi i}\int : h(z)X(\gamma', z) : \frac{dz}{z}z^a, \\
T_b^g(\gamma) &= \frac{1}{2\pi i}\int : g(w)X(\gamma, w) : \frac{dw}{w}w^b.
\end{aligned}
$$

Consider

$$
\begin{aligned}
[T_a^h(\gamma'), T_b^g(\gamma)] &= T_a^h(\gamma')T_b^g(\gamma) - T_b^g(\gamma)T_a^h(\gamma') \\
&= \frac{1}{(2\pi i)^2}\left(\int\int_{|w|<|z|} - \int\int_{|z|<|w|}\right)F(z, w)\frac{dz}{z}\frac{dw}{w}z^a w^b, \qquad (4.8)
\end{aligned}
$$

where

$$F(z,w) = F_1(z,w) + (h|\gamma)F_2(z,w)\left(\frac{z}{z-w}\right) - (\gamma'|g)F_3(z,w)\left(\frac{w}{z-w}\right)$$
$$+ ((h|g) - (h|\gamma)(g|\gamma'))F_4(z,w)\frac{zw}{(z-w)^2}\,,$$

and F_1, F_2, F_3 and F_4 analytic functions in z and w given by Lemma 4.5.

Now we note $\left(\iint_{|w|<|z|} - \iint_{|z|<|w|}\right)\frac{dz}{z}\frac{dw}{w} = \int_{C^1}\frac{dw}{w}\int_C\frac{dz}{z}$, where the contour C runs around w and does not have zero in its interior. The contour C^1 is around zero. These contour integrals have been explained in [KMPS]. Now it is straightforward to evaluate the inner integrals. Using the Cauchy integral formula the inner integral equals

$$(h|\gamma)F_2(z,w)z^a w^b]_{z=w} - (\gamma'|g)F_3(z,w)z^{a-1}w^{b+1}]_{z=w}$$
$$+ ((h|g) - (h|\gamma)(g|\gamma'))\frac{d}{dz}F_4(z,w)z^a w^{b+1}]_{z=w}\,. \tag{4.9}$$

However

$$\frac{d}{dz}F_4(z,w)z^a w^{b+1} = \frac{d}{dz}(X(\gamma',z)X(\gamma,w)z^a w^{b+1})$$
$$= z^{a-1}X(\gamma',z)(\gamma'(z)+a)X(\gamma,w)w^{b+1} \tag{4.10}$$

(by Lemma 4.6).

Hence (4.8) equals

$$\frac{1}{2\pi i}\int_{C'}\frac{dw}{w}w^{a+b}G(w)\,,$$

where

$$G(w) = (h|\gamma) : g(w)X(\gamma+\gamma',w) : -(\gamma'|g) : h(w)X(\gamma+\gamma',w) :$$
$$+ ((h|g) - (h|\gamma)(g|\gamma'))(\gamma'(w)X(\gamma+\gamma',w) + aX(\gamma+\gamma',w))\,.$$

Here we used (4.9) and (4.10) and we are also using the fact that $X(\gamma,w)X(\gamma',w) = X(\gamma+\gamma',w)$, which follows from the definition of vertex operators. Now clearly Proposition 4.7 follows. □

(4.11) Proposition. *Let $h \in \mathfrak{t}, \alpha \in Q$, let γ, γ' be null roots, and let $a, b \in \mathbb{Z}$. Then*

$$[T_a^h(\gamma'), X_b(\gamma+\alpha)] = (\gamma+\alpha|h)X_{a+b}(\gamma+\gamma'+\alpha)\,.$$

The proof is similar to the proof of the earlier proposition. We only have to use the following lemma (4.12) in the place of (4.5). □

(4.12) Lemma. *With the notation as above,*

$$: h(z)X(\gamma',z) : X(\gamma+\alpha,w) = \ : h(z)X(\gamma',z)X(\gamma+\alpha,w) :$$
$$+ (\gamma+\alpha|h)X(\gamma',z)X(\gamma+\alpha,w)\left(\frac{z}{z-w}\right),$$
$$|w|<|z|\,.$$

Proofs of Lemmas A and B. To see Lemma A, take $h, g \in \mathring{\mathfrak{h}}$, $\delta = \gamma$, $\delta' = \gamma'$ in Proposition 4.7, so that $(h, \delta') = (g, \delta) = 0$.

Lemma B follows from Proposition 4.11 by taking $h \in \dot{\mathfrak{h}}$, $\alpha = \alpha \in \dot{\Delta}$, $\delta = \gamma'$, and $\delta' = \gamma$.

Propositions 4.7 and 4.1 have been proved in a little more generality than is necessary than we needed here. The reason for this will become apparent in the next section.

Before we prove Lemma C, we recall the following from [MEY], which though stated for $n = 2$ case there, is true in our present situation.

(4.13) Lemma. *Let $\lambda \in \Gamma$, let δ be a null root, let $N = (\lambda \mid \delta)$ and let $m \in \mathbb{Z}$. Then*

$$X_m(\delta)(e^\lambda \otimes 1) = \begin{cases} \epsilon(\delta,\lambda)e^{\lambda+\delta} \otimes S_{-m-N}(\delta), & m+N<0 \\ \epsilon(\delta,\lambda)e^{\lambda+\delta} \otimes 1, & m+N=0 \\ 0, & m+N>0 \end{cases}$$

where the operators $S_p(\delta)$ are defined by $\exp T_-(\delta,z) = \sum_{p=0}^{\infty} S_p(\delta) z^p$.

Proof of Lemmas C. First observe that $\mathfrak{z}$ is graded by $\mathbb{Z} \times \Gamma$ and is invariant under the action of L_0 and $\mathfrak{h}$ (see 3.5). Hence we can assume the relations in $\mathfrak{z}$ are homogeneous. □

Let

$$\sum_{i=1}^{n-1} s_i T_a^{\delta_i}(\delta) + s_n X_a(\delta) = 0 \ ,$$

where $a \in \mathbb{Z}, \delta$ is a null root, and $s_1, s_2, \ldots, s_n$ are some complex numbers.

Choose $\lambda \in \Gamma$ such that $(\lambda, \delta) + a < 0$ and put $N = (\lambda \mid \delta)$ and $\gamma = \sum_{i=1}^{n-1} s_i \delta_i$. Consider

$$\begin{aligned} 0 &= \epsilon(\delta,\lambda)\,(T_a^{\gamma}(\delta) + s_n X_a(\delta))\, e^\lambda \otimes 1 \\ &= \sum_{a+N \leqq k \leqq 0} \gamma(k) e^{\lambda+\delta} \otimes S_{k-(a+N)}(\delta) \\ &\quad + s_n e^{\lambda+\delta} \otimes S_{-(a+N)}(\delta) \quad \text{(by Lemma 4.13)} \\ &= \sum_{a+N<k<0} \gamma(k) e^{\lambda+\delta} \otimes S_{k-(a+N)}\ (\delta) \\ &\quad + \gamma(a+N) e^{\lambda+\delta} \otimes 1 \\ &\quad + (\gamma, \lambda+\delta) e^{\lambda+\delta} \otimes S_{-(a+N)}(\delta) \\ &\quad + s_n e^{\lambda+\delta} \otimes S_{-(a+N)}(\delta) \\ &= A_1 + A_2 + A_3 + A_4, \qquad \text{say.} \end{aligned}$$

Now we have an equation in $V(\Gamma, \mathfrak{b})$. A_1 is linear combination of $e^{\lambda+\delta} \otimes a_1(n_1) a_2(n_2) \cdots a_\ell(n_\ell)$, where

$$a + N < n_1, n_2, \ldots, n_\ell < 0 \ .$$

Hence A_2 can only cancel with $A_3 + A_4$, and the corresponding term in $A_3 + A_4$ is

$$R := -\left(\frac{(\gamma \mid \lambda+\delta)}{a+N} + \frac{s_n}{a+N} \right) \delta_{(a+N)}$$

and $\gamma(a+N) + R = 0$. Hence

$$\gamma = B\delta$$

for some complex number B. Furthermore

$$B - \frac{(BN + s_n)}{a+N} = 0\,,$$
$$\Rightarrow Ba - s_n = 0\,,$$

i.e.

$$\sum_{i=1}^{n-1} s_i T_a^{\delta_i}(\delta) + s_n X_a(\delta) = B\left\{T_a^{\delta}(\delta) + aX_a(\delta)\right\}\,.$$

This completes the proof of Lemma C.

5. Generalized Virasoro Operators

As in the earlier sections, $A = A_{[n]} = \mathbb{C}[t_1^{\pm}, t_2^{\pm}, \ldots, t_n^{\pm}]$. It is well known that $\operatorname{Der} A$, the Lie algebra of derivations on A, is the $\mathbb{C}$-linear span of

$$\left\{D^i(\bar{r}) := t_1^{r_1} t_2^{r_2} \cdots t_i^{r_i+1} \cdots t_n^{r_n} \frac{d}{dt_i} \,|\, 1 \leqq i \leqq n, \bar{r} \in \mathbb{Z}^n\right\}\,,$$

where $\bar{r} = (r_1, r_2, \ldots, r_n)$ so that $\bar{r} = (\underline{r}, r_n)$.

The Lie bracket given by

$$[D^i(\bar{r}), D^j(\bar{s})] = s_i D^j(\bar{r} + \bar{s}) - r_j D^i(\bar{r} + \bar{s})\,.$$

A derivation D in $\operatorname{Der} A$ can be trivially extended to a derivation of the Lie algebra $\dot{g} \otimes A$. It is a simple fact about central extensions that any derivation of $\dot{\mathfrak{g}} \otimes A$ has a unique extension to the universal central extension of $\dot{\mathfrak{g}} \otimes A$ and leaves the centre invariant. It is easy to compute the action of $\operatorname{Der} A$ on the centre $\Omega_{[n]}$ of $\tau_{[n]}$:

$$D^i(\bar{r}) \cdot \overline{D_j(\bar{s})} = s_i \overline{D_j(\bar{s} + \bar{r})} + \delta_{ij} \sum_{\ell} r_\ell \overline{D_\ell(\bar{s} + \bar{r})}\,. \tag{5.1}$$

In the classical case $n = 1$, $\operatorname{Der} A_{[1]}$ is the Witt algebra. The Virasoro operators L_n (defined in 3.5) acting on the Fock space gives rise to a representation of Virasoro algebra, the universal central extension of Witt algebra.

In this section we construct operators for $\operatorname{Der} A$ which act on the Fock space generalizing the classical case. For $n \geqq 2$, it is known that $\operatorname{Der} A$ is centrally closed. In other words there are no non-trivial central extensions for $\operatorname{Der} A$, $(n \geqq 2)$. But our operators generate $\widetilde{\operatorname{Der}} A$ an abelian extension of $\operatorname{Der} A$. In other words, there is a surjective Lie algebra homomorphism ϕ from $\widetilde{\operatorname{Der}} A$ to $\operatorname{Der} A$ such that $\ker \phi$ is an abelian Lie algebra.

(5.2). As earlier, let $\underline{r} = (r_1, r_2, \ldots, r_{n-1}) \in \mathbb{Z}^{n-1}, r_n \in \mathbb{Z}$ and $\bar{r} = (\underline{r}, r_n)$. Recall $d_1, d_2, \ldots, d_{n-1} \in \mathfrak{t}$ from Sect. 3.2. The operators on V that will correspond to the $D^i(\bar{r})$ are given by:

$$T_{r_n}^{d_i}(\delta_{\underline{r}}) \quad \text{for} \quad i = 1, \ldots, n-1\,, \tag{5.3}$$

$$L_{r_n}(\delta_{\underline{r}}) := -\frac{1}{2\pi i} \int \sum : a_i(z) a^i(z) X(\delta_{\underline{r}}, z) : \frac{dz}{z} z^{r_n} \quad \text{for } i = n\,. \tag{5.4}$$

The following Propositions 5.5 and 5.8 describe how the operators at (5.3) and (5.4) act on $\tau_{[n]}$. In particular the action on the centre $\Omega_{[n]}$ is exactly as defined in (5.1).

(5.5) Proposition. *Let* $\underline{r}$, $\underline{s}$, $\underline{k} \in \mathbb{Z}^{n-1}$, $r_n, s_n, k_n \in \mathbb{Z}$.

(1) $[T^{d_i}_{r_n}(\delta_{\underline{r}}), T^g_{s_n}(\delta_{\underline{s}})] = s_i T^g_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}),\ g \in \dot{\underline{h}}$.

(2) $[T^{d_i}_{r_n}(\delta_{\underline{r}}), X_{s_n}(\delta_{\underline{s}} + \alpha)] = s_i X_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}} + \alpha),\ \alpha \in \dot{\Delta}$.

(3) $$\begin{aligned}&[T^{d_i}_{r_n}(\delta_{\underline{r}}), T^{\delta_{\underline{k}}}_{s_n}(\delta_{\underline{s}}) + k_n X_{s_n}(\delta_{\underline{s}})]\\ &= s_i(T^{\delta_{\underline{k}}}_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}) + k_n X_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}))\\ &\quad + k_i(T^{\delta_{\underline{r}}}_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}) + r_n X_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}})),\\ &\quad k \in \mathbb{Z}^{n-1},\ k_n \in \mathbb{Z}.\end{aligned}$$

Proof. (1) Choose $h = d_i,\ g \in \dot{\mathfrak{h}},\ \gamma' = \delta_{\underline{r}},\ \gamma = \delta_{\underline{s}},\ a = r_n$, and $b = s_n$ in Proposition 4.7. Then $(h|\gamma) = s_i,\ (h|g) = (\gamma'|g) = 0$ and (1) follows.

(2) Choose $h = d_i,\ \gamma' = \delta_{\underline{r}},\ \gamma = \delta_{\underline{s}},\ \alpha \in \dot{\Delta},\ a = r_n$ and $b = s_n$ in Proposition 4.11. Then $(\gamma + \alpha \mid h) = s_i$ and (2) follows.

(3) Choose $h = d_i,\ \gamma' = \delta_{\underline{r}},\ \gamma = \delta_{\underline{s}},\ g = \delta_{\underline{k}},\ a = r_n$ and $b = s_n$ in Proposition 4.7. Then $(h|\gamma) = s_i, (\gamma'|g) = 0, (h|g) = k_i$ and

$$\begin{aligned}[T^{d_i}_{r_n}(\delta_{\underline{r}}), T^{\delta_{\underline{k}}}_{s_n}(\delta_{\underline{s}})] &= s_i(T^{\delta_{\underline{k}}}_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}})) + k_i(T^{\delta_{\underline{r}}}_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}})\\ &\quad + r_n X_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}))\,. \end{aligned}\tag{5.6}$$

Now choose $h = d_i,\ \gamma' = \delta_{\underline{r}},\ r = \delta_{\underline{s}}, \alpha = 0,\ a = r_n$ and $b = s_n$ in Proposition 4.11.

Then $(\gamma + \alpha \mid d_i) = s_i$ and

$$[T^{d_i}_{r_n}(\delta_{\underline{r}}), X_{s_n}(\delta_{\underline{s}})] = s_i X_{s_n+r_n}(\delta_{\underline{r}} + \delta_{\underline{s}})\,. \tag{5.7}$$

(3) follows from (5.6) and (5.7). □

(5.8) Proposition.

(1) $[L_{r_n}(\delta_{\underline{r}}), T^g_{s_n}(\delta_{\underline{s}})] = s_n T^g_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}),\quad g \in \dot{\mathfrak{h}}$.

(2) $[L_{r_n}(\delta_{\underline{r}}), X_{s_n}(\delta_{\underline{s}} + \alpha)] = s_n X_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}} + \alpha),\quad \alpha \in \dot{\Delta}$.

(3) $$\begin{aligned}&[L_{r_n}(\delta_{\underline{r}}), T^{\delta_{\underline{k}}}_{s_n}(\delta_{\underline{s}}) + k_n X_{s_n}(\delta_{\underline{s}})]\\ &= s_n(T^{\delta_{\underline{k}}}_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}) + k_n X_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}))\\ &\quad + k_n(T^{\delta_{\underline{r}}}_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}) + r_n X_{r_n+s_n}(\delta_{\underline{r}} + \delta_{\underline{s}}))\,.\end{aligned}$$

The proof of Proposition 5.8 will be given towards the end of the section.

(5.9). The next natural question is to describe the Lie algebra generated by operators $T^{d_i}_{r_n}(\delta_{\underline{r}}), L_{r_n}(\delta_{\underline{r}})$. It turns out to be difficult in general. But it has a simple answer in the "zero moments" case, that is the Lie algebra generated by $T^{d_i}_0(\delta_{\underline{r}})\ (1 \leqq i \leqq n-1)$, corresponding to Der $A_{[n-1]}$ acting on $\tau_{[n-1]}$ (see Theorem 3.17).

First we will define an abelian extension of Der $A_{[n-1]}$.

(5.10). Let $A' = A_{[n-1]}$ and define $\widetilde{\mathrm{Der}}\, A' := \mathrm{Der}\, A' \oplus \Omega'_A/dA'$ with Lie bracket $[\ ]_\sim$.

(1) $[D^i(\underline{r}), D^j(\underline{s})]_\sim = s_i D^j(\underline{r} + \underline{s}) - r_j D^i(\underline{r} + \underline{s}) - s_i r_j(\sum_{\ell=1}^{n-1} r_\ell \overline{D_\ell(\underline{r} + \underline{s})})$.

(2) $[D^i(\underline{r}), \overline{D_j(\underline{s})}]_\sim = s_i \overline{D_j(\underline{s} + \underline{r})} + \delta_{ij} \sum_{\ell=1}^{n-1} r_\ell \overline{D_\ell(\underline{r} + \underline{s})}$.

(3) $[\overline{D_i(\underline{r})}, \overline{D_j(\underline{s})}]_\sim = 0$.

(5.11) Proposition. *The Lie algebra D_0 of operators on $V(\Gamma, \underline{b})$ generated by the operators $T_0^{d_i}(\delta_{\underline{r}})$, $1 \leqq i \leqq n-1$, $\underline{r} \in \mathbb{Z}^{n-1}$ is isomorphic to* $\widetilde{\mathrm{Der} A'}$.

(5.11) Proof. First we note the following:

$$(1)\ [T_0^{d_i}(\delta_{\underline{r}}), T_0^{d_j}(\delta_{\underline{s}})] = s_i T_0^{d_j}(\delta_{\underline{r}} + \delta_{\underline{s}}) - r_j T_0^{d_i}(\delta_{\underline{r}} + \delta_{\underline{s}}) - s_i r_j T_0^{\delta_{\underline{r}}}(\delta_{\underline{r}} + \delta_{\underline{s}}).$$

$$(2)\ [T_0^{d_i}(\delta_{\underline{r}}), T_0^{\delta_j}(\delta_{\underline{s}})] = s_i T_0^{\delta_j}(\delta_{\underline{r}} + \delta_{\underline{s}}) + \delta_{ij}(T_0^{\delta_{\underline{r}}}(\delta_{\underline{r}} + \delta_{\underline{s}})).$$

$$(3)\ [T_0^{\delta_i}(\delta_{\underline{r}}), T_0^{\delta_j}(\delta_{\underline{s}})] = 0.$$

(D1) follows from Proposition 4.7 by choosing $h = d_i$, $g = d_j$, $\gamma' = \delta_{\underline{r}}$, $\gamma = \delta_{\underline{s}}$ so that $(h|\gamma) = s_i$, $(g|\gamma') = r_j$, and $(h|g) = 0$.

(D2) also follows from Proposition 4.7 by choosing $h = d_i$, $g = \delta_j$, $\gamma' = \delta_{\underline{r}}$, $\gamma = \delta_{\underline{s}}$ so that $(h|\gamma) = s_i$, $(g|\gamma') = 0$, and $(h|g) = \delta_{ij}$.

(D3) is clear.

Now we have the isomorphism ψ from D_0 to $\widetilde{\mathrm{Der}}\, A'$ is defined by

$$T_0^{d_i}(\delta_{\underline{r}}) \mapsto D^i(\underline{r}),$$

$$T_0^{\delta_i}(\delta_{\underline{r}}) \mapsto \overline{D_i(\underline{r})}.$$

From (D1), (D2) and (D3) it is easy to see that D_0 contains the central operators $T_0^{\delta_i}(\delta_{\underline{r}})$ and ψ defines an isomorphism of Lie algebras. Also recall that the central operators $T_0^{\delta_i}(\delta_{\underline{r}})$, $1 \leqq i \leqq n-1$, $\underline{r} \in \mathbb{Z}^{n-1}$ can be identified with Ω'_A/dA' from Lemma 3.16. $\square$

Remark. The action of $\mathrm{Der}\, A'$ on $\Omega_{A'}/dA'$ has a simple interpretation in terms of cyclic homology. It is well known that $CH_1(R) \simeq \Omega_R/dR$ for any associative algebra R over $\mathbb{C}$ [KL]. Also it is easy to see from the definitions that $\mathrm{Der}\, R$ acts naturally on Ω_R/dR. This is precisely the action in (5.20) (ii).

(5.12) Towards the proof of Proposition 5.8. The techniques to be used in the proof of Proposition 5.8 are exactly as developed in Sect.4. First we normal order the products and then evaluate the integrals. We will only sketch the proofs as they are similar to the ones developed in Sect. 4. To start with we have the following lemmas. Their proofs are omitted, being straightforward but tedious.

(5.13) Lemma. *Let $\alpha \in Q, \gamma$ be a null root, $\{a_i\}, \{a^i\}$ dual bases for $\mathfrak{p}$, and z and w complex variables. Then*

$$(1)\ X(\alpha, z) \sum : a_i(w) a^i(w) X(\gamma, w) :$$

$$= \sum : a_i(w) a^i(w) X(\gamma, w) X(\alpha, z) : \ -2 : \alpha(w) X(\gamma, w) X(\alpha, z) : \frac{w}{z-w}$$

$$+ (\alpha \mid \alpha)\ : X(\gamma, w) X(\alpha, z) : \frac{w^2}{(z-w)^2},\ |w| < |z|;$$

(2) $\sum : a_i(w) a^i(w) X(\gamma, w) : X(\alpha, z) =$ *same as above*, $|z| < |w|$. $\square$

(5.14) Lemma. *Let $h \in \mathfrak{h}$, γ, γ' be null roots, and let $\{a_i\}, \{a^i\}$ be dual bases of $\mathfrak{p}$. Then*

$$\begin{aligned}(1)\ \ &: h(z)X(\gamma',z) :: \sum a_i(w)a^i(w)X(\gamma,w) :\\ &= : h(z)\sum a_i(w)a^i(w)X(\gamma,w)X(\gamma',z) :\\ &\quad -2 : h(z)\gamma'(w)X(\gamma',z)X(\gamma,w) : \frac{w}{z-w}\\ &\quad +2 : h(w)X(\gamma,w)X(\gamma',z) : \frac{zw}{(z-w)^2}, \qquad |w|<|z|;\end{aligned}$$

$$(2)\ \ : \sum a_i(w)a^i(w)X(\gamma,w) :: h(z)X(\gamma',z) : = \textit{same as above}\ |z|<|w|\,.$$

(5.15) Proposition. *Let $a,b\in\mathbb{Z}$, $\beta\in Q$, $h\in\mathfrak{h}$, and let γ, γ' be null roots.*

(1) $[X_a(\beta),L_b(\gamma)]$

$$= T^{\beta}_{a+b}(\beta+\gamma)-\frac{(\beta|\beta)}{2}\left(T^{\beta}_{a+b}(\beta+\gamma)+\left(a-1+\frac{(\beta|\beta)}{2}\right)X_{a+b}(\beta+\gamma)\right).$$

(2) $[T^h_a(\gamma'),L_b(\gamma)] = -aT^h_{a+b}(\gamma'+\gamma)$.

Proof. We will only sketch the proof of (2). The proof of (1) is similar.

Recall that

$$T^h_a(\gamma')=\frac{1}{2\pi i}\int : h(z)X(\gamma',z) : \frac{dz}{z}z^a$$

and

$$L_b(\gamma)=-\frac{1}{2\pi i}\frac{1}{2}\int\sum : a_i(w)a^i(w)X(\gamma,w)\frac{dw}{w} : w^b.$$

Consider

$$\begin{aligned}[T^h_a(\gamma'),L_b(\gamma)] &= T^h_a(\gamma')L_b(\gamma)-L_b(\gamma)T^h_a(\gamma')\\ &= \frac{1}{(2\pi i)^2}\left(\int\int_{|w|<|z|}-\int\int_{|z|<|w|}\right)F(z,w)\frac{dz}{z}\frac{dw}{w}z^aw^b, \qquad (5.16)\end{aligned}$$

where $-2F(z,w)=F_1(z,w)-2F_2(z,w)\frac{w}{z-w}+2F_3(z,w)\frac{zw}{(z-w)^2}$, F_1,F_2 and F_3 are analytic functions in z and w given by Lemma 5.14.

As in the Proof of Proposition 4.7, (5.16) equals $\frac{1}{2\pi i}\int\frac{dw}{w}G(w)$, where

$$G(w)=F_2(z,w)z^{a-1}w^b]_{z=w}-\frac{d}{dz}(F_3(z,w)z^aw^{b+1}\,)]_{z=w}$$

(by the Cauchy integral formula).

First consider

$$\begin{aligned}\frac{d}{dz}(F_3(z,w)z^{a-1}w^b) &= \frac{d}{dz}\left(h(w)X(\gamma,w)X(\gamma',z)z^aw^{b+1}\right)\\ &= h(w)X(\gamma,w)w^{b+1}\frac{d}{dz}[z^aX(\gamma',z)]\\ &= h(w)X(\gamma,w)w^{b+1}(z^{a-1}X(\gamma',z)(\gamma'(z)+a))\end{aligned}$$

(by Lemma 4.6).

Hence

$$\begin{aligned}G(w) &= h(w)\gamma'(w)X(\gamma',w)X(\gamma,w)w^{a-1}w^b\\ &\quad -h(w)X(\gamma,w)w^{b+1}(w^{a-1}X(\gamma',w)(\gamma'(w)+a))\\ &= -ah(w)X(\gamma+\gamma',w)w^{a+b}.\end{aligned}$$

Therefore

$$\frac{1}{2\pi i}\int G(w)\frac{dw}{w} = -\frac{a}{2\pi i}\int : h(w)X(\gamma+\gamma',w) : \frac{dw}{w}w^{a+b}$$
$$= -aT^h_{a+b}(\gamma+\gamma')\,.$$

Proof of Proposition 5.8. (1) Follows by choosing $b = r_n$, $\gamma = \delta_{\underline{r}}$, $h = g$, $a = s_n$, and $\gamma' = \delta_{\underline{s}}$ in Proposition 5.15(2).

(2) Choose $b = r_n$, $\gamma = \delta_{\underline{r}}$, $a = s_n$, and $\beta = \delta_{\underline{s}} + \alpha$ in Proposition 5.15(1). Then

$$\begin{aligned}&[X_{s_n}(\delta_{\underline{s}}+\alpha), L_{r_n}(\delta_{\underline{r}})]\\ &\quad = T^{\delta_{\underline{s}}+\alpha}_{r_n+s_n}(\delta_{\underline{r}}+\delta_{\underline{s}}+\alpha) - \left(T^{\delta_{\underline{s}}+\alpha}_{r_n+s_n}(\delta_{\underline{r}}+\delta_{\underline{s}}+\alpha) + s_nX_{r_n+s_n}(\delta_{\underline{r}}+\delta_{\underline{s}}+\alpha)\right)\\ &\quad = -s_nX_{r_n+s_n}(\delta_{\underline{r}}+\delta_{\underline{s}}+\alpha)\,.\end{aligned}$$

(3) Choose $a = s_n$, $\beta = \delta_{\underline{s}}$, $b = r_n$, and $\gamma = \delta_{\underline{r}}$ in Proposition 5.5(1). Then $(\beta \mid \beta) = 0$ and

$$\begin{aligned}[X_{s_n}(\delta_{\underline{s}}), L_{r_n}(\delta_{\underline{r}})] &= T^{\delta_{\underline{s}}}_{s_n+r_n}(\delta_{\underline{r}}+\delta_{\underline{s}})\\ &= -T^{\delta_{\underline{r}}}_{s_n+r_n}(\delta_{\underline{r}}+\delta_{\underline{s}}) - (r_n+s_n)X_{s_n+r_n}(\delta_{\underline{r}}+\delta_{\underline{s}})\,, \quad (5.17)\end{aligned}$$

where the last equality follows from Lemma 3.13. Now choose $a = s_n$, $h = \delta_{\underline{k}}$, $\gamma' = \delta_{\underline{s}}$, $b = r_n$, and $\gamma = \delta_{\underline{r}}$ in Proposition 5.15(2). Then

$$[T^{\delta_{\underline{k}}}_{s_n}(\delta_{\underline{s}}), L_{r_n}(\delta_{\underline{r}})] = -s_nT^{\delta_{\underline{k}}}_{r_n+s_n}(\delta_{\underline{r}}+\delta_{\underline{s}})\,. \quad (5.18)$$

Now combining (5.17) and (5.18) we have (3).

We record the following Proposition which determines the commutator products of the operators defined at (5.3) and (5.4). We are omitting the proof since it is computational and we have no need of it here.

(5.19) Proposition. *Let $a, b \in \mathbb{Z}$, and let γ, γ' be null roots. Then*

(1) $[L_a(\gamma), L_b(\gamma')] = (b-a)L_{a+b}\ (\gamma+\gamma') + \mathrm{Res}_{z=0}f(z)$,

$$\begin{aligned}\textit{where } f(z) := &-\left[\frac{d}{dz}(\gamma(z)X(\gamma',z)z^a)\right]\gamma'(z)X(\gamma',z)z^b\\ &-\frac{1}{2}\left[\frac{d^2}{dz^2}(\gamma'(z)X(\gamma',z)z^a)\right]X(\gamma,z)z^{b+1}\\ &+\frac{1}{2}\left[\frac{d^2}{dz^2}(X(\gamma',z)z^{a+1})\right]\gamma(z)X(\gamma,z)z^b\\ &+\frac{(\dim \mathfrak{p})}{12}\left[\frac{d^3}{dz^3}(X(\gamma',z)z^{a+1})\right]X(\gamma,z)z^{b+1}.\end{aligned}$$

(2) $$[T_a^{d_i}(\gamma'), L_b(\gamma)] = (\gamma\,|\,d_i)L_{a+b}(\gamma+\gamma') - aT_{a+b}^{d_i}(\gamma+\gamma') + \mathrm{Res}_{z=0}g(z),$$

$$where\, g(z) := (d_i\,|\,\gamma)\left[\frac{d}{dz}(X(\gamma',z)z^a)\right]\gamma'(z)X(\gamma,z)z^b + (d_i\,|\,\gamma')\frac{1}{2}\left[\frac{d^2}{dz^2}(X(\gamma',z)z^a)\right]X(\gamma,z)z^{b+1}.$$

(3) $$[T_a^{d_i}(\gamma'), T_b^{d_j}(\gamma)] = (\gamma\,|\,d_i)T_{a+b}^{d_j}(\gamma+\gamma') - (\gamma'\,|\,d_j)T_{a+b}^{d_i}(\gamma+\gamma') - (\gamma\,|\,d_i)(\gamma'\,|\,d_j)(T_{a+b}^{\gamma'}(\gamma+\gamma') + bX_{a+b}(\gamma+\gamma')).$$

(5.20) Note. (1) We see that for $\gamma = 0, L_a(\gamma) = L_a$. Thus, by taking $\gamma = \gamma' = 0$ in Proposition 5.19(1), we have 3.6(3).

(2) Proposition 5.19(3) can be deduced from Proposition (4.7).

(5.21) Open problem. Describe the Lie algebra generated by operators $T_a^{d_i}(\gamma)$, $1 \leqq i \leqq n-1$ and $L_a(\gamma)$.

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