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Super-derivations

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Abstract: It is shown that the square of a super-derivation can never be a generator (without taking its closure) if it is unbounded and self-adjoint.

1. Introduction

The notion of a quantum algebra has been introduced by A. Jaffe et al. [5–7] in connection with entire cyclic cohomology (cf. [3, 4, 8]). A key ingredient to this notion is a super-derivation, defined on a graded C^*-algebra, whose square *is* or *extends to* the generator of a one-parameter group of *-automorphisms. In this note we study the relationship between the super-derivation and the generator to seek the *right* definition of a quantum algebra and obtain among others the result stated in the abstract, i.e., if the square of a self-adjoint super-derivation is a generator then it is bounded.

We will state the main results in Sect. 2 and give their proofs in Sects. 3–6. Finally we will give a *spatial* example based on the algebra of bounded operators on a Hilbert space. One of the authors (A.K.) is grateful to C.J. K. Batty for many discussions.

In the rest of this section we will state the definition of a super-derivation and give some basic properties.

Let (A,γ) be a graded C^*-algebra; i.e., A is a C^*-algebra and γ is a *-automorphism of A of period two. Let

$$A_e = \{a \in A \mid \gamma(a) = a\},\ A_o = \{a \in A \mid \gamma(a) = -a\}\ .$$

Then it follows that A_e is a sub-C^*-algebra of A and that $A_e A_o \supset A_o$, $A_o^* = A_o$, and $A_o A_o \subset A_e$. The C^*-algebra A is the direct sum of A_e and A_o as a Banach space.

Let d be a super-derivation of A; i.e., its domain $D(d)$ is a (dense) γ-invariant subalgebra of A and d is a linear map of $D(d)$ into A such that

$$d(ab) = da \cdot b + \gamma(a) \cdot db,\ a, b \in D(d)\ ,$$

and $\gamma \circ d = -d \circ \gamma$. In particular

$$D(d) = D(d) \cap A_e + D(d) \cap A_o$$

and d maps $D(d) \cap A_e$ into A_o and $D(d) \cap A_o$ into A_e.

Let B be the crossed product $A \times_\gamma \mathbb{Z}_2$ of A by γ, and let U be the canonical unitary of B implementing γ. Define a linear map δ of $D(d) \subset A \subset B$ into $AU \subset B$ by

$$\delta(a) = U d(a) .$$

Then since $\delta(ab) = U d(ab) = U \cdot da \cdot b + U\gamma(a) \cdot db = \delta(a)b + a\delta(b)$, δ is a derivation. In particular if $D(d) = A$, then d is automatically bounded since the corresponding δ is bounded (see e.g. [2]).

Define a linear map d^+ on $D(d^+) = D(d)^*$ by

$$d^+(a) = \gamma(da^*)^* .$$

Then d^+, called the adjoint of d, is again a super-derivation.

An example of super-derivations is an inner one; if $q \in A_o$, the linear map defined by

$$\delta_q = qa - \gamma(a)q, \quad a \in A$$

is a super-derivation. Note that $(\delta_q)^+ = \delta_{q^*}$. Hence if γ is properly outer or freely acting (i.e., has no inner part [9]), δ_q being self-adjoint (i.e., $\delta_q^+ = \delta_q$) is equivalent to q being self-adjoint.

If γ is implemented by a unitary $u \in D(d)$, then it follows that d is inner. To see this apply d to the equality $ux = \gamma(x)u$ for $x \in D(d)$. Since $u \in A_e$, it follows that

$$du \cdot x + u \cdot dx = -\gamma(dx)u + xdu ,$$

which implies that $d = \delta_q$ with

$$q = -\frac{1}{2} u^* \cdot du .$$

If d is self-adjoint, then so is q, which follows from

$$0 = d(1) = d(u^*u) = du^* \cdot u + u^* du .$$

If d is a super-derivation, then d^2 is a derivation satisfying $d^2 \circ \gamma = \gamma \circ d^2$. If d is self-adjoint in addition, then d^2 is self-adjoint, i.e., $(d^2)^* = d^2$, where $(d^2)^*$ is defined by

$$(d^2)^* = -d^2(x^*)^*, \quad x \in D((d^2)^*) = D(d^2)^* .$$

(Note that $(d^2)^*$ is normally defined by the above equality without minus sign. See [1, 2, 12].) If $d = \delta_q$ with $q \in A_o$, then $d^2 = \delta_{q^2}$, where for $h = q^2 \in A_e$, δ_h is defined by

$$\delta_h(x) = hx - xh .$$

If $h = h^*$, which follows from, but does not imply, $q = q^*$, then $i \cdot \delta_h$ generates a one-parameter group α of *-automorphisms of A with $\alpha_t \circ \gamma = \gamma \circ \alpha_t$. We note that our one-parameter groups of *-automorphisms always preserve the grading (or commutes with γ).

2. Main Results

We call a linear operator L on the C^*-algebra A a generator if iL generates a strongly continuous one-parameter group α of *-automorphisms of A, where the strong continuity is defined by:

$$\|\alpha_t(x) - x\| \to 0, \quad \text{as } t \to 0, \quad \text{for } x \in A .$$

Note that if L is a generator, then L is a closed self-adjoint derivation.

Given a one-parameter automorphism group α of A and given an open subset U of $\mathbb{R}$, we denote by $A^\alpha(U)$ the spectral subspace defined as the closure of

$$\{\alpha_f(x) | x \in A, \operatorname{supp} \hat{f} \subset U\}$$

where, for a continuous $f \in L^1(\mathbb{R})$ and $x \in A$, $\alpha_f(x)$ is defined by

$$\alpha_f(x) = \int f(t)\alpha_t(x)\,dt$$

and $\hat{f}$ is defined by

$$\hat{f}(p) = \int f(t)e^{ipt}\,dt\,.$$

For $x \in A$, the α-spectrum $\mathrm{Sp}_\alpha(x)$ of x is defined by

$$\bigcap \{\{p | \hat{f}(p) = 0\} | \alpha_f(x) = 0\}\,.$$

It follows that $x \in A^\alpha(U)$ satisfies $\mathrm{Sp}_\alpha(x) \subset \bar{U}$ and that any $x \in A$ with α-spectrum in U belongs to $A^\alpha(U)$. Let L be the generator of α. If U is bounded, then $A^\alpha(U) \subset D(L)$. The union $\bigcup_{k>0} A^\alpha(-k,k)$, which is a dense *-subalgebra of A, is a core for L. See [10] for details.

Theorem 1. *Let (A,γ) be a graded C^*-algebra, and let d be a closed super-derivation of A and α a strongly continuous one-parameter group of *-automorphisms of A with $\alpha \circ \gamma = \gamma \circ \alpha$. Suppose that $d \circ \alpha_t = \alpha_t \circ d$ for all $t \in \mathbb{R}$, that $D(d^2)$ is dense in A, and that d^2 is a restriction of the generator L of α. Let $A_0 = \bigcup_{k>0} A^\alpha(-k,k)$. Then the following hold:*

(i) *$D(d) \cap A_0$ is a core for d and $D(d^2) \cap A_0$ is a core for d^2.*

(ii) *$D(d^2) \cap A_0$ is contained in $\bigcap_{k=1}^\infty D(d^k)$ and is invariant under d, and the closure of d^2 is L.*

(iii) *If $D(d^2)$ is a core for d, then $D(d) \cap A_0$ is contained in $\bigcap_{k=1}^\infty D(d^k)$ and is invariant under d.*

Remark 1. In the situation of the above theorem let d_1 be the closure of $d | D(d^2)$. Then d_1 commutes with α, and since $D(d^2) \cap A_0 \subset D(d_1^2)$, the closure of d_1^2 is L. Moreover $D(d_1^2)$ is a core for d_1 because $D(d^2) \cap A_0$ is a core for d_1.

We do not know in general whether the commutativity that $d \circ \alpha_t = \alpha_t \circ d$, $t \in \mathbb{R}$, can be derived from the other conditions in the above theorem. But it can if we assume that α is uniformly continuous or L is bounded, as the following theorem shows:

Theorem 2. *Let (A,γ) be a graded C^*-algebra, and let d be a closed super-derivation such that $D(d^2)$ is a core for d and d^2 is a restriction of an everywhere defined self-adjoint derivation L. Then d commutes with the strongly continuous one-parameter group α of *-automorphisms generated by iL.*

The following result concerns the problem of whether we have to take the closure of d^2 to get the generator L when d is unbounded. We have to restrict ourselves to self-adjoint super-derivations to prove:

Theorem 3. *Let (A,γ) be a graded C^*-algebra, and let d be a self-adjoint super-derivation of A and α a strongly continuous one-parameter group of *-automorphisms* *of A with $\alpha \circ \gamma = \gamma \circ \alpha$. Suppose that $d^2 \subset L$, where L is the generator of α. Then the following conditions are equivalent:*

(i) $d^2 = L$.

(ii) *d is everywhere defined (and hence bounded).*

(iii) *d is closed, $D(d^2)$ is dense, $d \circ \alpha_t = \alpha_t \circ d$ for all $t \in \mathbb{R}$, and $D(d) \supset A^\alpha(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.*

Hence we should not expect in general that $D(d)$ contains the entire analytic elements with respect to α (cf. [6, 7]),

Remark 2. Under the situation of the above theorem, if the super-derivation d is unbounded, the range of $\lambda \cdot 1 + d$ is not the whole A for any $\lambda \in \mathbb{C}$, and in particular the spectrum of d is the whole complex plane.

This may be proved as follows. Suppose that for some $\lambda \in \mathbb{C}$, $(\lambda + d)D(d) = A$. By applying γ one obtains that $(\lambda - d)D(d) = A$ and then $(\lambda^2 - d^2)D(d^2) = A$. By the above theorem d^2 cannot be the generator L; there must be a non-zero $x \in D(L)$ such that

$$(\lambda^2 - L)(x) = 0 .$$

Then, since $\alpha_t(x) = e^{itL}(x) = e^{i\lambda^2 t}x$, it follows that $\lambda^2 \in \mathbb{R}$. Since $(\lambda^2 - L)D(L) = A$, there is $y \in D(L)$ such that

$$x = (\lambda^2 - L)y .$$

Since $\alpha_t(y)$ satisfies

$$\alpha_0(y) = y, \quad \frac{d}{dt}\alpha_t(y) = \alpha_t(iLy) = i\lambda^2\alpha_t(y) - i\alpha_t(x) ,$$

it follows that

$$\alpha_t(y) = e^{i\lambda^2 t}y - ite^{i\lambda^2 t}x .$$

Since $\|\alpha_t(y)\| = \|y\|$ and $x \neq 0$, this is a contradiction.

If we further assume the situation of Theorem 1, the second part of the remark also follows from a general result (presented to us by C.J.K. Batty): If d is a closed operator with $\mathrm{Sp}\, d \neq \mathbb{C}$, then d^2 is closed. The proof of this goes as follows. Let $\{x_n\}$ be a sequence in $D(d^2)$ such that

$$\|x_n - x\| \to 0, \quad \|d^2 x_n - y\| \to 0$$

for some y. If $\lambda \notin \mathrm{Sp}\, d$ is non-zero,

$$d((\lambda - d)^{-1} - \lambda^{-1})x_n = (\lambda - d)^{-1}\lambda^{-1}d^2 x_n$$

converges to $(\lambda - d)^{-1}\lambda^{-1}y$. Since $((\lambda - d)^{-1} - \lambda^{-1})x_n$ converges to $((\lambda - d)^{-1} - \lambda^{-1})x$ and d is closed, it follows that $x \in D(d)$ and

$$((\lambda - d)^{-1} - \lambda^{-1})dx = (\lambda - d)^{-1}\lambda^{-1}y .$$

Hence $dx \in D(d)$ and $d^2x = y$. If $0 \notin \mathrm{Sp}\, d$, we just have to note that dx_n converges to $d^{-1}y$. Hence $x \in D(d)$ and $dx = d^{-1}y$, i.e., $x \in D(d^2)$ and $d^2x = y$.

Our final result concerns inner perturbations of a self-adjoint super-derivation d, which is used in [7]. Let q be a self-adjoint element of $D(d) \cap A_o$, and let

$$d_q = d + \delta_q$$

which is again a super-derivation with $D(d_q) = D(d)$. Then $D(d_q^2) = D(d^2)$ and

$$d_q^2 = d^2 + \delta_\Omega ,$$

where $\Omega = dq + q^2$ is a self-adjoint element of A_e. Thus if $\overline{d^2}$ is a generator, then $\overline{d_q^2}$ is also a generator as being just an inner perturbation of $\overline{d^2}$. Since $\Omega \in A_e$, the one-parameter group α_t^q generated by $i\overline{(d_q)^2}$ preserves the grading (i.e., commutes with γ).

Theorem 4. *Let (A, γ) be a graded C^*-algebra, and let d be a closed self-adjoint super-derivation of A and α a strongly continuous one-parameter group of $*$-automorphisms of A with $\alpha_t \circ \gamma = \gamma \circ \alpha_t$. Suppose that*

(i) *$d \circ \alpha_t = \alpha_t \circ d$ for all $t \in \mathbb{R}$,*
(ii) *$D(d^2)$ is a core for d, and*
(iii) *$d^2 \subset L$,*

where L is the generator of α (hence $\overline{d^2} = L$ due to Theorem 1).

If q is a self-adjoint element of $D(d) \cap A_0$, then the pair of the super-derivation $d_q = d + \delta_q$ and the generator $L_q = L + \delta_\Omega$ with $\Omega = dq + d^2$ satisfies the same conditions (i), (ii), (iii) *as for the pair of d and L.*

3. Proof of Theorem 1

Let f be a continuous function in $L^1(\mathbb{R})$ and $x \in D(d)$. Then it follows that $\alpha_f(x) \in D(d)$ and

$$d(\alpha_f(x)) = \alpha_f(dx) .$$

Suppose that supp $\hat{f}$ is compact with $\hat{f}(0) = 1$ and let $f_n(t) = f(nt)n$ for $n = 1, 2, \ldots$ Then

$$\alpha_{f_n}(x) \to x, \ d(\alpha_{f_n}(x)) = \alpha_{f_n}(dx) \to dx .$$

Since d is closed and

$$\mathrm{Sp}_\alpha(\alpha_{f_n}(x)) \subset \mathrm{supp}\, \hat{f}_n = n \cdot \mathrm{supp}\, \hat{f} ,$$

this implies that $D(d) \cap A_0$ is a core for d. By repeating this procedure once more one obtains that $D(d^2) \cap A_0$ is a core for d^2. Thus we have shown Theorem 1(i).

Lemma 1. *For $x \in D(d)$, it follows that* $\mathrm{Sp}_\alpha(dx) \subset \mathrm{Sp}_\alpha(x)$.

Proof. This is immediate since $\alpha_f(x) = 0$ implies $\alpha_f(dx) = 0$.

Lemma 2. *$D(d^2) \cap A^\alpha(-k, k)$ is dense in $A^\alpha(-k, k)$ for any $k > 0$.*

Proof. Let $x \in A^\alpha(-k, k)$ and $\varepsilon > 0$. Then there is a non-zero $f \in L^1(\mathbb{R})$ with $\mathrm{supp}\hat{f} \subset (-k, k)$ and $y \in A$ such that $\|x - \alpha_f(y)\| < \varepsilon/3$. Since $D(d^2)$ is dense, there is $z \in D(d^2)$ with $\|y - z\| < \varepsilon/2\|f\|_1$. Then $\alpha_f(z) \in D(d^2) \cap A^\alpha(-k, k)$ and $\|\alpha_f(z) - x\| < \varepsilon$.

Let $x \in D(d^2) \cap A_0$. Then $dx \in D(d) \cap A_0$ and $d \cdot dx = Lx$. Hence to prove that $d(D(d^2) \cap A_0) \subset D(d^2) \cap A_0$, we only have to show that $Lx \in D(d) \cap A_0$. By letting $t \to 0$ in the equality

$$d\left(\frac{1}{t}(\alpha_t(x) - x)\right) = \frac{1}{t}(\alpha_t(dx) - dx)$$

we obtain that $Lx \in D(d)$ and

$$dLx = Ldx .$$

Hence it follows that $D(d^2) \cap A_0$ is contained in $\bigcap_{k=1}^{\infty} D(d^k)$ and is invariant under d.

Since d^2 is bounded on $D(d^2) \cap A^\alpha(-k, k)$, it follows that

$$D(\overline{d^2}) \supset A^\alpha(-k, k) .$$

Thus $D(\overline{d^2}) \supset A_0$ and hence $\overline{d^2} = L$ because A_0 is a core for L and d^2 is a restriction of L, which completes the proof of Theorem 1(ii).

Theorem 1(iii) follows from Lemma 1 and the following:

Lemma 3. *Under the assumption of Theorem* 1(iii) *it follows that* $D(d) \cap D(L) \subset D(d^2)$.

Proof. Let $x \in D(d) \cap D(L)$. We have to show that

$$(dx, Lx) \in G(d) \equiv \{(a, da) | a \in D(d)\} .$$

Since $G(d)$ is a closed subspace of $A \oplus A$ as being the graph of the closed operator d, if $(dx, Lx) \notin G(d)$, there is a $(\varphi, \psi) \in A^* \oplus A^*$ such that

$$\varphi(a) + \psi(da) = 0, \quad a \in D(d) ,$$

$$\varphi(dx) + \psi(Lx) \neq 0 .$$

Let $f \in L^1(\mathbb{R})$ be such that supp $\hat{f}$ is compact with $\hat{f}(0) = 1$, and let $f_n(t) = f(nt)n$ as before. Since d and L commutes with α_{f_n}, $(\varphi \circ \alpha_{f_n}, \varphi \circ \alpha_{f_n})$ vanishes on $G(d)$ and

$$\lim\{\varphi \circ \alpha_{f_n}(dx) + \psi \circ \alpha_{f_n}(Lx)\} = \varphi(dx) + \psi(Lx) .$$

Thus $\varphi \circ \alpha_{f_n}$ and $\psi \circ \alpha_{f_n}$ in place of φ and ψ respectively still satisfies the above condition for a sufficiently large n. In particular we may suppose that $\psi \circ L$ (on $D(L)$) extends to a bounded functional on A, which we denote by $\overline{\psi \circ L}$.

For $a \in D(d^2)$ one has

$$\varphi(da) + \psi(d^2 a) = 0$$

which implies that $\varphi \circ d | D(d^2)$ is bounded. Since $D(d^2)$ is a core for d, it follows that $\varphi \circ d$ is also bounded, and

$$\overline{\varphi \circ d} + \overline{\psi \circ L} = 0 .$$

Hence $\overline{\varphi \circ d}(x) + \overline{\psi \circ L}(x) = 0$, which is a contradiction.

4. Proof of Theorem 2 (Commutativity)

It suffices to prove that there is a dense subalgebra $\mathscr{A}$ of $D(d)$ such that $\mathscr{A}$ is a core for d and $d(\mathscr{A})\subset\mathscr{A}$. Because then it follows that $L = d^2$ on $\mathscr{A}$ and

$$d\sum_{n=0}^{N}\frac{1}{n!}(itL)^n = \sum_{n=0}^{N}\frac{1}{n!}(itL)^n d$$

on $\mathscr{A}$ for any N, which implies that $d\circ\alpha_t = \alpha_t\circ d$ on $\mathscr{A}$, by taking the limit of $N\to\infty$ (cf. [6]). We can take $D(d)$ as $\mathscr{A}$ by the following:

Lemma 4. $D(d) = D(d^2)$.

Proof. This can be proved as Lemma 3, where we needed the commutativity that $d\circ\alpha_t = \alpha_t\circ d$ to make $\psi\circ L$ bounded, which is automatic in the present case.

5. Proof of Theorem 3 (Self-Adjoint Super-Derivation)

We have already remarked that if a super-derivation is everywhere defined then it is automatically bounded. Hence the implications (ii)$\Rightarrow$(i) and (ii)$\Rightarrow$(iii) follow immediately.

To prove (i)$\Rightarrow$(ii) we first note:

Lemma 5. *If* $D(d)\supset D(L)$, $d|D(L)$ *is relatively bounded with respect to* L.

Proof. Remember that δ defined by

$$\delta(a) = Ud(a), \quad a\in D(d)$$

as a map of $D(d)$ into $A\times_\gamma\mathbb{Z}_2$ is a derivation, where U is the canonical unitary implementing γ. Note that $\|\delta(a)\| = \|d(a)\|$ for $a\in D(d)$. For the two derivations δ and L of $D(L)$ into $A\times_\gamma\mathbb{Z}_2$, we adopt the same arguments as in [1] and conclude that δ is relatively bounded with respect to L. Hence the conclusion follows.

Suppose that $d^2 = L$. Since $D(d)\supset D(L)$, it follows by the above lemma that $d|A^\alpha(-k,k)$ is bounded for any $k>0$. Since L leaves $A^\alpha(-k,k)$ invariant, the left side of

$$d\sum_{n=0}^{N}\frac{1}{n!}(itL)^n = \sum_{n=0}^{N}\frac{1}{n!}(itL)^n d$$

on $A^\alpha(-k,k)$ converges to $d\circ\alpha_t$ as $N\to\infty$, and hence the right side should converge to $\alpha_t\circ d$. Thus we obtain that $d\circ\alpha_t = \alpha_t\circ d$ on A_0, and also that $d\circ\alpha_f = \alpha_f\circ d$ on A_0 for any continuous $f\in L^1(\mathbb{R})$. Since for an open bounded set U of $\mathbb{R}$, d is bounded on $A^\alpha(U)$, and $A^\alpha(U)$ is a closed span of $\alpha_f(x)$ with supp $\hat{f}\subset U$ and $\mathrm{Sp}_\alpha(x)$ compact, it follows that d leaves $A^\alpha(U)$ invariant. Then by Lemma 7 below it follows that L is bounded, which implies that d is everywhere defined. Thus we obtain that (i) implies (ii).

To show the lemma referred to above we first prove:

Lemma 6. *There is a constant* $c>0$ *such that for any* $\lambda\in\mathbb{R}$, $\varepsilon>0$, *and* $x\in A^\alpha(\lambda-\varepsilon,\lambda+\varepsilon)$,

$$\|Lx-\lambda x\|\leqq c\varepsilon\|x\| .$$

Proof. Let $f \in L^1(\mathbb{R})$ be a C^∞-function such that $\hat{f}(p) = 1$ for $p \in [-1, 1]$. Let

$$h(t) = e^{-i\lambda t} f(\varepsilon t)\varepsilon .$$

Then for any $g \in L^1$ with supp $\hat{g} \subset (\lambda - \varepsilon, \lambda + \varepsilon)$, it follows that $\hat{h}\hat{g} = \hat{g}$, since $\hat{h}(p) = \hat{f}(\varepsilon^{-1}(p - \lambda))$. Hence for $x \in A^\alpha(\lambda - \varepsilon, \lambda + \varepsilon)$, one has $\alpha_h(x) = x$ and

$$iL\alpha_h(x) = \int h(t) \frac{d}{dt} \alpha_t(x)\, dt = i\lambda \int h(t)\alpha_t(x)\, dt - \varepsilon \int e^{-i\lambda t} f'(\varepsilon t)\alpha_t(x)\varepsilon\, dt .$$

Thus one obtains

$$\| Lx - \lambda x \| \leqq c\varepsilon \| x \| ,$$

where

$$c = \int | f'(t) |\, dt .$$

This concludes the proof.

Lemma 7. *Suppose* (i) *or* (iii) *of Theorem* 3. *Then L is bounded.*

Proof. Suppose that L is unbounded. Then, since $\operatorname{Sp}\alpha = -\operatorname{Sp}\alpha$, there is a sequence $\{\lambda_n\}$ in $\operatorname{Sp}\alpha \cap (0, \infty)$ such that $\lambda_n \to \infty$.

Fix $\varepsilon_0 > 0$ such that $d | A^\alpha(-2\varepsilon_0, 2\varepsilon_0)$ is bounded. Note that $D(d^2) \cap A^\alpha(\lambda_n - \varepsilon_0, \lambda_n + \varepsilon_0)$ is dense in $A^\alpha(\lambda_n - \varepsilon_0, \lambda_n + \varepsilon_0)$ and is γ-invariant. Since $A^\alpha(\lambda_n - \varepsilon_0, \lambda_n + \varepsilon_0) \cap A_e \neq \{0\}$, there is a non-zero element x of $D(d^2) \cap A^\alpha(\lambda_n - \varepsilon_0, \lambda_n + \varepsilon_0) \cap A_e$ and let

$$y = x + \frac{1}{\sqrt{\lambda_n}} dx .$$

Since $x = (y + \gamma(y))/2$ one has $\| x \| \leqq \| y \|$; in particular $y \neq 0$. Since

$$\begin{aligned} dy - \sqrt{\lambda_n} y &= dx + \frac{1}{\sqrt{\lambda_n}} d^2 x - \sqrt{\lambda_n} x - dx \\ &= \frac{1}{\sqrt{\lambda_n}} (d^2 x - \lambda_n x) , \end{aligned}$$

it follows by Lemma 6 that

$$\| dy - \sqrt{\lambda_n y} \| \leqq \frac{c\varepsilon_0}{\sqrt{\lambda_n}} \| x \| \leqq \frac{c\varepsilon_0}{\sqrt{\lambda_n}} \| y \| .$$

Since $dy^* = \gamma(dy)^*$, it follows that

$$\| dy^* - \sqrt{\lambda_n} \gamma(y^*) \| = \| \gamma(dy - \sqrt{\lambda_n} y)^* \| \leqq \frac{c\varepsilon_0}{\sqrt{\lambda_n}} \| y \| .$$

Hence, since

$$\begin{aligned} d(yy^*) &= dy \cdot y^* + \gamma(y)\, dy^* \\ &= (dy - \sqrt{\lambda_n} y) y^* + \sqrt{\lambda_n} yy^* \\ &\quad + \gamma(y)(dy^* - \sqrt{\lambda_n} \gamma(y^*)) + \sqrt{\lambda_n} \gamma(y) \gamma(y^*) , \end{aligned}$$

we obtain

$$\| d(yy^*) \| \geqq \sqrt{\lambda_n} \| yy^* + \gamma(yy^*) \| - \frac{2c\varepsilon_0}{\sqrt{\lambda_n}} \| y \|^2$$

$$\geqq \sqrt{\lambda_n} \left(1 - \frac{2c\varepsilon_0}{\lambda_n} \right) \| y \|^2 .$$

Since $yy^* \in A^\alpha(-2\varepsilon_0, 2\varepsilon_0)$, it follows that

$$\| d | A^\alpha(-2\varepsilon_0, 2\varepsilon_0) \| \geqq \sqrt{\lambda_n} \left(1 - \frac{2c\varepsilon_0}{\lambda_n} \right) .$$

As $\lambda_n \to \infty$, this implies that d is unbounded on $A^\alpha(-2\varepsilon_0, 2\varepsilon_0)$, which is a contradiction.

Suppose (iii). Then by Lemma 7 it follows that L is bounded. Now we have to show that it follows then d is bounded.

Let $\varepsilon_0 > 0$ be such that $D(d) \supset A^\alpha(-3\varepsilon_0, 3\varepsilon_0)$ and let

$$M = \| d | A^\alpha(-3\varepsilon_0, 3\varepsilon_0) \| < \infty .$$

We shall show that $d | D(d^2) \cap A^\alpha(\lambda - \varepsilon_0, \lambda + \varepsilon_0)$ is bounded (by $M + \| L \|^{1/2}$) for any λ. From this the conclusion follows by the following lemma:

Lemma 8. *Let* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a finite sequence in* $\mathbb{R}$ *such that* $\bigcup_{i=1}^n (\lambda_i - \varepsilon_0, \lambda_i + \varepsilon_0) \supset \operatorname{Sp} \alpha$. *Then* $\sum_{i=0}^n D(d^2) \cap A^\alpha(\lambda_i - \varepsilon_0, \lambda_i + \varepsilon_0) = D(d^2)$.

Proof. If $x \in D(d^2)$, then for any continuous $f \in L^1(\mathbb{R})$ one has $\alpha_f(x) \in D(d^2)$ immediately. The rest is easy (see [10]).

Let $x \in D(d^2) \cap A^\alpha(\lambda - \varepsilon_0, \lambda + \varepsilon_0)$ be such that $\| x \| = 1$ and $dx \neq 0$, and let

$$y = dx / \| dx \| .$$

Then $y \in D(d)$ and

$$d(\gamma(y^*)x) = y^* dx - \gamma(dy^*)x = \frac{(dx)^*(dx)}{\| dx \|} - (dy)^* x$$

which implies that

$$\| d(\gamma(y^*)x) \| \geqq \| dx \| - \| Lx \| \, \| x \| / \| dx \| .$$

Since $y \in A^\alpha(\lambda - 2\varepsilon_0, \lambda + 2\varepsilon_0)$ and $\gamma(y^*)x \in A^\alpha(-3\varepsilon_0, 3\varepsilon_0)$, one obtains

$$\| dx \|^2 - M \| dx \| - \| L \| \leqq 0 .$$

Hence

$$\| dx \| \leqq \frac{M}{2} + \sqrt{\| L \| + \frac{M^2}{4}} \leqq M + \| L \|^{1/2} .$$

6. Proof of Theorem 4 (Inner Perturbations)

Since $d_q^2 = d^2 + \delta_\Omega$ with $\Omega = dq + q^2$ and $L = \overline{d^2}$, it is immediate that (ii) $D(d_q{}^2) = D(d^2)$ is a core for $d_q = d + \delta_q$ and that (iii) $d_q{}^2 \subset L_q = L + \delta_\Omega$. We only have to show that (i) $d_q \circ \alpha_t^q = \alpha_t^q \circ d_q$, where α^q is the one-parameter group of *-automorphisms generated by iL_q.

Define a family u_t of unitaries of A (adjoined by a unit if it is not unital) by

$$u_t = \sum_{n=0}^{\infty} i^n \int_{0 \leqq t_1 \leqq \ldots \leqq t_n \leqq t} \alpha_{t_1}(\Omega)\alpha_{t_2}(\Omega)\ldots\alpha_{t_n}(\Omega)\,dt_1\ldots dt_n .$$

for $t \in \mathbb{R}$. Then u_t is differentiable in $t \in \mathbb{R}$ and satisfies

$$u_0 = 1,$$

$$\frac{d}{dt}u_t = iu_t\alpha_t(\Omega) ,$$

$$\alpha_t^q(x) = u_t\alpha_t(x)u_t^*, \quad x \in A .$$

Lemma 9. *If* $A \ni 1$, *then* $D(d) \ni 1$ *and* $d(1) = 0$.

Proof. Note that the $\delta = Ud$ defined in the proof of Lemma 5 is a closed derivation. Hence this can be proved as for the derivations (see [2, 11]).

Supose that $q \in D(d^2)$. Then $\Omega \in D(d)$ and it easily follows that $u_t \in D(d)$. (If A is adjoined by a unit we can set $d(1) = 0$.) To prove that $d_q \circ \alpha_t^q = \alpha_t^q \circ d_q$, we have to show:

$$u_t\alpha_t(dx + \delta_q(x))u_t^* = (d + \delta_q)(u_t\alpha_t(x)u_t^*)$$

for $x \in D(d)$, which, by computation, follows from the following equality:

$$u_t\alpha_t(q)u_t^* = d(u_t)u_t^* + q, \ t \in \mathbb{R} .$$

When $t = 0$, this is correct. Now compute:

$$\begin{aligned}
\frac{d}{dt}u_t\alpha_t(q)u_t^* &= iu_t\alpha_t(\Omega)\alpha_t(q)u_t^* + iu_t\alpha_t(d^2q)u_t^* - iu_t\alpha_t(q)\alpha_t(\Omega)u_t^* \\
&= iu_t\alpha_t(\Omega q + d^2q - q\Omega)u_t^* \\
&= iu_t\alpha_t(dq\cdot q + d^2q - qdq)u_t^* \\
&= u_t\alpha_t(d\Omega)u_t^*
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}(d(u_t)u_t^*) &= id(u_t\alpha_t(\Omega))u_t^* - id(u_t)\alpha_t(\Omega)u_t^* \\
&= iu_t\alpha_t(d\Omega)u_t^* ,
\end{aligned}$$

where the first equality is easily justified by using the infinite series expansion of u_t. Hence

$$\frac{d}{dt}u_t\alpha_t(q)u_t^* = \frac{d}{dt}d(u_t)u_t^* .$$

Thus one obtains that $d_q \circ \alpha_t^q = \alpha_t^q \circ d_q$ for $q \in D(d^2)$. For a general $q \in D(d)$, since $D(d^2)$ is a core for d, we may choose a sequence $\{q_n\}$ in $D(d^2)$ such that $q_n \to q$, and $dq_n \to dq$. Then since $d_{q_n}(x) \to d_q(x)$ for $x \in D(d)$ and $\alpha_t^{q_n}(a) \to \alpha_t^q(a)$ for $a \in A$, one obtains the conclusion.

7. An Example

Let $\mathscr{H}$ be an infinite-dimensional Hilbert space and let U be a self-adjoint unitary on $\mathscr{H}$ such that both $(1+U)/2$ and $(1-U)/2$ are infinite-dimensional projections. Let Q be an *unbounded* self-adjoint operator on $\mathscr{H}$ such that $UQU = -Q$. We can define a self-adjoint super-derivation δ_Q on the graded C^*-algebra $(B(\mathscr{H}), \mathrm{Ad}\, U)$ as follows: $D(\delta_Q)$ consists of $x \in B(\mathscr{H})$ such that $xD(Q) \subset D(Q)$ and $Qx - \mathrm{Ad}\, U(x)Q$ is bounded on $D(Q)$, and $\delta_Q(x)$ is the bounded extension of $Qx - \mathrm{Ad}\, U(x)Q$ for $x \in D(\delta_Q)$.

Proposition 1. *δ_Q is a (not densely defined) closed-adjoint super-derivation on $(B(\mathscr{H}), \mathrm{Ad}\, U)$.*

Proof. Since $UQU = -Q$, it follows that $UD(Q) = D(Q)$, which implies that

$$U(Qx - \mathrm{Ad}\, U(x)Q)U = -Q\,\mathrm{Ad}\, U(x) + xQ$$

is well-defined on $D(Q)$. Hence one obtains that if $x \in D(\delta_Q)$ then $\mathrm{Ad}\, U(x) \in D(\delta_Q)$ and $\mathrm{Ad}\, U(\delta_Q(x)) = -\delta_Q(\mathrm{Ad}\, U(x))$.

Let $x \in D(\delta_Q)$ and $\xi, \eta \in D(Q)$. Since

$$\begin{aligned}(x^*\xi, Q\eta) &= (\xi, xQ\eta) = (\xi, (\mathrm{Ad}\, U(\delta_Q(x)) + Q\,\mathrm{Ad}\, U(x))\eta) \\ &= ((\mathrm{Ad}\, U(\delta_Q(x))^* + \mathrm{Ad}\, U(x^*)Q)\xi, \eta)\,,\end{aligned}$$

It follows that $x^*\xi \in D(Q)$, and

$$(Qx^* - \mathrm{Ad}\, U(x^*)Q)\xi = \mathrm{Ad}\, U(\delta_Q(x))^*\xi\,.$$

Hence it follows that $x^* \in D(\delta_Q)$ and

$$\delta_Q(x^*) = \mathrm{Ad}\, U(\delta_Q(x))^*\,.$$

The closedness of δ_Q easily follows from the closedness of Q. We omit the rest of the proof.

Let $H = Q^2$, which is a self-adjoint operator with $UHU = H$. We define a closed self-adjoint derivation δ_H in a similar way to δ_Q. (Remember that $\delta_H(x)$ is formally defined by $Hx - xH$ and satisfies that $\mathrm{Ad}\, U \circ \delta_H = \delta_H \circ \mathrm{Ad}\, U$.) Let B be the set of $x \in B(\mathscr{H})$ such that $t \to e^{itH}xe^{-itH}$ is continuous in norm. Then B is a C^*-algebra on which $\beta_t = \mathrm{Ad}\, e^{itH}$ acts as a strongly continuous one-parameter group of $*$-automorphisms of B, and δ_H is a generator of β. Note also that δ_Q commutes with $\mathrm{Ad}\, e^{itH}$.

Lemma 10. $\delta_Q{}^2 \subset \delta_H$.

Proof. Let $x \in D(\delta_Q{}^2)$ and $\xi \in D(H)$. Then it easily follows that

$$\begin{aligned}\delta_Q{}^2(x) &= (Q\delta_Q(x) - \mathrm{Ad}\, U\delta_Q(x)Q)\xi \\ &= (Q^2x - xQ^2)\xi\,,\end{aligned}$$

which concludes the proof.

Lemma 11. $D(\delta_Q) \cap D(\delta_H) = D(\delta_Q{}^2)$.

Proof. We only have to show that $D(\delta_Q) \cap D(\delta_H) \subset D(\delta_Q{}^2)$. Let $x \in D(\delta_Q) \cap D(\delta_H)$ and let $\xi \in D(Q)$. Since $D(H)$ is a core for Q, there is a sequence $\{\xi_n\}$ in $D(H)$ such that

$$\xi_n \to \xi, \quad Q\xi_n \to Q\xi .$$

Since $\delta_Q(x)\xi_n = (Qx - \operatorname{Ad} U(x)Q)\xi_n \in D(Q)$, it follows that

$$\{Q\delta_Q(x) - \operatorname{Ad} U(\delta_Q(x))Q\}\xi_n = (Q^2x - xQ^2)\xi_n$$

is well-defined, and converges to $\delta_H(x)\xi$. On the other hand $\operatorname{Ad} U(\delta_Q(x))Q\xi_n$ converges to $\operatorname{Ad} U(\delta_Q(x))Q\xi$, and hence $Q\delta_Q(x)\xi_n$ converges. Thus $\delta_Q(x)\xi \in D(Q)$ and

$$(Q\delta_Q(x)\xi - \operatorname{Ad} U(\delta_Q(x))Q)\xi = \delta_H(x)\xi .$$

Since ξ is an arbitrary vector in $D(Q)$, this implies that $\delta_Q(x) \in D(\delta_Q)$.

Proposition 2. *$D(\delta_Q{}^2)$ is a core for δ_Q.*

Proof. Since $D(\delta_Q) \cap D(\delta_H)$ is a core for δ_Q, which may be shown as Theorem 1(i), this follows from the above lemma.

Lemma 12. *$U \notin \overline{D(\delta_Q)}$ where the bar denotes the norm closure.*

Proof. We shall show that

$$\{x \in B(\mathscr{H}) \mid \|x - U\| < 1\} \cap D(\delta_Q) = \phi .$$

Let $E_+ = (1 + U)/2$ and $E_- = (1 - U)/2$. Since Q is unbounded and $UQU = -Q$, it follows that $Q^2E_+ = E_+Q^2$ is also unbounded. Hence there is a sequence $\{\xi_n\}$ in $D(Q^2) \cap E_+\mathscr{H}$ and $\{\lambda_n\}$ in $(1, \infty)$ such that $\|\xi_n\| = 1$ and

$$\|Q^2\xi_n - \lambda_n\xi_n\| \to 0, \quad \lambda_n \to \infty .$$

Let $\eta_n = Q\xi_n/\|Q\xi_n\| \in E_-\mathscr{H}$ and compute for $x \in D(\delta_Q)$,

$$\begin{aligned}(\delta_Q(x)\xi_n, \eta_n) &- \sqrt{\lambda_n}(x\xi_n, \xi_n) + \sqrt{\lambda_n}(x\eta_n, \eta_n) \\ &= (x\xi_n, (Q^2\xi_n/\|Q\xi_n\| - \sqrt{\lambda_n}\xi_n)) - (Q\xi_n - \sqrt{\lambda_n}\eta_n, x^*\eta_n) .\end{aligned}$$

Since $\|Q\xi_n\| - \sqrt{\lambda_n} \to 0$, the right side converges to zero. Hence if $\|x - U\| < 1$, then $\|E_+xE_+ - E_+\| < 1$ and $\|E_-xE_- + E_-\| < 1$ and thus

$$|(\delta_Q(x)\xi_n, \eta_n)| \geqq 2\sqrt{\lambda_n} - \sqrt{\lambda_n}(\|E_+xE_+ - E_+\| + \|E_-xE_- + E_-\|)$$

should hold as $n \to \infty$. This is a contradiction since the left side is bounded.

Let A be the closure of $D(\delta_Q)$. Then A is a β-invariant proper C^*-subalgebra of B. Let $\alpha = \beta|A$, and L the generator of α, and let $\gamma = \operatorname{Ad} U|A$. Note that $\delta_Q(D(\delta_Q)) \subset A$ since $D(\delta_Q{}^2)$ is a core for δ_Q. Thus $d = \delta_Q$ is a well-defined super-derivation on A. Now we sum up the result obtained:

Proposition 3. *The super-derivation d and the one-parameter group α of *-automorphisms defined on the graded C^*-algebra (A, γ) as above satisfies that $d \circ \alpha_t = \alpha_t \circ d$ for all $t \in \mathbb{R}$, $D(d^2)$ is a core for d, and $\overline{d^2} = L$, where L is the generator of α.*

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