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Higher Dimensional Classical W-Algebras

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Abstract: Classical W-algebras in higher dimensions are constructed. This is achieved by generalizing the classical Gel'fand–Dickey brackets to the commutative limit of the ring of classical pseudodifferential operators in arbitrary dimension. These W-algebras are the Poisson structures associated with a higher dimensional version of the Khokhlov–Zabolotskaya hierarchy (dispersionless KP-hierarchy). The two dimensional case is worked out explicitly and it is shown that the role of Diff $S(1)$ is taken by the algebra of generators of local diffeomorphisms in two dimensions.

1. Introduction

W-algebras play a prominent role in two dimensional physics. They first appeared in the context of integrable models (although under a different name) as Poisson structures associated with generalized KdV hierarchies [1–3], but their "popularity" dramatically increased after the work of Zamolodchikov. He showed in [4], using the bootstrap method, that the simplest extension of the Virasoro algebra by a field of spin 3 required the introduction of a nonlinear associative algebra, denoted since then by W_3. Soon after, Fateev and Lukyanov [5], using the formalism developed by Drinfel'd and Sokolov [6], which relates to each generalized KdV hierarchy a loop algebra, were able to generalize the results of Zamolodchikov to construct W_n-algebras, i.e. conformally extended algebras with fields of integer spins from 3 to n.

Before continuing any further, we should clarify some notational issues. In what follows, we will use the name W-algebras for the quantum algebras. These are the ones realized in a conformal field theory via operators acting on a Hilbert space. The Gel'fand–Dickey algebras and their reductions will be considered classical realizations of W-algebras. We will reserve the name classical (one-dimensional) W-algebras for nonlinear extensions of Diff $S(1)$.

Recently an unexpected connection has been unveiled between Gel'fand–Dickey algebras (and their associated integrable hierarchies), 2-D gravity, and, through their matrix model formulation, noncritical strings coupled to $c < 1$

matter [7]. In particular, it has been shown in [8] that the planar limit of these theories, in which only manifolds with the topology of a sphere are considered, is directly related to the Khokhlov–Zabolotskaya (KZ) hierarchy [9, 10] and its reductions. This hierarchy is also known as the dispersionless or classical KP-hierarchy, where the deformation parameter that takes us from the dispersive to the nondispersive case happens to be, in the context of noncritical strings, nothing but the renormalized string coupling constant. Moreover, the Poisson structure associated with the KZ-hierarchy (w_{KP}) was shown to be the universal W-algebra associated with the w_n-series, i.e. the classical limit of the W_n-algebras. We believe that this makes classical W-algebras a very interesting and fruitful field of study.

Nevertheless, as interesting as all this may be from the physical point of view, for example, the theory of 2-D surfaces, domain walls in 3-D, or critical phenomena in 2-D systems, from the point of view of a particle physicist, all these developments are little more than toy-models for the relevant higher dimensional case. It is the aim of this work to generalize some of these structures to higher dimensions. The hope is that they will be helpful in understanding topics such as nonperturbative gravity in $D > 2$ and even noncritical strings coupled to $c > 1$ matter.

We should also point out that the generalization of the Gel'fand–Dickey formalism to higher dimension comes as a little surprise. It has been repeatedly claimed in the literature that an essential ingredient in the one dimensional case was the existence of an invariant splitting compatible with the Adler trace, a property that disappears when one moves to higher dimensions. We will show, at least in the classical limit, that we can proceed without it.

The plan of the paper is as follows:

In Sect. 2, the required formalism to extend the usual one dimensional results for classical W-algebras to arbitrary dimension will be developed. This generalization will be shown to be quite straightforward, although it will require the introduction of some nonstandard machinery such as Guillemin's symplectic trace [11, 12], which we will discuss briefly.

In Sect. 3, we will revisit the one dimensional case. This is important because the generalization to higher dimensions will require the introduction of a splitting in the space of pseudodifferential symbols which does not reduce to the standard in one dimension. Nevertheless, we will prove that the induced algebras are nothing but the standard w_∞ and the $n \to \infty$ limit of the w_n series.[1]

In Sect. 4, we will explicitly display the algebras associated with the space of classical pseudodifferential symbols in two dimensions. We will show that the role of Diff$S(1)$ is taken by the algebra of local diffeomorphisms in two dimensions. We expect this connection with diffeomorphism algebras to extend to arbitrary dimension although we have not yet proved it so.

In Sect. 5, we will show how these Poisson structures are associated with a very natural generalization of the KZ-hierarchy in higher dimensions. We will also briefly discuss their integrability.

[1] These two algebras should not be confused. This is nothing but another example of the poor notational conventions that plague this field. In fact, w_∞ is nothing but a particular contraction of $w_{n \to \infty}$

Finally, in the conclusion we will recapitulate our results and also comment about deformations of these structures. We will explain why the standard deformation which takes w_n to W_n in one dimension is not directly applicable to the higher dimensional case.

2. General Formalism

The higher dimensional W-algebras are going to appear as Poisson structures in the space of pseudodifferential operators (ΨDO) of the classical type, so before going any further we will introduce the necessary concepts to deal with such objects [13, 12].

A ΨDO $\mathscr{B}$ is said to be of the classical type if $\mathscr{B}$: $C^\infty_{\mathrm{comp}}(X) \to C^\infty(X)$ can be expressed locally as

$$\mathscr{B}f(x) = \int_{\mathbf{R}^D_\xi} \int_{\mathbf{R}^D_y} e^{i(x-y)\xi} \Lambda(x, \xi) f(y) \frac{d^D \xi}{(2\pi)^D} d^D y + \mathscr{T} f(x) , \tag{2.1}$$

where D is the dimension of X and $\mathscr{T}$ is an operator with a smooth kernel[2], and $\Lambda(x, \xi) \in C^\infty(T^*X \backslash 0)$ (i.e. smooth functions on the cotangent bundle without the zero section) admits an asymptotic expansion for $|\xi| \to \infty$ of the form

$$\Lambda = \sum_{j=0}^{\infty} \tilde{u}_{m-j} , \tag{2.2}$$

with

$$\tilde{u}_{m-j}(x, t\xi) = t^{m-j} \tilde{u}_{m-j}(x, \xi) \quad t > 0 \tag{2.3}$$

which implies that

$$\tilde{u}_k(x, \xi) = u_k(x, \theta) |\xi|^k , \tag{2.4}$$

where θ stands for the angular variables in $\mathbf{R}^D$.

From now on when we refer to a ΨDO or its (smoothed) symbol, the reader should assume that it is of the classical type unless stated otherwise.

We will further restrict ourselves to ΨDO's with u_m equal to a constant, which we will take equal to one in order to simplify the notation, and $m \in \mathbf{Z}$. Then, M^m_D will denote the space of formal series of the form (2.2) with the above constraint.

We can equip the space of ΨDO's with a Lie algebra structure by defining the Lie bracket for any two symbols A and B by

$$[A, B] = A \circ B - B \circ A , \tag{2.5}$$

where $\circ$ denotes the usual composition of symbols given by

$$A \circ B = \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_\xi A \partial^\alpha_x B , \tag{2.6}$$

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[2] This roughly means that Λ is uniquely defined up to terms that decay faster than any power of $|\xi|$ when $|\xi| \to \infty$

where α is a multi-index, and a "nasty" factor of $-i$ has been absorbed in the definition of ∂_x.

Although, it would be very interesting to keep the full noncommutative structure of this space (see comments about deformations in the conclusions), in this paper we will restrict ourselves to the commutative limit, which we define below.

The Classical Limit of the Ring of Pseudodifferential Symbols. Let R^D denote the ring of pseudodifferential symbols in dimension D, where multiplication is defined by the composition of symbols. It is possible to define a degree in R^D as follows. Let the degree of a monomial be given by its degree of homogeneity,

$$\deg \tilde{u}(x, \xi) = j \Leftrightarrow \tilde{u}(x, t\xi) = t^j \tilde{u}(x, \xi) \tag{2.7}$$

for any $t > 0$. We can now define the degree of a polynomial as the degree of the leading term. Let us denote by R_p^D the subspace of all symbols with degree equal to or smaller than p. It is clear from (2.6) that

$$R_p^D \circ R_q^D = R_{p+q}^D , \tag{2.8}$$

making R^D into a filtered ring. Starting from any filtered ring we can define its associated graded ring as follows. Let

$$\mathrm{Gr}_p R^D \equiv R_p^D / R_{p-1}^D . \tag{2.9}$$

The multiplication in R^D induces a multiplication in $\mathrm{Gr}R^D \equiv \bigoplus_p \mathrm{Gr}_p R^D$. The reader can easily check that the composition

$$R_p^D \times R_q^D \xrightarrow{\circ} R_{p+q}^D \xrightarrow{\mathrm{gr}_{p+q}} \mathrm{Gr}_{p+q} R^D \tag{2.10}$$

induces a unique map

$$\mathrm{Gr}_p R^D \times \mathrm{Gr}_q R^D \to \mathrm{Gr}_{p+q} R^D , \tag{2.11}$$

converting $\mathrm{Gr}R^D$ into a graded ring. Moreover, this induced multiplication is nothing but the usual commutative multiplication of symbols as formal series. This would not be very interesting as it stands were if not for the fact that $\mathrm{Gr}R^D$ can be given a natural Poisson algebra structure.

The commutator $[A, B]$ in R^D induces a Lie bracket on $\mathrm{Gr}R^D$ as follows. Let $A \in R_p^D$ and $B \in R_q^D$. It is then easy to see that $[A, B] \in R_{p+q-1}^D$. The part in R_{p+q}^D has to vanish since it corresponds to the commutator in $\mathrm{Gr}R^D$, which is a commutative algebra. Therefore $[A, B]$ defines an element in $\mathrm{Gr}_{p+q-1}R^D$. Furthermore, this element only depends on the class of A modulo R_{p-1}^D, for if $A \in R_{p-1}^D$, then $[A, B] \in R_{p+q-2}^D$. Similarly, it only depends on the class of B modulo R_{q-1}^D. Therefore the composition

$$R_p^D \times R_q^D \xrightarrow{[\,,\,]} R_{p+q-1}^D \xrightarrow{\mathrm{gr}_{p+q-1}} \mathrm{Gr}_{p+q-1} R^D \tag{2.12}$$

induces a unique map

$$\{\,,\}: \mathrm{Gr}_p R^D \times \mathrm{Gr}_q R^D \to \mathrm{Gr}_{p+q-1} R^D . \tag{2.13}$$

The fact that $[\,,]$ is a Lie backet for R^D means that $\{\,,\}$ is a Lie bracket for $\mathrm{Gr}\,R^D$, but it has more structure. It turns out that it is a derivation over the commutative multiplication on $\mathrm{Gr}\,R^D$. In fact, let $A \in R^D_p$, $B \in R^D_q$, and $C \in R^D_s$. Then

$$\begin{aligned}
&\{\mathrm{gr}_p(A), \mathrm{gr}_q(B)\mathrm{gr}_s(C)\} = \{\mathrm{gr}_p(A), \mathrm{gr}_{q+s}(BC)\} \\
&= \mathrm{gr}_{p+q+s-1}[A, BC] \\
&= \mathrm{gr}_{p+q+s-1}([A, B]C) + \mathrm{gr}_{p+q+s-1}(B[A, C]) \\
&= \mathrm{gr}_{p+q-1}[A, B]\mathrm{gr}_s(C) + \mathrm{gr}_q(B)\mathrm{gr}_{p+s-1}[A, C] \\
&= \{\mathrm{gr}_p(A), \mathrm{gr}_q(B)\}\mathrm{gr}_s(C) + \{\mathrm{gr}_p(A), \mathrm{gr}_s(C)\}\mathrm{gr}_q(B)\,. \qquad (2.14)
\end{aligned}$$

In summary, this turns $\mathrm{Gr}\,R^D$ into a Poisson algebra. The reader will recognize that the Poisson bracket so defined is nothing but the canonical Poisson bracket on a $2n$-dimensional phase space with canonical coordinates (x^i, ξ_i).

We can also define this classical limit in a more "physical" way as follows. Let us introduce the formal parameter $\hbar$ in (2.6),

$$A \circ_\hbar B \equiv \sum_\alpha \frac{\hbar^{|\alpha|}}{\alpha!} \frac{\partial^\alpha A}{\partial \xi_\alpha} \frac{\partial^\alpha B}{\partial x^\alpha}\,, \qquad (2.15)$$

interpolating from the commutative multiplication for $\hbar = 0$ to the noncommutative composition of symbols for $\hbar = 1$. For $\hbar$ different from zero we can reabsorb it by rescaling ξ. This implies that $\circ_\hbar$ remains associative for all values of $\hbar$.

The classical limit is given by the leading term in the $\hbar \to 0$ limit. This implies that composition of symbols goes to standard commutative multiplication. However, notice that for the bracket, the leading term is already of order $\hbar$, therefore

$$\{A, B\} \equiv \lim_{\hbar \to 0} \frac{1}{\hbar}[A, B]_\hbar = \sum_{j=1}^{D} \left(\frac{\partial A}{\partial \xi_j} \frac{\partial B}{\partial x^j} - \frac{\partial B}{\partial \xi_j} \frac{\partial A}{\partial x^j} \right), \qquad (2.16)$$

as we found before.

Formal Geometry. The space M^m_D can be given the structure of an infinite dimensional manifold, but we will not need this machinery for our purposes. It will be sufficient to endow M^m_D with a formal geometry or algebraization of the strictly necessary geometric concepts. Our main goal is to define Poisson brackets on M^m_D. The geometrical objects we should define are: the class of functions on which we define the Poisson brackets, the vector fields, and 1-forms together with the map sending a function to its associated hamiltonian vector field.

We will define Poisson brackets on functions of the form:

$$F[A] = \int f(u)\,, \qquad (2.17)$$

where $f(u)$ is a polynomial of the u_j's and their derivatives. The precise meaning of the integration will depend on the particular context, i.e. what kind of functions the u's are or in which particular space they live. In what follows, we will only use the fact that $\int$ is a map which annihilates exact forms, i.e. there are no contributions from boundary terms.

The tangent space $T_\Lambda M^m_D$ at Λ is isomorphic to the infinitesimal deformations of Λ. These are clearly pseudodifferential symbols belonging to R^D_{m-1}. If $A \in R^D_{m-1}$ is of the form $A = \sum_{j \leqq m-1} a_j(x, \theta)\,|\xi|^j$, then the vector field D_A acting on a function F is defined by

$$D_A F = \frac{d}{d\varepsilon} \int f(u_j + \varepsilon a_j)\bigg|_{\varepsilon=0} . \tag{2.18}$$

The cotangent space $T^*_\Lambda M^m_D$ will be defined as the dual of $T_\Lambda M^m_D$ with respect to a nondegenerate inner product. The required inner product is supplied by Guillemin's symplectic trace [11, 12].

Symplectic Trace. Here we will adapt the general discussion of [12] to our somewhat restricted interest.

Let Y be a symplectic manifold of dimension 2D and ω the corresponding nondegenerate 2-form. Then the Poisson bracket for any two functions in Y is given by

$$\{f, g\} = \mathscr{L}_{H_f} g = \omega(H_f, H_g) = i(H_g)i(H_f)\omega , \tag{2.19}$$

where

$$\omega(H_f, \cdot) = -df . \tag{2.20}$$

In our particular case $Y \equiv X \times \mathbf{R}^D$ and ω is given locally by

$$\omega = \sum_{j=1}^{D} dx^j \wedge d\xi_j . \tag{2.21}$$

As we have already seen, there is a natural action of the multiplicative group of the real numbers on Y given by

$$\chi_t\colon \xi \mapsto t\xi , \tag{2.22}$$

where $t \in \mathbf{R}^\times_+$. This action is conformal, that is

$$\chi_t^* \omega = t\omega . \tag{2.23}$$

The infinitesimal action defines the Euler vector field

$$\sigma = \sum_{j=1}^{D} \xi_j \frac{\partial}{\partial \xi_j} . \tag{2.24}$$

Now we can define

$$\alpha = i(\sigma)\omega \quad \text{and} \quad \mu = \alpha \wedge (d\alpha)^{D-1} . \tag{2.25}$$

Notice that $d\alpha = di(\sigma)\omega = di(\sigma)\omega + i(\sigma)d\omega = \mathscr{L}_\sigma\omega = \omega$, where the last equality is obvious because σ is the vector field generating the flow induced by $\chi_{\exp t}$. This implies that α is nothing but the canonical 1-form of classical mechanics.

It is now clear that

$$\chi_t^* \alpha = t\alpha \quad \text{and} \quad \chi_t^* \mu = t^D \mu , \tag{2.26}$$

therefore

$$\mathscr{L}_\sigma \alpha = \alpha, \quad d\mu = \omega^D, \quad \mathscr{L}_\sigma \mu = D\mu . \tag{2.27}$$

We will now proceed to prove a couple of technical lemmas which will be required in what follows.

Lemma 2.28.

$$\{f, g\}\omega^D = Ddf \wedge dg \wedge \omega^{D-1} .$$

Proof.

$$\begin{aligned} 0 = i(H_f)i(H_g)\omega^{D+1} &= -(D+1)i(H_f)(dg \wedge \omega^D) \\ &= -(D+1)(i(H_f)dg \wedge \omega^D - dg \wedge i(H_f)\omega^D) \\ &= -(D+1)(dg(H_f)\omega^D + Ddg \wedge df \wedge \omega^{D-1}) \\ &= (D+1)(Ddf \wedge dg \wedge \omega^{D-1} - \{f, g\}\omega^D). \end{aligned}$$ ■

Lemma 2.29.

$$\{f, g\}\mu = d(gi(H_f)\mu) - (D-1)gdf \wedge \omega^{D-1} - \mathscr{L}_\sigma(gdf) \wedge \omega^{D-1} .$$

Proof.

$$\{f, g\}\mu = \frac{1}{D} i(\sigma)\{f, g)\omega^D .$$

Using the previous lemma this can be written as

$$\begin{aligned} \{f, g\}\mu &= i(\sigma)(df \wedge dg \wedge \omega^{D-1}) \\ &= \frac{1}{D} i(\sigma)(dg \wedge i(H_f)\omega^D) \\ &= \frac{1}{D} i(\sigma)d(gi(H_f)\omega^D) , \end{aligned}$$

where we have used $\mathscr{L}_{H_f}\omega = 0$ in the last line. From this we have,

$$\begin{aligned} \{f, g\}\,\mu &= \frac{1}{D}\mathscr{L}_\sigma(gi(H_f)\omega^D) - \frac{1}{D}d(gi(\sigma)i(H_f)\omega^D) \\ &= -\mathscr{L}_\sigma(gdf \wedge \omega^{D-1}) + \frac{1}{D}d(gi(H_f)i(\sigma)\omega^D) \\ &= -\mathscr{L}_\sigma(gdf) \wedge \omega^{D-1} - (D-1)gdf \wedge \omega^{D-1} + d(gi(H_f)\mu) . \end{aligned}$$ ■

From now on we will only deal with homogeneous functions on Y. If $f \in \mathrm{Gr}_p R^D$, then

$$\mathscr{L}_\sigma f = pf . \tag{2.30}$$

The following lemma will prove to be fundamental.

Lemma 2.31. *If $f \in \mathrm{Gr}_p R^D$ and $g \in \mathrm{Gr}_q R^D$,*

$$\{f, g\}\mu = d(gi(H_f)\mu) - (p+q+D-1)gdf \wedge \omega^{D-1} .$$

Proof. It follows immediately from the two previous lemmas and $\mathscr{L}_\sigma(gdf) = (p+q)gdf$. ■

Now we can define the symplectic residue by

$$\operatorname{Res} f = \begin{cases} f\mu & \text{if } f \in \mathrm{Gr}_{-D} R^D \\ 0 & \text{otherwise}. \end{cases} \tag{2.32}$$

In order to define the symplectic trace notice that Y is a principal $\mathbf{R}_+^\times$-bundle with base Z, i.e. $Y/\mathbf{R}_+^\times = Z$. The natural projection $Y \to Z$ will be denoted by ρ. Notice that $f\mu$ is a $\mathbf{R}_+^\times$ invariant form (because $f \in \mathrm{Gr}_{-D} R^D \Rightarrow \mathscr{L}_\sigma(f\mu) = 0$) and since it is always horizontal (i.e. annihilated by $i(\sigma)$), a unique form $f\mu$ on Z must exist such that

$$\rho^* f\mu = f\mu .$$

The symplectic Trace is then defined to be

$$\operatorname{Tr} f = \int_Z \operatorname{Res} f . \tag{2.33}$$

This definition is obviously independent on the section.

Now we can prove that the symplectic trace defined above has the usual trace property.

Theorem 2.34. *If $f \in \mathrm{Gr}_p R^D$ and $g \in \mathrm{Gr}_q R^D$, then $\operatorname{Tr}\{f, g\} = 0$ for all p and q.*

Proof.

$$\operatorname{Tr}\{f, g\} = \int_Z \{f, g\} \mu$$

if $\{f, g\} \in \mathrm{Gr}_{-D} R^D$. By Lemma 2.31

$$\operatorname{Tr}\{f, g\} = \int_Z d(gi(H_f)\mu) - (p + q - 1 + D) g df \wedge \omega^{D-1} = 0 .$$

The first term is zero because by assumption $\int$ annihilates exact forms. The second term also vanish by noticing that $\{f, g\} \in \mathrm{Gr}_{p+q-1} R^D$. ■

Generalized Adler Map. Now we have all the required ingredients to define Poisson brackets on M_D^m. Remember that one of the crucial properties in the standard one-dimensional classical case was that there were two closed subspaces on R^1 under Poisson brackets [14]. We will show that this is also true in arbitrary dimensions if we modify the old definitions slightly.

Define $R_{[q,p]}^D \equiv \bigoplus_{j=q}^p \mathrm{Gr}_j R^D$. It should be clear from our previous discussion that $R_+^D \equiv R_{[1,\infty)}^D$ and $R_-^D \equiv R_{(-\infty,0]}^D$ are closed under Poisson brackets. Notice that in one dimension this splitting differs from the standard one. It also will be convenient to define $R_\oplus^D \equiv R_{[-D,\infty)}^D$ and $R_\ominus^D \equiv R_{(-\infty,-D-1]}^D$. For any ΨDO symbol A, the symbols A_+, A_-, $A_\oplus$ and $A_\ominus$ will denote the projections of A on R_+^D, R_-^D, $R_\oplus^D$ and $R_\ominus^D$ respectively.

These $\oplus$ and $\ominus$ splittings will appear naturally because, in contrast with the usual one, ours is not compatible with the symplectic trace, i.e. $\operatorname{Tr} A_- B_-$ is in general different from zero. In particular $R_\ominus^D$ is the dual of R_+^D with respect the symplectic trace, and $R_\oplus^D$ is the dual of R_-^D.

Now we can define our 1-forms as parametrized by the dual space of R_{m-1}^D under the symplectic trace (recall that these symbols parametrized the vector fields on M_D^m). Therefore, 1-forms are parametrized by elements in $R^D_{[-D-m+1,\infty)}$. If $\Lambda \in R^D_{m-1}$ and $X \in R^D_{(-D-m+1,\infty)}$, the action of the 1-form X on ∂_Λ is given by

$$X(\partial_\Lambda) = \operatorname{Tr} \Lambda X \; . \tag{2.35}$$

This let us define the gradient of a function by

$$dF(\partial_\Lambda) = \partial_\Lambda F \; . \tag{2.36}$$

In analogy with the finite dimensional case, we should provide a map from one forms to vector fields in order to define the Poisson brackets. The required map is given by a suitable generalization of the standard Adler map [2], which reads in "components"

$$J(X) = \{\Lambda, X\}_\oplus \Lambda - \{\Lambda, (\Lambda X)_+\} \; . \tag{2.37}$$

First notice that $J(X) \in R^D_{m-1}$, so it parametrizes a vector field in $T_\Lambda M_D^m$. This is easily shown if we write (2.37) as

$$J(X) = -\{\Lambda, X\}_\ominus \Lambda + \{\Lambda, (\Lambda X)_-\} \; . \tag{2.38}$$

Also notice that because of the different splitting there is not a natural reduction to $\Lambda_- = 0$, in contrast to the standard case.[3]

Let Ω denote the map $X \mapsto \partial_{J(X)}$ from 1-forms to vectors fields induced by (2.37). In analogy with the finite dimensional case, it is convenient to introduce the symplectic form ω defined, on $\operatorname{Im} \Omega$, by

$$\omega(\Omega(X), \Omega(Y)) = \operatorname{Tr} J(X) Y \; . \tag{2.39}$$

Notice that, in contrast with the usual case in classical mechanics, this 2-form is not defined for all vector fields since, in general, the map J will not be an isomorphism. It follows from the definition of ω that the Poisson brackets will be given by

$$\{F, G\}_{GD} = \omega(\Omega(dF), \Omega(dG)) = \operatorname{Tr} J(dF) dG \; , \tag{2.40}$$

where we have introduced the suffix GD (for Gel'fand and Dickey) in order to avoid confusion with the canonical Poisson brackets in a finite-dimensional phase space used for the definition of the generalized Adler map.

It is now simple to check that this bracket is indeed antisymmetric. Explicitly,

$$\begin{aligned} \{F, G\}_{GD} &= \operatorname{Tr} J(dF) dG \\ &= \operatorname{Tr}(\{\Lambda, (dF\}_\oplus \Lambda dG - \{\Lambda, (\Lambda dF)_+\} dG) \\ &= \operatorname{Tr}(-\{\Lambda, (\Lambda dG)_-\} dF + \{\Lambda, dG\}_\ominus dF) \\ &= -\operatorname{Tr} J(dG) dF = -\{G, F\}_{GD} \; , \end{aligned} \tag{2.41}$$

[3] Of course, this does not necessarily imply that there is no possible reduction where only a finite subset of the u_i's are different from zero. It only means that if such reduction exists it will be more involved than in the usual case

where we have used $\operatorname{Tr} A_+ B_\oplus = \operatorname{Tr} A_- B_\ominus = 0$ for all A and B.

By analogy with the finite-dimensional case, we define $d\omega$ by

$$d\omega(\partial_{J(X)}, \partial_{J(Y)}, \partial_{J(Z)}) = \partial_{J(X)}\omega(\partial_{J(Y)}, \partial_{J(Z)}) \\ - \omega([\partial_{J(X)}, \partial_{J(Y)}], \partial_{J(Z)}) + \text{c.p.} \tag{2.42}$$

where c.p. is shorthand for cyclic permutations. But notice that the last term in (2.42) is not well defined unless $\operatorname{Im}\Omega$ forms a subalgebra of the vector fields. In fact, the proof of the following lemma is routine.

Lemma 2.43. *For any X and $Y \in R^{D}_{[-D-m+1,\infty)}$,*

$$[\partial_{J(X)}, \partial_{J(Y)}] = \partial_{J([\![X,Y]\!])} ,$$

where

$$[\![X, Y]\!] = \partial_{J(X)} Y - \partial_{J(Y)} X \\ + \{X, \Lambda\}_\ominus Y + \{(\Lambda X)_-, Y\} + \{Y, \Lambda\}_\oplus X + \{(\Lambda Y)_+, X\}$$

modulo the kernel of J.

It is easy to show [3] that closedness of ω, i.e. $d\omega = 0$, is equivalent to Jacobi identities for the bracket defined by (2.40). Now we can state the main result of this paper.

Theorem 2.44. *For any three vector fields $\partial_{J(X)}$, $\partial_{J(Y)}$, and $\partial_{J(Z)}$, in* $\operatorname{Im}\Omega$

$$d\omega(\partial_{J(X)}, \partial_{J(Y)}, \partial_{J(Z)}) = 0 ,$$

i.e. *ω is a closed 2-form.*

The proof of this theorem is given by a long, straightforward and explicit computation of $d\omega$ which we omit.

Bihamiltonian Structure. We can obtain another Poisson structure by deforming the previous one by $\Lambda \to \Lambda + \lambda$, for λ some constant parameter. Then

$$J(X) = J^2(X) + \lambda J^1(X) , \tag{2.45}$$

where J^2 is given by (2.37) and J^1 is given by

$$J^1(X) = -\{\Lambda, X\}_\ominus + \{\Lambda, X_-\} . \tag{2.46}$$

As usual the two kinds of Poisson brackets so obtained are coordinated. For "perverse" historical reasons the structure obtained by the deformation is called the first hamiltonian structure while the one given by (2.37) is called the second.

We will finish this section by giving a convenient prescription for computing the fundamental Poisson brackets among the $u_j(x, \theta)$. Although the "coordinates" u_j are not functions according to the definition we are using, we can still make sense of their Poisson bracket.

First notice that both structures are linear in X. This implies that $J(X)$ is necessarily of the form

$$J^{1,2}(X) = \sum_{i,j \geq 1} (J^{1,2}_{ij} \cdot X_j) |\xi|^{m-i} , \tag{2.47}$$

where $X = \sum_{j \geqq 1} X_j(x, \theta) |\xi|^{j-m-D}$, and the J_{ij}'s are certain differential operators acting on the X_j's.

We now choose two linear functionals of the form

$$l_A = \operatorname{Tr} A\Lambda \quad \text{and} \quad l_B = \operatorname{Tr} B\Lambda \,, \tag{2.48}$$

with $A = a(x, \theta) |\xi|^{i-m-D}$ and $B = b(x, \theta) |\xi|^{j-m-D}$. Their gradients are given by

$$dl_A = a |\xi|^{i-m-D} \quad \text{and} \quad dl_B = b |\xi|^{j-m-D} \,, \tag{2.49}$$

which implies

$$\{l_A, l_B\}^{1,2}_{\mathrm{GD}} = \int (J^{1,2}_{ij} \cdot a) b \,. \tag{2.50}$$

It is obvious that we would have obtained the same result if we had declared our fundamental Poisson brackets among the u's to be

$$\{u_i(x, \theta), u_j(x', \theta')\}^{1,2} = -J^{1,2}_{ij} \cdot \delta^D(x - x') \delta(\theta - \theta') \,, \tag{2.51}$$

where $\delta(\theta - \theta')$ is the "delta-function" associated with the standard measure in S^{D-1}.

3. $D = 1$ Revisited

In this section we will obtain the Poisson algebras induced by (2.37) in one dimension and associated with the Lax operator

$$\Lambda = \xi + \sum_{j \geqq 0} u_j \xi^{-j} \,. \tag{3.1}$$

Notice that they will differ from the standard w_{KP} [8] because of the different splitting. But, interestingly enough, the new algebra, denoted by w_{KZ}, is nothing but the limit when $n \to \infty$ of the standard w_n after making the reduction of setting $u_0 = 0$.

A straightforward computation yields

$$\begin{aligned} J_{00} &= \partial \,, \\ J_{j0} &= u_{j-1} \partial \,, \\ J_{0j} &= \partial u_{j-1} \,. \end{aligned} \tag{3.2}$$

And if $i, j \geqq 0$, then

$$\begin{aligned} J_{ij} = {} & i u_{i+j-1} \partial + j \partial u_{i+j-1} + (1 - j) u_{i-1} \partial u_{j-1} \\ & + \sum_{k=0}^{j-2} ((i - k - 1) u_{i+j-k-2} \partial u_k + (j - k - 1) u_k \partial u_{i+j-k-2}) \,. \end{aligned} \tag{3.3}$$

Imposing now the constraint $u_0 = 0$, the associated Dirac brackets give

$$\begin{aligned} J^{(0)}_{ij} = {} & i u_{i+j-1} \partial + j \partial u_{i+j-1} - j u_{i-1} \partial u_{j-1} \\ & + \sum_{k=1}^{j-2} ((i - k - 1) u_{i+j-k-2} \partial u_k + (j - k - 1) u_k \partial u_{i+j-k-2}) \,. \end{aligned} \tag{3.4}$$

The explicit form of the classical Gel'fand–Dickey algebras with the standard splitting is given in [14]. It reads

$$\begin{aligned} J_{n-1,n-1} &= -n\partial\,, \\ J_{i,n-1} &= -(i+1)u_{i+1}\partial\,, \\ J_{n-1,j} &= -(j+1)\partial u_{j+1}\,, \\ J_{ij} &= (n-j-1)\partial u_{i+2+j-n} + (n-i-1)u_{i+2+j-n}\partial \\ &\quad + \sum_{l=j+2}^{n-1} [(l-i-1)u_{i+j+2-l}\partial u_l + (l-j-1)u_l\partial u_{i+j+2-l}] \\ &\quad - (i+1)u_{i+1}\partial u_{j+1}\,, \end{aligned} \tag{3.5}$$

where $i, j = 0, 1, \ldots, n-2$ and with the proviso that $u_{l<0} = 0$ in the above formulas.

If we now do the following field redefinition $u_{n-j-1} \mapsto u_j$. We obtain

$$\begin{aligned} J_{00} &= -n\partial\,, \\ J_{j0} &= (j-n)u_{j-1}\partial\,, \\ J_{0j} &= (j-n)\partial u_{j-1}\,. \end{aligned} \tag{3.6}$$

And if $i, j \geqq 0$, then

$$\begin{aligned} J_{ij} &= iu_{i+j-1}\partial + j\partial u_{i+j-1} + (n-i)u_{i-1}\partial u_{j-1} \\ &\quad + \sum_{k=0}^{j-2} ((i-k-1)u_{i+j-k-2}\partial u_k + (j-k-1)u_k\partial u_{i+j-k-2})\,. \end{aligned} \tag{3.7}$$

If we again impose the constraint $u_0 = 0$, we have

$$J_{ij}^{(0)} = J_{ij} + \frac{(j-n)(i-n)}{n} u_{i-1}\partial u_{j-1}\,. \tag{3.8}$$

And now we can take the limit $n \to \infty$ in the expression above and recover (3.4). Therefore

$$w_{\mathrm{KZ}} \simeq w_{n\to\infty}\,. \tag{3.9}$$

Let us now focus our attention on the first structure. After the natural reduction of setting $u_0 = u_1 = 0$, the Adler map simply becomes

$$J(X) = -\{\Lambda, X_+\}_\ominus\,. \tag{3.10}$$

From this we obtain

$$J_{ij} = (i-1)u_{i+j-2}\partial + (j-1)\partial u_{i+j-2}\,, \tag{3.11}$$

and this is nothing but the w_∞ algebra, as the reader can check in [15].

All of this seems to indicate that the new splitting is quite "natural" in one dimension.

4. $D = 2$

Now we are going to use the tools developed in Sect. 2 in order to compute an explicit example of the new higher dimensional classical W-algebras. We will limit ourselves to the two-dimensional case because, as our reader will see in what follows, explicit expressions soon become very cumbersome.

Let us first compute the Poisson brackets associated with the second hamiltonian structure in M_D^m. The fundamental Poisson brackets between the u's, after imposing the constraint $u_1(x,\theta) = u_2(x,\theta) = 0$, are given by

$$\{u_i^t(z,\bar z),\, u_j^p(z',\bar z')\} = -J_{ij}^{tp}\cdot\delta(z-z')\delta(\bar z-\bar z'), \tag{4.1}$$

where we have introduced complex coordinates $z = \frac{1}{2}(x^2 + ix^1)$ and its complex conjugate $\bar z$, and where

$$u_j(x,\theta) = \sum_{p\in Z} u_j^p e^{ip\theta},$$

with

$$\begin{aligned} J_{ij}^{tp} = {} & (j+p-2)\partial_{\bar z} u_{i+j-3}^{t+p-1} + (i+t-2)u_{i+j-3}^{t+p-1}\partial_{\bar z} \\ & + (j-p-2)\partial_z u_{i+j-3}^{t+p+1} + (i-t-2)u_{i+j-3}^{t+p+1}\partial_z \\ & - \sum_{s\in Z}\sum_{k\geqq j}\big((i+t-k-s-2)u_{i+j-k-3}^{t+p-s-1}\partial_{\bar z}u_k^s \\ & + (j+p-k-s-2)u_k^s\partial_{\bar z}u_{i+j-k-3}^{t+p-s-1} \\ & \times(i-t-k+s-2)u_{i+j-k-3}^{t+p-s+1}\partial_z u_k^s \\ & + (j-p-k+s-2)u_k^s\partial_z u_{i+j-k-3}^{t+p-s+1}\big) \\ & + \sum_{s\in Z}\sum_{j-2\leqq k<j}\big((2m+k+s-t-p-i-j-4)u_{i+j-k-3}^{t+p-s-1}\partial_{\bar z}u_k^s \\ & + (2m+k-s+t+p-i-j-4)u_{i+j-k-3}^{t+p-s+1}\partial_z u_k^s \\ & + (k+s-p-j+2)u_k^s(\partial_{\bar z}u_{i+j-k-3}^{t+p-s-1}) \\ & + (k-s+p-j+2)u_k^s(\partial_z u_{i+j-k-3}^{t+p-s+1})\big), \end{aligned} \tag{4.2}$$

and the proviso that $u_{j<3}^t = 0$ for all $t\in Z$. We have chosen to complexify the algebra, i.e. consider complex u's, because expressions become more transparent. Of course a real section can be taken if desired.

Now it is simple to check that there is a finite-dimensional subalgebra generated by u_3^1 and u_3^{-1}. If we use the suggestive notation $u_3^1 \equiv 2P_{\bar z}$ and $u_3^{-1} \equiv 2P_z$, we obtain

$$\{P_{\bar z}(z,\bar z),\, P_{\bar z}(z',\bar z')\} = -(\partial_{\bar z}P_{\bar z}(z,\bar z) + P_{\bar z}(z,\bar z)\partial_{\bar z})\cdot\delta(z-z')\delta(\bar z-\bar z'),$$

$$\{P_z(z,\bar z),\, P_z(z',\bar z')\} = -(\partial_z P_z(z,\bar z) + P_z(z,\bar z)\partial_z)\cdot\delta(z-z')\delta(\bar z-\bar z'),$$

$$\{P_{\bar z}(z,\bar z),\, P_z(z',\bar z')\} = -(\partial_z P_{\bar z}(z,\bar z) + P_z(z,\bar z)\partial_{\bar z})\cdot\delta(z-z')\delta(\bar z-\bar z').$$

This is nothing but the algebra of loal diffeomorphisms of a two dimensional manifold in complex coordinates.

By inspecting expression (4.2), the reader can check that the u_j^t fall in two different representations of the diffeomorphism algebra. The sets composed by $u_j^{j-2}, u_j^{j-4}, \ldots, u_j^{-j+2}$ form finite dimensional-representation, which correspond to $(j-2)$-symmetric covariant tensor 1-densities. The others fall in two infinite-dimensional representations, corresponding to an odd or even upper index. It would be very interesting to find a reduction which will leave us only with the finite-dimensional representations, but for the time being we have not been able to find it.

In what concerns the first structure, we will only say that after the natural reduction of setting all the $u_j(x, \theta)$ with $j \leqq m + 2$ to zero, the algebra obtained is isomorphic to the linear part of (4.2). In this case, it is worth noticing that there is a subalgebra spanned by the finite-dimensional representations of the diffeomorphism algebra.[4]

5. Associated Integrable Hierarchies

In this section we will show that these new Poisson structures give a bi-hamiltonian formulation of higher-dimensional KZ-hierarchies.

Let us first recall the standard one-dimensional formulation of these hierarchies. The KP-hierarchy [16] can be defined as the Lax-type evolution equations given by

$$\frac{\partial \Lambda}{\partial t_n} = [\Lambda_+^n, \Lambda] = [\Lambda, \Lambda_-^n] , \tag{5.1}$$

where $\Lambda = \xi + \sum_{i \geqq 0} u_i \xi^{-i}$, and the $+$ and $-$ stand for the standard projections over the differential and "integral" parts. The KZ-hierarchy [9, 10] is nothing but the classical limit of (5.1), so it reads

$$\frac{\partial \Lambda}{\partial t_n} = \{\Lambda_+^n, \Lambda\} = \{\Lambda, \Lambda_-^n\} . \tag{5.2}$$

From the point of view of the differential equations, the classical limit is such that higher derivative terms are disregarded or, equivalently, fields are taken to be slowly varying in their spatial coordinate.

It is now a simple exercise to show that all these flows for different n commute.

Proposition 5.3. *For all $i, j \in \mathbf{N}$,*

$$\frac{\partial^2 \Lambda}{\partial t_i \partial t_j} = \frac{\partial^2 \Lambda}{\partial t_j \partial t_i} . \tag{5.4}$$

---

[4] This subalgebra is closely related to the symmetric Schouten bracket. We hope to come back to this issue in a future publication

Proof.
$$\begin{aligned}
\frac{\partial^2 \Lambda}{\partial t_i \partial t_j} &= \frac{\partial}{\partial t_i}\{\Lambda^j_+, \Lambda\} \\
&= \left\{\left(\frac{\partial \Lambda^j}{\partial t_i}\right)_+, \Lambda\right\} + \{\Lambda^j_+, \{\Lambda^i_+, \Lambda\}\} \\
&= \{\{\Lambda^i_+, \Lambda^j\}_+, \Lambda\} + \{\Lambda^j_+, \{\Lambda^i_+, \Lambda\}\} \\
&= \{\{\Lambda^i_+, \Lambda^j\}_+, \Lambda\} + \{\Lambda^i_+, \{\Lambda^i_+, \Lambda\}\} + \{\{\Lambda^j_+, \Lambda^i_+\}, \Lambda\} \\
&= \{\{\Lambda^i_+, \Lambda^j_-\}_+, \Lambda\} + \{\Lambda^i_+, \{\Lambda^j_+, \Lambda\}\} \\
&= \{\{\Lambda^i, \Lambda^j_-\}_+, \Lambda\} + \{\Lambda^i_+, \{\Lambda^j_+, \Lambda\}\} \\
&= \{\{\Lambda^j_+, \Lambda^i\}_+, \Lambda\} + \{\Lambda^i_+, \{\Lambda^j_+, \Lambda\}\} \\
&= \frac{\partial^2 \Lambda}{\partial t_j \partial t_i}\,. \qquad \blacksquare
\end{aligned}$$

This is an important result because if these flows are hamiltonian, Proposition 5.3 implies that there are an infinite number of commuting conserved charges, thus proving formal Liouville integrability of the system. In the case of the KZ-hierarchy, these flows are bihamiltonian with respect to w_{KP} and $w_{1+\infty}$ [8].

As the reader has probably already noticed, the key property in the proof of Proposition 5.3 was that there is an invariant splitting with respect to Poisson brackets. But in Sect. 2 it has already been shown that such splitting exist in arbitrary dimension if the definitions are slightly modified. This implies that (5.2) defines a Liouville integrable hierarchy in higher dimensions whenever Λ belongs to M^1_D and the $+$ part stands for the projection over $R^D_{[1,\infty)}$. Then the proof of Proposition 5.3 goes step-by-step to the higher-dimensional case.

It is now trivial to show that

$$\frac{\partial \Lambda}{\partial t_k} = J^1(dH_{k+1}) = J^2(dH_k)\,, \tag{5.5}$$

where $H_k = \frac{1}{k}\operatorname{Tr}\Lambda^k$, and its gradient is given by

$$dH_k = \Lambda^{k-1} \bmod R^D_{-D}\,. \tag{5.6}$$

The H_k are clearly conserved charges for any of the KZ-flows, that is

$$\frac{\partial H_k}{\partial t_j} = \operatorname{Tr}\{\Lambda^j_+, H_k\} = 0 \tag{5.7}$$

because the symplectic trace annihilates Poisson brackets. The proofs of non-triviality and independence of these conserved charges are identical to the usual ones, so we will not repeat them here.

Standard procedures also yield

$$\{H_k, H_j\}^1_{\mathrm{GD}} = \{H_k, H_j\}^2_{\mathrm{GD}} = 0\,, \tag{5.8}$$

for all $k, j \in \mathbf{N}$.

6. Conclusions

We have shown how to construct classical W-algebras in higher dimensions. The standard one dimensional formalism has been shown to extend to higher dimensions with minor modifications and the use of some new tools such as the symplectic trace.

Of course much remains to be done. We have not proved that the connection between these new classical W-algebras and the algebra of generators of local diffeomorphisms goes beyond dimension two, although we firmly believe that it is so. Neither have we studied potentially interesting reductions of these algebras, in particular if we can restrict ourselves to fields falling in finite dimensional representations of the diffeomorphism algebra. It would also be an interesting problem to see the connection between w_{KZ} and the $c = 1$ matrix model formulation of matrix models. The fact that $w_{KZ} \simeq w_{n\to\infty}$ seems to strongly indicate that such a connection indeed exists. It would also be interesting to construct Lagrangian field theories with these new algebras as their algebra of symmetries, and last but not least, to "quantize" them.

We would not like to finish this paper without a word about why we have restricted ourselves to the classical case.

A Comment on Deformations. It would look very natural to try to extend the present formalism to the ring of ΨDO's in arbitrary dimension without restricting ourselves to the classical limit. There is a noncommutative generalization of the symplectic trace due to Wodzicki [12], so we could simply try to substitute our Poisson brackets in the underlying 2D-dimensional symplectic space for commutators. Unfortunately, without such a restriction we rapidly get into trouble, the main problem being that there is no invariant splitting for $D > 1$ with respect to the commutator of ΨDO's, i.e. it is not possible to define a $+$ and $-$ subalgebras closed under commutation and such that $R^D = R^D_+ \oplus R^D_-$. The interested reader can check that this is a required ingredient in the proofs of Lemma 2.43 and Theorem 2.44.

Nevertheless, it is interesting to point out that the hierarchy defined by (5.1) for a ΨDO $\Lambda \in M^1_D$ still possesses an infinite number of conserved charges, which are given by the Wodciki–Trace [12] of the integer powers of Λ. But in this case it is simple to prove that the flows do not commute, as this would necessitate the existence of an invariant splitting. The reader is invited to verify this in the proof of Proposition 5.3. This implies that the hierarchy so defined would not be integrable, at least in the sense of Liouville.

The above discussion suggests that deformations of these structures is a "tricky" business in $D > 1$.

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